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Entropy fluctuations for parabolic maps

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Abstract
We prove log-normal fluctuations and the weak-invariance principle for the convergence to the entropy in the Ornstein–Weiss theorem for a class of parabolic maps of the interval. For such maps, we also compute the Lyapunov exponent using the linear recurrence of the returns of cylinders into themselves.

Mathematics Subject Classification: 37B20, 37D25, 37D50, 60F05, 94A17

1. Introduction

One of the most remarkable applications of the exponential statistics for the first return time in dynamical systems is, as first pointed out in [5] and [12] and successively in [19], the possibility of evaluating the fluctuations in the Ornstein–Weiss computation of metric entropy [18]. We briefly recall this last result. Let us suppose that $\mathcal{C}$ is a finite or countable measurable partition of the measurable dynamical system (X, β, μ, T), where β is the σ-algebra over X, and μ a T-invariant probability ergodic measure, with T a measurable application on X.

Let us denote with $C_n(x)$, the unique element of (the nth join) $\mathcal{C}_n = \bigvee_{i=1}^{n} T^{-(i-1)}\mathcal{C}$, which contains the point $x \in X$, and finally define $R_n(x) = \inf\{k \geqslant 1 : T^k(x) \in C_n(x)\}$. This quantity is sometimes called the n-repetition time of x, since, as in the original paper of Ornstein and Weiss, given an ergodic stationary sequence, it represents the first moment at which the initial n-block of the sample sequence is repeated.

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Ornstein and Weiss proved in [18] that for μ-a.e. $x \in X$ one has:

$$\lim_{n\to\infty} \frac{\log R_n(x)}{n} = h_\mu(T, \mathcal{C}),$$

where $h_\mu(T, \mathcal{C})$ is the metric entropy of the partition $\mathcal{C}$. From now on we will write $h = h(\mu) = h_\mu(T, \mathcal{C})$, when $\mathcal{C}$ is generating. For strongly mixing stationary processes [12, 5], for a large class of non-Markovian maps of the interval [19], for unimodal maps [2] and finally for the class of (ϕ, f) mixing measures introduced in [7], the following fluctuation result has been proved:

$$\mu\left(\left\{x \in X : \frac{\log R_n(x) - nh}{\sigma(\phi)\sqrt{n}} > u\right\}\right) \longrightarrow \frac{1}{\sqrt{2\pi}} \int_u^\infty \mathrm{e}^{-x^2/2}\,\mathrm{d}x, \tag{1}$$

where $\sigma(\phi)$ is the variance of the potential ϕ associated with the equilibrium state $\mu = \mu_\phi$. In the case of (ϕ, f)-mixing measures [7], the variance is given by a limit involving some moments of the information function $-\log \mu(C_n(x))$.

The first result of this paper is to prove a similar result for the class of parabolic maps of the interval introduced in section 2. Our proof relies on a useful result recently proved by Saussol [21]: inspired by the works of Collet *et al* [5] and Paccaut [19], he showed that whenever the Shannon–McMillan–Breiman convergence to the metric entropy exhibits log-normal fluctuations, the same is true for Ornstein–Weiss, provided the first return times are exponentially distributed over cylinders. To be more precise, let us define the error to the asymptotic distribution of the first return times into cylinders as:

$$E_\mu(C_n(x)) = \sup_{t\geqslant 0} |\mu_{C_n(x)}(z : \tau_{C_n(x)}(z)\mu(C_n(x)) > t) - \mathrm{e}^{-t}|,$$

where $\tau_{C_n(x)}(z)$ denotes the first return of the point $z \in C_n(x)$ into the cylinder[5] $C_n(x)$ and $\mu_{C_n(x)}$ is the conditional measure on $C_n(x)$. Suppose that:

(i) $E_\mu(C_n(x)) \to 0$ for μ-almost every x as $\mu(C_n(x)) \to 0$;

(ii) the fluctuations in Shannon–McMillan's theorem are log-normal, i.e.

$$\mu\left(\left\{x \in X : \frac{-\log \mu(C_n(x)) - nh}{\sigma\sqrt{n}} > u\right\}\right) \longrightarrow \frac{1}{\sqrt{2\pi}} \int_u^\infty \mathrm{e}^{-x^2/2}\,\mathrm{d}x,$$

where $0 < \sigma < \infty$.

Then the limit in distribution (1) follows.

For the class of parabolic transformations considered in this paper, the first item, the exponential distribution of the first return time, was proved in [8]. In section 2, we will provide log-normal fluctuations for Shannon–McMillan through a weak-Gibbs characterization of the absolutely continuous invariant measure.

We will moreover establish the weak-invariance principle (WIP) for the process $\log R_n(x)$, which means that the sequence $(\log R_{[nt]} - [nt]h)/\sigma(\phi)\sqrt{n}$ $(t \in [0, 1],\ n \geqslant 1)$, converges in distribution to standard Brownian motion. This follows from the same principle stated for the process $\log \mu(C_n(x))$, which in turn will be a consequence of the WIP for the random variable $\log |DT(x)|$, which is a piecewise Hölder continuous function. We will also present in the appendix an extension of the central limit theorem (CLT) and of the WIP for a large class of non-Hölder functions.

In section 3, we will show how to compute the Lyapunov exponent of the invariant measure by means of the first return of a ball into itself. This technique has been proposed in [22] in the case of maps of the interval with the derivative of p bounded variation and

[5] $\tau_{C_n(x)}(z) = \min\{n > 0 : T^n(z) \in C_n(x); z \in C_n(x)\}$.

successively applied to $C^{1+\alpha}$ diffeomorphisms of surfaces in any dimension [23]. In the latter case, one obtains bounds involving symmetric couples of Lyapunov exponents. This technique relies on the asymptotic behaviour of the first return of a cylinder into itself defined as: $\liminf_{n\to\infty}(\tau_{C_n(x)}/n)$, where $\tau_{C_n(x)} = \inf\{\tau_{C_n(x)}(y) : y \in C_n(x)\}$. It has been proved in [22] that, whenever the metric entropy of the system is positive, the above limit is greater or equal to 1 almost everywhere[6]. This was already proved for the class of maps considered in this paper with a more direct computation [8]. We now improve this result by showing that the limit exists and equals 1 under some conditions and then we apply it for the computation of the Lyapunov exponent.

2. Fluctuations

2.1. Central limit theorem

We now introduce the class of non-uniform maps of the interval for which we will compute the fluctuations of the entropy. For $0 < \alpha < 1$ let us consider the following map of the unit interval:

$$T(x) = \begin{cases} x(1+2^\alpha x^\alpha) & \text{for } x \in [0, \frac{1}{2}], \\ 2x-1 & \text{for } x \in (\frac{1}{2}, 1]. \end{cases}$$

The statistical properties of this transformation have been widely studied in the last few years; see for instance the abundant bibliography listed in [16] and the recent paper [6] which quotes the very latest achievements[7]. This map is the prototype of parabolic behaviour (the derivative is equal to 1 at some fixed point) and it was the first for which an algebraic rate for the decay of correlations was proved. We now recall some properties of it and add new ones. The transformation T has a countable Markov partition ξ generated by the preimages a_n of 1: $\xi = \{A_m : m \in N\}$, with $A_m = (a_{m+1}, a_m]$ and $A_0 = (\frac{1}{2}, 1]$.

We can associate with each point $x \in (0, 1]$ a unique infinite sequence $\omega = \omega_1\omega_2\cdots$ with the property that $T^{m-1}x \in A_{\omega_m}$ for all integer $m \geqslant 1$; the sequence ω satisfies the admissibility condition: $\omega_m\omega_{m+1}$ appears in ω iff $\omega_m = 0$ or $\omega_{m+1} = \omega_m - 1$. A cylinder

$$C_n = \bigcap_{i=1}^{n} T^{-(i-1)}A_{\omega_i} \in \xi_n = \bigvee_{i=1}^{n} T^{-(i-1)}\xi,$$

will be equivalently written in its symbolic representation: $(\omega_1, \omega_2, \ldots, \omega_n)$. We also recall the notion of maximal n-cylinder, whenever the letter $\omega_n = 0$, which means that C_n is sent over I exactly after n iterations. If the cylinder $C_n = (\omega_1 \cdots \omega_n)$ is not maximal, we extend it into the maximal cylinder $C_{n+\omega_n} = (\omega_1 \cdots \omega_n(\omega_n - 1)(\omega_n - 2)\cdots 0) \in \xi_{n+\omega_n}$, which is topologically equal to C_n. The map T preserves an absolutely continuous invariant measure μ whose decreasing density behaves like $\rho(x) \sim x^{-\alpha}$ near the parabolic fixed point 0 (a more precise bound on such a density will be used in a moment; we recall here that the case of σ-finite invariant measures was studied in [4]).

In what follows, we will need a CLT for the logarithm of the derivative of the map. Let us recall that for values of $\alpha \in (0, \frac{1}{2})$ the CLT has been proved in [9, 24] for the set of Hölder

[6] This result implies, in particular, that in the formulation of the Ornstein–Weiss theorem we could take in the definition of $R_n(x)$, $k \geqslant 1$, instead of $k \geqslant n$ as in the original paper [18].

[7] We recall in particular the contributions of Fisher-Lopes, H Hu, S Isola, Liverani–Saussol–Vaienti, M Mori, O Sarig, H Takesaki, M Thaler, L-S Young, M Yuri, S Gouëzel etc.

continuous function ϕ on the unit interval, with the finite variance given by

$$\sigma^2(\phi) = \int \phi^2\,\mathrm{d}\mu - \left(\int \phi\mathrm{d}\mu\right)^2 + 2\sum_{n=1}^{\infty}\left(\int \phi\cdot\phi\circ T^n\,\mathrm{d}\mu - \left(\int \phi\,\mathrm{d}\mu\right)^2\right). \tag{2}$$

The fact that the variance of the process $S_n = \sum_{i=0}^{n-1}\phi(T^i(x))$ grows linearly as $\mathrm{Var}\, S_n = \sigma^2(\phi)n + \mathrm{o}(n)$ is a consequence of the convergence in $L_1(\lambda)$ of the sum $\sum_{n=0}^{\infty} P^n\phi$, where λ denotes the Lebesgue measure over X and P is the Perron–Frobenius operator associated with the potential $-\log|DT(x)|$ [15]. We will come back to this point in section 2.2.

We apply in our case the CLT to the function $\phi = \log|DT(x)|$; it can easily be shown to be piecewise Hölder with exponent α on the two intervals $[0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$, the discontinuity being placed at the point $\frac{1}{2}$. The fact that the discontinuity is located on the boundary of the Markov partition allows us to extend the previous result about the CLT to our piecewise Hölder function, as it has been recently proved in [6] and [10]. Alternatively, we could observe that the function $P\phi$ becomes Hölder continuous with exponent α on the whole interval and this is sufficient to get the CLT for the function ϕ. We will moreover assume that ϕ is not a coboundary, which implies that $\sigma(\phi) > 0$ [8].

Technically, it is advantageous to work with cylinders which become maximal with a prescribed rank. For this purpose define

$$I_{n,\gamma} = \{(\omega_1\cdots\omega_n) : \omega_n > [n^\gamma]\},$$

where $0 < \gamma < 1$ will be determined later, and denote by $\overline{I_{n,\gamma}}$ the complementary set.

Lemma 2.1. *There exists a constant c_1 depending only on the map T for which $\mu(\overline{I_{n,\gamma}}) \geqslant 1 - c_1[n^\gamma]^{1-(1/\alpha)}$.*

Proof. Since $(\omega_1, \ldots, \omega_n) \in T^{-(n-1)}A_{\omega_n}$, we have $\mu(I_{n,\gamma}) \leqslant \sum_{i=[n^\gamma]}^{\infty}\mu(A_i)$, where $A_i = (a_{i+1}, a_i]$. Since the density ρ is bounded by: $\rho(x) \leqslant ax^{-\alpha}$ and $a_i \leqslant ci^{-1/\alpha}$, where a and c are constants independent of x [16], we get:

$$\mu(I_{n,\gamma}) \leqslant \int_0^{a_{[n^\gamma]}} \rho(x)\,\mathrm{d}x \leqslant \int_0^{c[n^\gamma]^{-1/\alpha}} ax^{-\alpha}\,\mathrm{d}x \leqslant c_1[n^\gamma]^{1-(1/\alpha)},$$

where c_1 is a constant independent of n and dependent only on c and a. Therefore, $\overline{I_{n,\gamma}} = \{(\omega_1\ldots\omega_n) : \omega_n \leqslant [n^\gamma]\}$, has measure: $\mu(\overline{I_{n,\gamma}}) \geqslant 1 - c_1[n^\gamma]^{1-(1/\alpha)}$. □

Lemma 2.2. *Let C be a maximal cylinder of the partition ξ_n, then there exist two constants $c_3 > c_4$ depending only on the map T, such that:*

$$c_4\frac{1}{|DT^n(y)|} \leqslant \mu(C) \leqslant c_3\frac{n+1}{|DT^n(y)|} \qquad \forall\, y \in C.$$

Proof. We have, for $C \in \xi_n$:

$$\mu(C) = \int_0^1 \rho\chi_C\,\mathrm{d}x = \int_0^1 \rho(T_C^{-n}(y))\frac{1}{|DT^n(T_C^{-n}(y))|}\,\mathrm{d}y,$$

where $T_C^{-n} : [0, 1] \to C$ and χ_C denotes the characteristic function of the set C. Observe that the cylinder $C \subset A_{\omega_1}$ and the biggest value of ω_1 compatible with the maximality condition and for which A_{ω_1} is closest to the neutral fixed point is $\omega_1 = n - 1$. In this case, $C \subset (a_n, a_{n-1}]$ and we will need an upper bound for ρ on such an interval.

[8] This (standard) assumption seems very reasonable. For example, the variance is zero for the full quadratic map $x \to 4x(1 - x)$, which is differentiable conjugate to the full tent-map $x \to 1 - 2|x|$: in this case the cylinders are too regular and the fluctuations are no longer normal, but they converge to a finite mixture of exponential times [5].

Therefore: $D^{-1}\inf\rho \leqslant \mu(C)|DT^n(y)| \leqslant D\rho(a_n)$, for all $y \in C$, where D is the distortion constant given in proposition 3.3 [16]. Since $\inf\rho > 0$ and $\rho(x) \leqslant ax^{-\alpha}$, we obtain: $D^{-1}\inf\rho \leqslant \mu(C)|DT^n(y)| \leqslant Daa_n^{-\alpha}$. But in a way similar to the proof of lemma 3.2 in [16][9], it is easy to see that there exists a constant c_5 such that $a_n^\alpha \geqslant c_5/(n+1)$, which concludes the proof of lemma 2. □

The next corollary is an immediate consequence of the preceding lemma.

Corollary 2.1. *If $C \in \xi_n \cap \overline{I_{n,\gamma}}$, then:*

$$c_4\frac{1}{2^{n^\gamma}} \leqslant \frac{\mu(C)}{|DT^n(y)|^{-1}} \leqslant c_3(n+n^\gamma+1), \qquad y \in C.$$

Proof. If $C \in C_n \cap \overline{I_{n,\gamma}}$, it can be viewed as a maximal cylinder of the partition $C_{n+\omega_n}$. By the preceding lemma we have:

$$c_4\frac{1}{|DT^{n+\omega_n}(y)|} \leqslant \mu(C) \leqslant c_3\frac{n+\omega_n+1}{|DT^{n+\omega_n}(y)} \qquad \forall\, y \in C.$$

Using the factorization $DT^{n+\omega_n}(y) = DT^{\omega_n}(T^n y)DT^n(y)$ and the fact that $|DT^{\omega_n}(T^n y)| \leqslant \sup_{x\in A_0}|DT(x)|^{\omega_n} \leqslant 2^{n^\gamma}$, we get immediately the result. □

Remark 2.1. What we actually proved is a sort of weak Gibbs property for the measure μ. It will be clear in a moment that in order to use the CLT for the potential $-\log|DT(x)|$, we will need to control the quantity $(1/\sqrt{n})\log 2^{n^\gamma}$. It reduces to zero in the limit of large n provided $\gamma < \frac{1}{2}$.

Remark 2.2. In what follows we will study the fluctuations of the two processes $\log\mu(C_n(x))$ and $\log R_n(x)$, with respect to the probability invariant measure μ. These two processes are defined with respect to the partition ξ, which means that $C_n \in \xi_n$.

We are now ready to prove the main result of this section.

Theorem 2.1. *For the parabolic map T and $\alpha \in (0, \frac{1}{2})$, the process $\log R_n(x)$ satisfies the convergence in law (1).*

Proof. As we said in the introduction it will be sufficient to prove the log-normal fluctuations for Shannon–McMillan's theorem that is equivalent to show that:

$$\mu\left(\left\{x : \mu(C_n(x)) < \mathrm{e}^{-nh-\sigma(\phi)u\sqrt{n}}\right\}\right) \to \frac{1}{\sqrt{2\pi}}\int_u^\infty \mathrm{e}^{-x^2/2}\,\mathrm{d}x.$$

By lemma 2.1:

$$\begin{aligned}&\left|\mu\left(\left\{x : \mu(C_n(x)) < \mathrm{e}^{-nh-\sigma(\phi)u\sqrt{n}}\right\}\right) - \sum\left\{\mu(C_n) : C_n \in \overline{I_{n,\gamma}}, \mu(C_n) < \mathrm{e}^{-nh-\sigma(\phi)u\sqrt{n}}\right\}\right|\\ &\qquad \leqslant \sum_{C_n\in I_{n,\gamma}}\mu(C_n)\\ &\qquad \leqslant c_1[n^\gamma]^{1-(1/\alpha)},\end{aligned}$$

[9] The proof works out the same induction argument as in [16]; this lower bound has already been used in [21] too.

so that it will be sufficient in the following to restrict ourselves to the cylinder in $\overline{I_{n,\gamma}}$. Corollary 2.1 implies that:

$$\begin{aligned}
\mu\left(\left\{x : \mu(C_n(x)) < \mathrm{e}^{-nh-\sigma(\phi)u\sqrt{n}}\right\}\right) \\
&\geqslant \mu\left(\left\{x : c_3(n+n^\gamma+1)|DT^n(x)|^{-1} < \mathrm{e}^{-nh-\sigma(\phi)u\sqrt{n}}\right\}\right) \\
&\geqslant \mu\left(\left\{x : \frac{\sum_{i=0}^{n-1}\log|DT(T^i(x))| - nh}{\sigma(\phi)\sqrt{n}} > u + \frac{\log c_3 + \log(n+n^\gamma+1)}{\sigma(\phi)\sqrt{n}}\right\}\right) \\
&\geqslant \mu\left(\left\{x : \frac{\sum_{i=0}^{n-1}\log|DT(T^i(x))| - nh}{\sigma(\phi)\sqrt{n}} > u + \delta\right\}\right),
\end{aligned}$$

where δ is any positive number bigger than $(\log c_3 + \log(n+n^\gamma+1))/\sigma(\phi)\sqrt{n}$ for n sufficiently large. The CLT for the function[10] $\log|DT(x)|$ guarantees that:

$$\liminf_{n\to\infty} \mu\left(\left\{x : \mu(C_n(x)) < \mathrm{e}^{-nh-\sigma(\phi)u\sqrt{n}}\right\}\right) \geqslant \frac{1}{\sqrt{2\pi}}\int_{u+\delta}^{\infty} \mathrm{e}^{-x^2/2}\,\mathrm{d}x,$$

which gives the desired result for the lower bound when δ goes to zero. To get a similar result for the upper bound we proceed as above and we find easily:

$$\begin{aligned}
\mu\left(\left\{x : \mu(C_n(x)) < \mathrm{e}^{-nh-\sigma(\phi)u\sqrt{n}}\right\}\right) \\
&\leqslant \mu\left(\left\{x : c_4 2^{-n^\gamma}|DT^n(x)|^{-1} < \mathrm{e}^{-nh-\sigma(\phi)u\sqrt{n}}\right\}\right) \\
&\leqslant \mu\left(\left\{x : \frac{\sum_{i=0}^{n-1}\log|DT(T^i(x))| - nh}{\sigma(\phi)\sqrt{n}} > u + \frac{\log c_4 - n^\gamma\log 2}{\sigma(\phi)\sqrt{n}}\right\}\right) \\
&\leqslant \mu\left(\left\{x : \frac{\sum_{i=0}^{n-1}\log|DT(T^i(x))| - nh}{\sigma(\phi)\sqrt{n}} > u - \delta'\right\}\right)
\end{aligned}$$

provided that for any $\delta' > 0$ one has

$$\left|\frac{\log c_4 - n^\gamma\log 2}{\sigma(\phi)\sqrt{n}}\right| < \delta',$$

which is possible by the remark above. By taking the lim sup on both sides, using again the CLT for $\log|DT(x)|$ and by sending finally δ' to zero, we get the desired upper bound. □

2.2. *Invariance principle*

The CLT could be improved to get what is called the weak invariance principle. Such a principle has been obtained for the piecewise version of our map, and for a large class of observables, by Chernov [3]. After our paper was finished, we discovered a very recent article by Pollicott and Sharp [20], where they proved the WIP for the nonlinear map T in the case of Hölder functions and in the range $0 < \alpha < \frac{1}{3}$. We provide here a proof for the function $\log|DT(x)|$ which is piecewise Hölder; other generalizations to non-Hölder functions will be given in theorem 2.3 and in the appendix (see also remark 2.3). As we will see at the end of this section, the WIP

[10] Notice that $(1/n)\log|DT^n(x)|$ goes, when $n \to \infty$ and μ-a.e., to the μ-Lyapunov exponent which in our case coincides with h [14].

for the function $\log |DT(x)|$, namely for the random variable $(\log |DT^n(x)| - nh)/\sigma(\phi)\sqrt{n}$, allows us to translate it to the random variable $(-\log \mu(C_n(x)) - nh)/\sigma(\phi)\sqrt{n}$, and therefore to $(\log R_n(x) - nh)/\sigma(\phi)\sqrt{n}$. Let us first recall what the WIP means applied, for instance, to the process $\log R_n$.

For each $x \in [0, 1]$ we construct the random variable $W_{n,x}(t)$ for $t \in [0, 1]$ as: $W_{n,x}(k/n) = (\log R_k(x) - nh)/\sigma(\phi)\sqrt{n}$ for $k = 0, 1, \ldots, n$ and it extends linearly on each of the subintervals $[k/n, (k+1)/n]$. For each x, $W_{n,x}$ is therefore an element of the space $\mathcal{I}$ of the continuous function on [0, 1] topologized with the supremum norm. If we denote with D_n the distribution of $W_{n,x}$ on $\mathcal{I}$, namely

$$D_n(H) = \mu(\{x : W_{n,x} \in H\})$$

where H is a Borel subset of $\mathcal{I}$, then the WIP asserts that the distribution D_n converges weakly to the Wiener measure. This means that $\log R_n(x) - nh$ is for large n, and after a suitable normalization, distributed approximately as the position at time $t = 1$ of a particle in Brownian motion [1].

We begin to prove the WIP for the function $\log |DT(x)|$; in this regard, we adapt to our case theorem 1.4 in [3], which gives sufficient conditions to get the WIP for $L_2(\mu)$ functions ϕ with positive and finite variance $\sigma(\phi)$. Note that in our case $\phi = \log |DT(x)|$ is a piecewise Hölder continuous function with exponent α. A basic assumption in Chernov's theory is that the first moment of the autocorrelation function is finite:

$$\sum_{n=1}^{\infty} n \left| \int \phi \cdot \phi \circ T^n \, \mathrm{d}\mu - \left(\int \phi \, \mathrm{d}\mu \right)^2 \right| < \infty. \tag{3}$$

This guarantees the asymptotic linearity of $\operatorname{Var} S_n = \sigma^2(\phi)n + \mathrm{o}(n)$, where $S_n = \sum_{i=0}^{n-1} \phi(T^i(x))$ and $\sigma^2(\phi)$ is given by formula (2). In our case, we already have this asymptotic behaviour for $\operatorname{Var} S_n$, as pointed out in section 2.1 (i.e. we do not have to check the assumption above).

Let us now assume that for any $N \geqslant 1$ we can find a partition $\mathcal{A} \equiv \mathcal{A}^{(N)}$ of X such that:

- $\|\phi - \mathbb{E}(\phi|\mathcal{A})\|_{L_2(\mu)} = \mathrm{o}(N^{-1})$.
  In [3], it was shown that in order to prove this condition it is sufficient to verify that: $\mathcal{H}_F(\operatorname{diam} \mathcal{A}) = \mathrm{o}(N^{-1})$, where
  $$\mathcal{H}_F(d) = \sup_{\operatorname{diam}\mathcal{A} \leqslant d} \|\phi - \mathbb{E}(\phi|\mathcal{A})\|_{L_2(\mu)}.$$
  For Hölder continuous functions of exponent α, or for piecewise Hölder continuous functions with the discontinuities located on the border of the Markov partition, $\mathcal{H}_F(d) \leqslant \mathrm{const}\, d^{\alpha}$ and therefore we will have simply to verify that: $\operatorname{diam} \mathcal{A} = \mathrm{o}(N^{-1/\alpha})$
- $\mathcal{L}_F(n/N) = \mathrm{o}(1/n)$,
  where $\mathcal{L}_F(d) = \sup_B \int_B (\phi - \mathbb{E}(\phi))^2 \, \mathrm{d}\mu(x)$, and the supremum is over all measurable subsets $B \subset X$ such that $\mu(B) \leqslant d$. When $\phi \in L_\infty(\mu)$, $\mathcal{L}_F(d) \leqslant \mathrm{const}\, d$, then the above assumption reduces to $n/N = \mathrm{o}(1/n)$.[11]
- Assume that there exists an integer valued function $n = n(N) = \mathrm{o}(N)$ such that $n \to \infty$ when $N \to \infty$ satisfying the condition:
  $$\beta_N(n) = \mathrm{o}\left(\frac{n}{N}\right), \tag{4}$$

[11] This condition explains why in [3] n is chosen as $n = [N^{1/2} \log^{-\epsilon} N]$, $\epsilon > 0$. In the following we instead choose $n = N^z$, $0 < z < 1$, which forces z to be smaller than $\frac{1}{2}$ in order to satisfy the assumption of the item. The preceding weaker choice will not improve our final result.

where $\beta_N(n)$ is defined by:

$$\beta_N(n) = \max_{0\leqslant k\leqslant N-n-1} \sum_i \sum_j |\mu(B_i \cap D_j) - \mu(B_i)\mu(D_j)|, \tag{5}$$

where $B_i \in \mathcal{A}_{k+1}$ and $D_j \in T^{-(k+n)}\mathcal{A}_{N-k-n}$ ($\mathcal{A}_k = \bigvee_{i=1}^{k} T^{-(i-1)}\mathcal{A}$ is the kth join of $\mathcal{A}$). Whenever the three items above are satisfied, the function ϕ verifies the WIP.

We will construct the partition $\mathcal{A}$ in two steps following the strategy in [3] for the piecewise linear version of T, which was based on a Markov-like approximation introduced in [13]. We first consider the family $\mathcal{F}_{n_1}$ of cylinders in $\xi_{n_1} = \bigvee_{i=1}^{n_1} T^{-(i-1)}\xi$ satisfying the condition:

$$C \in \mathcal{F}_{n_1} \Longleftrightarrow C = (\omega_1, \ldots \omega_{n_1} : \omega_i \leqslant n_2,\ i = 1, \ldots, n_1),$$

where $n_2 = \mathrm{o}(n_1)$ and $n_1 = \mathrm{o}(n)$ will be determined later. Since:

$$\mathrm{diam}(\mathcal{F}_{n_1}) \equiv \sup_{C\in\mathcal{F}_{n_1}} (\mathrm{diam}\ C) \leqslant (\min DT_{|A_{n_2}})^{-n_1} \leqslant (1 + 2\alpha(\alpha+1)a_{n_2+1})^{-n_1},$$

we can use the lower bound on a_n provided in section 2 to get

$$\mathrm{diam}(\mathcal{F}_{n_1}) = \mathrm{O}(\mathrm{e}^{-\tilde{C}n_1/n_2}),$$

where the constant $\tilde{C}$ is independent of n. We then take the cylinders in ξ_{n_1} which are not in $\mathcal{F}_{n_1}$ (we call this family $\mathcal{F}^c_{n_1}$) and cut them into smaller cylinders of diameter $\leqslant \mathrm{diam}(\mathcal{F}_{n_1})$. The union of these cylinders, which we call $\mathcal{G}_{n_1}$, with those of $\mathcal{F}_{n_1}$ forms our initial partition $\mathcal{A}$: $\mathcal{A} = \mathcal{F}_{n_1} \cup \mathcal{G}_{n_1}$. Let us now treat the quantity $\beta_N(n)$ in the third item above. Note that the sum in (5) can be estimated as follows:

$$\sum_{i,j} |\mu(B_i \cap D_j) - \mu(B_i)\mu(D_j)|$$
$$\leqslant \sum_i (|\mu(B_i \cap D_i^+) - \mu(B_i)\mu(D_i^+)| + |\mu(B_i \cap D_i^-) - \mu(B_i)\mu(D_i^-)|), \tag{6}$$

where D_i^+ is the union of those elements D_j for which $\mu(B_i \cap D_j) > \mu(B_i)\mu(D_j)$ and, similarly, $D_i^- = \bigcup\{D_j : \mu(B_i \cap D_j) \leqslant \mu(B_i)\mu(D_j)\}$. This decomposition allows us to bound the left-hand side of (5) by summing four times over the measures of the D_i. Let us now consider the family $\mathcal{G}(n,k)$ of all the cylinders $B_i \in \mathcal{A}_{k+1}$ of the form:

$$B_i = B_{i_1} \cap T^{-1}B_{i_2} \cap \cdots \cap T^{-k}B_{i_{k+1}},$$

where at least one of the B_{i_l}, $l = 1, \ldots, k+1$, belongs to $\mathcal{G}_{n_1}$. We have:

$$\begin{aligned}\mu(\mathcal{G}(n,k)) &\leqslant \sum_{l=1}^{k+1} \sum_{C\in\mathcal{F}^c_{n_1}} \sum_{B\subset C} \mu(T^{-l}B)\\ &\leqslant (k+1) \sum_{C\in\mathcal{F}^c_{n_1}} \mu(C)\\ &\leqslant (k+1)n_1 \sum_{i=n_2}^{\infty} \mu(A_i),\end{aligned}$$

where $A_i = (a_{i+1}, a_i]$. A computation similar to that in lemma 2.1 gives $\mu(\mathcal{G}(n,k)) \leqslant (k+1)n_1c_1n_2^{1-1/\alpha}$. Altogether this term will give a contribution to (5) of order:

$$\mathrm{O}(Nn_1n_2^{1-1/\alpha}). \tag{7}$$

Let us now consider in the first sum defining (5) all the cylinders which are obtained by taking the pull-back of elements in $\mathcal{F}_{n_1}$. These cylinders belong to the partition ξ_{n_1+k} while the second sum in (5) is taken over $T^{-(k+n)}\mathcal{A}_{N-k-n}$. We observe that the sum $\beta_N(n)$ is exactly what defines the speed of weak-Bernoullicity for the two partitions; we can therefore follow straightforwardly the proof of theorem 3.3 in [8] (see also [21] for more details) to get:

$$\sum_{B\in\xi_{n_1+k}} \sum_{D\in T^{-(k+n)}\mathcal{A}_{N-k-n}} |\mu(B\cap D)-\mu(B)\mu(D)| \leqslant 2 \sum_{B\in\xi_{n_1+k}} \|P^{k+n}((\chi_B-\mu(B))\rho\|_{L_1(\lambda)},$$

where P is the Perron–Frobenius operator associated with the potential $-\log|DT(x)|$. Note that this bound depends, after the application of (6), only on the power of the pull-back $T^{-(k+n)}$ and not on the length of the cylinders in $\mathcal{A}_{N-k-n}$. We then continue as in [8] by splitting ξ_{n_1+k} into two families of cylinders: those $M(k,n_1,n)$ for which $B\in M(k,n_1,n)$ becomes maximal ($B\in\xi_{p_B}$) for $n_1+k\leqslant p_B<k+n/2$, and the complementary set $M(k,n_1,n)^c$. For $B\in M(k,n_1,n)$, the Perron–Frobenius operator factorizes as $P^{k+n}=P^{k+n-p_B}P^{p_B}$ and the function $P^{p_B}(\rho\chi_B)$ will belong to the right cone[12] upon which the powers of P act with the following (up to a logarithmic correction) polynomial decay established in [16]:[13]

$$\|P^{k+n-p_B}(P^{p_B}(\rho\chi_B)-\mu(B))\|_{L_1(\lambda)} \leqslant \mu(B)\mathrm{O}_L((k+n-p_B)^{1-1/\alpha}) = \mu(B)\mathrm{O}_L(n^{1-1/\alpha}).$$

The cylinders in $M(k,n_1,n)^c$ sum up to the set $T^{-(k+n_1)+1}[0,a_{n/2-n_1}]$ whose measure can be computed as in lemma 2.1 giving: $\mu((M(k,n_1,n)^c)=\mathrm{O}(n^{1-1/\alpha})$, remembering that $n_1=\mathrm{o}(n)$. In conclusion we obtain:

$$\beta_N(n)=\mathrm{O}_L((n^{1-1/\alpha})+\mathrm{O}(Nn_1n_2^{1-1/\alpha}))=\mathrm{O}(Nn_1n_2^{1-1/\alpha}). \tag{8}$$

We now define the various integers according to the rules:

$$n=N^z, \qquad n_1=n^z, \qquad n_2=n_1^z, \qquad 0<z<1. \tag{9}$$

The assumptions $\beta_N(n)=\mathrm{O}(Nn_1n_2^{1-1/\alpha})=\mathrm{o}(n/N)$ and $\operatorname{diam}\mathcal{A}=\mathrm{O}(\mathrm{e}^{-\tilde{C}n_1/n_2})=\mathrm{o}(N^{-1/\alpha})$ are verified for $\alpha<(z^3/z^3+z^2-z+2)$ which allows us to get $0<\alpha<\frac{1}{3}$ sending $z\to 1$. But the last condition $n/N=\mathrm{o}(1/n)$ imposes that $z<\frac{1}{2}$, as we said in the footnote (11), so that we finally get the following theorem.

Theorem 2.2. *For $0<\alpha<\frac{1}{15}$, the weak invariance principle holds for the function* $\log DT(x)$*, or equivalently for the process* $\log DT^n(x)$.

Remark 2.3. The three assumptions quoted in the items in the preceding section are sufficient conditions to prove the WIP for *any* Hölder continuous function over X or for a piecewise Hölder continuous function with discontinuities on the borders of ξ . Indeed the Hölder exponent enters only in the negative power of $N^{-1/\alpha}$, which surely dominates the subexponential decay of the diameter of $\mathcal{A}$. We can then state the following general theorem.

Theorem 2.3. *Let F be an Hölder continuous function on the unit interval X (or piecewise Hölder with discontinuities on the borders of ξ), for which the variance $\sigma(F)>0$. Then for $0<\alpha<\frac{1}{15}$, the process $\sum_{i=0}^{n-1}F(T^ix)$ verifies the WIP.*

Remark 2.4. It is not impossible that our proof could be improved in order to get the WIP in the same interval as the CLT, namely for $0<\alpha<\frac{1}{2}$. A first step in this direction, but with a different technique, has been done in the already quoted paper [20] for Hölder continous functions, where the range of the parameter α was pushed to $\frac{1}{3}$. Our proof covers the more general case of piecewise Hölder functions (with discontinuities on the borders of ξ); moreover, we will show in the appendix how to improve this result for a larger class of non-Hölder continuous function, even to compute the CLT.

[12] This is one of the main reasons to introduce the notion of maximal cylinder.
[13] The symbol O_L means: $\mathrm{O}_L(\epsilon)=O(\epsilon(\log\epsilon^{-1})^r)$ in the limit $\epsilon\to 0$, for any constant r.

We now show how to apply theorem 2.2 to prove the WIP for the two processes $-\log\mu(C_n(x))$ and $\log R_n(x)$, still with respect to the partition ξ_n. As far as we know, the WIP for the first return time has been proved up to now (Kontoyiannis [12]) only in the case of finite-valued stationary strongly mixing processes with some sort of finite-order Markov chain approximation (the assumption on the coefficient 'γ' introduced by Ibragimov [11], see also section 1.1 in [12]). The mixing properties of our map are much weaker (it satisfies a property close to the α-mixing condition, see [8], lemma 3.1), as a consequence of its lack of uniform hyperbolicity.

Theorem 2.4. *The WIP holds for the process* $-\log\mu(C_n(x))$ *provided* $\alpha < \frac{1}{15}$.

Proof. According to theorem 4.1 in Billingsley [1] it will be enough to prove that:

$$\mu\left(x;\max_{l\leqslant n}|\frac{-\log\mu(C_l(x))-\log|DT^l(x)|}{\sigma\sqrt{n}}|\geqslant\epsilon\right)\to 0, \tag{10}$$

when n goes to infinity and ϵ being any positive number (from now on we simply write $\sigma=\sigma(\phi)$). Let us consider the family $\mathcal{C}_{l,n}$ of all the cylinders belonging to the partitions ξ_l, $1\leqslant l\leqslant n$ of the form: $C_l\in\mathcal{C}_{l,n}\Leftrightarrow C_l=(\omega_1,\ldots,\omega_l)$, with $\omega_l>n^\gamma$, $\forall l=1,\ldots,n$. By an argument already used in the proof of lemma 2.1, we easily get that $\mu(\mathcal{C}_{l,n})\leqslant 1/(n^{\gamma(1/\alpha-1)-1})$, which goes to 0 when n goes to infinity provided $\alpha<\frac{1}{3}$. Then it will be sufficient to consider in the left hand side of (10) only those x which are in the complement of $\mathcal{C}_{l,n}$. For such points and by using corollary 2.1 we have, for $1\leqslant l\leqslant n$:

$$\frac{c_4}{2^{[n^\gamma]}|DT^l(x)|}\leqslant\mu(C_l(x))\leqslant\frac{c_3(l+1+n^\gamma)}{|DT^l(x))}, \tag{11}$$

which implies that

$$\left|\frac{-\log\mu(C_l(x))-\log|DT^l(x)|}{\sigma\sqrt{n}}\right|\leqslant\left|\frac{c_4+n^\gamma\log 2}{\sigma\sqrt{n}}\right|$$

for n large and this gives us the desired result since $\gamma<\frac{1}{2}$. □

Theorem 2.5. *For* $0<\alpha<\frac{1}{15}$, *the weak invariance principle holds for the process* $\log R_n(x)$.

Proof. The proof uses again the criterion (10), where we compare this time the two processes $\log R_n(x)$ and $-\log\mu(C_n(x))$ and it is a consequence, by standard measure theoretical arguments, of the following result which is of independent interest[14]. □

Theorem 2.6. *For* $\alpha<\frac{1}{5}$ *and for any* $\beta>0$ *we have for* μ*-almost every* x

$$\lim_{n\to\infty}\frac{\log[R_n(x)\mu(C_n(x)]}{n^\beta}=0.$$

[14] Let us sketch this argument. Take a large subset $X_{\epsilon'}\subset[0,1]$ of measure $>(1-\epsilon')$ where the limit (2.6) is uniform. Then for n bigger than a certain $n_{\epsilon'}$, we have: $|(-\log\mu(C_l(x))-\log R_l(x))/\sigma\sqrt{n}|<\epsilon'$ for all $x\in X_{\epsilon'}$ and for all $n_{\epsilon'}<l<n$, and this part vanishes in (10) when ϵ' goes to 0. The other contribution: $\mu(x\in X_{\epsilon'};\max_{l\leqslant n_{\epsilon'}}|(-\log\mu(C_l(x))-\log R_l(x))/\sigma\sqrt{n}|\geqslant\epsilon)$ goes to 0 by Chebyshev's inequality when $n\to\infty$.

Proof. The proof follows if we could show that eventually for μ-almost every x:

(i) $\log[R_n(x)\mu(C_n(x)] \geqslant -r(n)$,
(ii) $\log[R_n(x)\mu(C_n(x)] \leqslant r(n)$,

where $r(n)$ is an arbitrary sequence of non-negative constants such that $\sum e^{-r(n)} < \infty$.[15] We will prove in detail the point (i), the other follows in the same way after the inspection of equation (12). We note that the analogues of (i) and (ii), and consequently the limit (2.6), have been proved by Kontoyiannis [12] in the context of the finite-valued stationary strongly mixing processes with the finite-order Markov chain approximation quoted above. We use here a completely different approach based on a fine analysis of the statistics of the first return time. This will allow us to weaken the hypothesis on the constants $r(n)$ which were taken in [12] as $\sum n e^{-r(n)} < \infty$. If we introduce the measurable set $Z_n = \{x; \log[R_n(x)\mu(C_n(x)] \leqslant -r(n)\}$, the point (i) holds whenever $\sum \mu(Z_n) < \infty$, by the Borel–Cantelli lemma. By introducing the conditional measure $\mu_A(B) = \mu(A \cap B)/\mu(A)$, for measurable sets A and B, and by summing over the cylinders $C_n \in \xi_n$, we can write

$$\mu(Z_n) = \sum_{C_n} \mu(C_n)\mu_{C_n}(x; R_n(x)\mu(C_n) \leqslant e^{-r(n)}).$$

We said in the introduction that for our map the distribution $\mu_{C_n}(x; R_n(x)\mu(C_n) \leqslant t)$ converges, when n goes to infinity and for cylinders around almost all points, to $1 - e^{-t}$. What we need now is the rate of convergence for a wide class of cylinders. We first observe that $\mu(Z_n)$ can be bounded as:

$$\mu(Z_n) \leqslant \sum_{C_n} \mu(C_n) \sup_{t\geqslant 0} |\mu_{C_n}(x; R_n(x)\mu(C_n) > t) - e^{-t}| + \sum_{C_n} \mu(C_n)(1 - e^{-e^{-r(n)}}). \tag{12}$$

The second sum in (12) is clearly summable in n and we now handle the first sum. We begin by restricting this sum over the family $C'_{s,n}$ of cylinders in ξ_n satisfying:

$$C_n = (\omega_1, \ldots, \omega_n) \in C'_{s,n} \quad \Leftrightarrow \quad \omega_i \leqslant n^{\gamma}, \qquad i = 1, \ldots, n.$$

The contribution to $\sum_{C_n}$ of the cylinders just discarded is of order (see section 2.2) $1/n^{\gamma(1/\alpha-1)-1}$, which is summable for $\alpha < \frac{1}{5}$, since $\gamma < \frac{1}{2}$ (in the proof of this theorem we do not require $\gamma < \frac{1}{2}$ from the very beginning; this condition will be necessary at the end of the proof).

We now recall a general bound for the statistics of the first return time proved in [8] for any measure preserving transformation on a probability space. If we denote with $R_U(x)$ the first return into the measurable set U, then ([8], theorem 2.1):

$$\sup_{t\geqslant 0} |\mu_U(x; R_U(x)\mu(U) > t) - e^{-t}| \leqslant d(U),$$

where

$$d(U) = 4\mu(U) + c(U)(1 + \log c(U)^{-1})$$

and

$$c(U) \leqslant \inf\{a_M(U) + b_M(U) + M\mu(U) | M \text{ integer}\},$$

being

$$a_M(U) = \mu_U(x; R_U(x) \leqslant M)$$

[15] Put $r(n) = \nu n^{\beta}$, take the limits and then send ν to zero.

and

$$b_M(U) = \sup\{|\mu_U(T^{-M}V) - \mu(V)|; V \text{ measurable}\}.$$

If we now call, as in the introduction, τ_U the first return of the set U into itself, then it has been proved for our map T that ([8], lemma 3.5):

$$a_M(C_n) \leqslant \frac{4D}{\inf \rho} \frac{M\mu(C_n)}{\lambda(T^{\tau_{C_n}} C_n)}, \tag{13}$$

where C_n is any cylinder in ξ_n, D is the distortion constant already used in section 2.1, and λ denotes the Lebesgue measure on the unit interval. Our next approximation will consist in keeping in $C'_{s,n}$ only those cylinders for which $\tau_{C_n} > [n/2]$: let us call $C''_{s,n}$ this family. We now bound from above the measure of $(C''_{s,n})^c$ and we show that it is summable; at this regard we will follow straightforwardly the proof of the point (1) of proposition 3.7 in [8]. This proof gives the summability of the measure of all cylinders $C_n \in \xi_n$ belonging to the interval $A_0 = (\frac{1}{2}, 1]$ and for which $\tau_{C_n} \leqslant [n/2]$. This measure is of order $1/n^{1/\alpha}$. In our situation the cylinders in $C'_{s,n}$ will be at a distance bigger than $a_{[n^\gamma]}$ from the neutral fixed point. This will introduce two differences with respect to the proof of [8]: first we need to bound from below the derivative $DT^k(x)$, with $x \in A_{[n^\gamma]}$ (and k smaller or equal than a certain constant k_0), instead of $x \in A_0$. Second, we have to introduce a factor $\mathrm{O}(n^\gamma)$ in order to replace the measure μ with the Lebesgue measure λ; that factor is an upper bound for the density ρ up to $a_{[n^\gamma]}$. These two facts are related to some bounds on the Perron–Frobenius operator, and we defer to the quoted paper for the details. Taking into account these slight modifications we get that the measure of $(C''_{s,n})^c$ is of order $\mathrm{O}(1/n^{(1/\alpha)-\gamma})$, which is summable for $\alpha < \frac{2}{3}$. We are now ready to bound $a_M(C_n)$ for cylinders $C_n \in C'_{s,n} \cap C''_{s,n}$. We first observe that $\lambda(T^{\tau_{C_n}} C_n) \geqslant \lambda(T^{[n/2]} C_n)$; then, by using the weak-Gibbs bounds (11) and corollary 2.1 and the upper bound on the density ρ on $C'_{s,n}$, and by neglecting the algebraic powers of n (which will be of lower order), we check easily that the ratio $\mu(C_n)/\lambda(T^{\tau_{C_n}} C_n)$ is of order

$$\frac{2^{n^\gamma}}{\inf_{A_{[n^\gamma]}} DT^{[n/2]}} = \mathrm{O}\left(\frac{2^{n^\gamma}}{(1 + c_5 1/n^\gamma)^{[n/2]}}\right),$$

where c_5 is a constant dependent on T. If we now chose $M = (1 + c_5(1/n^{1-\gamma-\psi}))^{[n/2]}$, where $0 < \psi < 1 - \gamma$, we get that $a_M(C_n)$ goes exponentially fast to zero provided that $\gamma < \frac{1}{2}$. We then recall that for maximal cylinders C_n the quantity $b_M(C_n)$ is bounded by $\mathrm{O}_L((M-n)^{1-1/\alpha})$ ([8], lemma 3.1). The maximal cylinders in the family $C'_{s,n}$ are at most of order $n + n^\gamma$. For the preceding choice of M, $b_M(C_n)$ and $M\mu(C_n)$ go exponentially fast to zero and therefore $\sum_{C_n \in C'_{s,n} \cap C''_{s,n}} \mu(C_n) d(C_n)$ is summable in n. We have thus showed that $\sum \mu(Z_n) < \infty$, which implies the point (i) stated at the beginning of the proof. Note that the range of values of α for which this theorem holds is larger than those of theorems 2.4 and 2.5. □

3. Lyapunov exponent

We show in this section how to compute the Lyapunov exponent of the ergodic measure μ by using recurrence of balls. Let us define the first return of a ball $B_r(x)$ of centre x and radius r into itself as: $\tau_{B_r(x)} = \inf\{k > 0 : T^k B_r(x) \cap B_r(x) \neq \emptyset\}$. It has been proved in [22] that for one-dimensional maps with a finite number of branches and the derivative of p-bounded variation ($p > 0$) and equipped with a measure μ of positive metric entropy h_μ, the Lyapunov exponent λ_μ verifies the lower bound:

$$\liminf_{r \to 0} \frac{\tau_{B_r(x)}}{-\log r} \geqslant \frac{1}{\lambda_\mu}, \qquad \text{for } \mu\text{-almost every } x.$$

We first observe that our parabolic map has the derivative of p-bounded variation for $1 < p < 1/(1-\alpha)$, so that the preceding bound applies to it[16]. We now prove that the limit exists and is equal to the inverse of the Lyapunov exponent, provided $0 < \alpha < \frac{1}{2}$. We first observe that we can replace the limit r with a sequence r_n going to zero for $n \to \infty$ and such that $(\log r_{n+1}/\log r_n) \to 1$. We then consider the set of cylinders $I_{n,\gamma}$ introduced above. Since $\sum_n \mu(I_{n,\gamma}) < \infty$ for $\alpha < \frac{1}{2}$,[17] by the Borel–Cantelli lemma almost all points x will belong to cylinders $(\omega_1, \omega_2, \ldots, \omega_n)$ in $\overline{I_{n,\gamma}}$ with $\omega_n < [n^\gamma]$, for n big enough. The proof of lemma 2.2 gives that the length of the image of such a cylinder (say $C_n(x)$), on the unit interval will be bounded, uniformly by distortion, between $D^{-1}|DT^{[n^\gamma]+n}(x)|^{-1}$ and $D|DT^n(x)|^{-1}$. Take now a ball centred at x and of radius $r_n = D|DT^n(x)|^{-1}$. Since $B_r(x) \supset C_n(x)$ and $\tau_{C_n(x)}$ (the first return of the cylinder into itself) is greater or equal to $\tau_{B_r(x)}$, we have that:

$$\limsup_{n\to\infty} \frac{\tau_{B_{r_n(x)}}}{-\log r_n} \leqslant \limsup_{n\to\infty} \frac{\tau_{A_n(x)}}{\log D|DT^n(x)|} \leqslant \limsup_{n\to\infty} \frac{\tau_{A_n(x)}}{n} \frac{n}{\log D|DT^n(x)|}.$$

The second factor in the last limit goes to λ_μ^{-1}; the return of the cylinder $A_n(x)$ is bounded at most by $([n^\gamma]+n)$ (by the maximality of the cylinder), so that the first factor in the above limit tends to 1. We have thus proved more than expected, namely theorem 3.1.

Theorem 3.1. *Let T be the parabolic map introduced above in the interval $0 < \alpha < \frac{1}{2}$; then for μ-almost every x:*

(i) $\lim_{n\to\infty} \frac{\tau_{A_n(x)}}{n} = 1$,

(ii) $\lim_{r\to 0} \frac{\tau_{B_r(x)}}{-\log r} = \frac{1}{\lambda_\mu}$.

We stress again that point (i) improves the result in [8], where only the lower bound for $0 < \alpha < 1$ was proved. It should be pointed out that some statistical properties of this map, like the CLT and our theorem 3.1, can usually be proved in the range $0 < \alpha < \frac{1}{2}$ (see the conclusions below). On the other hand, it can be shown that the rate of convergence to the exponential law for the distribution of the first return times, still valid in the whole range $0 < \alpha < 1$, becomes not optimal when $\alpha \in (\frac{1}{2}, 1)$ [21].

4. Concluding remarks and open questions

- We stressed above that the CLT has been proved for the map T and for smooth observables (usually Lipschitz or Hölder) in the range $0 < \alpha < \frac{1}{2}$. Recently, Gouëzel [6] and Hu [10] have shown examples of functions (respectively vanishing in a neighbourhood of 0 and with zero average [6], and with the property that $\phi(0) = \int \phi \, d\mu$ [10]), for which the CLT holds even in the range $\frac{1}{2} \leqslant \alpha < 1$. Gouëzel also announced (private communication), that for a more special class of functions of zero average, there is convergence to a stable law, different from the normal one, in the range $\frac{1}{2} \leqslant \alpha < 1$. It would be interesting to investigate if such a stable law gives the fluctuations for our process $\log R_n$ in such a range.

[16] We recall that a function $g : [0, 1] \to \mathbb{R}$ is of p-bounded variation if:

$$\sup \left(\sum_{i=1}^{m} |g(x_{i-1}) - g(x_i)|^p : m \in \mathbb{N}, 0 \leqslant x_0 < x_1 < \cdots < x_m \leqslant 1 \right) < \infty,$$

where $g(x) = 0$ on the points where T is discontinuous. In our case $g(x) = DT(x)$. By replacing the sum with the integral of the second derivative of T, we get a finite variation provided p is, at least, in the range given above.

[17] In fact $\alpha < \gamma/(\gamma+1)$, but in this section γ can be chosen in the interval (0,1), contrarily to remark 2.1, where the choice of γ was determined by the CLT.

- The natural step after having established the convergence of the process $\log R_n(x)$ to the Gaussian variable, is the computation of the speed of such a convergence. This is usually called a Berry–Essen estimate and it provides a bound of the type $n^{-1/2}$ for systems with strong mixing properties. The lack of uniform hyperbolicity in our map T could give a weaker approximation of order $n^{-\theta}$, with $\theta < \frac{1}{2}$, see [7] for a discussion of this point in the context of (ϕ, f)-mixing systems.
- There is another statistical property that can be associated with the random variable $R_n(x)$, namely the large deviations around the metric entropy. This can be settled in the following way: does a real function f exist for which the following limits hold

$$\lim_{n\to\infty} \frac{1}{n} \log \mu \left(\left\{ x \in X : \frac{1}{n} \log R_n(x) > h + u \right\} \right) = f(h+u),$$

$$\lim_{n\to\infty} \frac{1}{n} \log \mu \left(\left\{ x \in X : \frac{1}{n} \log R_n(x) < h - u \right\} \right) = f(h-u),$$

where u belongs to some open interval around h and $f(u)$ is zero for $u = h$? For aperiodic and irreducible subshifts of finite type endowed with a Gibbs measure μ associated with an Hölder potential ψ, it has been proved in [5] that f, the free energy, is, for u belonging to an open interval $[0, u_0]$, the Legendre transform of the deviation function $G(\beta)$ defined by

$$G(\beta) = \lim_{n\to\infty} \frac{1}{n} \log \sum_C \mu(C)^{\beta+1}, \tag{14}$$

where the sum is over all the cylinders C of length n.

It would be interesting to investigate the existence of the free energy and of the limit (14) for our class of parabolic maps. For the subshifts of finite type quoted above, the function $G(\beta)$ is related to the topological pressure $P(\psi)$ of the potential ψ: $G(\beta) = -(\beta+1)P(\psi) + P((\beta+1)\psi)$. It has been recently proved [17], that for the map T the pressure of the function $\beta\psi$ admits a phase transition for $\beta = 1$. This could therefore affect the large deviations of the variable $R_n(x)$ around h and this could be another characterization of the lack of hyperbolicity for such maps.

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Appendix

We stated in theorem 2.3 a general result to get the WIP for Hölder continuous functions for the parabolic map T provided $0 < \alpha < \frac{1}{15}$. We now show how to relax the Hölder regularity in order to get not only the WIP but also the CLT for a larger class of functions not covered by the other methods quoted in the references. In this regard, Chernov's technique is particularly useful since it distinguishes in a clear way the contribution of the regularity of the function from the mixing properties of any (good) generating partition. The class of functions $\mathcal{S}$ that we will consider is defined throughout the function $\mathcal{H}_F$ introduced in section 2.2, in the following way.

Definition A.1 (space of functions $\mathcal{S}$). *$\mathcal{S}$ is the space of $L_\infty(\mu)$ functions for which $\mathcal{H}_F(d) \leqslant$ const$/|\log d|^q$, where the exponent $q > 0$ will be determined later.*

The exponent q can be chosen bigger than 2 for systems with exponential mixing rate [3]; for our map we could take $q > 8$ (see below). As pointed out by Chernov, $\mathcal{S}$ contains, among others, all the functions of bounded p-variation (see footnote (13)) on [0, 1]; in this case: $\mathcal{H}_F(d) \leqslant \text{const}\, d^a$ with $a = \min\{\frac{1}{2}, 1/p\}$. Moreover [3] 'even if $\mathcal{H}_F(d) \leqslant \text{const}\, d^a$ with some $a > 0$, the function F may be everywhere discontinuous... in every open set in [0,1]'.

Theorem A.1. *Let F be a function in $\mathcal{S}$ for $q > 8$. Suppose moreover that $0 < \alpha < \frac{1}{15}$. Then the CLT and the WIP hold for the process $\sum_{i=0}^{n-1} F(T^i x)$ and the map T provided $\sigma(F) > 0$.*

Proof. The proof is a straightforward verification of the assumptions in theorems 1.2 and 1.4 in Chernov's paper [3]; we enumerate them as C1, ..., C4 and some have already been checked in the proof of our theorem 2.3. In contrast to theorem 2.3, we have now first to assure that:

$*$ **C1**: the first moment of the autocorrelation function (3) is finite.

To do that, we will use the following bound on correlations, proved in theorem 1.1 in [3]. $\square$

Theorem A.2 (Chernov). *For any function $F \in L_\infty(\mu)$, any $n \geqslant 1$ and any partition $\mathcal{A}$ we have:*

$$\left| \int F \cdot F \circ T^n \, \mathrm{d}\mu - \left(\int F \mathrm{d}\mu \right)^2 \right| \leqslant 2\|F\|_{L_\infty}^2 \beta(n) + 2\|F\|_{L_\infty} \mathcal{H}_F(d) + \mathcal{H}_F(d)^2$$

where $d = \operatorname{diam} \mathcal{A}$ and $\beta(n)$ is defined as:

$$\beta(n) = \sum_{i,j} |\mu(B_i \cap D_j) - \mu(B_i)\mu(D_j)|,$$

where $B_i \in \mathcal{A}$ and $D_j \in T^{-n}\mathcal{A}$.

We will use in the following the partition $\mathcal{A} = \mathcal{A}^{(N)}$ constructed in section 2.2 and the integers N, n_1, n_2 will be related to each other as in (9). A proof similar to that which gave us the upper bound on $\beta_N(n)$, allows us to get now:

$$\beta(n) = \mathrm{O}(n_1 n_2^{1-1/\alpha}) + \mathrm{O}_L(n^{1/\alpha}) \tag{A.1}$$

and the dominant term is easily seen to be $\mathrm{O}(n_1 n_2^{1-1/\alpha})$.

Since by assumption $\mathcal{H}_F(\operatorname{diam}\mathcal{A}) \leqslant \text{const.}|\log(\operatorname{diam}\mathcal{A})|^{-q}$, and using the subexponential decay of the diameter of $\mathcal{A}$ found in section 2.2, we get that $\mathcal{H}_F(d) \leqslant \mathrm{O}((n_2/n_1)^q)$, where $d = \operatorname{diam}\mathcal{A}$. By neglecting the quadratic term $\mathcal{H}_F(d)^2$, we therefore see that the sum giving the first moment of the autocorrelation function is composed of two terms respectively of order: $n^{1+z+z^2(1-1/\alpha)}$ and $n^{(z^2-z)q+1}$. The first will be summable for $\alpha < \frac{1}{11}$ and the second for $q > 8$, by sending $z \to \frac{1}{2}$.

We have then to check the three other conditions:

$*$ **C2**: $\beta_N(n) = \mathrm{o}(n/N)$,
$*$ **C3**: $\mathcal{L}_F(n/N) = \mathrm{o}(1/n)$,
$*$ **C4**: $\mathcal{H}_F(\operatorname{diam}\mathcal{A}) = \mathrm{o}(N^{-1})$.[18]

The first two were worked out in section 2.2 giving $z < \frac{1}{2}$ and $0 < \alpha < \frac{1}{15}$. The last one requires, remembering the preceding upper bound on $\mathcal{H}_F(\operatorname{diam}\mathcal{A})$ and the scalings (9), that $N^{z^3 q - z^2 q + 1} \to 0$, which is achieved for $q > 8$ provided z is sent to $\frac{1}{2}$. The theorem then follows by collecting all these bounds.

[18] To prove the CLT it is sufficient to ask for the condition $\mathcal{H}_F(\operatorname{diam}\mathcal{A}) = \mathrm{o}(N^{-1/2})$. The condition which is used above to prove the WIP is stronger.

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