Research Report *Ergodic Theory* team, 2006-2010

1. Introduction

The Ergodic Theory team of CPT is composed by three permanent professors and researchers: E. Lanneau, S. Troubetzkoy and S.Vaienti. PhD students and several guests contribute to the scientific activity and animation of the team. According to the three senior members, three main topics characterize the work of the team: *Statistical properties of dynamical systems* (S. Troubetzkoy and S. Vaienti); *Teichmüller Theory and Polygonal Billiards* (E. Lanneau and S. Troubetzkoy).
In the next sections we will describe in detail the research and the other scientific activities of the three subgroups in the period 2006-2010.

2. Statistical properties of dynamical systems

This area involves S. Troubetzkoy and S. Vaienti. The following **PhD students** worked under Vaienti's direction: Luca Rossi (2004-2006); Johan Nilsson (2005-2007); Ph. Marie (2006-2009) and Jean-Francois Bertazzon (2006-2010) under Troubetzkoy's direction. Starting in 2010 Vaienti will co-direct the PhD of Hale Aytac the with the University of Porto.

Several **guests** have been invited: N. Haydn (USC Los Angeles, one month par year, three months in 2008-CNRS grant): H. Hu (Michigan State University, one month par year); H. Bruin (University of Surrey, one month in 2009); C. Liverani (one month in 2007); G. Turchetti and his students (University of Bologna, in the framework of the Socrates-Erasmus project with the University of Toulon); G. Mantica (University of Como, in the framework of the Socrates-Erasmus project with the University of Toulon); M. Abadi (in the framework of the project "Modélisation et récurrence stochastique dans les systèmes en évolution", accord France-Brésil CAPES-COFECUB); J. Alves and J. Freitas (University of Porto, in the framework of different kinds of exchanges: CNRS, Socrates-Erasmus, etc.), J. Bobok (1 month in 2007, 1 month in 2008 ("Poste Tcheque", University of Toulon).
Starting in 2009, S. Vaienti is a member of the project *DynEurBra*, between France and Brasil and in the *Seventh Framework Program* "Marie Curie Actions", 2009-2012.

S. Vaienti was co-organizer of the five weeks conference at CIRM *Session résidentielle sur les thémes du GDRE Franco-Italien "GREFI-MEFI"*(2008).
He will organize also the Conference at CIRM in 2011 on *Large Deviations in Dynamical Systems* with D. Volny, I. Melbourne and M. Nicol.

We now describe the scientific topics investigated in the last four years with some perspectives.

2.1. **Ergodic properties of non-uniformly expanding systems.** In the paper [H. HU, S. VAIENTI, Absolutely continuous invariant measures for non-uniformly expanding maps, *Ergodic Theory and Dynamical Systems*, **29**, 1185-1215, (2009)], we introduced a large class of multidimensional nonuniformly expanding maps, with indifferent fixed points and unbounded distortion, and not necessarily markovian. Unbounded distortion away from the indifferent fixed points means that there are uncountably many points x whose neighbourhoods contain points y, arbitrary close to x, such that the distortion of $|\, detDT|$ is unbounded along the backward orbits

converging to the indifferent fixed point. We constructed absolutely continuous invariant measures (a.c.i.m.) for these maps by first replacing the transformation with the first return map with respect to a domain outside a small region around the indifferent fixed point. On this induced space, we got an a.c.i.m. as the fixed point of the transfer operator and we successively extended this measure to get an a.c.i.m. on the whole space. This extension is compatible with the existence of σ-finite components.
For these class of maps the problem of estimating the rate of decay of correlation is still open. The basic reason is that such maps do not admit an inducing scheme given by a Gibbs-Markov induced map, which would have been allowed to use the Lai-Sai Young theory : we remind that Young showed that the rates of decay of correlations are directly related to the tail of the return time function of the associated Gibbs-Markov induced map. We will return to this approach below. Even if the induced scheme in our previous paper is not Gibbs-Markov, it can be proved that it is a fibred system in the sense of [J. Aaronson, M. Denker, O. Sarig, R. Zweimueller,Aperiodicity of cocycles and conditional local limit theorems, *Stoch. Dyn.* **4** (2004), 31–62] (actually skew-product rigid). In the latter paper conditions are given to show the aperiodicity of cocycles, which is a basic tool, together with the renewal equation, to show an optimal polynomial decay of correlations on the induced space, as it was firstly proved by Sarig. In the forthcoming paper [H. HU, S. VAIENTI, Polynomial Decay for Non-Markov maps], we begin to extend those conditions to prove aperiodicity to higher-dimensional maps, with the aim of establishing a general theory of polynomial decay for Non Markov maps.
We said above that in the Lai-Sai Young framework, the existence of an absolutely continuous invariant measure and its statistical properties, notably the decay of correlations, can be deduced from the geometry of the map, namely from the existence and properties of a Young tower, or induced Gibbs-Markov map. In the forthcoming paper [J. ALVES, J. FREITAS, S. LUZZATTO, S. VAIENTI, From rates of mixing to recurrence times], we observed that in the context of nonuniformly expanding systems, the hypothesis of being modelled by a Young tower is essentially necessary as well sufficient for the validity of statistical properties such as decay of correlations and large deviations. This has interesting implications for the ubiquity of Young towers in nonuniformly expanding systems and implies that the assumption of a Young tower is without loss of generality. One of our results explicitly states that if one knows the large deviations of the time average of the potential of the map with a polynomial, subexponential or exponential decay, then there exists a Gibbs-Markov induced map with a distribution of the first return time which obeys the same kind of decays. Another result quantifies the link between the rate of decay of correlations, for the observables in certain functional spaces, and the rate of decay of the deviations for the time average of an observable in the same class. This is an interesting subject of research in itself; we were led to use and adapt to our situations a theorem by Azuma and Hoeffding, which by the way allowed us to prove the large deviations for the Viana map.
We conclude this section by quoting our paper [G. CRISTADORO, N. HAYDN,Ph. MARIE, S. VAIENTI, Statistical properties of intermittent maps with unbounded derivative, which should appear in Nonlinearity after revision), where we studied the ergodic and statistical properties of a class of maps of the circle and of the interval of Lorenz type which present indifferent fixed points and points with unbounded derivative. These maps have been previously investigated in the physics literature. We prove in particular that correlations decay polynomially, and that

suitable Limit Theorems (convergence to Stable Laws or Central Limit Theorem) hold for Hölder continuous observables. We moreover show that the return and hitting times are in the limit exponentially distributed.

2.2. **Random perturbation of dynamical systems.** In the paper [C. LIVERANI, Ph. MARIE, S. VAIENTI, Random Classical fidelity, *Journal of Statistical Physics* **128**, 4, 1079-1091, (2007)], we introduced a random perturbed version of the classical *fidelity* and we show that it converges with the same rate of decay of correlations, but not uniformly in the noise. We succesively discovered very interesting applications of this result, in particular it allowed us to study the effect of a random perturbation on the orbit of a discrete dynamical system. In the paper [Ph. MARIE, G. TURCHETTI, S. VAIENTI, F. ZANLUNGO, Error Distribution in randomly perturbed orbits, *Chaos*, to appear], we analyzed the statistics of the global errors given by the algebraic difference at iteration n between the exact orbit and an orbit perturbed at each step with a random error of order ε. We provided exact results for two model maps, regular and chaotic respectively, and stating a general theorem on their asymptotics. This analysis suggests the existence of a time scale depending on ε below which the error spread around zero remains comparable with the local error. The scale is basically $\log(1/\varepsilon)$ for chaotic maps and ε^{-1} for regular maps and is related to the interplay of the noise with the exponential or linear divergence of nearby orbits. In other two related papers [G. TURCHETTI, S. VAIENTI, F. ZANLUNGO, Relaxation to the asymptotic distribution of global errors due to rounf off, soumis to appear in *Europhys. Lett.* and G. TURCHETTI, S. VAIENTI, F. ZANLUNGO, Asymptotic distribution of global errors in the numerical computations of dynamical systems, submitted], we applied the previous results to pure round off noise in computers and we shown in particular that for chaotic maps our methods allows us to find a threshold value below which the numerically simulated system can be considered as equivalent as the exact one and moreover this threshold linearly grows as the number of bits used to represent real numbers. These works could be of some interest to state the reliability of numerical computations of dynamical systems.
We continue this section by quoting our recent paper [H. HU, Ph. MARIE, S. VAIENTI, Attractors, Pseudo-orbits and Stationary Measures], where we classified the ergodic components of stationary measures in terms of equivalence classes when a suitable equivalence relation is introduced in the space of pseudo-orbits, following a seminal work by Ruelle in the eighties. There is in fact a natural link between pseudo-orbits and randomly perturbed orbits: in particular one can show that the stationary measures have support on the basin of attraction of pseudo-orbits. Our analysis is based on the existence of a Lasota-Yorke inequality for the transfer operator acting on suitable functional spaces containing the densities of the stationary measures. This allowed us to treat non-invertible dynamical systems. It would be interesting to generalize those ideas to diffeomorphisms, eventually with singularities, in such a way to classify the ergodic components of the SRB-measures.
Recently Ph. Marie, PhD Student of S. Vaienti and J. Rousseau, PhD student of B. Saussol, obtained a result which could be considered as the first step to establish a theory of recurrence for randomly perturbed systems [Ph. Marie, J. Rousseau, Recurrence for random dynamical systems, to appear in *Discrete and Continuous Dynamical Systems* (2009)]. They introduced the concepts of quenched and annealed return times for systems generated by the composition of random maps and finally proved that for super-polynomially mixing systems, the random recurrence rate is equal to the local dimension of the stationary measure.

We conclude this section by adressing a few questions which we planned to study in the future, namely: (i) the stochastic stability of the non-uniformly expanding maps described in the previous section. This will be probably require to understand the link between stationary measures and induction; (ii) still for the previous non-uniformly expanding systems: generalize the entropy formula under random perturbations; (iii) prove the strong stochastic stability (convergence of the density in L^1) for the parabolic maps of the interval of Pomeau-Manneville type. A first step in this direction is in the forthcoming paper [J. ALVES, J. FREITAS, S. VAIENTI, Statistical stability of intermittent maps]; (iv) develop the theory of recurrence in presence of noise and develop also a theory of extreme values in presence of noise.

2.3. **Recurrence.** It has been shown by several authors that some classes of mixing dynamical systems have limiting return times distributions that are almost everywhere Poissonian. In the paper [N. HAYDN, S. VAIENTI, The compound Poisson distribution and return times in dynamical systems, *Probability Theory and Related Fields*, **144**, (2009), 517-542], we studied the behaviour of return times at periodic points and show that the limiting distribution is a compound Poissonian distribution. We also derived error terms for the convergence to the limiting distribution. We also proved a very general theorem that can be used to establish compound Poisson distributions in many other settings. The theoretical results of this paper were used in the work [N. HAYDN, E. LUNEDEI, S. VAIENTI, Averaged number of visits, *Chaos*, **17**, 033119, 13 pages, (2007)], where we introduced a new indicator for dynamical systems, the *averaged number of visits*, to estimate the frequency of visits in small regions, when a map is iterated up to the inverse of the measure of this region. We computed this quantity analytically and numerically for various systems and we show that it depends on the ergodic properties of the systems and on their topological properties like the presence of periodic points.
Another aspect of the recurrence that we investigated in a series of paper, was the large deviation properties of the process given by the first return of a set into itself when its measures converges to zero and the set is centered around a given point. Typically in the choice of this point, and whenever the target set is a cylinder of lenght n, the first return of this cylinder divided by n converges to 1 for systems with strong mixing systemes. In the paper [M. ABADI, S. VAIENTI, Large deviations for short returns, *Discrete and Continuous Dynamical Systems A*, **21**, (2008), 729-747], and essentially for Bernoulli systems, we showed that the decay rate for the large deviation of the return time to cylinder sets is exponential with a rate given by the Rényi entropy function. In a subsequent paper [N. HAYDN, S. VAIENTI, The Rényi Entropy Function and the Large Deviation of Short Return Times, *Ergodic Theory and Dynamical Systems*, **30**, 2010,159-179] we generalized that result to weakly ψ-mixing systems and we explicityly proved the existence and regularity properties of the Rényi entropy function for such systems. We also obtain bounds for the free energy of the process described above. Those results have been generalized to the more interesting physical situation of sets given by balls [G. MANTICA, S. VAIENTI, On the statistical distribution of first-return times of balls and cylinders in chaotic systems, to appear in *International Journal of Bifurcations and Chaos*]. In this work we described the statistical distributionn of these first returns times in various settings: when phase space is composed of sequences of symbols from a finite alphabet (with applications for instance to biological problems) and when phase space is a two-dimensional manifold. We derived

relations linking these statistics with entropies, as we said above, and with Lyapunov exponents.

2.4. **Orthogonal polynomials.** The Fourier transform of orthogonal polynomials with respect to their own orthogonality measure defines the family of Fourier-Bessel functions. In the paper [G. MANTICA, S. VAIENTI, The asymptotic behavior of the Fourier transform of Orthogonal Polynomials I: Mellin transform techniques, *Ann. Inst. H. Poincaré*, **8**, 2, (2007)]we studied the asymptotic behaviour of these functions and of their products, for large values of the argument. By employing a Mellin analysis we constructed a general framework to exhibit the realation of the asymptotic decay laws to certain dimensions of the orthogonality measure, thyat are definied via the divergence abscissa of suitable integrals. We underlined the unifying rôle of Mellin transform techniques in deriving classical and new results.

2.5. **Diophantine approximation and the mass transference principle.** In the article [A.H. FAN, J. SCHMELING, S. TROUBETZKOY, A multifractal mass transference principle for Gibbs measures with applications to dynamical diophantine approximation; to be submitted to *Acta Math.*] A.H. Fan, J. Schmeling and S. Troubetzkoy have studied diophantine approximation for a Gibbs measure μ. For a μ-generic point x, and a sequence $\{r_n\}_{n\geq 1}$ we consider the intervals $]T^n x - r_n \pmod 1, T^n x + r_n \pmod 1[$. We studied the covering properties of these intervals in analogy to the classical covering problem of Dvoretzky. We obtained a mass transference principle for Gibbs measures. These are multi-fractal measures, a similar principle has been shown for mono-fractal measures by Beresnevich and Velani. We use this principle to completely describe the combinatorial structure of typical relatively short sequences and we describe the occurece of relatively long "atypical" word. This description allows us to calculate the Hausdorff dimension of the set of points covered infinitely often by the intervals.

2.6. **Quantum ergodicity.** Marklof et Rudnick have asked if it is possible that a quantum ergodic map is not quantum uniquely ergodic. C.-H. Chang, T. Krueger et R. Schubert and S. Troubetzkoy gave a positive answer to this question in [C.H. CHANG, T. KRUEGER, R. SCHUBERT, S. TROUBETZKOY, Quantisations of piecewise affine maps on the torus and their quantum limits *Communications Mathematical Physics* 282 (2008) 395–418]. We constructed quantum ergodic maps which have singular quantum limits and non-quantum ergodic maps with convex combinations of absolutely continuous invariant measures as quantum limits.

2.7. **Open systems.** S. Bundfuss, T. Krueger, and S. Troubetzkoy have studied the coding properties of open systems [S. BUNDFUSS, T. KRUEGER, S. TROUBETZKOY, Topological and symbolic dynamics for hyperbolic systems with holes, *Ergodic Theory Dyn. Sys.* (2010)]. Generically we showed that transitive components are of finite type. In dimension 1, these components are always codes, there are only finitely many of them, and our upper bound is optimal. We gave partial generalizations to higher dimensions.

2.8. **Discretizations.** C. Rojas and S. Troubetzkoy have considered the statistical properties of discretizations of continuous functions [C. ROJAS, S. TROUBETZKOY, Coding Discretizations of Continuous Functions, submitted to *Discrete Math.*] We show that generically every word appears with arbitrary frequency.

3. Teichmüller theory and Billiards

The area is studied by E. Lanneau and S. Troubetzkoy. The following **PhD students** are involved: Sylvie Jourdan (2006-2010) (co-director Lanneau). Again in this topics many **guests** have been invited, and in particular among them researchers awarded by the field medal : J.-C. Yoccoz, C. McMullen and A. Okounkov. A. Avila and H. Masur are also regular guests. E.Lanneau is member of the ANR project "Teichmüller", the project Franco-Israelien and PICS (France-USA).

E. Lanneau was co-organizer of several conferences in CIRM (School ergodic theory in 2006 and conference in honour of Masur in 2009). E. Lanneau also organized a conference in Roscoff in 2008 (dynamical systems). S. Troubetzkoy is organizing an Arbeitsgemeinschaft on Mathematical Billiards at Oberwolfach in 2010.

It would be also interested to notice that there will be a conference in US (Madison, April 2010) around a conjecture of Lanneau and Thiffeault. Thurston has confirmed that he would be interested to participate to this conference around *small dilatations of pseudo-Anosov homeomorphisms.*

We now describe the scientific topics investigated in the last four years with some perspectives.

3.1. **Closures of the Teichmüller discs.** For an arbitrary dynamical system, it is very hard in general to give information on the behaviour of a particular orbit. Nevertheless the situation for unipotent flows in homogeneous spaces is very well-understood. Ratner proved the striking result that the closure of *any* orbit of any group generated by unipotent elements acting on a homogenous space is also a nice homogeneous space.

The cotangent bundle of the moduli space of curves (points of this bundle are flat surfaces) is preserved by the Lie group $\mathrm{SL}_2(\mathbb{R})$. There is a strong hope to believe that the closure of any is an algebraic suborbifold (Kontsevich). This is the main conjecture in Teichmüller dynamics.

This conjecture has been recently proven, for genus two surfaces, by McMullen. With P. Hubert and M. Möller, E. Lanneau extends McMullen's techniques to higher genera. We give a description of the closure of orbits stabilised by pseudo-Anosov element [HUBERT LANNEAU MOELLER Completely periodic directions and orbit closures of many pseudo-Anosov Teichmller discs, *Math. Ann.*, to appear] and [HUBERT LANNEAU MOELLER The Arnoux-Yoccoz Teichmller disc, *Geom. Funct. Anal.* **18** (2009), no. 6]. We also wrote a survey of these technics: [HUBERT LANNEAU MOELLER Orbit closures via topological splitting, *volume 14 of Surveys in Differential Geometry* (2010)].

A common tools in these technics is the used of pseudo-Anosov maps. To date there are two methods to produce pseudo-Anosov diffeomorphisms in the coordinates of the flat surface. In the first one, due to Thurston, a pseudo-Anosov diffeomorphism is obtained as a product of two parabolic elements. The second one is due to Veech, based on the Rauzy induction of interval exchange transformations. In [HUBERT LANNEAU MOELLER The Arnoux-Yoccoz Teichmller disc, *Geom. Funct. Anal.* **18** (2009), no. 6] we provide a new construction of such maps. In the article [Pseudo-Anosov homeomorphisms on hyperelliptic surfaces have large entropy (with C. Boissy)] we also generalize the Veech's construction to half-translation surfaces.

E. Lanneau and P. Hubert also proved that some pseudo-Anosov diffeomorphisms are not given by Thurston's construction [HUBERT LANNEAU Veech groups with no parabolic element, *Duke Math. J.* **133** (2006)].

It would be very interesting to have a similar description in full generality. The question whether there exists or not a cyclic Veech group, generated by a single hyperbolic element, is still an open problem.

3.2. **Rauzy-Veech induction for linear involutions.** Interval exchange transformations are closely related to Abelian differentials on Riemann surfaces. It is very well known that the continued fractions encode cutting sequences of hyperbolic geodesics on the Poincaré upper half-plane. Similarly, one can encode the Teichmüller geodesic flow using the Rauzy-Veech induction (analogous to Euclidean algorithm).

But the Rauzy-Veech induction initially elaborated to prove ergodicity of the Teichüller flow (Masur and Veech in 1982) is actually very efficient far beyond these initial problems (there are many very recent results from Kontsevich and Zorich; Avila, Gouëzel, and Yoccoz; Avila and Viana, Avila and Forni, etc.).

However, all the aforementioned results concern only the moduli space of Abelian differentials. The corresponding questions for strata of strict quadratic differentials remain open. With C. Boissy, E. Lanneau gave a discrete representation of the Teichmüller flow on *quadratic differentials* [BOISSY LANNEAU Dynamics and geometry of the Rauzy-Veech induction for quadratic differentials, *Ergodic Theory Dynam. Systems* **29** (2009), no. 3].

In the article [LANNEAU An Infinite sequence of fixed point free pseudo-Anosov homeomorphisms on a genus two surface, *preprint* (2010)] E. Lanneau used the Rauzy-Veech induction in order to construct pseudo-Anosov homeomorphisms without a fixed separatrix, answering a question a Avila.

In would be very interesting to use the Lanneau-Boissy's construction in order to prove spectral properties of the Teichmüller flow. There are still partial results in this direction due to Avila and Resende,

3.3. **Dilatations of pseudo-Anosov homeomorphisms.** Study of surfaces homeomorphisms starts with Thurston in the seventies. The concept of pseudo-Anosov is very important and one can think of such maps as the elementary maps in order to understand the mapping class group.

The dilatations of theses maps (related to the topological entropy) are special algebraic integers, called Perron numbers. The set of logarithms of dilatations equals the set of Teichmüller lengths of geodesics on the moduli space of complex curves. We know very little on these dilatations and the principal conjecture in this domain is to understand the least dilatation (when the genus is fixed). There is a conjecture of McMullen about the asymptotic of these dilatations.

With J.-L. Thiffeault, E. Lanneau calculated the least dilatation for genus two surfaces, which is the **first** result in this direction [LANNEAU THIFFEAULT On the minimum dilatation of pseudo-Anosov diffeomorphisms on surfaces of small genera, *Annales de l'institut Fourier* to appear (2010)].

E. Lanneau and J.-L. Thiffeault also gave several inequalities on these dilatations, answering questions of Farb.

In two articles [LANNEAU THIFFEAULT Enumerating Pseudo-Anosov Homeomorphisms of the Punctured Disc, *preprint* (2009)] and [LANNEAU THIFFEAULT

On the minimum dilatation of braids on the punctured disc,*preprint* (2009)] E. Lanneau and J.-L. Thiffeault investigated the case of the punctured disc. They produce an algorithm to obtain the dilatations.

If we fixe the combinatorial type of the pseudo-Anosov, it is not clear whether the least dilatation goes to one with the genus. In the article [BOISSY LANNEAU Pseudo-Anosov homeomorphisms on hyperelliptic surfaces have large entropy,*preprint* (2010)] E. Lanneau, in collaboration with C. Boissy, gave an answer. They show that the least dilatations, on hyperelliptic surfaces (in hyperelliptic components), are bounded above by $\sqrt{2}$ (this bound being sharp). E. Lanneau and C. Boissy are working on a related problem if we relax the condition on hyperellipticity.

3.4. **Mathematical billiards.** In [S. TROUBETZKOY, Periodic billiard orbits in right triangles II, *Annales de l'Institut Fourier* (2010)] Troubetzkoy demonstrated the density of periodic orbits for billiards in right triangles. Previously, density was only known for rational polygons. In addition he proved a local density result which is stronger than that know for rational polygons. This result is obtained through the analysis of the symmetries of the infinite flat surface corresponding to the billiards. We should mention that these results were presented in an article in La Recherche [B. RITTAUD, Les rebonds d'une boule de billard, *La recherche* No. 389 09/2005].

One would like to know if a billiard table is determined by the combinatorics of the points of collision of an orbit with the boundary of the table. J. Bobok and Troubetzkoy call two tables order equivalent is there exists, in each of the tables, a point whose orbit projects to a sequence dense in the respective boundary, and the two sequences have the same combinatorial order [J. BOBOK, S. TROUBETZKOY, Does a billiard orbit determine its (polygonal) table? submitted to *Fundementa Math.*]. We showed that an irrational polygon can not be order equivalent to a rational polygon, and that two rational polygons which are order equivalent have the same number of sides and the same angles at corresponding corners. In particular two triangle which are order equivalent are similar. All rectangles are order equivalent, thus in general one can not say more, but if two rational polygons are order equivalent and have greatest common denominator at least 3, then they are order equivalent. Recently we obtained similar results by replacing order equivalence by polygons having dense orbits with the same coding. These results were also presented in an article in La Recherche [B. RITTAUD, Une boule pour lire la forme du billiard, *La recherche* No. 427 02/2009].

Does a point source of light illuminate a room (planar domain) whose walls are mirrors? Except for the trivial results that a convex room is completely illuminated by any point, all the known results are negative: i.e. examples of rooms and positions of the light sources which do not illuminate everything. P. Hubert, M. Schmoll and Troubetzkoy have shown the first positive results for a class of polygons [P. HUBERT, M. SCHMOLL, S. TROUBETZKOY, Modulars fibers and illumination problems *International Mathematics Research Notices* (2008) no. 8 Art. ID rnn011, 42 pp.]. We established a quantative version of Kronecker's theorem, and used this result to prove that for prelattice polygons, every point illuminates all points except for an exceptional class which is at most countable. For a subclass of these polygons, the Veech polygons, the exceptional set is finite. To prove this second result we charaterized the $Aff^+(X, \omega)$-invariant subspaces of $X \times X$ for a Veech surface Veech (X, ω).

In 1912 the Ehrenfests proposed the "wind-tree" model to study diffusion. Since this time there have been few mathematically rigorous results on this model. In [P.HUBERT, S. LELIEVRE, S. TROUBETZKOY, The Ehrenfest wind-tree model: periodic directions, recurrence, diffusion submitted to *Journal fr die reine und angewandte Mathematik (Crelle's Journal)*] P. Hubert, S. Lelievre and S. Troubetzkoy have shown that generically this model is recurrent and have given a lower bound on the diffusion rate. To prove this we described the periodic orbit structure using the symmetries of the model for cetain parameters of the model. Then the recurrence and diffusion results for these parameter values were obtained using metric approximation techniques, and finally all the results were extended to generic parameter values again by approximation

A. Stepin and S. Troubetzkoy have given a new characterization of weak mixing which can be applied to study approximations [S. TROUBETZKOY, Approximation and billiards, *Systèmes dynamiques et approximations diophantiennes, Séminaires et congrès,* no. 22 Socit Mathmatique de France (2010), article with Stepin in preparation]. They showed that in the C^1 topology, the generic billiard is weak mixing. This result also holds for generic convex tables, for which KAM theory implies that sufficiently smooth tables (C^6) are never ergodic. This theorem improved a result of Gruber who shown that in the C^0 topology, C^1 convex tables generically have a dense orbit.

S. Troubetzkoy has studied the "Fagnano" dual billiard periodic orbit Q for a polygon P [S. TROUBETZKOY, Dual billiards, Fagnano orbits and regular polygons *American Math. Monthly* 116:3 (2009) 251–260]. The notion of a Fagnano orbit generalizes that fact that a triangle Q is a periodic orbit of its median triangle P. He characterized regular polygons and affinely regular polygons in terms of there Fagnano orbits and gave a comple desicription of the map $Q \to P$.