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TARGETS AND HOLES

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ABSTRACT. We address the extreme value problem of a one-dimensional dynamical system approaching a fixed target while constrained to avoid a fixed set, which can be thought of as a small hole. The presence of the latter influences the extremal index which depends explicitly on the escape rate.

This work is motivated by the appearance of extreme events in specific real world contexts. We are interested in the statistical description of a perishable dynamics (i.e., an open system) approaching a fixed target. As examples one can think of the process describing an environmental catastrophic event or a pandemic outbreak (with the underlying space being the spatial distribution of the epidemic) approaching a critical extension before it disappears. Thus, the dynamical setting is novel in that it has two main features: in the phase space, on one hand there is a target point which will be approximated by small balls around it, on other hand there is an absorbing region which terminates the process on entering it.

A one dimensional prototype of such situation can be formulated as an *extreme value problem for an open system*, thus allowing a rigorous study. Similar setups, in the presence of shrinking targets or absorbing regions, have already been studied in many situations; see [4, 8, 10, 11, 16, 27, 29] for an account of the literature. Recurrence in open systems has also been considered by Kifer [25, 26].

We consider a dynamical system where there is an absorbing region, a *hole* H, such that an orbit entering terminates its evolution (i.e., it is lost forever). The introduction of systems with holes dates back, at least, to Pianigiani and Yorke [31]. By considering the orbits of the whole state space, it is possible to construct a *surviving set*. Within this set, we fix a point and a small ball around it, the *target set* B. We investigate the probability of hitting B for the first time after n steps while avoiding H, in the $n \to \infty$ limit. We will show that this question can be formulated in a precise probabilistic manner by introducing conditionally invariant probability measures for the open system.

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In our investigation, we will call the entrance of the system trajectory into the target an extreme event, and the closest approach of the trajectory to the target is measured by so-called extreme values (of a suitable function of the distance). An extreme value distribution (EVD) will be obtained by means of a spectral approach on suitably perturbed transfer operators (see, among others, [1, 5, 6, 12, 22, 30]). The choice of conditionally invariant measures makes the process nonstationary; obtaining an EVD in such a framework is a nonnegligible improvement (see [17] for a general discussion of this matter).

The boundary levels and the extremal index of the EVD will be expressed in terms of the Hausdorff dimension of the surviving set and of the escape rate, respectively. The EVD will explicitly depend on whether the target point in the surviving set is periodic or not, cf. our main result, Proposition 3.1. The theory above can also be adapted to handle a sequence of target sets which shrink to a point outside the surviving set. In this case, it predicts correctly that the EVD is degenerate, i.e., the dynamics cannot approach the target point indefinitely. These three cases together thus define a *trichotomy* of possible EVDs.

Our approach also links parameters of the EVD to other dynamical quantities; thus it provides tools of computing dynamical indicators through approximating the limiting distribution by the so-called Generalized Extreme Value (GEV) distribution, and vice versa; this will be the object of future investigations. For example, it is an interesting problem to investigate, theoretically and numerically, the Hausdorff dimension of the survival set via extreme value theory.

In Section 1 we will detail the systems we will consider. Section 2 will present the deduction of the extreme value distribution by using a well-established spectral approach. In Section 3 we will compute explicitly the extremal index. The full statement of the result is Proposition 3.1 in Section 3.2. Last, in Section 4 we show how a degenerate EVD arises when the target set becomes disjoint from the surviving sets. For the sake of simplicity, we will restrict ourselves to uniform expanding maps of the intervals, although generalizations are possible following the same approach: the remarks after Proposition 3.1 discuss possible extensions.

1. The open system

To access open systems through an operator-theoretic framework, we will adapt the theory developed by C. Liverani and V. Maume-Deschamps [29]. They considered Lasota–Yorke maps[1] $T : I \circlearrowleft$ on the unit interval I and a transfer operator with a potential g of bounded variation (BV).

We denote with $\mathcal{L}$ the transfer (Perron–Frobenius) operator associated with T and g; it acts on functions $f \in BV \cap L^1(\mu_g)$ as

$$\mathcal{L}f(x) = \sum_{T(y)=x} f(y)g(y), \tag{1.1}$$

where μ_g is the conformal measure left invariant by the dual $\mathcal{L}^*$ of the transfer operator,

$$\mathcal{L}^*\mu_g = e^{P(g)}\mu_g,$$

where $P(g)$ is the topological pressure of the potential g (see, among others, [21]).

[1] I.e., uniformly expanding maps, $\inf_I |T'| = \beta > 1$, such that there exists a finite partition of the interval I with the property that T restricted to the closure of each element is C^1 and monotone.

For simplicity, we will restrict ourselves to the potential $g = \frac{1}{|T'|}$; however we refer to Remarks 3.2–3.3 for possible extensions of the result to other potentials. First of all note that, in this case, the conformal measure $\mu_{|T'|^{-1}}$ will be Lebesgue (denoted by m) and $P(g) = 0$. Recall that if we equip the space of BV functions with the norm given by the total variation plus the L^1 norm,[2] then the unit ball of BV is compact in L^1; this will allow us to make good use of the spectral decomposition of transfer operators. Moreover, our probability distributions will be explicitly written in terms of the Lebesgue measure and therefore they will be accessible to numerical computations. We will use later on the quantity $\Theta(g)$ defined as

$$\log \Theta(g) := \lim_{n\to\infty} \frac{1}{n} \log \sup_I g_n, \text{ where } g_n = g(x) \times \cdots \times g(T^{n-1}x);$$

in our case it simply becomes $\Theta(g) = \beta$.

We then consider a proper subset $H \subset I$ of measure $0 < m(H) < 1$, called the *hole*, and its complementary set $X_0 = I \backslash H$. We denote by $X_n = \bigcap_{i=0}^n T^{-i} X_0$ the set of points that have not yet fallen into the hole at time n. The *surviving set* will be denoted by $X_\infty = \bigcap_{n=1}^\infty X_n$. The key objects in our study are conditionally invariant probability measures.

Definition 1.1. A probability measure ν which is absolutely continuous with respect to Lebesgue is called a conditionally invariant probability measure if it satisfies for any Borel set $A \subset I$ and for all $n > 0$

$$\nu(T^{-n}A \cap X_n) = \nu(A)\, \nu(X_n). \tag{1.2}$$

We use for it the abbreviation a.c.c.i.p.m.

The measure ν is supported on X_0, $\nu(X_0) = 1$, and moreover

$$\nu(X_n) = \alpha^n, \text{ where } \nu(X_1) = \nu(T^{-1}X_0) = \alpha < 1.$$

Apart from being absolutely continuous with respect to Lebesgue, this measure is numerically accessible in simulations.

Remark 1.2. Our a.c.c.i.p.m. plays a role analogous to a quasi-stationary measure defined in a stochastic framework. However, for stochastic dynamics, the role of (the characteristic function of) X_∞ would be played by a hitting probability, also called a committor, namely that of reaching a vicinity of the target before falling into the hole (see [19]).

The existence of a.c.c.i.p.m. in our setting is achieved by Theorem A in [29]. Note that α contains, at the same time, the information about $m(H)$ and the expansion of the system (see equations (3.1) and (3.2)). We now introduce our first perturbed transfer operator defined on bounded variation function f as

$$\mathcal{L}_0(f) = \mathcal{L}(f \mathbb{1}_{X_0}). \tag{1.3}$$

We will use the following facts which are summarized in [29, Lemma 1.1, Lemma 4.3]:

- Let $\nu = \mathbb{1}_{X_0} h_0 m$ a probability measure with $h_0 \in L^1$; then ν is an a.c.c.i.p.m. if and only if $\mathcal{L}_0 h_0 = \alpha h_0$, for some $\alpha \in (0, 1]$.

[2]From now on we will denote $L^1(m)$ and $L^\infty(m)$ by L^1 and L^∞. The L^1 norm will be written as $|\cdot|_1$.

- Let α, h_0 be as above. Moreover, let μ_0 be a probability measure on I such that $\mathcal{L}_0^*\mu_0 = \alpha\mu_0$. Then μ_0 is supported in X_∞[3] and the measure Λ with
$$\Lambda = h_0\,\mu_0 \quad \text{is } T\text{-invariant.}$$
- The measure μ_0 satisfies the conformal relation:
$$\mu_0(TA) = \alpha \int_A |T'| d\mu_0, \tag{1.4}$$
for every measurable set $A \subset I$ on which T is one to one.
- For any $v \in L^1(\mu_0)$ and $w \in L^\infty(\mu_0)$ we have the duality relationship:
$$\int \mathcal{L}_0 v\ w\ d\mu_0 = \alpha \int v\ w \circ T\ d\mu_0. \tag{1.5}$$
Actually, this duality formula will only be used to rewrite the integral (2.8) and in that case w will be the characteristic function of a measurable set and $v = h_0$ which is μ_0-integrable.

We are now strengthening our assumptions by considering small holes since in this case we can use the results in [29, Section 7] and that will allow us to apply the spectral approach of extreme value theory. Later on, in Section 3.1 we will detail a constrain on the size of the hole, which has to be taken into account along the requirements of perturbative theorems. We first need a few preparatory results which will be also essential for the next considerations.

1.1. **Lasota–Yorke inequalities.** Lemma 7.4 in [29] states that for each $\chi \in (\beta = \Theta(g), 1)$ there exists $a, b > 0$, independent of H, such that, for each w of bounded variation:
$$\|\mathcal{L}^n w\|_{BV} \leqslant a\chi^n \|w\|_{BV} + b|w|_1 \tag{1.6}$$
$$\|\mathcal{L}_0^n w\|_{BV} \leqslant a\chi^n \|w\|_{BV} + b|w|_1. \tag{1.7}$$
The proof of the first inequality is standard; the second one relies on the fact that the jumps in the total variation norm of the backward images of the hole grow linearly with n and they are dominated by the exponential contraction of the derivative; see also the proof of [2, Theorem 2.1].

1.2. **Closeness of the transfer operators and their spectra.** We introduce a so-called triple norm, defined by $|||\mathcal{P}|||_1 := \sup_{\|w\|_{BV} \leqslant 1} |\mathcal{P}w|_1$, where $w \in \mathrm{BV}$ and the linear operator $\mathcal{P}$ maps into L^1.[4] It is easily proven in [29, Lemma 7.2] that
$$|||\mathcal{L} - \mathcal{L}_0|||_1 \leqslant e^{P(g)} m(H) = m(H). \tag{1.8}$$
The idea is now to take a hole of small m-measure in such a way that even the spectra of the two operators are close. This is achieved next.

The following result is proved in [29, Theorem 7.3]. For each $\chi_1 \in (\chi, 1)$ and $\delta \in (0, 1-\chi_1)$, there exists $\epsilon_0 > 0$ such that if $|||\mathcal{L}_0 - \mathcal{L}|||_1 \leqslant \epsilon_0$ then the spectrum of $\mathcal{L}_0$ outside the disk $\{z \in \mathbb{C}, |z| \leqslant \chi_1\}$ is δ-close, with multiplicity, to the one of $\mathcal{L}$. This result will allow us to get a very useful quasi-compactness representation for the two operators, which will be the starting point of the perturbation theory of extreme values.

[3] By the hypothesis of [29], X_∞ is not empty. Note that this fact follows trivially by compactness whenever all the X_n are closed; however, this is not always the case (for example if one branch of T is not onto).

[4] If we use a different measure "meas" instead of m we will write $|||\mathcal{P}|||_{\text{meas}}$.

1.3. **Quasi-compactness of the transfer operators.** First of all we should add a further restriction for our unperturbed system, namely we will require that T has a unique invariant measure μ absolutely continuous with respect to m with density h and moreover the system (I, T, μ) is mixing. Therefore, $\mathcal{L}h = h$ and since $\mathcal{L}^* m = m$, we have that $\mu = hm$. Moreover, recalling [3], for any function v of bounded variation, there exists a linear operator $\mathcal{Q}$ with spectral radius $\mathrm{sp}(\mathcal{Q})$ strictly less than 1, such that

$$\mathcal{L}v = h \int v\, dm + \mathcal{Q}v. \tag{1.9}$$

By the closeness of the spectra the same representation holds for $\mathcal{L}_0 : BV \to BV$, namely there will be a number λ_0, a non-negative function $\tilde{h}_0 \in BV$, a probability measure μ_0 and a linear operator $\mathcal{Q}_0$ such that $\mathcal{Q}_0(h_0) = 0$, i.e. $\mathcal{Q}_0$ projects on the complement of $\mathrm{Span}\{h_0\}$, with spectral radius strictly less than 1 such that for any $v \in \mathrm{BV}$:

$$\mathcal{L}_0 \tilde{h}_0 = \lambda_0 \tilde{h}_0, \ \mathcal{L}_0^* \mu_0 = \lambda_0 \mu_0 \tag{1.10}$$

$$\lambda_0^{-1} \mathcal{L}_0 v = h_0 \int v\, d\mu_0 + \mathcal{Q}_0 v. \tag{1.11}$$

Notice that we normalize $\tilde{h}_0$ in such a way that $\int \tilde{h}_0 d\mu_0 = 1$. Thus h_0 in the expression of ν will be given by $h_0 = \tilde{h}_0 / d$ where $d = \int \tilde{h}_0 dm$. Therefore, in the framework of small holes we will have $\lambda_0 = \alpha$; moreover the measure $\Lambda = h_0 \mu_0$ will be T-invariant and $\Lambda(X) = \frac{1}{d}$.

2. Extreme value distribution

For a fixed *target point* $z \in X_\infty$ let us consider the observable

$$\phi(x) = -\log |x - z| \quad \text{for } x \in I,$$

and the function

$$M_n(x) := \max\{\phi(x), \cdots, \phi(T^{n-1}x)\}.$$

For $u \in \mathbb{R}_+$, we are interested in the probabilities of $M_n \leqslant u$, where M_n is now seen as a random variable on a suitable (yet to be chosen) probability space $(\Omega, \mathbb{P})$. First of all we notice that the set of $x \in I$ for which it holds $\{M_n \leqslant u\}$ is equivalent to the set $\{\phi \leqslant u, \ldots, \phi \circ T^{n-1} \leqslant u\}$. In turn this is the set $E_n := (B^c \cap T^{-1}B^c \cdots \cap T^{-(n-1)}B^c)$ where, for simplicity of notation, we denote with B^c the complement of the open ball $B := B(z, e^{-u})$, which we call the *target* (set). So far we are following points which will enter the ball B for the first time after at least n steps, but we should also guarantee that they have not fallen into the hole before entering the target. Therefore we should consider the event: $E_n \cap X_{n-1}$ conditioned on X_{n-1}, i.e., conditioned on the event of not terminating at least for $n-1$ steps. To assure that, the natural sequence of probability measures is given by the following.

Definition 2.1. For any Borel set $A \subset I$ and any $n \geqslant 1$ we introduce the sequence of probability measures:

$$\mathbb{P}_n(A) := \frac{\nu(A \cap X_{n-1})}{\nu(X_{n-1})}.$$

Suppose now that, rather than taking one ball B, we consider a sequence of balls $B_n := B(z, e^{-u_n})$ centered at the target point z and of radius e^{-u_n}. Therefore:

$$\mathbb{P}_n(M_n \leqslant u_n) = \frac{1}{\nu(X_{n-1})} \int_I \mathbb{1}_{B_n^c \cap X_0}(x) \cdots \mathbb{1}_{B_n^c \cap X_0}(T^{n-1}x) d\nu, \tag{2.1}$$

and we will consider the limit for $n \to \infty$, where u_n is a *boundary level* sequence which guarantees the existence of a non-degenerate limit. We anticipate that such a sequence will be dictated directly by the proof below and it must satisfy for a given τ

$$n\, \Lambda\big(B(z, e^{-u_n})\big) \to \tau \quad \text{as } n \to \infty. \tag{2.2}$$

By introducing our second perturbed operator $\tilde{\mathcal{L}}_n : BV \to BV$ acting as

$$\tilde{\mathcal{L}}_n v = \mathcal{L}_0(v \mathbb{1}_{B_n^c}) = \mathcal{L}(v \mathbb{1}_{B_n^c} \mathbb{1}_{X_0}),$$

it is straightforward to check that

$$\mathbb{P}_n(M_n \leqslant u_n) = \frac{1}{\alpha^{n-1}} \int_I \tilde{\mathcal{L}}_n^n h_0 \, dm. \tag{2.3}$$

Roughly speaking, when $n \to \infty$, the operator $\tilde{\mathcal{L}}_n$ converges to $\mathcal{L}_0$ in the spectral sense as $\mathbb{1}_{B_n^c}$ becomes less and less relevant in $\mathcal{L}_0(v\mathbb{1}_{B_n^c})$. In particular, the top eigenvalue of $\tilde{\mathcal{L}}_n$ will converge to that of $\mathcal{L}_0$ and this will allow us to control the asymptotic behavior of the integral on the right hand side of (2.3). We now make these arguments rigorous by adapting the perturbative strategy put forward in [22, 24]. We will work with the following hypothesis.

Standing assumptions. Assume that $h_- := \operatorname{ess\,inf}_{\mathrm{supp}(\Lambda)} h_0 > 0$, i.e., the essential infimum is taken with respect to Λ. Let

$$r_{k,n} := \frac{\Lambda(B_n \cap T^{-1}B_n^c \cap \cdots \cap T^{-k}B_n^c \cap T^{-(k+1)}B_n)}{\Lambda(B_n)},$$

where $r_{k,n}$ is the conditional probability with respect to Λ that we return to B_n exactly after $k+1$ steps. Assume that

$$r_k = \lim_{n\to\infty} r_{k,n} \quad \text{exists for all } k.$$

We will now prove that we satisfy the necessary assumptions A1–A4 of [22, 24].

Assumption 1. The operators $\tilde{\mathcal{L}}_n$ enjoy the same Lasota–Yorke inequalities (1.6) with the same expansion constant χ and b in front of the weak norm. It is sufficient to adapt the arguments of [29] by replacing $\mathbb{1}_{X_0}$ with $\mathbb{1}_{X_0 \cap B_n^c}$.

Assumption 2. We now compare the two operators; here the weak and strong Banach spaces will be again L^1 and BV. We have:

$$\int |(\mathcal{L}_0 - \tilde{\mathcal{L}}_n)v| \, dm = \int |\mathcal{L}_0(v\mathbb{1}_{B_n})| \, dm \leqslant \|v\|_{BV}\, m(B_n \cap X_0), \tag{2.4}$$

by expressing $\mathcal{L}_0$ in terms of $\mathcal{L}$ and since the L^∞ norm of v is bounded by $\|v\|_{BV}$ in one dimensional systems; see [3, Section 2.3]. Then, for the triple norm, $|||\mathcal{L}_0 - \tilde{\mathcal{L}}_n|||_1 \leqslant m(B_n \cap X_0)$ and therefore for n large enough (see Section 1.2), we get the following spectral properties, analogously to (1.10), namely:

$$\tilde{\mathcal{L}}_n h_n = \lambda_n h_n, \ \tilde{\mathcal{L}}_n^* \mu_n = \lambda_n \mu_n \tag{2.5}$$

$$\lambda_n^{-1} \tilde{\mathcal{L}}_n g = h_n \int g \, d\mu_n + \tilde{\mathcal{Q}}_n g, \tag{2.6}$$

where $h_n \in \mathrm{BV}$, μ_n a Borel measure such that $\int h_n d\mu_n = 1$ and $\tilde{\mathcal{Q}}_n$ a linear operator with spectral radius less than one; moreover $\sup_n \mathrm{sp}(\tilde{\mathcal{Q}}_n) < \mathrm{sp}(\mathcal{Q}) < 1$.

Assumption 3. Next, we need to show that

$$\sup\left\{ \int (\mathcal{L}_0 - \tilde{\mathcal{L}}_n) v \, d\mu_0 : v \in \mathrm{BV}, \|v\|_{\mathrm{BV}} \leqslant 1 \right\} \times \|\mathcal{L}_0(h_0 \mathbb{1}_{B_n})\|_{\mathrm{BV}} \leqslant C_\sharp \Delta_n, \tag{2.7}$$

where

$$\Delta_n := \int \mathcal{L}_0(\mathbb{1}_{B_n} h_0) \, d\mu_0 = \alpha \Lambda(B_n)$$

and $C_\sharp$ is a constant. Notice that the first term on the left hand side of (2.7) is the triple norm $|||\mathcal{L}_0 - \tilde{\mathcal{L}}_n|||_{\mu_0}$.[5] This is bounded by $\alpha\mu_0(B_n)$, as can be obtained by an argument analogous to (2.4), combined with (1.5).[6] The second factor is bounded by the Lasota–Yorke inequality with a constant C_{h_0} depending on h_0. Then by the first standing assumption $\alpha C_{h_0} \mu_0(B_n) \leqslant \frac{\alpha C_{h_0}}{h_-} \Lambda(B_n)$.

Assumption 4. We now define the following quantity for $k \geqslant 0$:

$$q_{k,n} := \frac{\int (\mathcal{L}_0 - \tilde{\mathcal{L}}_n) \tilde{\mathcal{L}}_n^k (\mathcal{L}_0 - \tilde{\mathcal{L}}_n)(h_0) \, d\mu_0}{\Delta_n}. \tag{2.8}$$

By the duality properties enjoyed by the transfer operators with respect to our standing assumption, it is easy to show that

$$q_{k,n} = \alpha^{k+1} r_{k,n}. \tag{2.9}$$

We observe that by the Poincaré Recurrence Theorem with respect to the invariant measure Λ, as $r_{k,n}$ is the probability that the system returns to B_n in exactly $k+1$ steps, we have

$$\sum_{k=0}^{\infty} \alpha^{-(k+1)} q_{k,n} = \sum_{k=0}^{\infty} r_{k,n} = 1.$$

We denote by θ the *extremal index* (EI), which will be therefore between 0 and 1:

$$\theta := 1 - \sum_{k=0}^{\infty} r_k.$$

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[5] The reader could wonder why we used two different triple norms, the first in (2.4) with respect to m and the second in (2.7) with respect to μ_0. The first was used to get the quasi-compactness representation for the operator $\tilde{\mathcal{L}}_n$ given in (2.5) and we should use there the same couple of adapted function spaces L^1 and BV as prescribed by the main theorem in [24]. The second allowed us to compare the maximal eigenvalues of $\mathcal{L}_0$ and $\tilde{\mathcal{L}}_n$ and it requires the eigenfunction of the dual of $\mathcal{L}_0$, which is μ_0 as prescribed in [23].

[6] We used here that $\sup_I v \leqslant v(0) + |v|_{\mathrm{TV}}$, where $|\cdot|_{\mathrm{TV}}$ denotes the total variation seminorm. Since μ_0 is not atomic (see next section), we can take $v(0) = 0$. A similar estimate was used in the bound given in the proof of [29, Lemma 7.2].

In order to apply the perturbation theorem by Keller and Liverani [23], we need that the eigenfunction of $\tilde{\mathcal{L}}_n$ be chosen in such a way that $\int h_n d\mu_0 = 1$ and $\int h_n d\mu_n = 1$. This can be accomplished by replacing the previous quantities in [23] with $\hat{h}_n = \frac{h_n}{\int h_n d\mu_0}$ and $\hat{\mu}_n = \mu_n \int h_n d\mu_0$. With our standing assumption, since we satisfy A1–A4, the mentioned perturbation theorem gives (we recall the top eigenvalue of $\mathcal{L}_0$, λ_0, is equal to α)

$$\lambda_n = \alpha - \theta\ \Delta_n + o(\Delta_n) = \alpha \exp\left(-\tfrac{\theta}{\alpha}\Delta_n + o(\Delta_n)\right), \text{ as } n \to \infty, \tag{2.10}$$

or equivalently,

$$\lambda_n^n = \alpha^n \exp\left(-\tfrac{\theta}{\alpha} n\Delta_n + o(n\Delta_n)\right). \tag{2.11}$$

We now substitute (2.11) in the right hand side of (2.3) and use (2.5) to get

$$\begin{aligned}\mathbb{P}_n(M_n \leqslant u_n) &= \frac{1}{\alpha^{n-1}} \int \lambda_n^n \hat{h}_n\, dm \int h_0\, d\hat{\mu}_n + \lambda_n^n \int \widetilde{\mathcal{Q}}_n^n h_0\, dm \\ &= \alpha \exp(-\tfrac{\theta}{\alpha} n\Delta_n + o(n\Delta_n)) \int \hat{h}_n\, dm \int h_0\, d\hat{\mu}_n + \lambda_n^n \int \widetilde{\mathcal{Q}}_n^n h_0\, dm.\end{aligned}$$

It has been proved in [23, Lemma 6.1] that $\int \tilde{h}_0\, d\hat{\mu}_n \to 1$ for $n \to \infty$. Therefore $\int h_0\, d\hat{\mu}_n = \frac{1}{d} \int \tilde{h}_0\, d\hat{\mu}_n \to \frac{1}{d}$.

Now we observe that by (2.4) and by the perturbative theorem in [24], we have that $|h_n - \tilde{h}_0|_1 \to 0$ as $n \to \infty$. Analogously $\tilde{\mathcal{L}}_n$ can be considered as a perturbation of $\mathcal{L}_0$ acting this time on the weak space $L^1(\mu_0)$. The proof of Lemma 7.4 in [29] shows that uniform Lasota-Yorke inequalities still hold for the operators $\mathcal{L}_0$ and $\tilde{\mathcal{L}}_n$, and moreover they are close in the triple norm with respect to μ_0, see (2.7). Therefore the spectral projectors on the one-dimensional eigenspace generated by $\tilde{h}_0$ and h_n will converge in the $L^1(\mu_0)$ norm still by [24], which implies that $|h_n - \tilde{h}_0|_{\mu_0} \to 0$. Then $\int \hat{h}_n dm = \frac{\int h_n dm}{\int h_n d\mu_0} \to \frac{\int \tilde{h}_0 dm}{\int \tilde{h}_0 d\mu_0} = d \int h_0 dm$. Moreover

$$\int h_0\, dm = \frac{1}{\alpha} \int \mathcal{L}_0 h_0\, dm = \frac{1}{\alpha} \int \mathcal{L}(h_0 \mathbb{1}_{X_0})\, dm = \frac{1}{\alpha} \int h_0 \mathbb{1}_{X_0}\, dm = \frac{1}{\alpha} \nu(X_0) = \frac{1}{\alpha},$$

and this term will compensate the α in the numerator in the equality above. Note that the choice given by (2.2) is equivalent to $n\Delta_n \to \alpha\tau$. In this case λ_n^n will be simply bounded in n and $\int |\widetilde{\mathcal{Q}}_n^n(h_0)|\, dm \leqslant \mathrm{sp}(\mathcal{Q})^n \|h_0\|_{BV} \to 0$. In conclusion we have

$$\lim_{n\to\infty} \mathbb{P}_n(M_n \leqslant u_n) = e^{-\tau\theta}, \tag{2.12}$$

which is the Gumbel's law.

3. The extremal index

3.1. **Smallness of the hole.** We briefly return to Section 1.2 to quantify the distance between the maximal eigenvalue of $\mathcal{L}$, which is 1, and that of $\mathcal{L}_0$, which is $\alpha \leqslant 1$. In the previous section we described the asymptotic deviation of λ_n from α as $n \to \infty$. For the next considerations we will compare α to 1. This is given in [23, formula (2.3)], and with our notation reads as (see (1.8)):

$$1 - \alpha \leqslant \hat{C} \|\!|\mathcal{L}_0 - \mathcal{L}|\!\|_1 \leqslant \hat{C} m(H), \tag{3.1}$$

where the constant $\widehat{C}$ is computed explicitly in [22, Section 2.1] and depends on the density h_0. We now strengthen the assumption on the "smallness" of the hole by requiring that $m(H)$ is such that for a fixed $1 < D < \beta = \inf_I |T'|$ it holds

$$\alpha > \frac{D}{\beta}; \tag{3.2}$$

for instance take $m(H) \leqslant \frac{1}{\widehat{C}}(1 - \frac{D}{\beta})$. This has two interesting consequences; one will be established at the end of this section when we will compute the extremal index for periodic points. The other one states that the measure μ_0, and therefore Λ, is not atomic. The proof is a straightforward adaption of [20, Lemma 2], where the conformal structure of μ_0 is used and their "d" is replaced by our "D". Another proof of the non-atomicity of Λ for more general holes is given in [29, Lemma 4.3].

3.2. **Position of the target point.** We now return to the computation of the extremal index θ, which relies on the $r_{k,n}$. By using the fact that we restricted our considerations to the potential $\frac{1}{|T'|}$ we can easily reproduce the arguments on the invariant set X_∞. These give two types of behavior according to the nature of the target point z; see [1, 14, 15] for similar computations for different kinds of dynamical systems. Recall that we write B_n instead of $B(z, e^{-u_n})$. By recalling the definition of Lasota–Yorke maps, let z be a non-periodic point and not belonging to the countable union S of the preimages of the boundary points of the domains of local injectivity of T. On $I \backslash S$, the maps $T^n, n \geqslant 1$, are all continuous and moreover $\Lambda(I\backslash S) = \frac{1}{d}$. Now, we fix k and go to the limit for large n in (2.9). By exploiting the continuity of T^k and by taking n large enough, all the points in B_n will be around $T^k(z)$ and at a positive distance from B_n, so that $r_{k,n}$ is zero and no limit in n is required any more.

Suppose now z is a periodic point of minimal period p; all the $r_{k,n}$ with $k \neq p-1$ are zero for the same reason exposed above. When $k = p-1$ any point in B_n will be at a positive distance from B_n when iterated $p-2$ times; this again is a consequence of continuity for large n. But for $k = p-1$, $T^{k+1}(z) = T^p(z) = z$; by taking again n large enough there will be only one preimage of $T^{-p}B_n$, denoted $T_z^{-p}B_n$ intersecting B_n. Since the map T^p is uniformly expanding, such a preimage will be properly included in B_n. We are thus led to compute

$$\frac{\Lambda(T_z^{-p}B_n)}{\Lambda(B_n)} = \frac{\int_{T_z^{-p}B_n} h_0 d\mu_0}{\int_{B_n} h_0 d\mu_0}. \tag{3.3}$$

We now make an additional *assumption*, namely that h_0 is continuous at z; we recall that the set of discontinuity points is countable, since $h_0 \in \mathrm{BV}$. Since z is periodic with period p we have to compare the density at the numerator and at the denominator in (3.3) in two close points and both close to z. Therefore

$$\frac{\Lambda(T_z^{-p}B_n)}{\Lambda(B_n)} \sim \frac{\int_{T_z^{-p}B_n} d\mu_0}{\int_{B_n} d\mu_0},$$

and the equality will be restored in the limit of large n when the previous two close points will converge to z. So we are left with estimating the ratio $\frac{\mu_0(T_z^{-p}B_n)}{\mu_0(B_n)}$; we point out again that $B_n = T^p(T_z^{-p}B_n)$ and that T^p is one-to-one on $T_z^{-p}B_n$.

Therefore, by considering T^p and iterating (1.4), we obtain

$$\frac{\mu_0(T_z^{-p}B_n)}{\mu_0(B_n)} = \frac{\mu_0(T_z^{-p}B_n)}{\int_{T_z^{-p}B_n} \alpha^p |(T^p)'|(y) d\mu_0(y)}.$$

Passing to the limit and exploiting again the continuity of T^p at z, we finally have

$$r_{p-1} = \frac{1}{\alpha^p |(T^p)'|(z)}, \text{ and } \theta = 1 - \frac{1}{\alpha^p |(T^p)'|(z)}$$

where $\alpha|T'(z)| > D > 1$. By collecting the previous result we have proved the following:

Proposition 3.1. *Let T be a uniformly expanding map of the interval I preserving a mixing measure. Let us fix a small absorbing region, a hole $H \subset I$; then there will be an absolutely continuous conditionally invariant measure ν, supported on $X_0 = I \backslash H$ with density h_0. Write $\alpha = \nu(T^{-1}X_0)$. If the hole is small enough there will be a probability measure μ_0 supported on the surviving set X_∞ such that the measure $\Lambda = h_0\mu_0$ is T-invariant; we will assume that h_0 is bounded away from zero. Having fixed the positive number τ, we take the sequence u_n satisfying $n\Lambda(B(z, \exp(-u_n))) = \tau$, where $z \in X_\infty$. Then, we take the sequence of conditional probability measures $\mathbb{P}_n(A) = \frac{\nu(A \cap X_{n-1})}{\nu(X_{n-1})}$, for $A \subset I$ measurable, and define the random variable $M_n(x) := \max\{\phi(x), \cdots, \phi(T^{n-1}x)\}$, where $\phi(x) = -\log|x - z|$. Moreover we will suppose that all the iterates $T^n, n \geqslant 1$ are continuous at z and also that h_0 is continuous at z when the latter is a periodic point. Then we have:*

- *If z is not a periodic point:*

$$\mathbb{P}_n(M_n \leqslant u_n) \to e^{-\tau}.$$

- *If z is a periodic point of minimal period p, then*

$$\mathbb{P}_n(M_n \leqslant u_n) \to e^{-\tau\theta},$$

  where the extremal index θ is given by:

$$\theta = 1 - \frac{1}{\alpha^p |(T^p)'|(z)}$$

Note that in the literature the *escape rate* η for our open system is usually defined as $\eta = -\log \alpha$; thus we can see the extremal index as

$$\theta = 1 - \frac{1}{e^{-p\eta} |(T^p)'|(z)}.$$

Remark 3.2. We presented here the simplest possible case. However, starting again from the transfer operator (1.1), it could be possible to perform the same analysis with a different potential, adapting the construction of the spaces to handle different weights. As a starting point, [29] contains elements to treat conditional measures in such situations.

Remark 3.3. In light of [4, 11], it would be interesting to construct a statement analogous to our main Proposition 3.1, when either the hole is not of a given size or the dynamics generated is mixing at a subexponential rate.

Remark 3.4. An analogous billiard statement, following [9], could be constructed from the above provided there is enough hyperbolicity to beat the complexity growth. In a nutshell, given a billiard, one can consider the Poincaré map given by the collision with the scatterers. One has then the freedom to choose absorbing scatterers and target scatterers as long as the absorbing part is not too wide.

Remark 3.5. As the approach to study the extremal index is perturbative in nature, it should not come as a surprise that one could consider a one-parameter family of maps T_ε which are small perturbations of T. It could be possible, following some of the techniques of [1, 18], to establish the behavior of the extremal index with respect to deterministic perturbations or noisy perturbations in our framework of targets and holes.

For simplicity we restricted ourselves to one target point z and "an" observable, i.e., the logarithmic distance. These hypotheses are not essential, and are merely there to simplify the presentation. The theory could be rewritten after considering a finite set of target points $\{z_1, z_2, z_3, \ldots, z_n\}$ and adjusting the sequence of balls B_n to have multiple connected components as long as each point satisfies the same kind of law: we avoid doing so not to clutter the exposition. Moreover, suppose that one is not interested in reaching a certain target point or avoiding a region, but one has an observable of interest (say for example the speed of some object, or the depth of some path) and wants to compute the probabilities with respect to the values of such observable. Obviously, before applying our setup, one has the extra problem of identifying regions of the phase space that correspond to such values of the observables: in one dimension this does not create additional difficulties (see also [7]).

3.3. **On the choice of the boundary sequence.** Let us now comment on (2.2), i.e., the scaling behavior $n\Lambda(B(z, e^{-u_n})) \to \tau$. As we already argued, the measure Λ is not atomic, $\Lambda(B)$ varies continuously with the radius of the ball. Therefore, for any fixed τ and n we could choose u_n so that

$$\Lambda(B(z, e^{-u_n})) = \frac{\tau}{n}. \tag{3.4}$$

Unluckily, the measure Λ is often not computationally accessible. However, we can use the following approximation scheme to construct a sequence of u_n which still satisfies (2.2). Let

$$d_n(z) := \frac{\log \Lambda(B(z, e^{-u_n}))}{\log e^{-u_n}}.$$

Since the density h_0 is bounded away from zero by the standing assumptions, for δ arbitrarily small and n large enough we have that

$$d_n(z) \geqslant \frac{\log \mu_0(B(z, e^{-u_n}))}{\log e^{-u_n}} - \delta.$$

By [29, Theorem B], whenever the map T has large images and large images with respect to the hole H (see the discussion before [29, Theorem B]), then for all $z \in X_\infty$, there exists $t_0 > 0$ such that

$$\liminf_{n\to\infty} \frac{\log \mu_0(B(z, e^{-u_n}))}{\log e^{-u_n}} \geqslant t_0$$

and the Hausdorff dimension of the surviving set $HD(X_\infty)$ satisfies

$$HD(X_\infty) \geqslant t_0.$$

Therefore, if we fix again δ and take correspondingly n large enough we have that $d_n(z) \geqslant t_0 - \delta - \delta \geqslant t_0 - 2\delta$ which implies $\Lambda(B(z, e^{-u_n})) \leqslant e^{-u_n(t_0-2\delta)}$, and, together with (3.4), finally $\tau \leqslant ne^{-u_n(t_0-2\delta)}$. In other words, $u_n \leqslant -\frac{\log \tau}{t_0-2\delta} + \frac{\log n}{t_0-2\delta}$, which can also be written as

$$\sup_n \left\{u_n - \tfrac{\log n}{t_0}\right\} \leqslant -\frac{\log \tau}{t_0}, \tag{3.5}$$

as long as (3.4) still holds true. In the computational approach to extreme value theory, the boundary level u_n is chosen with the help of an affine function (see [28]):

$$u_n = \frac{\log \tau^{-1}}{a_n} + b_n.$$

The sequences a_n and b_n can be obtained with the help of the Generalized Extreme Value (GEV) distribution in order to fit Gumbel's law. The inequality (3.5) suggests that for n large $a_n \sim t_0$ and $b_n \sim \frac{\log n}{t_0}$, therefore we could attain a lower bound for the Hausdorff dimension of the surviving set. We defer, for instance, to [13] to show how to use the GEV distribution to estimate the sequences a_n, b_n, and we will show in future studies how to use such estimates to approach $HD(X_\infty)$.

4. How far are we from the surviving set? The degenerate limit

We noted several times that the support of μ_0 is the surviving set X_∞. This means that if we pick the open ball $B_n = B(z, e^{-u_n})$ centered in a point $z \notin X_\infty$ or even in the hole, then when the radius of the ball is sufficiently small, we have $\mu_0(B_n) = 0$, since X_∞ is a closed set. This immediately implies by the argument similar to that we used in (3.1) that

$$|\lambda_n - \alpha| \leqslant \text{const} \times |||\mathcal{L}_0 - \tilde{\mathcal{L}}_n|||_{\mu_0} \leqslant \text{const} \times \alpha\mu_0(B_n) = 0.$$

The fact that the perturbed eigenvalue could become equal to the unperturbed one for a finite size of the perturbation, is already a part of [23, Theorem 2.1] and is also detailed in [22, Footnote (3)]. Therefore, if we call $\hat{n}$ the first n for which $B_n \cap X_\infty = \varnothing$, for any $n \geqslant \hat{n}$ we have that

$$\mathbb{P}_n(M_n \leqslant u_n) = \alpha \int \hat{h}_n \, dm \int h_0 \, d\hat{\mu}_n + \alpha^n \int \tilde{\mathcal{Q}}_n^n h_0 \, dm.$$

As explained above, for $n \to \infty$ the first term on the right goes to 1 and $\int \tilde{\mathcal{Q}}_n^n h_0 \, dm \to 0$; we thus have that

$$\mathbb{P}_n(M_n \leqslant u_n) \to 1, \;\; n \to \infty. \tag{4.1}$$

Trivially, (4.1) states that if the target point is off the surviving set, then the trajectories will not be able to approach it arbitrary close. This result has two interesting consequences for applications, in particular the second one will provide a full description of the extreme value distribution (EVD) for any choice of the target set.

First, we observe that the limit (4.1) holds for any sequence u_n going to infinity, and for simplicity we now put $u_n = \log n$. Then we could reasonably argue that for the smallest $\hat{n}$ for which

$$\mathbb{P}_{\hat{n}}(M_{\hat{n}} \leqslant \log \hat{n}) \approx 1,$$

then

$$\text{dist}(z, X_\infty) \approx \frac{1}{\hat{n}},$$

where $\approx$ means "approximately equal".

Second, let us return to the statement of our main Proposition 3.1. Whenever we take the point $z \in X_\infty$ and by a suitable choice of the sequence u_n as we explained in Section 3.3, we get a non-degenerate limit for our EVD, in particular different from 1. Instead, if we pick the point z outside the surviving set and no matter what the sequence u_n is, provided it goes to infinity, we get a degenerate limit equal to one for the EVD.

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