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Statistics of temperature increments in fully developed turbulence
Part I. Theory

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Abstract

Using a method recently proposed by Sinai and Yakhot, we obtain the equation for the probability density function (pdf) of the temperature increments in fully developed turbulence. The closure problem is reduced to the determination of two conditional expectations and can be carried out mainly by requiring isotropy and a weak correlation of the physical variables. As a consequence of our analysis, we show that the pdf is nearly Gaussian about zero and then has a stretched exponential behavior for large fluctuations.

1. Introduction

The intermittent nature of small scales in fully developed turbulence has been widely investigated over the last decade either experimentally, e.g. Refs. [1–3], or theoretically. Indeed, following Kolmogorov's refined theory [4], a new generation of models based on fractal and multifractal approaches has been recently worked out, e.g. Refs. [5,6]. Models based on modified versions of the log-normal assumption have also been proposed; these are based on ad hoc probability distributions of the small scales and provide relatively good agreement with experimental data. One of the main properties inferred from all of these investigations is that small-scale statistical distributions associated with either the velocity field or a passively advected scalar are very far from Gaussianity, e.g. Refs. [2,7]. Another characteristic feature of the problem is that so far it was not possible to directly assess the intermittent nature of the small scales from the transport equations for the turbulent velocity or scalar fields. Indeed, most of the models are based on global statistical approaches describing the mechanisms involved in the inertial transfer of energy from the larger scales to the smaller ones.

On the contrary, in this paper, we are directly focusing on the conservation equation for an advected passive scalar in a turbulent flow. We generalize the Yakhot–Sinai technique recently

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introduced to compute the probability density function (pdf) for temperature advected in a random velocity field [8] and in Rayleigh–Bénard convection [9,10]. Attention is paid hereafter to the determination of the pdf equation for the temperature increment $\Delta\theta(\boldsymbol{r},\boldsymbol{x}) = \theta(\boldsymbol{x}+\boldsymbol{r}) - \theta(\boldsymbol{x})$ between two points separated by a vector distance $\boldsymbol{r}$. We now briefly sketch the strategy of this method: starting from the conservation equation for the temperature field $\theta(\boldsymbol{x},t)$, we generalize the Yaglom formula [11] to the determination of the mixed temperature–velocity structure functions $\langle(\Delta\theta)^n\,\Delta u_L\rangle$ for homogeneous and isotropic turbulence (Δu_L denotes the longitudinal velocity increment). Therefore, we replace the spatial mean $\langle\cdot\rangle$ by the ensemble average and introduce a probability density function which is the product of the pdf for the temperature difference, say $P(r,X)$, where X is the value of the stochastic variable, and the conditional probability densities of the velocity difference Δu_L and the square of the temperature gradient $(\nabla\theta)^2$.

The above generalization of the Yaglom formula for all the moments of $\Delta\theta$ then allows us to write the partial differential equation for $P(r,X)$ which turns out to be a linear PDE whose coefficients are expressed in terms of the conditional expectations of Δu_L and $(\nabla\theta)^2$ respectively. Let us call $q_1(r,X)$ and $q_2(r,X)$ these two conditional expectations. The closure hypothesis of our theory is then just lying in the determination of the functions $q_1(r,X)$ and $q_2(r,X)$. We think that the main novelty of the Yakhot–Sinai approach consists in replacing any closure assumption on *conditional probabilities* by an equivalent assumption on *conditional expectations*, which is clearly a weaker statistical request. We will give some theoretical arguments to determine q_1 and q_2; basically, we will perturb our conditional expectations around the situation of statistical independence (the two variables are uncorrelated) and use the local isotropy of the fields. Our predictions for q_1 and q_2 are confirmed by experimental observations [12]. We will finally solve the partial differential equation for $P(r,X)$ in the asymptotic regions $X\approx 0$ and $|X|\to\infty$.

The main results of our analysis are:

First, we show that the pdf is fairly Gaussian for small $|X|$, but has a well stretched-exponential behavior for large $|X|$, as confirmed by several experimental observations [13,14]. The analysis is carried out both in the inertial range and in the dissipative one.

Second, an interesting fact appears in the experimental determination of $q_1(r,X)$ in the inertial range, namely, the q_1 we find for a large interval of values of r is in contradiction with the local isotropy of the velocity and temperature fields. This anomaly disappears for small values of r where we recover q_1 in agreement with local isotropy. It is just this last q_1 we used in the solution of our differential equations, but we think that the discrepancy observed in the inertial range needs further investigations and understanding.

During the preparation of this work, we discovered that similar problems were addressed in Ref. [15]: Eq. (8) in Ref. [15] is similar to our starting equation (2.8), but the successive derivations are quite different, since, for example, we derive a partial differential equation instead of an ordinary one, where the scale r plays a fundamental role. Moreover, we use the well-established statistical calculus of turbulence [11] and take care to guarantee the existence of the solution of our equation.

The experimental analysis of the model developed in this article will be reported in a forthcoming paper [12].

2. The model

In this section, we generalize the Yaglom formula for the determination of the mixed moment $\langle(\Delta\theta)^2\,\Delta u_L\rangle$ and extend it to any moment of the type $\langle(\Delta\theta)^n\,\Delta u_L\rangle$. We will use this result to write down a PDE for the probability density function of $\Delta\theta$, in section 3.

Let us consider the conservation equation for temperature:

$$\frac{\partial\theta(\boldsymbol{x},t)}{\partial t}+\boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla}\theta(\boldsymbol{x},t)=k_0\,\nabla^2\theta(\boldsymbol{x},t)\,, \tag{2.1}$$

where k_0 is the thermal diffusivity. The temperature and velocity fields will be also computed in the point $(\boldsymbol{x}+\boldsymbol{r},t)$ and in that case the operations of derivation respectively read

$$\boldsymbol{\nabla}\theta(\boldsymbol{x}+\boldsymbol{r},t)=\left(\frac{\partial\theta(\boldsymbol{\xi},t)}{\partial\xi_i}\bigg|_{\xi_i=x_i+r_i}\right)_{i=1,2,3}, \tag{2.2}$$

$$\nabla^2\theta(\boldsymbol{x}+\boldsymbol{r},t)=\sum_{i=1}^{3}\frac{\partial^2\theta(\boldsymbol{\xi},t)}{\partial\xi_i^2}\bigg|_{\xi_i=x_i+r_i}, \tag{2.3}$$

where $\boldsymbol{\xi}$ denotes the first set of spatial coordinates. The quantity we are interested in is the temperature increment or

$$\Delta\theta(\boldsymbol{r},\boldsymbol{x},t)=\theta(\boldsymbol{x}+\boldsymbol{r},t)-\theta(\boldsymbol{x},t)\,. \tag{2.4}$$

Following the approach developed in Ref. [8], we first define the following quantity D_n for any integer $n>1$, which is a function of the independent variables $\boldsymbol{r}$, $\boldsymbol{x}$ and of the time t:

$$\begin{aligned}D_n(\boldsymbol{r},\boldsymbol{x},t)\equiv{}&\frac{\partial}{\partial t}\{[\Delta\theta(\boldsymbol{r},\boldsymbol{x},t)]^n\}\\&+\boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla}\{[\Delta\theta(\boldsymbol{r},\boldsymbol{x},t)]^n\}\\&-k_0\boldsymbol{\nabla}^2\{[\Delta\theta(\boldsymbol{r},\boldsymbol{x},t)]^n\}\,.\end{aligned} \tag{2.5}$$

By developing $D_n(\boldsymbol{r},\boldsymbol{x},t)$, we meet terms of the type

$$\boldsymbol{\nabla}\theta(\boldsymbol{x}+\boldsymbol{r},t)\quad\text{and}\quad\nabla^2\theta(\boldsymbol{x}+\boldsymbol{r},t)\,. \tag{2.6}$$

These terms should be written as

$$\boldsymbol{\nabla}_x\theta(\boldsymbol{x}+\boldsymbol{r},t)\quad\text{and}\quad\nabla_x^2\theta(\boldsymbol{x}+\boldsymbol{r},t)\,, \tag{2.7}$$

meaning derivation with respect to the $\boldsymbol{x}$ variables of the composite function $\theta(\boldsymbol{\xi},t)\circ(\boldsymbol{x}+\boldsymbol{r},t)$.

However, by a trivial application of the chain rule, the quantities (2.7) computed in the point $\boldsymbol{x}$ (with $\boldsymbol{r}$ regarded as a fixed parameter independent from $\boldsymbol{x}$) are the same as the corresponding quantities (2.2) and (2.3) computed in the point $\boldsymbol{x}+\boldsymbol{r}$. Then, an easy calculation shown in the appendix gives

$$\begin{aligned}D_n(\boldsymbol{r},\boldsymbol{x},t)={}&-n[\Delta\theta(\boldsymbol{r},\boldsymbol{x},t)]^{n-1}\\&\times\Delta\boldsymbol{u}(\boldsymbol{r},\boldsymbol{x},t)\cdot\boldsymbol{\nabla}\theta(\boldsymbol{x}+\boldsymbol{r},t)\\&-n(n-1)k_0[\Delta\theta(\boldsymbol{r},\boldsymbol{x},t)]^{n-2}\\&\times\{\nabla[\Delta\theta(\boldsymbol{r},\boldsymbol{x},t)]\}^2\,,\end{aligned} \tag{2.8}$$

where $\Delta\boldsymbol{u}(\boldsymbol{r},\boldsymbol{x},t)=\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t)-\boldsymbol{u}(\boldsymbol{x},t)$. We then average $D_n(\boldsymbol{r},\boldsymbol{x},t)$ over all $\boldsymbol{x}$ within $\mathbb{R}^3$ in the stationary state. We could first average and then take the time derivative: this derivative would be zero according to the Kolmogorov similarity hypothesis for which the statistical characteristics of our fields are independent of time (see Monin and Yaglom [11, Vol. 2, p. 401]). In the regime of homogeneous and isotropic turbulence, and using also the property of divergencelessness of the velocity field and zero boundary conditions for all the variables and their derivatives at infinity, we easily get $\langle D_n(\boldsymbol{r},\boldsymbol{x})\rangle=0$, which implies

$$\begin{aligned}&\frac{1}{k_0(n-1)}\langle[\Delta\theta(\boldsymbol{r},\boldsymbol{x})]^{n-1}\,\Delta\boldsymbol{u}(\boldsymbol{r},\boldsymbol{x})\cdot\boldsymbol{\nabla}\theta(\boldsymbol{x}+\boldsymbol{r})\rangle\\&\quad=-\langle[\Delta\theta(\boldsymbol{r},\boldsymbol{x})]^{n-2}\{\nabla[\Delta\theta(\boldsymbol{r},\boldsymbol{x})]\}^2\rangle\,,\end{aligned} \tag{2.9}$$

where we have dropped the dependence on time t in all the variables. The detailed derivation of Eqs. (2.8) and (2.9), with further comments, is given in the appendix, where we also indicate a different starting point to get Eq. (2.12) below.

When $n=0$, $D_0(\boldsymbol{r},\boldsymbol{x})$ is trivially zero; for $n=1$, $D_1(\boldsymbol{r},\boldsymbol{x})$ is equal to $-\Delta\boldsymbol{u}(\boldsymbol{r},\boldsymbol{x})\cdot\boldsymbol{\nabla}\theta(\boldsymbol{x}+\boldsymbol{r})$ and its mean is zero since $\boldsymbol{u}$ is solenoidal. We further modify (2.9) according to the well-known tensorial statistical calculus of homogeneous and isotropic turbulence (see e.g. Ref. [11, Vol. 2]). The goal is to project Eq. (2.9) along the direction of $\boldsymbol{r}$. We need two additional assumptions which seem very reasonable from a physical point of view:

(H1) First, we require that all the fields (we use the notation of Ref. [11]) $\{\theta^k(\boldsymbol{x}),\,\theta^l(\boldsymbol{x})\,\boldsymbol{u}(\boldsymbol{x})\}$, with k, $l \geq 0$, are isotropic and therefore $\langle \theta^l(x)\, u(x) \rangle = 0$ for $l \geq 0$.

(H2) Second, we require that

$$\langle \theta^j(x)\, \theta^k(x+r)\, \xi_i(x)\, \xi_i(x) \rangle = \langle \theta^j(x+r)\, \theta^k(x)\, \xi_i(x+r)\, \xi_i(x+r) \rangle ,$$

where $\xi_i(x)$ is any component of the gradient field $\partial\theta(x)/\partial x_i$ (see section 12.4 in Ref. [11, Vol. 2] for similar results).

The first hypothesis (H1) is necessary to prove that (see demonstration in appendix) $\forall n \geq 0$:

$$\langle [\Delta\theta(r,x)]^n\, \Delta u_i(r,x) \rangle = \begin{cases} \langle [\Delta\theta(r,x)]^n\, \Delta u_{\mathrm{L}}(r,x) \rangle \dfrac{r_i}{r} , & n \text{ even} , \\ 0 , & n \text{ odd} , \end{cases} \tag{2.10}$$

which is known to hold in the regime of local isotropy for $n = 1,2$ [11, Vol. 2, pp. 103, 104]. $\Delta u_{\mathrm{L}}(r,x)$ represents the projection along the direction of r, and we set $r = |r|$.

With the help of the second hypothesis (H2), we prove in the appendix that

$$\langle [\Delta\theta(r,x)]^n [\nabla\theta(x+r)]^2 \rangle + \langle [\Delta\theta(r,x)]^n [\nabla\theta(x)]^2 \rangle = \begin{cases} 2\langle [\Delta\theta(r,x)]^n [\nabla\theta(x)]^2 \rangle , & n \text{ even} , \\ 0 , & n \text{ odd} . \end{cases} \tag{2.11}$$

Using (2.10) and (2.11) and with straightforward calculus, we prove in the appendix that Eq. (2.9) is equivalent to the following one where only the even moments are kept:

$$\frac{1}{\bar{N} \cdot 2n(2n-1)} \left(\frac{2}{r} + \frac{\partial}{\partial r} \right) \langle [\Delta\theta(r,x)]^{2n}\, \Delta u_{\mathrm{L}}(r,x) \rangle = -2\langle [\Delta\theta(r,x)]^{2n-2} [\nabla\theta(x)]^2 \rangle + \frac{2k_0}{\bar{N} \cdot 2n(2n-1)} \left(\frac{2}{r} + \frac{\partial}{\partial r} \right) \frac{\partial}{\partial r} \langle [\Delta\theta(r,x)]^{2n} \rangle , \tag{2.12}$$

where we used the notation $\bar{N} = k_0 \langle [\nabla\theta(x)]^2 \rangle$ and, consequently, rescaled and renamed the temperature gradient according to $[\nabla\theta(x)]^2 \to [\nabla\theta(x)]^2 / \langle [\nabla\theta(x)]^2 \rangle$. Note that we could equivalently have written: $\langle [\Delta\theta(r,x)]^{2n}\, \Delta u_{\mathrm{L}}(r,x) \rangle = -2\langle [\Delta\theta(r,x)]^{2n}\, u_{\mathrm{L}}(x) \rangle$. As explicitly recalled in the appendix, the isotropy of the fields implies that all the averaged quantities in (2.12) depend only on r.

In the case $n = 1$, Eq. (2.12) reduces to the well-known Yaglom formula [11, Vol. 2, p. 400]

$$\langle [\Delta\theta(r,x)]^2\, \Delta u_{\mathrm{L}}(r,x) \rangle - 2k_0 \frac{\mathrm{d}}{\mathrm{d}r} \langle [\Delta\theta(r,x)]^2 \rangle = -\tfrac{4}{3} \bar{N} r . \tag{2.13}$$

A solution of (2.13) is given in the inertial range, i.e. for separations such that $\eta \ll r \ll L$ where η and L are the Kolmogorov and integral scales respectively, by neglecting the second term in the LHS since this term describes the effect of molecular thermal conductivity [11, Vol. 2, p. 400]. Actually, the ratio of the last term in Eq. (2.12) to the one in the LHS of the same equation is dimensionally proportional to Pr/Re, where Pr is the Prandtl number (here close to 1) and Re is the Reynolds number associated with the considered scale r. Thus, when r lies in the inertial range, $\mathrm{Pr}/\mathrm{Re} \ll 1$ and then we get the scaling

$$\langle \Delta\theta(r,x)]^2\, \Delta u_{\mathrm{L}}(r,x) \rangle = -\tfrac{4}{3} \bar{N} r . \tag{2.14}$$

We make the same assumption in the inertial range by neglecting the last term in the RHS of (2.12); the equation, in the dissipative range, will be treated in section 5.

We have seen that in Eq. (2.12) we have normalized the square of the gradient of temperature. Equivalently, we could have used the normalized fields

$$\overline{\overline{\Delta\theta}}(r,x) = \frac{\Delta\theta(r,x)}{\langle [\Delta\theta(r,x)]^2 \rangle^{1/2}} ,$$

$$\overline{\overline{\Delta u}}_{\mathrm{L}}(r,x) = \frac{\Delta u_{\mathrm{L}}(r,x)}{\langle [\Delta u_{\mathrm{L}}(r,x)]^2 \rangle^{1/2}} ,$$

but, in this case, Eq. (2.12) is more complicated, since the variance of the two fields depend on r and the derivative $\partial/\partial r$ acts explicitly on them. However, if we set ourselves again in the inertial range and neglect terms of the type $\mathrm{d}\langle[\Delta\theta(r,x)]^2\rangle/\mathrm{d}r$ and $\mathrm{d}\langle[\Delta u_L(r,x)]^2\rangle/\mathrm{d}r$ (see Ref. [11, Vol. 2, pp. 397, 400]), and if for the two terms $\langle[\Delta\theta(r,x)]^2\rangle$ and $\langle[\Delta u_L(r,x)]^2\rangle$ we assume scalings respectively of the type $c_1 r^{2/3}$ and $c_2 r^{2/3}$ where c_1 and c_2 depend on $\bar{N}$ and on the mean kinetic energy dissipation rate $\bar{\epsilon}$ [11, Vol. 2, pp. 397, 400], then it is easy to show that Eq. (2.12) without the last term on its RHS remains unchanged provided that the operator $[2/r + \partial/\partial r]$ is replaced by the following one: $c_3(2 + r\,\partial/\partial r)$, where c_3 is a function of c_1 and c_2. In this paper, we prefer to work with the general equation (2.12) since it does not require any scaling for $\langle[\Delta\theta(r,x)]^2\rangle$. Such a result should be deduced from our analysis, once the pdf is known.

3. The statistical approach and the closure hypothesis

We now introduce, following Yakhot and Sinai [8,9], the following stochastic variables on the physical space $\mathbb{R}^3$ endowed with the usual volume measure:

$$X = \Delta\theta(r,x)\,, \quad Y = \Delta u_L(r,x)\,,$$
$$Z = (\nabla\theta(x))^2\,, \tag{3.1}$$

and write the corresponding pdf as

$$P(r,X,Y,Z) = P(r,X)\,Q(r,Y,Z|X)\,, \tag{3.2}$$

which explicitly depends on the parameter $r > 0$ (and not on r, again by isotropy) and where $Q(\cdot\,|\,\cdot)$ denotes a conditional probability. By replacing the spatial mean by the ensemble averaging, we can then write (2.12) as

$$\int X^{2n}\Big[\Big(\frac{2}{r}+\frac{\partial}{\partial r}\Big)[q_1(r,X)\,P(r,X)] + 2\bar{N}\frac{\partial^2}{\partial X^2}[q_2(r,X)\,P(r,X)] - 2k_0\Big(\frac{2}{r}+\frac{\partial}{\partial r}\Big)\frac{\partial}{\partial r}P(r,X)\Big]\,\mathrm{d}X = 0\,, \tag{3.3}$$

where the conditional expectations $q_1(r,X)$ and $q_2(r,X)$ are defined by

$$q_1(r,X) = \int YQ(r,Y|X)\,\mathrm{d}Y\,, \tag{3.4}$$

$$q_2(r,X) = \int ZQ(r,Z|X)\,\mathrm{d}Z\,. \tag{3.5}$$

By the definition of the stochastic variables (3.1), the integrals in Z (respectively Y) in (3.4) (respectively (3.5)) can be factorized and are equal to 1: we therefore only need a two-variable conditional probability $Q(\cdot\,|\,\cdot)$.

To justify Eq. (3.3) and the consequences we will derive from it, we make the following assumptions: first of all, we ask for a symmetric pdf in X. Then, if we return to Eq. (2.10), we see, looking at the proof in the appendix, that we proved more, namely

$$\langle[\Delta\theta(r,x)]^n\,\Delta u_L(r,x)\rangle = 0\,, \quad n \text{ odd}\,, \tag{3.6}$$

that, translated into the statistical language of this section, means

$$\int X^n q_1(r,X)\,P(r,X)\,\mathrm{d}X = 0\,, \quad n \text{ odd}\,. \tag{3.7}$$

This implies, by the growing properties of q_1 and P stated below, that the odd part of the product $q_1 P$ is zero, so that $q_1 P$ is an even function of X and a fortiori q_1 is even. Unfortunately we do not have an analog of (3.7) for $q_2(r,X)$ and therefore we assume that q_2 is an even function of X too. Then, we suppose that q_1 and q_2 have a polynomial growth in X and restrict ourselves to pdfs which are functions of a rapid decrease of X with, at least, a dominant exponential-decay behavior at infinity of the type: $P(r,X) \sim \mathrm{e}^{-b|X|}$. These hypotheses allow us to derive with respect to r inside the integral when r ranges in any domain bounded away from zero; moreover they are sufficient to conclude that, as a consequence of the Paley–Wiener theorem, the vanishing of the integral (3.3) for all the even powers of X implies that

$$\left(\frac{2}{r}+\frac{\partial}{\partial r}\right)[q_1(r,X)\,P(r,X)] + 2\bar{N}\frac{\partial^2}{\partial X^2}[q_2(r,X)\,P(r,X)] - 2k_0\left(\frac{2}{r}+\frac{\partial}{\partial r}\right)\frac{\partial}{\partial r}P(r,X) = 0\,. \tag{3.8}$$

This equation is the main result of this paper; it controls the shape of the pdf of X through the scale r.

Let us now suppose that an exponentially decaying pdf satisfies (3.8) with the RHS equal to an even $F(X)$. Then, $F(X)$ will decay exponentially fast too and its Fourier transform $\hat{F}(\lambda)$ will be real analytic and, by condition (3.3), will satisfy

$$\left.\frac{\partial^J \hat{F}(\lambda)}{\partial \lambda^J}\right|_{\lambda=0} = \frac{1}{2\pi}\int (-\mathrm{i})^J X^J F(X)\,\mathrm{d}X = 0\,, \quad \forall J \geq 0\,,$$

thus implying that $F(X)=0$ identically.

To further illustrate this point, which is not sufficiently considered in the other works related to the Yakhot–Sinai technique, we now give a counter-example. Let us consider the even function $\hat{F}(\lambda)=\exp(-\lambda^{-2}-\lambda^2)$, which is $C^\infty(\mathbb{R})$ and is zero with all its derivatives in $\lambda=0$ (but not analytic there). The Fourier transform of $\hat{F}(\lambda)$ is a function $F(X)$ which is analytic on the real axis, of rapid decrease and satisfying $\int X^{2n}F(X)\,\mathrm{d}X=0$, $\forall n\geq 0$. Therefore, we could have a rapidly decreasing pdf satisfying Eq. (3.8) with the RHS equal to $F(X)\neq 0$. To avoid this problem it is sufficient to restrict the class of our pdf as explained before, or otherwise require further conditions on the pdf which allow to discard those cases like the counter-example sketched above. We will return concretely to these questions at the end of section 5.2.

An observation of experiments [12] is that q_1 and q_2 and P are sometimes not even functions of X, but show instead a weak asymmetry around the origin. This is probably due to a lack of the complete local-isotropy regime: this effect is more pronounced for large scales and the symmetry is almost completely restored for small scales, see ref. [12] and a further discussion below. In the forthcoming sections, we will consider q_1 and q_2 as even functions of X, but we think that the question of their symmetry must be better investigated.

3.1. Determination of q_1

If the joint-probability distribution function of the variables $\Delta\theta$ and Δu_L were Gaussian, it is well known that q_1 would be equal to $q_1(r,X)=C(r)\,X$, where $C(r)$ is the correlation coefficient between $\Delta\theta$ and Δu_L, generally a function of r. This ansatz can also be viewed as the first-order expansion of q_1 in X, whenever a weak dependence of the variables Y and X is taken into account in the conditional probability $Q(r, Y|X)$, and this is in agreement with the original spirit of the paper by Yakhot and Sinai. Our primary experimental result [12] leads us to conjecture for q_1 an expression of the form

$$q_1(r,X)=\gamma_1 r^\epsilon X\,, \quad 0<\epsilon<1\,, \quad \gamma_1>0 \tag{3.9}$$

at least for small X, thus recovering the same conditional expectation as given by joint-Gaussian statistics.

Clearly, the scaling (3.9) is not in agreement with the local isotropy of our fields since q_1 should be an even function of X. Nevertheless, some experimental observations for r varying in a large interval of values in the inertial range confirm the validity of (3.9) also for large $|X|$, with the additional fact that the slope γ_1 has a different value according to whether X is positive or negative; this effect becomes more and more prominent when r decreases and, by approaching the dissipative range, the slope γ_1 changes its sign for $X<0$ so that q_1 then assumes the form

$$q_1(r,X)=-d(r)+\gamma_1^{\pm} r^\epsilon X\,, \tag{3.10}$$

with $\gamma_1^- > 0$ for $X<0$ and $\gamma_1^+ < 0$ for $X>0$, where $d(r)$ is a positive function of r necessary to insure the zero mean property of $q_1(r, X)$. As anticipated before, we take in the following $\gamma_1^- = |\gamma_1^+|$ just to guarantee the evenness of $q_1(r, X)$ and therefore to completely restore the local isotropy. Furthermore, we will consider $d(r)$ of the form d/r^ν, with d and ν positive constants. This choice is suggested for separations lying in the inertial range by analytical considerations and experimental observations [12]. This form of $d(r)$ will be only important in the asymptotic region $|X| \to 0$; besides, we will take $d(r) \approx d = \text{const.}$ in the dissipative range. This last assumption is not really necessary for two reasons (and we adopt it only for mathematical convenience): (i) first, if we also keep $\nu > 0$, we will simply get a steeper decaying solution in r; (ii) we could study the case of small $|X|$ directly on Eq. (5.5), where $q_1(r, X)$ is absent and the solution then decays as r^{-1}. As a last remark, we will also consider q_1 growing to infinity as X^2 in the inertial range, to show the influence on the exponential decay in X of the solution of (3.8). We summarize the different choices for q_1 in Table 1.

3.2. Determination of q_2

We now pass to the determination of q_2, the expectation of $Z = |\nabla\theta|^2 / \langle |\nabla\theta|^2 \rangle$ conditioned to $X = \Delta\theta$. Due to the physical nature of these variables, it is reasonable to expect a certain degree of correlation among them even for small $\Delta\theta$. This, and the positivity of Z, suggests for q_2 the form

$$q_2(r, X) = \gamma_2 r^\rho + \gamma_3 \frac{X^2}{r^\sigma}, \quad \rho, \sigma, \gamma_2, \gamma_3 > 0 . \tag{3.11}$$

The term $\gamma_2 r^\rho$ must be less than one since q_2 satisfies the normalization condition $\int q_2(r, X)\, P(r, X)\, \mathrm{d}X = 1$. The power r^σ in the denominator corresponds to the experimental observation [12] that the parabola (3.11) closes sharply whenever r decreases.

Eq. (3.11) can again be viewed as the lowest-order symmetric expansion of q_2, and we conjecture, supported by experimental evidence [12], such a behavior also for large $|X|$ with eventually a different exponent for X. It will be clear from Eq. (4.11) below, that the exponent 2 in X in the scaling (3.11) gives only a subexponential decay of the pdf at infinity and therefore the pdf does not answer the asymptotic request we made on it. This problem will be solved by requiring for q_2 an asymptotic behavior, for large X, as $|X|^\beta$, $0 \le \beta \le 1$; we note that such a scaling for q_2 has been also proposed in ref. [15] with β ranging in $0 \le \beta \le 2$. For our model, β belonging to the interval $]1,2]$ is not sufficient to guarantee the unicity of the solution of (3.8), since the pdf will decay in the subexponential way. Whenever $|X|$ is small, it will be useful to consider a slightly different form for q_2, which allows us to separate

Table 1
Modelling hypotheses for the conditional expectation q_1

	$q_1(r, X)$	
	$\|X\|$ small	$\|X\|$ large
Inertial range	$+\frac{d}{r^\nu} + \gamma_1^+ r^\epsilon \|X\| \approx + \frac{d}{r^\nu}$	$+\frac{d}{r^\nu} + \gamma_1^+ r^\epsilon \|X\|^\eta \approx \gamma_1^+ r^\epsilon \|X\|^\eta$; $\eta = 1, 2$
Dissipative range	$+\frac{d}{r^\nu} + \gamma_1^+ r^\epsilon \|X\| \approx d$	$+\frac{d}{r^\nu} + \gamma_1^+ r^\epsilon \|X\| \approx \gamma_1^+ r^\epsilon \|X\|$

the variables in the solution of (3.8), as explained in the next section. Since $q_2(r, X)$ can be written as

$$q_2(r, X) = r^{\rho}\left(\gamma_2 + \frac{\gamma_3}{r^{\sigma+\rho}} X^2\right),$$

we will suppose that the quantity $\gamma_3/r^{\sigma+\rho}$ is a slowly varying function of r both in the inertial and dissipative ranges. In the dissipative case, we will require that γ_3 is much larger than the corresponding quantity in the inertial range and we will simply rename $\gamma_3/r^{\sigma+\rho} \equiv \gamma_3 = \text{const}$. At the end of section 5.1 we will, however, reconsider q_2 in the original form (3.11), since we will neglect the first term on the RHS of (3.11), when $r \to 0^+$.

Whenever $|X|$ is large, and since q_2 can be written as $q_2(r, X) = r^{-\sigma}(\gamma_2 r^{\rho+\sigma} + \gamma_3 X^{|\beta|})$, we will suppose that the quantity $\gamma_2 r^{\rho+\sigma}$ varies slowly in r and can be considered as a constant that we simply call γ_2. This assumption is not necessary in the separation-of-variables technique for large $|X|$, but it will be adopted to keep the same notation of the preceding ansatz for small $|X|$. In fact we could simply neglect the first term in the RHS of (3.11) with respect to the second one for large $|X|$.

We summarize the different choices for q_2 in Table 2; note that, as explained before, the parameter γ_3 is larger in the dissipative range. We again point out that the values of q_2 in Table 2 are those used, for technical reasons, in the analytic solution of Eq. (3.8) by separating the variables. Other methods, including the numerical one, could well use the original scaling (3.11).

4. Asymptotic solutions: inertial range

4.1. General considerations

In order to clarify all the questions addressed in section 3, we now solve analytically Eq. (3.8) in the inertial range by neglecting the last term on the LHS as anticipated at the end of section 2. The solution of (3.8) in the dissipative range will be treated in section 5. The method we use is the separation-of-variables technique and it consists in writing a solution of (3.8) as $P(r, X) = \phi_\alpha(X)\,\psi_\alpha(r)$, where both ϕ_α and ψ_α depend on a parameter, say α, upon which we have to integrate in order to get, by superposition, the most general solution. Therefore,

$$P(r, X) = \int \phi_\alpha(X)\,\psi_\alpha(r)\,C(\alpha)\,\mathrm{d}\alpha\,, \qquad (4.1)$$

where the coefficient $C(\alpha)$ must be determined by specifying, for example, the function $P(r, X)$ at given r, say r_0.

The function $P(r_0, X)$ is clearly an initial condition of the problem and its choice, essentially of physical nature, will determine the asymptotic behavior of the general solution of (3.8). Instead we should be able to predict such a behavior directly from our model. We therefore proceed in the following way: we invite the reader to come back to these considerations after

Table 2
Modelling hypotheses for the conditional expectation q_2

	$q_2(r, X)$	
	$\lvert X\rvert$ small	$\lvert X\rvert$ large
Inertial range	$r^{\rho}(\gamma_2 + \gamma_3 X^2)$	$r^{-\sigma}(\gamma_2 + \gamma_3\lvert X\rvert^{\beta})$
Dissipative range	$r^{\rho}(\gamma_2 + \gamma_3 X^2)$ and $\gamma_3 X^2 r^{-\sigma}$	$r^{-\sigma}(\gamma_2 + \gamma_3\lvert X\rvert^{\beta})$

sections 4 and 5, but we prefer to anticipate them to justify the forthcoming analysis.

We ask, first of all, for a function $\phi_\alpha(X)$ bounded around zero and decaying to zero for $|X| \to +\infty$. This last assumption will be sufficient to find an exponentially decaying $\phi_\alpha(X)$ for $\alpha > 0$, thus guaranteeing the existence of all the moments, and this both in the inertial and dissipative ranges (cf. Eqs. (4.11) and (5.8) below). We now show how to get some information about the factor $C(\alpha)$ in (4.1); a more complete analysis of this type will be performed in ref. [12] with the numerical solution of $\phi_\alpha(X)$, $\forall X \in \mathbb{R}$. We keep for $\phi_\alpha(X)$ in (4.1) directly the exponential leading term in the solution (4.11) below (the same argument holds in the dissipative range by taking (5.8)), and compute $\psi_\alpha(r)$ in the point $r = r_0$. Then, we assume that $P(r_0, X)$ has a leading asymptotically decaying term of type: $\exp(-cX^q)$, c, $q > 0$. It is therefore easy to see that $P(r_0, X)$ can be expressed as the Laplace transform of a function $\tilde{C}(\alpha)$, essentially proportional to $C(\alpha)$. Moreover, a straightforward change of variable shows that $\tilde{C}(\alpha)$ can be obtained by antitransforming the function:

$$\text{const.} \times \exp\{-c[\tfrac{1}{2}(3-\beta)X]^{2q/(3-\beta)}\},$$

where $\frac{1}{2}(3-\beta)$ is the exponent of $|X|$ in (4.11). Now, from a well-known theorem on the Laplace inverse transform [18, p. 310], only when $q \le \frac{1}{2}(3-\beta)$, we can get a (distribution) solution for $\tilde{C}(\alpha)$; this shows that the various exponents of $|X|$ found in (4.11), (4.12) and (5.8) give the steepest exponential stretching compatible with an exponentially decaying initial solution. In this sense we can claim that our model predicts the exponents of the stretched exponential once the parameter β is given by the asymptotic behavior of $q_2(r, X)$. Note, however, that also a different asymptotic behavior of $q_1(r, X)$ at infinity can change the stretched exponential, as is illustrated by Eq. (4.12) below. We conclude these preliminary considerations observing that a mathematical constraint on β is that it should not give a subexponential decay, as explained at the end of section 3.2. Probably a different analytic deduction of Eq. (3.8) instead of using moments could relax such a constraint (see also section 5.3) [19].

4.2. Small $|X|$

We now solve the restricted Eq. (3.8) in the inertial range for small $|X|$, assuming that $\beta = 2$. We keep $q_1 = +d(r)$ (at order zero in X) with $d(r) = d/r^\nu$. The differential equations satisfied by $\phi_\alpha(X)$ and $\psi_\alpha(r)$ are

$$\frac{d^2\phi_\alpha(X)}{dX^2} + \frac{4\gamma_3}{\gamma_2 + \gamma_3 X^2} X \frac{d\phi_\alpha(X)}{dX} + \left(\frac{2\gamma_3}{\gamma_2 + \gamma_3 X^2} - \frac{\alpha}{2(\gamma_2 + \gamma_3 X^2)}\right)\phi_\alpha(X) = 0 \tag{4.2}$$

and

$$\frac{d\psi_\alpha(r)}{dr} = \left(\frac{\alpha r^{\nu+p}\bar{N}}{d} - \frac{2-\nu}{r}\right)\psi_\alpha(r). \tag{4.3}$$

The solution of (4.2) can be written as

$$\phi_\alpha(X) = \exp\left(-\frac{\gamma_3}{\gamma_2} X^2\right) U_\alpha(X),$$

where $U_\alpha(X)$ satisfies

$$\frac{d^2 U_\alpha(X)}{dX^2} - \frac{\alpha}{2\gamma_2} U_\alpha(X) = 0, \tag{4.4}$$

and a local series expansion of the solution of (4.4) about zero gives, having chosen $\alpha < 0$,

$$U_\alpha(X) = \text{const.} \times \left(1 - \frac{|\alpha|}{2\gamma_2} X^2 + \mathcal{O}(X^4)\right), \tag{4.5}$$

where we have neglected the other linear independent solution by requiring that $dU_\alpha(X)/dX|_{X=0} = 0$. Returning to (4.2), we thus have at the second order in X

$$\phi_\alpha(X) = \text{const.} \times \left(1 - \frac{1}{\gamma_2}(\gamma_3 + \tfrac{1}{2}|\alpha|)X^2\right), \tag{4.6}$$

which can be considered as a Gaussian smooth-ing of the pdf in the neighborhood of zero. The radial part $\psi_\alpha(r)$ is easily obtained by solving (4.3):

$$\psi_\alpha(r) = \text{const.} \times \frac{1}{r^{2-\nu}} \exp\left(\frac{-\bar{N}\alpha r^{\nu+\rho+1}}{d(\nu+\rho+1)}\right). \quad (4.7)$$

Eq. (5.4) below shows that the smooth scaling (4.6) drastically changes with a different q_2 in the dissipative range.

4.3. *Large* $|X|$

The ordinary differential equations satisfied respectively by $\psi_\alpha(r)$ and $\phi_\alpha(X)$ are

$$\frac{d\psi_\alpha(r)}{dr} = \left(-\frac{\alpha\bar{N}}{\gamma_1^{\pm}} r^{-\epsilon-\sigma} - \frac{2+\epsilon}{r}\right)\psi_\alpha(r) \quad (4.8)$$

and

$$\frac{d^2\phi_\alpha(X)}{dX^2} + \eta_X \frac{2\gamma_3\beta|X|^{\beta-1}}{\gamma_2+\gamma_3|X|^\beta}\frac{d\phi_\alpha(X)}{dX} + \left(\frac{\gamma_3\beta(\beta-1)|X|^{\beta-2}}{\gamma_2+\gamma_3|X|^\beta} - \eta_X\frac{\alpha|X|}{2(\gamma_2+\gamma_3|X|^\beta)}\right)\phi_\alpha(X) = 0, \quad (4.9)$$

where $\eta_X = 1$ for $X>0$ and $\eta_X=-1$ for $X<0$ and we adopted for q_2 the scaling quoted in Table 2.

The solution of (4.8) is

$$\psi_\alpha(r) = \text{const.} \times \frac{1}{r^{2+\epsilon}} \exp\left(-\frac{\alpha\bar{N}}{\gamma_1^+}\frac{r^{1-\epsilon-\sigma}}{1-\epsilon-\sigma}\right). \quad (4.10)$$

We take the parameter α positive; this choice is consistent with the decaying solution of (4.9) in the interval $X>0$ which, in fact, by the standard change of variables already used in Eq. (4.4) and by formula (3.4.28) in Ref. [16], asymptotically reads

$$\phi_\alpha(X) = \text{const.} \times \left(\frac{2\gamma_3}{\alpha}\right)^{1/4} X^{-3(\beta+1)/4} \times \exp\left[-\left(\frac{\alpha}{2\gamma_3}\right)^{1/2}\frac{2}{3-\beta}X^{(3-\beta)/2}\right], \quad X\to+\infty. \quad (4.11)$$

A function like (4.11) with X and α replaced respectively by $|X|$ and $|\alpha|$, $\alpha<0$, is also the asymptotically decaying solution of (4.9) when $X\to-\infty$. But a negative α does not change the solution (4.10), since for $X<0$ we have to replace γ_1^+ by γ_1^- which is negative, and therefore we get, by superposition, a symmetric pdf.

We conclude this section with a remark: if we assume a different asymptotic behavior at infinity for q_1, we change the leading decaying term in the solution (4.11). For example, if we take $q_1 \sim \gamma_1^+ r^\epsilon X^2$, for large $|X|$, and use the same technique that gave (4.11), we get for the leading term of the solution of (4.9)

$$\phi_\alpha(X) \approx \exp\left[-\left(\frac{\alpha}{2\gamma_3}\right)^{1/2}\frac{2}{4-\beta}X^{(4-\beta)/2}\right], \quad (4.12)$$

and then, whenever $\beta=0$, which corresponds to $q_2=$ const. at infinity, we get a Gaussian pdf also at infinity (q_2 constant at infinity has been discussed in ref. [15]). Note that with this choice of q_1, β can now range in the interval [0, 2].

5. Asymptotic solutions: dissipative range

If we want to solve Eq. (3.8) in the dissipative range, the equation is not separable for large $|X|$ anymore and we will see in a moment how to supply for. On the contrary, when $|X|$ is small, we can separate the variables. Both cases will be treated in the following subsections.

5.1. *Small* $|X|$

We choose q_2 as given in Table 2, with $\rho=1$ to simplify the computations. Moreover, assuming that $d(r)$ is a slowly varying function of r when $r \to 0$, allows us to take $q_1 = +d = \text{const.}$; finally our analysis will also be asymptotic for r tending to zero.

The spatial factor $\phi_\alpha(X)$ has the same solution as in Eq. (4.6), with the radial factor now satisfying the differential equation

$$\frac{d^2\psi_\alpha(r)}{dr^2} + \frac{2}{r}\frac{d\psi_\alpha(r)}{dr} + \left(\frac{\bar{N}\alpha r}{2k_0} - \frac{d}{rk_0}\right)\psi_\alpha(r) = 0\,, \tag{5.1}$$

whose solution can be written as $\psi_\alpha(r) = r^{-1}u_\alpha(r)$, where $u_\alpha(r)$ satisfies

$$\frac{d^2u_\alpha(r)}{dr^2} + \left(\frac{\bar{N}\alpha r}{2k_0} - \frac{d}{rk_0}\right)u_\alpha(r) = 0\,. \tag{5.2}$$

To solve (5.2) we apply the classical Frobenius method to find a local series expansion about $r = 0$; after a long but straightforward calculation and multiplying finally by r^{-1}, we find for the leading term

$$\psi_\alpha(r) = \text{const.} \times \left(\frac{1}{r} - \frac{d}{k_0}\ln r + \mathcal{O}(|r \ln r|)\right)\,. \tag{5.3}$$

Note that the parameter α does not appear in this leading-terms expansion: it enters for the first time as a coefficient of r^2.

The assumption (3.11) for q_2 is particularly close to the experimental observation that in the dissipative range the minimum of the parabola (3.11) approaches zero when $r \to 0$ [12]. If we directly assume this and write $q_2 = r^{-\sigma}\gamma_3 X^2$, $|X|$ small, we get a new completely different solution for the spatial part $\phi_\alpha(x)$ with a singularity in zero. In fact, the differential equation satisfied by $\phi_\alpha(x)$ is now

$$\frac{d^2\phi_\alpha(X)}{dX^2} + \frac{4}{X}\frac{d\phi_\alpha(X)}{dX} + \left(\frac{2}{X^2} - \frac{\alpha}{2\gamma_3 X^2}\right)\phi_\alpha(X) = 0\,, \tag{5.4}$$

which produces a diverging solution at the origin. This pathology agrees, in some sense, with the experimental observation that the pdf becomes more and more peaked around zero when $r \to 0$ [2,12].

5.2. Large $|X|$

We now return to the case of large $|X|$. We make the following observation: in the inertial range, the last term on the LHS of (3.8) was discarded, since it was neglected in Eq. (2.13), when $n = 1$. But still for the physical considerations, quoted after Eq. (2.13), it is now the first term in the LHS of (2.8) that is neglected in the dissipative range [11, Vol. 2, p. 400]. We make the same assumption for $n > 1$, and therefore our pdf in the dissipative range will satisfy the equation

$$\bar{N}\frac{\partial^2}{\partial X^2}[q_2(r, X)\, P(r, X)] - k_0\left(\frac{2}{r} + \frac{\partial}{\partial r}\right)\frac{\partial}{\partial r}P(r, X) = 0\,. \tag{5.5}$$

Taking for q_2 the scaling quoted in Table 2, with $\sigma = 1$ to simplify the computations, we immediately separate the variables in (5.5) as

$$\frac{d^2\phi_\alpha(X)}{dX^2} + \eta_X \frac{2\gamma_3\beta|X|^{\beta-1}}{\gamma_2 + \gamma_3|X|^\beta}\frac{d\phi_\alpha(X)}{dX} + \left(\frac{\gamma_3\beta(\beta-1)|X|^{\beta-2}}{\gamma_2 + \gamma_3|X|^\beta} - \frac{\alpha}{2(\gamma_2 + \gamma_3|X|^\beta)}\right)\phi_\alpha(X) = 0 \tag{5.6}$$

and

$$\frac{d^2\psi_\alpha(r)}{dr^2} + \frac{2}{r}\frac{d\psi_\alpha(r)}{dr} - \frac{\alpha\bar{N}}{2rk_0}\psi_\alpha(r) = 0\,. \tag{5.7}$$

Eq. (5.7) gives a solution dominated by r^{-1} as in the solution (5.3) and with the parameter α entering as a coefficient of $\ln(r)$. Besides, the decaying solution of (5.6) for large $|X|$ can be obtained as in (4.11) for $\alpha > 0$ (with the same sign now for $X > 0$ and $X < 0$) and reads

$$\phi_\alpha(X) = \text{const.} \times \left(\frac{2\gamma_3}{\alpha}\right)^{1/4} |X|^{-3\beta/4} \times \exp\left[-\left(\frac{\alpha}{2\gamma_3}\right)^{1/2} \frac{2}{2-\beta} |X|^{(2-\beta)/2}\right]. \tag{5.8}$$

5.3. Comments

We now make a few comments on the solutions (4.11) and (5.8). Let us first suppose that the parameter β is the same for both ranges. Then the exponent of $|X|$ in the exponential factor in the inertial range is larger than the corresponding one in the dissipative range and this fact was experimentally observed in Ref. [13, Fig. 6]. Now, as explained in section 3, the fact that we need an exponentially decaying solution to justify the derivation of Eq. (3.8) (in particular its unicity) forces us to choose $\beta = 0$ in (5.8), which corresponds to q_2 constant at infinity. If we relax this assumption as discussed at the end of section 4.1, we see that the value $\beta = 1$ is consistent with the exponent of $|X|$ found by Gagne [2] and Ching [13] in the dissipative range, about 0.5. It is also in agreement with the average value, about 1 given by Ching [13] in the inertial range. This parameter β has been taken as a constant both in the dissipative and inertial range, eventually with different values. This allows us to find explicit asymptotic solutions for Eq. (3.8). One could consider β as a function of r and try to solve Eq. (3.8), for example, numerically. In this case, one expects to find the stretching exponential varying with r such as in Ref. [13].

6. Conclusions and perspectives

In this paper, we have presented a method to determine the pdf for the temperature-difference function in isotropic and homogeneous turbulence. This pdf is the solution of a PDE that contains two conditional expectations q_1 and q_2. The function q_1 quantifies the influence of the turbulent velocity field on the transfer of temperature from the large scales to the small ones, whereas q_2 represents the link between temperature and its dissipation, which is known to play an important role in turbulence modelling [17]. We showed that it is possible to conjecture the analytical form of the closure functions with physical arguments, but also by requiring a certain regularity of the solution of Eq. (3.8). We also gave the asymptotic form of this solution in the ranges $X \approx 0$ and $|X|$ large (respectively Gaussian and stretched exponential); in this case one only needs the asymptotic behaviors of q_1 and q_2 which are experimentally more accessible.

A careful experimental analysis is needed to better investigate the asymmetry of the pdf and to find the precise values of the various exponents ϵ, σ, ρ, β and ν and the relations among them, that could give different exponential stretching for different values of r, as pointed out experimentally in Ref. [13]. A further analysis is also needed in the dissipative range, eventually injecting into the equation a small random noise simulating the presence of heat sources.

Moreover, the inconsistency, clearly pointed out in this paper, between joint Gaussian statistics for Δu_L and $\Delta\theta$ and local isotropy for the inertial range scales must be studied in more detail.

Finally, numerical computations should extend our asymptotic solutions to the whole X and r axes.

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Appendix. Derivation of Eq. (2.8), (2.9) and (2.12)

To derive Eq. (2.8), let us define the quantity

$$(\Delta\theta)^n = [\theta(\boldsymbol{x}+\boldsymbol{r},t)-\theta(\boldsymbol{x},t)]^n\,. \tag{A.1}$$

We then have

$$D_n(\boldsymbol{r},\boldsymbol{x},t) \equiv \frac{\partial}{\partial t}[(\Delta\theta)^n] + \boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla[(\Delta\theta)^n] - k_0\nabla^2[(\Delta\theta)^n]\,, \tag{A.2}$$

where all the operators of derivation are with respect to $\boldsymbol{x}$ as prescribed in (2.7). Note that our result equivalently holds if we replace $\boldsymbol{u}(\boldsymbol{x},t)$ with $\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t)$ in (A.2). Therefore

$$\begin{aligned} D_n(\boldsymbol{r},\boldsymbol{x},t) &= n(\Delta\theta)^{n-1}\frac{\partial}{\partial t}[\theta(\boldsymbol{x}+\boldsymbol{r},t)] \\ &\quad - n(\Delta\theta)^{n-1}\frac{\partial}{\partial t}[\theta(\boldsymbol{x},t)] \\ &\quad + n(\Delta\theta)^{n-1}\boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla\theta(\boldsymbol{x}+\boldsymbol{r},t) \\ &\quad - n(\Delta\theta)^{n-1}\boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla\theta(\boldsymbol{x},t) \\ &\quad - k_0 n(\Delta\theta)^{n-1}\nabla^2\theta(\boldsymbol{x}+\boldsymbol{r},t) \\ &\quad + k_0 n(\Delta\theta)^{n-1}\nabla^2\theta(\boldsymbol{x},t) \\ &\quad - k_0 n(n-1)(\Delta\theta)^{n-2}|\nabla(\Delta\theta)|^2\,. \end{aligned} \tag{A.3}$$

By adding and subtracting the quantity

$$n(\Delta\theta)^{n-1}\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t)\cdot\nabla\theta(\boldsymbol{x}+\boldsymbol{r},t) \tag{A.4}$$

and by using the conservation equation (2.1) taken in the points $\boldsymbol{x}$ and $\boldsymbol{x}+\boldsymbol{r}$, under the identification of the derivations (2.2), (2.3) and (2.7), we finally get Eq. (2.8), namely

$$\begin{aligned} D_n(\boldsymbol{r},\boldsymbol{x},t) &\equiv \frac{\partial}{\partial t}[(\Delta\theta)^n] + \boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla[(\Delta\theta)^n] \\ &\quad - k_0\nabla^2[(\Delta\theta)^n] \\ &= -n(\Delta\theta)^{n-1}\,\Delta\boldsymbol{u}(\boldsymbol{r},\boldsymbol{x},t)\cdot\nabla\theta(\boldsymbol{x}+\boldsymbol{r},t) \\ &\quad - n(n-1)k_0(\Delta\theta)^{n-2}\{\nabla[\Delta\theta]\}^2\,. \end{aligned} \tag{A.5}$$

To derive (2.9) we now average (A.5) over the space. The expectation of the last two terms in the LHS of (A.5) is zero by divergenceless of $\boldsymbol{u}$, zero boundary conditions and by isotropy. Our last assumption is therefore

$$\left\langle \frac{\partial}{\partial t}[(\Delta\theta)^n]\right\rangle = 0 \tag{A.6}$$

and to justify it, we invoked, in section 2, stationarity and the Kolmogorov similarity hypothesis in the regime of fully developed turbulence. Notice that a direct calculation of $\langle\partial[(\Delta\theta)^2]\rangle/\partial t$ leads to the Yaglom formula (2.13) only if (A.6) holds (cf. the remark at the end of this appendix).

The same assumption is made in Ref. [11, Vol. 2, p. 399] where it is explicitly written (with our notation): "$\partial\langle(\Delta\theta)^2\rangle/\partial t = 0$ for $r \ll L$ and sufficiently large Re and Pe (Peclet number)."

In the derivation of Eq. (2.12) we will use the following properties of the four-dimensional isotropic field $\{\theta(\boldsymbol{x}), \boldsymbol{u}(\boldsymbol{x})\}$, where $\boldsymbol{u}(\boldsymbol{x})$ is a vector and θ a scalar (cf. formulae (12.52) and (12.53) in ref. [11, Vol. 2]):

$$\langle u_i(\boldsymbol{x})\theta(\boldsymbol{x}+\boldsymbol{r})\rangle = \langle u_{\rm L}(\boldsymbol{x})\theta(\boldsymbol{x}+\boldsymbol{r})\rangle\,\frac{r_i}{r}\,, \tag{A.7}$$

$$\langle u_i(\boldsymbol{x}+\boldsymbol{r})\theta(\boldsymbol{x})\rangle = \langle u_{\rm L}(\boldsymbol{x}+\boldsymbol{r})\theta(\boldsymbol{x})\rangle\,\frac{r_i}{r}\,, \tag{A.8}$$

and $\langle u_{\rm L}(\boldsymbol{x})\theta(\boldsymbol{x}+\boldsymbol{r})\rangle = -\langle u_{\rm L}(\boldsymbol{x}+\boldsymbol{r})\theta(\boldsymbol{x})\rangle$ are functions depending solely on r; the subscript L means projection along the $\boldsymbol{r}$-direction.

We start by modifying the LHS of (2.9), which gives

$$\langle(\Delta\theta)^{n-1}\Delta\boldsymbol{u}(\boldsymbol{r},\boldsymbol{x})\cdot\nabla\theta(\boldsymbol{r}+\boldsymbol{x})\rangle = \frac{1}{n}\frac{\partial}{\partial r_i}\langle(\Delta\theta)^n\,\Delta u_i(\boldsymbol{r},\boldsymbol{x})\rangle\,, \tag{A.9}$$

where we used the convention on the summation of repeated indices and the property of divergencelessness of the velocity field. Now,

$$\begin{aligned} \langle(\Delta\theta)^n\,\Delta u_i(\boldsymbol{r},\boldsymbol{x})\rangle &= \sum_{k=0}^{n}(-1)^k\binom{n}{k} \\ &\quad \times\{\langle\theta^k(\boldsymbol{r}+\boldsymbol{x})\,[\theta(\boldsymbol{x})]^{n-k}u_i(\boldsymbol{r}+\boldsymbol{x})\rangle \\ &\quad + (-1)^{n+k}\langle\theta^k(\boldsymbol{x})\,[\theta(\boldsymbol{x}+\boldsymbol{r})]^{n-k}u_i(\boldsymbol{x})\rangle\}\,. \end{aligned} \tag{A.10}$$

By applying the first hypothesis (H1) of section 2 to the fields $\{[\theta(x)]^{n-k},\ \theta^k(x)\,u(x)\}$ and $\{\theta^k(x),\ [\theta(x)]^{n-k}u(x)\}$ and using (A.7) and (A.8), we can transform (A.10) when n is even as

$$\langle(\Delta\theta)^n\,\Delta u_i(r,x)\rangle = \langle(\Delta\theta)^n\,\Delta u_{\rm L}(r,x)\rangle\,\frac{r_i}{r}\,, \tag{A.11}$$

and we immediately see that the LHS is zero when n is odd.

Taking the derivative $\partial/\partial r_i$ as prescribed by (A.9), we finally get for n even

$$\begin{aligned}\frac{\partial}{\partial r_i}\langle(\Delta\theta)^n\,\Delta u_i(r,x)\rangle &= \frac{\partial}{\partial r_i}\left(\frac{r_i}{r}\langle(\Delta\theta)^n\,\Delta u_{\rm L}(r,x)\rangle\right)\\ &= \left(\frac{2}{r}+\frac{\partial}{\partial r}\right)\langle(\Delta\theta)^n\,\Delta u_{\rm L}(r,x)\rangle\,.\end{aligned} \tag{A.12}$$

We now transform the RHS of (2.9); by using the second hypothesis (H2) of section 2 and the same trick as in (A.10), we immediately get

$$\begin{aligned}-\langle(\Delta\theta)^{n-2}\{\nabla\theta(r,x)]\}^2\rangle &= -2\langle(\Delta\theta)^{n-2}[(\nabla\theta(x)]^2\rangle\\ &\quad+\frac{2}{n-1}\frac{\partial}{\partial r_i}\left\langle(\Delta\theta)^{n-1}\frac{\partial}{\partial x_i}\theta(x)\right\rangle\,,\end{aligned} \tag{A.13}$$

where the first term in the RHS is zero for $n-2$ odd.

Now it is straightforward to check that

$$\frac{1}{n}\frac{\partial}{\partial r_i}\langle(\Delta\theta)^n\rangle = \langle(\Delta\theta)^{n-1}\frac{\partial}{\partial x_i}\theta(x)\rangle\,, \tag{A.14}$$

and therefore

$$\begin{aligned}\frac{\partial}{\partial r_i}\left\langle(\Delta\theta)^{n-1}\frac{\partial}{\partial x_i}\theta(x)\right\rangle &= \frac{1}{n}\frac{\partial}{\partial r_i}\left(\frac{\partial}{\partial r_i}\langle(\Delta\theta)^n\rangle\right)\\ &= \frac{1}{n}\frac{\partial}{\partial r_i}\left(\frac{r_i}{r}\frac{\partial}{\partial r}\langle(\Delta\theta)^n\rangle\right)\\ &= \frac{1}{n}\left(\frac{2}{r}+\frac{\partial}{\partial r}\right)\frac{\partial}{\partial r}\langle(\Delta\theta)^n\rangle\,,\end{aligned} \tag{A.15}$$

which concludes the proof of Eq. (2.12).

Remark. It is interesting to note, although it is not very surprising, that Eq. (2.12) can be obtained by directly imposing condition (A.6) above and then using the machinery developed in the second half of this appendix.

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