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LARGE DEVIATIONS FOR SHORT RECURRENCE

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Abstract. Over a ψ-mixing dynamical system we consider the function $\tau(C_n)/n$ in the limit of large n, where $\tau(C_n)$ is the first return of a cylinder of length n to itself. Saussol et al. ([30]) proved that this function is constant almost everywhere if the C_n are chosen in a descending sequence of cylinders around a given point. We prove upper and lower general bounds for its large deviation function. Under mild assumptions we compute the large deviation function directly and show that the limit corresponds to the Rényi's entropy of the system. We finally compute the free energy function of $\tau(C_n)/n$. We illustrate our results with a few examples.

1. Introduction.

In the statistical analysis of the Poincaré recurrence it is classical to study the first return or the first hitting time of a point (in some space Ω and under the iteration of a map T) to a cylinder and to get exponential limit laws for these distributions (cylinders are defined in the next section as the elements of the backward iteration of a suitable partition of Ω). We refer the reader to Abadi and Galves [4] for an up-to-date on this subject. It appears recently that to deal with the above quantities it is fundamental to describe the first return of a cylinder (not of a point) to itself. See for instance Galves and Schmitt [18], Abadi [1], [2] and [3], Abadi and Vergne [5], Hirata, Saussol and Vaienti [22] and Haydn and Vaienti [20]. If we denote with $\tau(C_n)$ the first return of an n-cylinder to itself, it was proved by Saussol et al. [30] and independently by Afraimovich et al. [7] that for an ergodic measure μ of positive metric entropy, the ratio $\tau(C_n)/n$ verifies

$$\liminf_{n\to\infty} \frac{\tau(C_n)}{n} \geq 1 \ , \tag{1}$$

whenever $\cap_n C_n = z$ and z is chosen μ-almost everywhere. To stress the fact that the decreasing sequence of cylinders C_n is taken around the point z, we will sometimes write $C_n(z)$. If the system satisfies the specification property [23], the same authors show that the above limit exists and is 1 almost everywhere. This result has also been proved for a class of non-uniformly expanding maps of the interval [22].

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We stressed above that the first return of sets plays a crucial role in establishing the exponential limit law for the distribution of the first return and entry times. Here we briefly quote some other nice applications of such a quantity.
In the case of zero-entropy systems, the asymptotic behavior of $\frac{\tau_{C_n(z)}}{n}$ reveals interesting features, usually related to the arithmetic properties of the map; we defer to the papers [24, 25, 12] where that analysis was carried out for Sturmian shifts and for substitutive systems.

In a different context, the first return of a set has been used to define the *recurrence dimension*, since it was used as a set function to construct a suitable outer measure in the Carathéodory scheme [6, 27].

The recurrence of sets has been also related to the Algorithmic Information Content in [10] and used to characterize the statistical behavior of the system.

Some of these results have a nice counterpart whenever we replace cylinders with balls $B_r(z)$ of radius r around $z \in \Omega$ (supposed to be a metric space), and we consider the first return of the ball into itself, $\tau_{B_r(z)}$. The natural generalization of the limit (1) is now the quantity:

$$\lim_{r\to 0^+} \frac{1}{-\log r} \tau_{B_r(z)} \ . \tag{2}$$

For a large class of maps of the interval, it has been proved that [30]:

$$\lim_{r\to 0^+} \frac{\tau_{B_r(z)}}{-\log r} = \frac{1}{\lambda} \qquad z,\ \mu-\text{a.e.} \tag{3}$$

where λ is the Lyapunov exponent of the measure μ. The limit (3) can be generalized to multidimensional transformations, in particular for smooth diffeomorphisms of a compact manifold, and in these cases it becomes a suitable combination of the reciprocal of the largest and smallest Lyapunov exponents [31].

In this paper we describe the large deviation properties of $\tau(C_n)/n$. Besides the motivations addressed above, we would like to point out the link between the large deviation of $\tau(C_n)/n$ and the Rényi's entropy. These entropies have been extensively studied in the last years for their connection with thermodynamic and multi-fractal formalism of dynamical systems [8, 16, 19]. They have also been applied to intermittency since one can show a phase transition in the spectrum of these entropies [14, 26]. We recall also the close connection between the Rényi entropies and the topological pressure, whenever we consider potential of the form $\beta\phi$, where ϕ is usually the jacobian of the map T, see for instance [9].

The large deviation function of a certain limiting process which is constant almost everywhere is defined as the measure of all points that deviate from the typical value. Of course this quantity goes to zero, in many cases exponentially fast. Therefore one is interested in considering the lower deviation function

$$\lim_{n\to\infty} \frac{1}{n} \ln \mu \left(z; \frac{\tau(C_n(z))}{n} < 1-\varepsilon \right) \ , \tag{4}$$

for positive ε. Through this paper ln stands for natural logarithm. Very often the upper deviation is not interesting since one has $\tau(C_n) \le n + n_0$ for all n-cylinders C_n and n_0 a fixed integer, which implies that:

$$\lim_{n\to\infty} \frac{1}{n} \ln \mu \left(z; \frac{\tau(C_n(z))}{n} > 1+\varepsilon \right) = 0 \ ,$$

This happens for instance for aperiodic sub-shifts of finite type (and in this case n_0 is the lowest power for which the transition matrix becomes strictly positive) and a fortiori for systems which can be coded with a complete grammar, and this will be actually the case in our paper, as a consequence of a particular assumption stated in Lemma (2.1), see also Remark 2.

In this work we consider therefore the limit (4) and:
(a) prove lower and upper bounds for it
(b) compute explicitly the lower large deviation function (4)
(c) compute its free energy function.

The bounds we prove in (a) are given in terms of the maximum and minimum rate of exponential decay of the measure of cylinders. The principal result of this paper is with respect to (b). It says that the large deviation function exists and is a piecewise linear function over (0,1). Moreover, it is linear over each interval $(1/m, 1/(m+1))$ with $m \in I\!N$. Further, at the sequence $\{1/m\}$ it coincide with the Rényi entropy of the process.

For completeness, we compute the free energy function of $\tau(C_n)/n$. As it can be expected by (b), it is not differentiable on the whole real line. We find that, if γ_μ denote the maximum rate of exponential decay of the measure of the cylinders, then for $t \geq -\gamma_\mu$ the free energy is the identity and for $t \leq -\gamma_\mu$ it is constantly γ_μ. Therefore, by applying the large deviation principle by Ellis [17] or Planchky and Steinbach [28] we recover the bounds that we establish directly in (a).

We illustrate our results with several examples.

Our framework is the class of ψ-mixing processes. For instance, irreducible and aperiodic finite state Markov chains are known to be ψ-mixing with exponential decay. Moreover, Gibbs states which have summable variations are ψ-mixing (see [32]). They have exponential decay if they have Hölder continuous potential (see [11]). We refer the reader to [15] for a source of examples of mixing processes.

2. **Bounds for the large deviation function.** Let $(\Omega, \mathcal{F}, \mu, T)$ be a measurable dynamical systems over the space Ω, with μ a probability measure on the σ-algebra $\mathcal{F}$ and T a measurable map preserving μ. We put $\mathcal{C}$ a finite generating partition of Ω and we write $\mathcal{C}^n = \bigvee_{j=0}^{n-1} T^{-j}\mathcal{C}$ for its n-join, whose elements will be called cylinders (of length n) and denoted with C_n or with $C_n(z)$ to specify that they contain the point z. Let $\overline{\Omega} = \mathcal{C}^{Z}$, where $\mathcal{C}^{Z}$ denotes the countable product, over $Z\!\!\!Z$, of $\mathcal{C}$ with itself. For each $x = (x_m)_{m \in Z} \in \overline{\Omega}$ and $m \in Z\!\!\!Z$, let $X_m : \overline{\Omega} \to \mathcal{C}$ be the m-th coordinate projection, that is $X_m(x) = x_m$.

We can equivalently (and with a more probabilistic language) say that a subset $C_n \subseteq \overline{\Omega}$ is a n-cylinder if $C_n \in \mathcal{C}^n$ and

$$C_n = \{X_0 = a_0; \ldots ; X_{n-1} = a_{n-1}\} \ ,$$

with $a_i \in \mathcal{C},\ i = 0, \ldots, n-1$.

The σ-algebra $\mathcal{F}$ is that generated by the algebra of finite disjoint reunion of cylinders. We shall assume without loss of generality that there is no subsets of measure 0 in the partition $\mathcal{C}$.

We say that the process $(\Omega, \mathcal{F}, \mu, T)$ is ψ-mixing if the sequence

$$\psi(l) = \sup \left| \frac{\mu(B \cap C)}{\mu(B)\mu(C)} - 1 \right| \ ,$$

converges to zero. The supremum is taken over B and C such that $B \in \sigma(X_0^n), C \in \sigma(X_{n+l+1}^{\infty})$ with $\mu(B)\mu(C) > 0$.

Wherever it is not ambiguous we will write C for different positive constants even in the same sequence of equalities/inequalities.

Let us now define:

$$\tau(C_n) = \inf\{k \geq 1 \mid T^k C_n \cap C_n \neq \emptyset\} \ .$$

As we did before we will write $C_n(z)$ if we need to be precise that the cylinder C_n contains the point z. Since $\tau(C_n)$ can take only integer values, the set involved in (4) is equivalent to $\{z; \tau(C_n(z)) \leq [\delta n]\}$ with $\delta \in (0,1)$. Since $\tau(C_n(z))$ is obviously the same for all the points z in the cylinder, the set $\{z; \tau(C_n(z)) \leq [\delta n]\}$ coincides with the following one

$$\mathcal{C}_{\delta,n} = \{C_n \in \mathcal{C}^n \mid \tau(C_n) \leq [\delta n]\} \ .$$

In this section we will provide lower and upper bounds for the measure of the sets $\mathcal{C}_{\delta,n}$; these results hold for ψ-mixing measures equipped with some mild assumption. First we establish exponential bounds for the measure of any fixed cylinder. The upper bound is a well-known result [18]; for the lower bound we assume the extra condition $\psi(0) < 1$.

Lemma 2.1. *Let $(\Omega, \mathcal{F}, \mu, T)$ be a ψ-mixing dynamical system. Then there exist positive constants C_1, c_1 such that for all $x \in \Omega$ and $n \in I\!N$*

$$\mu(C_n(x)) \leq C_1 \exp(-c_1 n) \ . \tag{5}$$

Moreover, if $\psi(0) < 1$, then there exist positive constants C_2, c_2 such that for all $n \in I\!N$

$$\mu(C_n(x)) \geq C_2 \exp(-c_2 n) \ .$$

Remark 1. In the following we will *define* γ_μ as the supremum of all the constants c_1 which verifies (5), namely

$$\gamma_\mu = \sup\{c_1 > 0; \exists C_1 > 0 \text{ such that } \forall n \text{ and } \forall x \in \Omega : \mu(C_n(x)) \leq C_1 \exp(-c_1 n)\} \ .$$

Remark 2. As a consequence of the preceding lemma, in particular of the condition $\psi(0) < 1$, we get that all cylinders have positive measure, namely for any string $\{a_0, \ldots, a_{n-1}\} \in \mathcal{C}^n$, $n \geq 1$, the cylinder $C_n = \{a_0, \ldots, a_{n-1}\}$ has positive measure. Although this condition is strong, but satisfied in several interesting situations which we will quote later on, it will allow us to compute the large deviation function for our process. By relaxing that condition one could only prove bounds on the limsup of $\frac{1}{n} \ln \mu(\mathcal{C}_{\delta,n})$. We will return to this point in the concluding remarks.

The next proposition establishes a linear upper and lower bound for the exponential decay rate of the measure of cylinders with return before δn. We need the following definition and lemma which describe some symbolic structure of the cylinders in $\mathcal{C}_{\delta,n}$. With the notation y/x, we will mean that x divides y.

Definition 2.2. Let us define, for $n \in I\!N$ and $1 \leq j \leq n$:

$$B_n(j) = \{C_n \in \mathcal{C}^n \mid j/\tau(C_n)\} \ .$$

Remark 3. If $j \leq n/2$ then the symbolic representation of $C_n \in B_n(j)$ is

$$C_n = (\underbrace{a_{i_1}, \ldots, a_{i_j}}_{1} \underbrace{a_{i_1}, \ldots, a_{i_j}}_{2}, \ldots, \underbrace{a_{i_1}, \ldots, a_{i_j}}_{[n/j]} \underbrace{a_{i_1}, \ldots, a_{i_r}}_{1}) \ .$$

Thus, $B_n(j)$ is the set of n-cylinders whose symbolic representation consists in any symbol for the first j-block and then this initial block repeats to complete the n-string. For $j \geq n/2$, the symbolic representation of an n-cylinder is

$$C_n = (\underbrace{a_{i_1}, \ldots, a_{i_{n-j}}}_{1}, \underbrace{a_{i_1}, \ldots, a_{i_{n-2(n-j)}}}_{2}, \underbrace{a_{i_1}, \ldots, a_{i_{n-j}}}_{3}) \ ,$$

namely, it consists in the first and last block of equal symbols of length $n - j$ and whatever else for the central $n - 2(n - j)$ block.

Lemma 2.3.

$$B_n([\delta n]) \subseteq \mathcal{C}_{\delta,n} \subseteq \bigcup_{[\delta n]/2}^{[\delta n]} B_n(j) \ . \tag{6}$$

Remark 4. To make the notation less heavy we write $[\delta n]/2$ to mean $[[\delta n]/2]$.

Proposition 1. *Let μ be a ψ-mixing measure. Take $\delta \in (0, 1)$.*

(a) *Then*

$$\limsup \frac{1}{n} \ln \mu(\mathcal{C}_{\delta,n}) \leq -\gamma_\mu(1 - \delta) \ .$$

(b) *Assume $\psi(0) < 1$. Then*

$$\liminf \frac{1}{n} \ln \mu(\mathcal{C}_{\delta,n}) \geq -h_\mu(1 - \delta) \ .$$

where h_μ denotes the metric entropy of μ.

Remark 5. (a) extends a result by Collet et al.[13] which establishes that for Gibbs measures $\mu(\mathcal{C}_{1/3,n}) \leq Ke^{-Kn}$ and by Abadi [1] which shows that for the more general ϕ-mixing systems, there exists $\delta_0 \in (0, 1)$ such that $\mu(\mathcal{C}_{\delta_0,n}) \leq Ke^{-Kn}$.

The next assumption will be used to get a general lower bound for the measure of the set $\mathcal{C}_{\delta,n}$ and also to compute the free energy of the process $\tau(C_n)/n$ in section 4.

Hypothesis 1. There exists a sequence of n-cylinders (not necessarily around the same point) $\{P_n\}_n$ such that

$$\lim_{n\to\infty} -\frac{1}{n} \ln \mu(P_n) = \gamma_\mu \ , \tag{7}$$

and

$$\lim_{n\to\infty} \frac{\tau(P_n)}{n} = 0 \ . \tag{8}$$

The above properties mean that among the sequences of cylinders whose measure converges to the largest possible, one can be chosen such that the cylinders have short return. We will give in Sect. 4.2 a few examples which verify this assumption. It is interesting to note that the example 4.5 (a two state Markov chain), could be arranged in such a way that $\psi(0) \geq 1$, still verifying formulas (7) and (8). Instead, when the condition $\psi(0) < 1$ is verified, the existence of the limit (7) implies the existence of the limit (8). This is the content of the next Proposition which has been suggested to us by the referee. We warmly thank him.

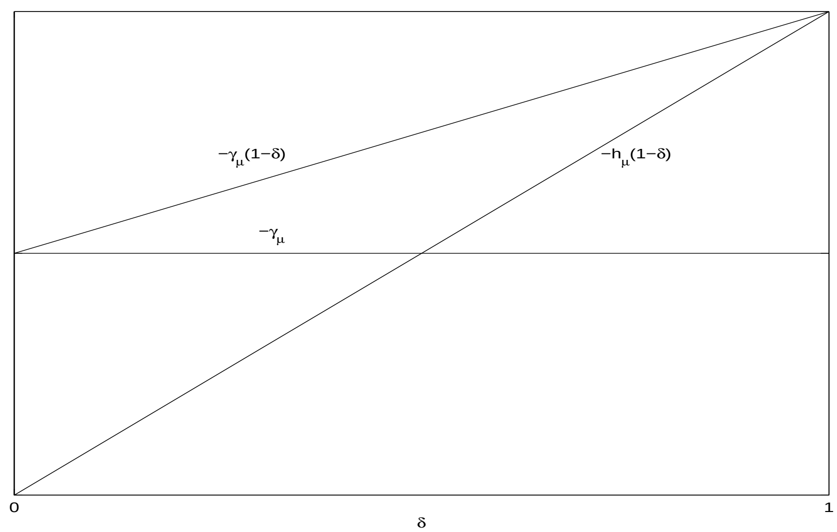

FIGURE 1. Propositions 1 and 3

Proposition 2. *Let us suppose that* $\{P_n\}_n$ *is a sequence of cylinders verifying (7). Let us suppose moreover that* $\psi(0) < 1$. *Then there exists a new sequence of cylinders* P'_n, $n \in \mathbb{N}$ *such that:*

$$\lim_{n\to\infty} -\frac{1}{n} \ln \mu(P'_n) = \gamma_\mu \ ,$$

and

$$\lim_{n\to\infty} \frac{\tau(P'_n)}{n} = 0 \ .$$

Proposition 3. *Let* $(\Omega, \mathcal{F}, \mu, T)$ *be a* ψ*-mixing dynamical system which verifies Assumption 1. Then the following inequality holds:*

$$\liminf \frac{1}{n} \ln \mu(\mathcal{C}_{\delta,n}) \geq -\gamma_\mu \ .$$

So far, collecting Propositions 1 and Proposition 3 we have proved that the measure of cylinders with short returns is contained in the triangle determined by $-\gamma_\mu(1-\delta)$, $-h_\mu(1-\delta)$ and $-\gamma_\mu$, as shown in the picture.

2.1. **Proofs.**

Proof of Lemma 2.1. As we said above, the upper bound has been proved by Galves and Schmitt [18]. We now show the lower bound. By the ψ-mixing property and under the assumption $\psi(0) < 1$ we have

$$\begin{aligned} \mu(C_n(x)) &\geq (1-\psi(0))\mu(C_1(x))\mu(T^{-1}C_{n-1}(x)) \qquad (9)\\ &\geq (1-\psi(0))^{n-1} \prod_{i=0}^{n-1} \mu(T^{-i}C_1(x)) \ . \end{aligned}$$

Since we assumed that there are no cylinders in $\mathcal{C}$ of measure 0 we have

$$0 < b := (1-\psi(0)) \min_{C_1 \in \mathcal{C}} \mu(C_1) < 1 \ .$$

Thus
$$\mu(C_n(x)) \geq \frac{1}{(1-\psi(0))} b^n \ .$$
This shows that the measure of n-cylinders has exponential lower bound. In particular, if $\psi(0) < 1$, all cylinders have positive measure. This ends the proof. □

Proof of Lemma 2.3. By definition we have the following inclusions where the first reunion is a disjoint one
$$B_n([\delta n]) \subseteq \mathcal{C}_{\delta,n} = \bigcup_{j=1}^{[\delta n]} \{\tau(C_n) = j\} \subseteq \bigcup_{j=1}^{[\delta n]} B_n(j) \ . \tag{10}$$
Hence the first inequality follows by the first inclusion. Set for a moment $x = [\delta n]$. For any integer $1 \leq j \leq x/2$ one can write $x = [x/j]j + r_j$ with $0 \leq r_j \leq x/2$ and so $x/2 \leq [x/j]j \leq x$. Therefore, for each $1 \leq j \leq x/2$ one has that $B_n(j) \subseteq B_n(i)$ where $[\delta n]/2 \leq i = [x/j]j \leq [\delta n]$. Thus
$$\bigcup_{j=1}^{[\delta n]} B_n(j) = \bigcup_{j=[\delta n]/2}^{[\delta n]} B_n(j) \ .$$
This ends the proof. □

Proof of Proposition 1. We will avoid the dependence on x in $C_n(x)$ whenever it is clear. First we prove (a). By (6)
$$\mu(\mathcal{C}_{\delta,n}) \leq \sum_{j=[\delta n]/2}^{[\delta n]} \mu(B_n(j)) \ .$$
Further, by the ψ-mixing property
$$\mu(C_n) \leq (1+\psi(0))\mu(C_j)\mu(T^{-j}C_{n-j}) \ .$$
Therefore using (i): the exponential decay of cylinders for those of length $n-j$ in the next inequality; (ii) the symbolic construction of cylinders in $B_n(j)$, we have (here and in the sequel of the proof the constant C will denote the factor $(1+\psi(0))$)
$$\begin{aligned}
\sum_{C_n \in B_n(j)} \mu(C_n) &\leq \sum_{C_n \in B_n(j)} C\mu(C_j)\mu(T^{-j}C_{n-j}) \\
&\leq C\exp(-\gamma_\mu(n-j)) \sum_{C_j \in \mathcal{C}^j} \mu(C_j) \\
&= C\exp(-\gamma_\mu(n-j)) \ .
\end{aligned}$$
Taking ln and dividing by n we get
$$\frac{\ln \mu(\mathcal{C}_{\delta,n})}{n} \leq \frac{\ln C}{n} - \gamma_\mu \frac{n-j}{n} + \frac{1}{n}\ln\frac{[\delta n]}{2} \leq \frac{\ln C}{n} + \frac{1}{n}\ln\frac{[\delta n]}{2} - \gamma_\mu \frac{n-[\delta n]}{n} \ .$$
The last inequality follows since $j \leq [\delta n]$. The most right term in the above inequality converges to $-\gamma_\mu(1-\delta)$. This ends the proof of (a).

The second inequality is obtained analogously to the previous one. By (6)
$$\mu(\mathcal{C}_{\delta,n}) \geq \mu(B_n([\delta n])) \ .$$
By using $\psi(0) < 1$ in the ψ-mixing property we get
$$\mu(C_n) \geq (1-\psi(0))\mu(C_{[\delta n]})\mu(T^{-[\delta n]}C_{n-[\delta n]}) \ .$$

We now remark on two facts:
(i) if we fix the cylinder C_n in the left hand side of the previous lower bound, then the cylinders on the right hand side verify: $C_{[\delta n]} \supset C_n$ and $T^{[\delta n]}C_n \subset C_{n-[\delta n]}$;
(ii) we observe that by Egorov's theorem and given $\epsilon > 0$ we can find a measurable subset $\Omega_\epsilon \subset \Omega$ of measure $\mu(\Omega_\epsilon) > 1 - \epsilon$, such that the convergence of the limit (Shannon, Mc-Millan, Breiman theorem): $\lim_{n\to\infty} -\frac{1}{n}\log\mu(C_n(x)) = h_\mu$ is uniform whenever $x \in \Omega_\epsilon$.

Therefore by the exponential uniform lower bound of the measure of the cylinders we have

$$\begin{aligned} & \sum_{C_n \in B_n([\delta n])} \mu(C_n) \\ \geq & \sum_{C_n \in B_n([\delta n]),\ C_n \cap \Omega_\epsilon \neq \emptyset,\ T^{[\delta n]}C_n \cap \Omega_\epsilon \neq \emptyset} C\mu(C_{[\delta n]})\mu(T^{-[\delta n]}C_{n-[\delta n]}) \ . \end{aligned}$$

Since $\mu\{C_n \in \mathcal{C}^n; C_n \subset \Omega_\epsilon^c \ \textit{and}\ T^{[\delta n]}C_n \subset \Omega_\epsilon^c\} \leq 2\epsilon$, and observing that for the item (i) above the cylinders $C_{n-[\delta n]}$ in the preceding sum will contain at least one point of Ω_ϵ, we continue the preceding lower bound as:

$$\begin{aligned} \geq & \quad C\exp(-(h_\mu+\epsilon)(n-[\delta n])) \sum_{C_j \in \mathcal{C}^{[\delta n]},\ C_j \cap \Omega_\epsilon \neq \emptyset,\ T^{[\delta n]}C_j \cap \Omega_\epsilon \neq \emptyset} \mu(C_j) \\ \geq & \quad C\exp(-(h_\mu+\epsilon)(n-[\delta n]))(1-2\epsilon) \ , \end{aligned}$$

provided n is taken large enough (depending on ϵ). Taking ln and dividing by n we get

$$\frac{\ln\mu(\mathcal{C}_{\delta,n})}{n} \geq \frac{\ln C(1-2\epsilon)}{n} - (h_\mu+\epsilon)n - [\delta n]n \ .$$

The most right term in the above inequality converges to $-h_\mu(1-\delta)$ by sending first n to infinity and then ϵ to zero. This ends the proof of (b). □

Proof of Proposition **2.** Let us take $k(n)$ an integer increasing function of n and construct by concatenation the cylinder of length $k(n)n$: $P_{k(n)n} \equiv P_nP_n\ldots P_n$, $k(n)$ times, where P_n satisfies (7). By using the ψ-mixing condition we immediately get that $\mu(P_{k(n)n}) \geq (1-\psi(0))^{k(n)}\mu(P_n)^{k(n)n}$ and a similar upper bound with $(1-\psi(0))$ replaced by $(1+\psi(0))$. This implies that $\lim_{n\to\infty} -\frac{1}{k(n)n}\log\mu(P_{k(n)n}) = \gamma_\mu$. The cylinders $P_{k(n)n}$ have first return which is at most n. We now begin to construct the new sequence P'_j, $j \in \mathbb{N}$. When $j = k(n)n$ we put $P'_j = P_{k(n)n}$. For $nk(n) < j < (n+1)k(n+1)$, write $j = [j/n] + r$, $r < n$, and build up P'_j by concatenating $[j/n]$ times the cylinder P_n followed by the first r symbols of P_n; the first return of P'_j is again at most n. Therefore: $\lim_{j\to\infty} \frac{\tau(P'_j)}{j} = 0$. Moreover by applying to the cylinders P'_j with $nk(n) < j < (n+1)k(n+1)$ the ψ-mixing condition $[j/n]$ times as above we immediately get that $\lim_{j\to\infty} -\frac{1}{j}\log\mu(P'_j) = \gamma_\mu$, which concludes the proof of the Proposition. □

Proof of Proposition **3.** By Assumption 1 one has that $P_n \in \{C_n \mid \tau(C_n) \leq \delta n\}$ for n large enough. Therefore $\mu\{\tau(C_n) \leq \delta n\} \geq \mu(P_n)$. By definition of γ_μ the proposition follows. □

3. **Computation of the large deviation function.** The goal of this section is to present the next theorem which links the measure of short returns with the generalized Rényi's entropy.

Definition 3.1. For any $\beta \in I\!R$ the generalized Rényi entropy of a measure μ relative to the partition $\mathcal{C}$ are denoted by $H_R(\beta)$ and defined as

$$\begin{aligned} H_R(\beta) &= -\lim_{m\to\infty} \frac{1}{m\beta} \ln \sum_{C_m \in \mathcal{C}^m} \mu(C_m)^{\beta+1} \\ &= -\lim_{m\to\infty} \frac{1}{m\beta} \ln \int_{\mathcal{C}^m} \mu(C_m)^{\beta} d\mu \ , \end{aligned}$$

for $\beta \neq 0$ and

$$H_R(0) = h_\mu = -\lim_{m\to\infty} \frac{1}{m} \sum_{C_m \in \mathcal{C}^m} \mu(C_m) \ln \mu(C_m) \ ,$$

and provided the limits exist.

These limits can be computed in several circumstances: Bernoulli shifts, irreducible and aperiodic Markov shifts, Gibbs measures for Hölder continuous potentials ϕ, etc. [8, 16, 19]. In the latter case, it is possible to establish a link between the Rényi entropies and the topological pressure of the function $\beta\phi$ and therefore one could use the whole machinery of thermodynamic formalism (see, e.g. Ruelle's book [29]) to get the existence and the smoothness property of the function $H_R(\beta)$ [9]. In the following we will assume that the limit defining the Rényi entropies exists for positive integer numbers; moreover we will need a further, monotonicity, property which is a direct consequence of the definition:

$$H_R(\beta_1) \leq H_R(\beta_2), \qquad \beta_1 \geq \beta_2.$$

One could wonder whether for a given ψ-mixing measure, the Rényi entropies exist and are smooth as a function of the parameter β and for a given generating partition. We already mentioned a few examples where it is the case. In general we do not know. Instead for any ψ-mixing measure one could prove the existence of the metric entropy $H_R(0)$ for any finite generating partition [21]. We now state our main result.

Theorem 3.2. *Let $(\Omega, \mathcal{F}, \mu, T)$ be a ψ-mixing dynamical system. Assume $\psi(0) < 1$ and suppose that the Rényi entropies relative to the partition $\mathcal{C}$ exist for any $\beta \in \mathbb{N}$. Then, for $\delta \in (0,1]$, the limit*

$$M(\delta) := \lim_{n\to\infty} \frac{1}{n} \ln \mu(\mathcal{C}_{\delta,n}) \ ,$$

exists. Moreover, its explicit form is

$$\begin{aligned} M(\delta) = \ - & \left[\frac{1}{\delta}\right] \left(1 - \delta\left[\frac{1}{\delta}\right]\right) H_R\left(\left[\frac{1}{\delta}\right]\right) \\ + & \left(1 - \delta - \delta\left[\frac{1}{\delta}\right]\right)\left(\left[\frac{1}{\delta}\right] - 1\right) H_R\left(\left[\frac{1}{\delta}\right] - 1\right) \ , \end{aligned}$$

which is a non-decreasing continuous piece-wise affine function.

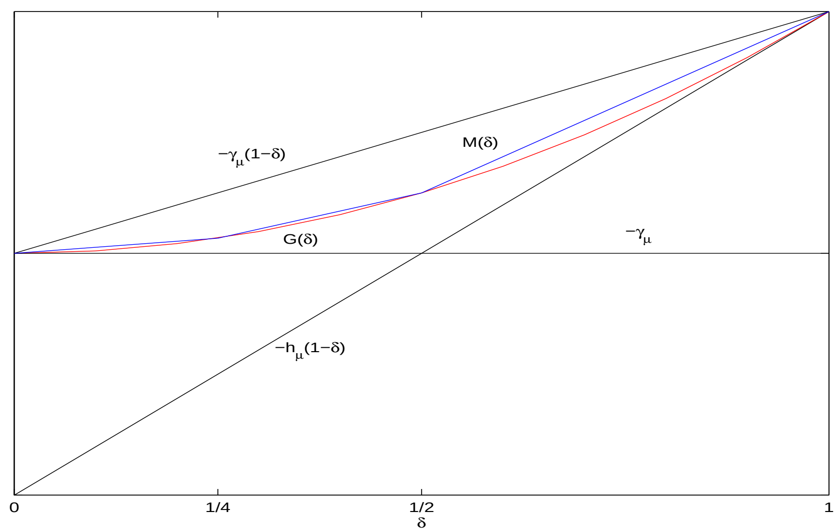

FIGURE 2. Theorem 3.2

Remark 6. The (horrendous) explicit expression of $M(\delta)$ has a simple interpretation. For $\delta = 1/k$ with $k \in I\!N$ the most left term disappears and the most right one simply becomes

$$-\frac{1}{k}(k-1)H_R(k-1)\ .$$

For values of $\delta \in (1/(k+1), 1/k)$ it is just a linear function of δ which interpolates $-H_R(k)k/(k+1)$ and $-H_R(k-1)(k-1)/k$.

Whenever the Rényi entropies are defined for $\beta > 0$, $M(\delta)$ interpolates the function

$$G(\delta) = -H_R\left(\frac{1}{\delta} - 1\right)(1-\delta)\ , \tag{11}$$

over the set $\{1/k \mid k \in I\!N\}$.

Note that $H_R(0)$ is the metric entropy. In the case when G is convex (see picture below, and this is what happens in the aforementioned examples where $H_R(\beta)$ is a real analytic function of β), $M(\delta)$ provides a lower bound for $H_R(0)$. Namely, one has

$$\frac{M(\delta)}{\delta - 1} \le H_R\left(\frac{1}{\delta} - 1\right) \le H_R(0)\ .$$

Let us give two examples.

Example 1. Consider an automorphism T over the space Ω with a finite partition $\mathcal{E} = \{a_1, \ldots, a_\alpha\}$ and which has maximal entropy measure μ. Therefore for any n-cylinder set $C = C_n = (a_{i_1}, \ldots, a_{i_n}) \in \mathcal{E}^n$ one has $\mu(C_n) = Ce^{-h_M n}$. In such a case $\gamma_\mu = \rho_\mu = h_M$. Moreover $H_R(\delta)$ is constant and for all $\delta \in (0,1)$

$$\lim_{n\to\infty} \frac{1}{n} \ln \mu(\mathcal{C}_{\delta,n}) = -h_M(1-\delta)\ .$$

Example 2. Consider a Bernoulli automorphism T over the space Ω with finite generating partition $\mathcal{C} = \{0,1\}$ such that $\mu(1) = p$ and $\mu(0) = p$. The Rényi entropies are easily seen to be, for $\beta \neq 0$: $H_R(\beta) = \frac{1}{\beta}\ln(p^\beta + (1-p)^\beta)$. An immediate application of the preceding theorem gives:

$$\lim_{n\to\infty} \frac{1}{n} \ln \mu(\mathcal{C}_{\delta,n}) = \begin{cases} \delta \ln\left(p^{1/\delta} + (1-p)^{1/\delta}\right) & \delta = \frac{1}{k} \\ (1-\delta)\ln\left(p^2 + (1-p)^2\right) & \frac{1}{2} \leq \delta \leq 1 \end{cases} . \tag{12}$$

For $1/(k+1) < \delta < 1/k$, $M(\delta)$ is a linear function between $M(1/(k+1))$ and $M(1/k)$. Namely

$$M(\delta) = \frac{M(1/k) - M(1/(k+1))}{1/k - 1/(k+1)}\,(\delta - 1/k+1) + M\left(1/(k+1)\right) \ .$$

Moreover

$$\lim_{\delta\to 0}\lim_{n\to\infty} \frac{1}{n}\ln \mu(\mathcal{C}_{\delta,n}) = \ln p = -\gamma_\mu \ .$$

Example 3. The previous Bernoulli measure is the very simple example of an equilibrium (Gibbs) state on the full shift. We could also consider Markov measures on aperiodic finite Markov chains (an explicit formula for the Renyi entropies exists in this case as well). These two measures are particular cases of Gibbs equilibrium states for Hölder continuous potential ϕ on subshifts of finite type. Let us denote with $P(\phi)$ the topological pressure of the function ϕ. Then the equilibrium state μ_ϕ verifies the variational principle: $P(\phi) = h_{\mu_\phi} + \int \phi d\mu_\phi$. In [9] it is proved that $H_R(\beta)\beta = (1+\beta)P(\phi) - P((1+\beta)\phi)$, where we *used* the measure μ_ϕ to compute the Renyi entropies. An easy change of variable shows that $G(\lambda) = -P(\phi) + \frac{P(\lambda^{-1}\phi)}{\lambda^{-1}}$.

3.1. **Proof of Theorem 3.2 and corollaries.** We divide the proof in Proposition 4 and Proposition 5 below which prove a lower and upper bound respectively.

Proposition 4. *Let $(\Omega, \mathcal{F}, \mu, T)$ be a ψ-mixing dynamical system. Assume $\psi(0) < 1$ and suppose that the Rényi entropies relative to the partition $\mathcal{C}$ exist for any $\beta \in \mathbb{N}$. Then, for $\delta \in (0,1]$ the following holds*

$$\liminf_{n\to\infty} \frac{1}{n}\ln \mu(\mathcal{C}_{\delta,n}) \geq M(\delta) \ .$$

Proof. Write $n = [\delta n][1/\delta] + r$. Thus

$$r = n\left(1 - \delta\left[\frac{1}{\delta}\right]\right) + (\delta n - [\delta n])\left[\frac{1}{\delta}\right] \ .$$

Therefore

$$L := n\left(1 - \delta\left[\frac{1}{\delta}\right]\right) \leq r \leq n\left(1 - \delta\left[\frac{1}{\delta}\right]\right) + \left[\frac{1}{\delta}\right] =: U \ . \tag{13}$$

Now we divide the proof in $0 < \delta < 1/2$ and $1/2 \leq \delta < 1$.

-Case $0 < \delta < 1/2$. We put $n = [1/\delta][\delta n] + r$; then

$$\begin{aligned} \mu(\mathcal{C}_{\delta,n}) &\geq \mu(B_n([\delta n])) \\ &= \sum_{C_n \in B_n([\delta n])} \mu(C_n) \\ &\geq \sum_{C_{[\delta n]} \in \mathcal{C}^{[\delta n]}} (1-\psi(0))^{[1/\delta]} \mu(C_{[\delta n]})^{[1/\delta]} \mu(T^{-[\delta n][1/\delta]} C_r) \ . \end{aligned}$$

For any $C_{[\delta n]} \in \mathcal{C}^{[\delta n]}$ we write $C_{[\delta n]} = C_r \cap T^{-r} C_{[\delta n]-r}$. Thus, again by the mixing property

$$\mu(C_{[\delta n]}) \geq (1-\psi(0))\mu(C_r)\mu(T^{-r}C_{[\delta n]-r}) \ . \tag{14}$$

Moreover $\mathcal{C}^{[\delta n]} = \mathcal{C}^r \times \mathcal{C}^{[\delta n]-r}$, thus

$$\mu(\mathcal{C}_{\delta,n}) \geq (1-\psi(0))^{2[1/\delta]} \sum_{C\in\mathcal{C}^r} \mu(C_r)^{[1/\delta]+1} \sum_{C\in\mathcal{C}^{[\delta n]-r}} \mu(C_{[\delta n]-r})^{[1/\delta]} \ .$$

Applying (13) we get

$$\mu(\mathcal{C}_{\delta,n}) \geq (1-\psi(0))^{2[1/\delta]} \sum_{C_U\in\mathcal{C}^U} \mu(C_U)^{[1/\delta]+1} \sum_{C_{[\delta n]-L}\in\mathcal{C}^{[\delta n]-L}} \mu(C_{[\delta n]-L})^{[1/\delta]} \ .$$

Notice that

$$\lim_{n\to\infty} \frac{U}{n} = 1-\delta\left[\frac{1}{\delta}\right] \qquad ; \qquad \lim_{n\to\infty} \frac{[\delta n]-L}{n} = \delta - (1-\delta\left[\frac{1}{\delta}\right]) \ .$$

Then, by the exchanges of variables $n' = U$ and $n" = [\delta n] - L$, we conclude that

$$\frac{1}{n}\ln \mu(\mathcal{C}_{\delta,n}) \ ,$$

is bounded from below by $M(\delta)$. This ends the first case.

-Case $1/2 \leq \delta < 1$. One has

$$\mu(\mathcal{C}_{\delta,n}) \geq \mu(B_n([\delta n])) \ .$$

Since $[\delta n] \geq n/2$ the first and the last $n-[\delta n]$ symbols of C_n are equal. Thus, dividing C_n in its first and last $n-[\delta n]$ symbols and its $n-2(n-[\delta n])$ central symbols and then using the ψ-mixing property one has

$$\begin{aligned}\sum_{C_n\in B_n([\delta n])} \mu(C_n) &\geq \sum_{C\in\mathcal{C}^{n-[\delta n]}} \sum_{C\in\mathcal{C}^{2[\delta n]-n}} (1-\psi(0))^2 \mu(C_{n-[\delta n]})^2 \mu(C_{2[\delta n]-n}) \\ &\geq K \sum_{C_{n-[\delta n]}\in\mathcal{C}^{n-[\delta n]}} \mu(C_{n-[\delta n]})^2 \ .\end{aligned}$$

Further,

$$\begin{aligned}&\frac{1}{n}\ln \sum_{C_{n-[\delta n]}\in\mathcal{C}^{n-[\delta n]}} \mu(C_{n-[\delta n]})^2 \\ = \ &\frac{n-[\delta n]}{n}\frac{1}{n-[\delta n]} \ln \sum_{C_{n-[\delta n]}\in\mathcal{C}^{n-[\delta n]}} \mu(C_{n-[\delta n]})^2 \ .\end{aligned}$$

Thus the proposition follows. □

Proposition 5. *Let $(\Omega, \mathcal{F}, \mu, T)$ be a ψ-mixing dynamical system. Suppose that the Rényi entropies relative to the partition $\mathcal{C}$ exist for any $\beta \in \mathbb{N}$. Then, for $\delta \in (0,1]$ the following holds*

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mu(\mathcal{C}_{\delta,n}) \leq M(\delta) \ .$$

Proof. We first need the following useful lemma.

Lemma 3.3. *Let* $c_{i,n} > 0$ *with* $1 \le i \le n$, $n \in \mathbb{N}$. *Then (if the limits exist, otherwise it holds for* $\limsup$ *and* $\liminf$*) the following equality holds*

$$\lim_{n\to\infty} \frac{1}{n} \ln \sum_{i=1}^{n} c_{i,n} = \lim_{n\to\infty} \frac{1}{n} \ln \max_{1\le i\le n} \{c_{i,n}\} .$$

Proof. Observe that $\max_{1\le i\le n}\{c_{i,n}\} \le \sum_{i=1}^{n} c_{i,n} \le n \ \max_{1\le i\le n}\{c_{i,n}\}$. Taking the logarithm, dividing by n and then taking the limit, the lemma follows. □

We return to the proof of the Proposition. As we show in Proposition 1 we have

$$\mu(\mathcal{C}_{\delta,n}) \le \sum_{j=[\delta n]/2}^{[\delta n]} \mu(B_n(j)) \le n \max_{[\delta n]/2\le j\le[\delta n]} \mu(B_n(j)) .$$

Therefore by Lemma 3.3 in order to get an upper bound for $\limsup_{n\to\infty} \ln \mu(\mathcal{C}_{\delta,n})/n$ it is enough to find an upper bound for

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mu(B_n(j(n))) , \tag{15}$$

where $j(n)$ is (one of) the j that realizes the following maximum

$$\max_{[\delta n]/2\le j\le[\delta n]} \mu(B_n(j)) .$$

Now, we also divide this proof in two cases: (a) $0 < \delta < 1/2$ and (b) $1/2 \le \delta < 1$.

Assume (a). Since $j(n) < n/2$, for each $C_n \in B_n(j(n))$ every contiguous and non-overlapped block of $j(n)$ symbols of C_n is equal to the first one. Thus, write $n = [n/j]j + r$ recalling that j depends on n and r depends on j. By the ψ-mixing property

$$\mu(B_n(j)) \le \sum_{C_j\in\mathcal{C}^j} (1+\psi(0))^{[n/j]} \mu(C_j)^{[n/j]} \mu(C_r) .$$

As in (14) one has

$$\mu(C_j) \le (1+\psi(0))\mu(C_r)\mu(T^{-r}C_{j-r}) .$$

Thus, since $\mathcal{C}^j = \mathcal{C}^r \times \mathcal{C}^{j-r}$, we get that $\mu(B_n(j))$ is bounded from above by

$$(1+\psi(0))^{2[n/j]} \sum_{C_r\in\mathcal{C}^r} \mu(C_r)^{[n/j]+1} \sum_{C_{j-r}\in\mathcal{C}^{j-r}} \mu(C_{j-r})^{[n/j]} .$$

Hence $\limsup_{n\to\infty} \ln \mu(\mathcal{C}_{\delta,n})/n$ is bounded from above by the sum of the following three limits

$$\limsup_{n\to\infty} \frac{1}{n} \ln \sum_{C_r\in\mathcal{C}^r} \mu(C_r)^{[n/j]+1} , \tag{16}$$

$$\limsup_{n\to\infty} \frac{1}{n} \ln \sum_{C_{j-r}\in\mathcal{C}^{j-r}} \mu(C_{j-r})^{[n/j]} , \tag{17}$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \ln(1+\psi(0))^{2[n/j]} . \tag{18}$$

Since $[\delta n]/2 \le j$, the limit (18) is zero. We now consider the limits (16), (17) and the sequence $\{j(n)/n\}_{n\in\mathbb{N}}$. Remind that $n = [n/j(n)]j(n) + r(n)$. We have two cases according to the sequence $\{j(n)/n\}_{n\in\mathbb{N}}$ converges or diverges.

A. If it converges and since $[\delta n]/2 \le j(n) \le [\delta n]$, it goes to some $\lambda \in [\delta/2, \delta]$. We still have two cases.

-If $\lambda = 1/k$ with $k \in I\!N$ then $\lim_{n\to\infty} r(n)/n = 0$. Thus (16) goes to zero and (17) converges to

$$-H_R\left(\frac{1}{\lambda}-1\right)\lambda(\frac{1}{\lambda}-1) = H_R\left(\frac{1}{\lambda}-1\right)(\lambda-1)\ .$$

-If $\lambda \ne 1/k$ we can write $1 = \lambda[1/\lambda] + r_\lambda$. So $\lim_{n\to\infty} r(n)/n = r_\lambda = 1-\lambda[1/\lambda]$. Thus (16) converges to

$$-H_R\left(\left[\frac{1}{\lambda}\right]\right)(1-\lambda\left[\frac{1}{\lambda}\right])\left[\frac{1}{\lambda}\right]\ ,$$

and (17) converges to

$$-H_R\left(\left[\frac{1}{\lambda}\right]-1\right)\left(\lambda-(1-\lambda\left[\frac{1}{\lambda}\right])\right)(\left[\frac{1}{\lambda}\right]-1)\ .$$

The sum of the last two quantities is nothing but $M(\lambda)$. But as we previously said, the latter function interpolates the function $G(\lambda)$, see (11). It is straightforward to check that $G(\lambda)$ is increasing [8], and then the maximum of the possible limits of $M(\lambda)$ is attained when $\lambda = \delta$.

B. If $j(n)/n$ does not converge, then consider all its converging subsequences. The possible limits are those obtained in **A**. The maximum is yet attained when $\lambda = \delta$.

Thus we conclude the proof of the first case.
Proof of (b). We proceed by estimating

$$\mu(\mathcal{C}_{\delta,n}) \le \sum_{j=[\delta n/]2}^{[\delta n]} \mu(B_n(j))\ .$$

We recall the different structure of $B_n(j)$ for $j \le [n/2]$ and $j > [n/2]$. Using Lemma 3.3 we have only to identify (one of) the maximum of $\mu(B_n(j))$ for $[\delta n]/2 \le j \le [\delta n]$. Call it again $j(n)$.

Consider first that $[n/2] < j(n) \le [\delta n]$. By the definition of $B_n(j)$ one has

$$\begin{aligned} &\mu(B_n(j)) \\ \le\ & \sum_{C_{n-j(n)}\in\mathcal{C}^{n-j(n)}}\ \sum_{C_{2j(n)-n}\in\mathcal{C}^{2j(n)-n}} (1+\psi(0))^2\mu(C_{n-j(n)})^2\mu(C_{2j(n)-n}) \\ \le\ & K \sum_{C_{n-j(n)}\in\mathcal{C}^{n-j(n)}} \mu(C_{n-j(n)})^2\ . \end{aligned}$$

Therefore we consider

$$\frac{n-j(n)}{n}\ \frac{1}{n-j(n)}\ln\sum_{C\in\mathcal{C}^{n-j(n)}}\mu(C_{n-j(n)})^2\ . \tag{19}$$

We have two cases according to the sequence $\{j(n)/n\}_{n\in I\!N}$ converges to $[1/2,\delta]$ or $[\delta/2, 1/2)$.

A. We still divide this case in two: $\delta/2 \le \lambda \le 1/2$ and $1/2 < \lambda \le \delta$. If $\delta/2 \le \lambda \le 1/2$ then (19) converges to the same limit established in case **A** of (a). If $1/2 < \lambda \le \delta$ then (19) converges to

$$-(1-\lambda)H_R(1)\ .$$

The maximum of both cases is still when $\lambda = \delta$.

B. Consider that $[\delta n]/2 \leq j(n) \leq [n/2]$. This case is treated verbatim like the case when $\lambda \leq 1/2$.

As in the proof of (a), if $j(n)/n$ does not converges, consider all its converging subsequences. The possible limits are those obtained in **A** and **B**. The maximum is yet attained when $\lambda = \delta$. Now the maximum at $\lambda = \delta$ is $-(1-\delta)H_R(1)$.

This ends the proof of the proposition. □

4. **Computation of the free energy.** We show in this section that for ψ-mixing dynamical systems the free energy function of $\tau(C_n)/n$ can be easily computed; the free energy is defined as

$$
\begin{aligned}
F(\beta) &\stackrel{def}{=} \lim_{n\to\infty} \frac{1}{n} \ln \int_{\Omega} \exp(\beta\tau(C_n)) d\mu \\
&= \lim_{n\to\infty} \frac{1}{n} \ln \sum_{i=1}^{n} \exp(\beta j) \mu(\tau(C_n) = j) \ , \qquad (20)
\end{aligned}
$$

wherever the limit exists.

In the next proposition we invoke Assumption 1.

Proposition 6. *Let $(\Omega, \mathcal{F}, \mu, T)$ be a ψ-mixing dynamical system and suppose Assumption 1 holds. Then the following equality holds:*

$$
F(\beta) = \begin{cases} \beta & -\gamma_\mu \leq \beta < 0 \\ -\gamma_\mu & \beta \leq -\gamma_\mu \end{cases} . \qquad (21)
$$

Remark 7. Therefore, the free energy function is continuous. It is not differentiable only at the point $\beta = -\gamma_\mu$. Notice that $-\gamma_\mu \geq -h_\mu$, where h_μ is the metric entropy. The equality holds only when μ is the maximal entropy measure. In that case one has $\gamma_\mu = h_\mu$. Since $F(\beta)$ is not differentiable, by applying the classical theorem of large deviations (see for instance Ellis' book [17]), we can only get an upper bound for $M(\delta)$ by means of the Legendre transform of the free energy function $F(\beta)$, denoted by $\mathcal{L}(F)(\delta)$. It is a straightforward computation to show that

$$
M(\delta) \leq \mathcal{L}(F)(\delta) = -\gamma_\mu(1-\delta) \ .
$$

Thus we recover the upper bound we get directly in Proposition 1 (a).

4.1. **Proof of Proposition 6.**

Proof. By Lemma 3.3, in order to obtain the limit (20) it is enough to obtain the limit $\lim_{n\to\infty}(\ln m_n(\beta))/n$, where

$$
m_n(\beta) \stackrel{def}{=} \max_{1\leq j\leq n} \exp(\beta j)\ \mu(\tau(C_n) = j), \qquad \forall n \in I\!N. \qquad (22)
$$

Take any cylinder C_n such that $\tau(C_n) = j$. Put $n = [n/j]j + r$. By definition of $B_n(j)$

$$
\mu(\tau(C_n) = j) \leq \mu(B_n(j)) \ .
$$

As in the proof of Proposition (1) and with the same notations used there:

$$
\begin{aligned}
\mu(B_n(j)) &\leq \sum_{C_j \in \mathcal{C}^j} (1+\psi(0))\mu(C_j)\mu(T^j C_{n-j}) \\
&\leq (C \exp(-\gamma_\mu(n-j)) \ .
\end{aligned}
$$

Applying the above inequality to (22) we have

$$\begin{aligned} \frac{1}{n}\ln m_n(\beta) &\leq \frac{1}{n}\ln \max_j \{C_1 \exp\left(\beta j - \gamma_\mu(n-j)\right)\} \\ &= C_2 \max_j \left\{\beta\frac{j}{n} - \gamma_\mu\frac{n-j}{n}\right\} . \end{aligned} \tag{23}$$

where C_1 and C_2 are suitable constants. For $1 \leq j \leq n$, and $n \in I\!N$, define the numbers

$$a_{j,n} = \frac{j}{n} \qquad \text{and} \qquad b_{j,n} = -\gamma_\mu \frac{n-j}{n} ,$$

and the linear functions $g_{j,n}(\beta) = a_{j,n}\beta + b_{j,n}$.

Observe that $g_{j,n}(-\gamma_\mu) = -\gamma_\mu$ for all $1 \leq j \leq n \in I\!N$. Moreover $1/n < a_{1,n} < \cdots < a_{n,n} = 1$. Therefore

$$\max_j g_{j,n}(\beta) = g_{n,n}(\beta) = \beta \qquad \forall \beta \geq -\gamma_\mu \ ; \ \forall n \in I\!N \ .$$

We conclude that

$$\limsup_{n\to\infty} \frac{1}{n}\ln m_n(\beta) \leq \beta \ .$$

On the other hand

$$\max_j g_{j,n}(\beta) = g_{1,n}(\beta) \qquad \forall \beta \leq -\gamma_\mu \ ; \ \forall n \in I\!N \ .$$

Moreover, $\lim_{n\to\infty} g_{1,n}(\beta) = -\gamma_\mu$. We conclude that

$$\limsup_{n\to\infty} \frac{1}{n}\ln m_n(\beta) \leq -\gamma_\mu \ ; \ \forall n \in I\!N \ .$$

This ends the proof of the upper bound for $F(\beta)$.

Now we proceed to prove the lower bound for $\liminf_{n\to\infty}(\ln m_n(\beta))/n$. We divide its proof in two cases.

Case (i): $-\gamma_\mu \leq \beta < 0$. Since $\beta < 0$ one has

$$\sum_{j=1}^{n} \exp(\beta j)\mu(\tau(C_n) = j) \geq \exp(\beta n) \ , \tag{24}$$

which obviously gives:

$$\liminf_{n\to\infty} \frac{1}{n}\ln \int_\Omega \exp(\beta\tau(C_n))d\mu \geq \beta \ .$$

Case (ii): $\beta < -\gamma_\mu$. For each $n \in I\!N$, consider the n-cylinder P_n which verifies Assumption 1. We have the inequality:

$$\sum_{j=1}^{n} \exp(\beta j)\mu(\tau(C_n) = j) \geq \exp\left(\beta\tau(P_n)\right)\mu(P_n) \ .$$

Thus, by limit (8)

$$\liminf_{n\to\infty} \frac{1}{n}\ln \int_\Omega \exp(\beta\tau(C_n))d\mu \geq -\gamma_\mu \ .$$

This ends the proof of formula (21). $\square$

4.2. **Examples.** In this section we present some examples of systems for which the Assumption 1 is verified. Moreover, in all of them a stronger property holds: there exists a positive constant $C > 0$ such that $\tau(P_n) \le C$ for all $n \in I\!\!N$.

Example 4. Consider a Bernoulli automorphism T over the space Ω with a finite partition $\mathcal{C} = \{a_1, \dots, a_\alpha\}$ and which has a non maximal entropy measure μ. So that, for any n-cylinder set $C = C_n = (a_{i_1}, \dots, a_{i_n}) \in \mathcal{C}^n$ one has $\mu(a_{i_1}, \dots, a_{i_n}) = \prod_{j=1}^n p_{i_j}$. Choose

$$P_n = (\underbrace{1, \dots, 1}_{n}) \qquad \forall n \in I\!\!N \ .$$

Therefore, if we put $p = \max_{i\in\{1,\dots,\alpha\}} \{p_i\}$, then $\mu(P_n) = p^n$. It follows immediately that $\gamma_\mu = -\ln p$. Further, $\tau(P_n) = 1$ for all $n \in I\!\!N$. Thus condition (1) holds.

Example 5. Consider an automorphism T over the space Ω with a finite partition $\mathcal{C} = \{a_1, \dots, a_\alpha\}$ and which has maximal entropy measure μ. Hence for any n-cylinder $C_n = (a_{i_1}, \dots, a_{i_n}) \in \mathcal{C}^n$ one has $\mu(C_n) = Ce^{-h_M n}$. It is immediate to check that $\gamma_\mu = -h_M$. Further, if x is any fixed point and we choose $P_n = C_n(x)$, we have that

$$\mu(P_n) = Ce^{-h_M n}. \qquad \text{and} \qquad \tau(P_n) \le C \ .$$

Thus condition (1) holds.

Example 6. Consider an irreducible and aperiodic Markov chain over the space Ω with a two-state partition $\mathcal{C} = \{0, 1\}$ and transition probabilities given by the matrix

$$Q = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} ,$$

with $Q(0,0) = \mu_0(0) = 1-p$. Thus the measure of any n-cylinder is

$$\mu(a_0, \dots, a_{n-1}) = \mu(a_0) \prod_{i=0}^{n-1} Q(a_i, a_{i+1}) = \mu(a_0)(1-p)^{n_p}(1-q)^{n_q} p^{n_0} q^{n_1} ,$$

satisfying the constrains

$$n_p + n_q + n_0 + n_1 = n-1 \qquad \text{and} \qquad |n_0 - n_1| = 0 \text{ or } 1 \ .$$

Thus taking the limit

$$-\liminf_{n\to\infty} \frac{1}{n} \ln \mu(a_0, \dots, a_{n-1}) = -l_p \ln(1-p) - l_q \ln(1-q) - 2l_{0,1} \frac{\ln p + \ln q}{2} ,$$

where

$$l_p + l_q + 2l_{0,1} = 1 \quad \text{and} \quad 0 \le l_p \le 1, \ 0 \le l_q \le 1, \ 0 \le l_{0,1} \le 1 \ .$$

Therefore

$$\gamma_\mu = \min\left\{ -\ln(1-p), -\ln(1-q), -\frac{\ln p + \ln q}{2} \right\} \ .$$

Now

- if $\gamma_\mu = -\ln(1-p)$ we take $P_n = (0, \dots, 0)$
- if $\gamma_\mu = -\ln(1-q)$ we take $P_n = (1, \dots, 1)$
- if $\gamma_\mu = -(\ln p + \ln q)/2$ we take $P_n = (\underbrace{1, 0, 1, 0, \dots}_{n})$

In any case $\tau(P_n) \le 2$. Thus Assumption 1 holds.

5. **Concluding remarks.** The main results of this paper have been to compute the large deviations and the free energy of the process $\tau(C_n)/n$, by showing that the limits defining those quantities exist. As we pointed out in the Remark 2, the condition $\psi(0) < 1$ was necessary to implement our approach, in particular to prove the lower bounds for the two limits:

$$\liminf_{n\to\infty} \frac{1}{n} \ln \mu(\mathcal{C}_{\delta,n})$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \ln \int_\Omega \exp(\beta\tau(C_n))d\mu.$$

To get similar bounds without the condition $\psi(0) < 1$ would probably require a different strategy. Instead we would like to point out that the upper bounds for the preceding limits, with the $\limsup$ at the place of the $\liminf$, do not need the preceding condition, so that they remain true for a general ψ-mixing measure: they are exactly the contents of Proposition 5 and of Proposition 6 (upper bound). In this case of course one would get an upper large deviation bound for the process $\tau(C_n)/n$, which is weaker than our complete result but still interesting, especially for its connection with the Rnyi entropies.

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