EXTREME VALUE THEORY WITH SPECTRAL TECHNIQUES: APPLICATION TO A SIMPLE ATTRACTOR.

JASON ATNIP, NICOLAI HAYDN, AND SANDRO VAIENTI

Abstract. We give a brief account of application of extreme value theory in dynamical systems by using perturbation techniques associated to the transfer operator. We will apply it to the baker's map and we will get a precise formula for the extremal index. We will also show that the statistics of the number of visits in small sets is compound Poisson distributed.

1. Introduction

Extreme value theory (EVT) has been widely studied in the last years in application to dynamical systems both deterministic and random. A review of the recent results with an exhaustive bibliography is given in our collective work [25]. As we will see, there is a close connection between EVT and the statistics of recurrence and both could be worked out simultaneously by using perturbations theories of the transfer operator. This powerful approach is limited to systems with quasi-compact transfer operators and exponential decay of correlations; nevertheless it can be applied to situations where more standard techniques meet obstructions and difficulties, in particular to:
- non-stationary and random dynamical systems,
- observable with non-trivial extremal sets,
- higher-dimensional systems.

Another big advantage of this technique is the possibility of defining in a precise and universal way the extremal index (EI). We defer to our recent paper [7] for a critical discussion of this issue with several explicit computations of the EI in new situations. The germ of the perturbative technique of the transfer operator applied to EVT is in the fundamental paper [23] by G. Keller and C. Liverani; the explicit connection with recurrence and extreme value theory has been done by G. Keller in the article [22], which contains also a list of suggestions for further investigations. We successively applied this method to i.i.d. random transformations in [5, 7], to randomly quenched dynamical systems in [2], to coupled maps on finite lattices in [14], and to open systems with targets and holes in [17].

The object of this note is to illustrate this technique by presenting a new application to a bi-dimensional invertible system. We will see that the perturbative technique could be applied in this case as well provided one could find the good functional spaces where the transfer operator exhibits quasi-compactness.

We will find a few limitations to a complete application of the theory and to its generalization to wider class of maps in higher dimensions, see Remarks 3.2 and 3.3.

Date: February 27, 2024.

J Atnip, School of Mathematics and Statistics, University of New South Wales, Sydney, NSW 2052, Australia. E-mail: j.atnip@unsw.edu.au.

N Haydn, Mathematics Department, USC, Los Angeles, 90089-2532. E-mail: nhaydn@usc.edu.

S Vaienti, Aix Marseille Université, Université de Toulon, CNRS, CPT, UMR 7332, Marseille, France. E-mail: vaienti@cpt.univ-mrs.fr.

When the first version of this paper circulated, the spectral technique discussed above did not allow us to get another property related to limiting return and hitting times distribution in small sets, namely the statistics of the number of visits, which takes usually the form of a compound Poisson distribution. This has been recently achieved in the paper [3], and it could be easily applied to the system under investigation in this paper. We will briefly quote this technique in section 5. As for the EVT, such a technique suffers of the limitation imposed by the choice of the parameters, see remark 3.3. In particular, it does not allow us to treat the case of the *fat* Baker's map, where the invariant set is the full square. This is instead possible with another technique developed by two of us, see [20], which allows to recover compound Poisson distributions for invertible maps in higher dimension and arbitrary small sets. By using this approach, we will be able to construct an example for the fat baker map with a compound Poisson distribution which is neither the standard Poisson nor the Pòlya-Aeppli,which are the most common compound distributions. We will finally discuss the extension to compound Poisson point process on the real line.

2. A pedagogical example: the generalized baker's map

We now treat an example for which there are not apparently established results for the extreme value distributions. This example, the generalized baker's map, from now on simply abbreviated as baker's map, is a prototype for uniformly hyperbolic transformations in more than one dimension, two in our case, and in order to study it with the transfer operator, we will introduce suitable anisotropic Banach spaces. Our original goal was to investigate directly larger classes of uniformly hyperbolic maps, including Anosov ones, but, as we said above, the generalizations do not seem straightforward; we will explain the reason later on. With the usual probabilistic approaches extreme value distributions have been obtained for the linear automorphisms of the torus in [8].

We will refer to the baker's transformation studied in Section 2.1 in [10], but we will write it in a particular case in order to make the exposition more accessible. The baker's transformation $T(x_n, y_n)$ is defined on the unit square $X = [0,1]^2 \subset \mathbb{R}^2$ into itself by:

$$x_{n+1} = \begin{cases} \gamma_a x_n & \text{if } y_n < \alpha \\ (1-\gamma_b) + \gamma_b x_n & \text{if } y_n > \alpha \end{cases}$$

$$y_{n+1} = \begin{cases} \frac{1}{\alpha} y_n & \text{if } y_n < \alpha \\ \frac{1}{\upsilon}(y_n - \alpha) & \text{if } y_n > \alpha, \end{cases}$$

with $\upsilon = 1-\alpha$, $\gamma_a + \gamma_b \leq 1$, see Fig. 1. To simplify some of the next formulae, we will take $\alpha = \upsilon = 0.5$ and $\gamma_a = \gamma_b < 0.5$. This last value must be strictly less than 1/2 since Lemma 3.1 requires the stable dimension d_s strictly less than one, which corresponds to a fractal invariant set (*thin baker's map*). This condition will be relaxed in the example 5.3 (*fat baker's map*), but using an approach different of the spectral one leading to Lemma 3.1.

The map T is discontinuous at the horizontal line $\Gamma : \{y = \alpha\}$. The singularity curves for $T^l, l > 1$ are given by $T^{-l}\Gamma$ and they are constructed in this way: take the preimages $T_Y^{-l}(\alpha)$ of $y = \alpha$ on the y-axis according to the map:

$$T_Y(y) = \begin{cases} \frac{1}{\alpha} y, y < \alpha \\ \frac{1}{\upsilon} y - \frac{\alpha}{\upsilon}, y \geq \alpha. \end{cases} \quad (1)$$

Then $T^{-l}\Gamma = \{y = T_Y^{-l}(\alpha)\}$. Any other horizontal line will be a stable manifold of T. The invariant non-wandering set Λ will be at the end an attractor foliated by vertical

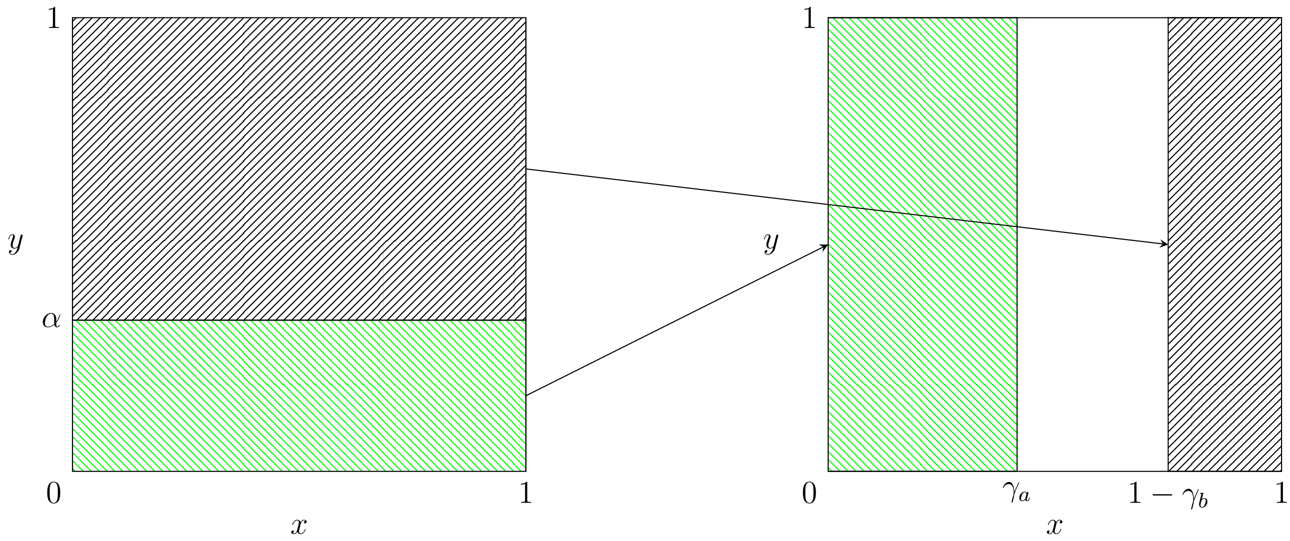

FIGURE 1. Action of the baker's map on the unit square. The lower part of the square is mapped to the left part and the upper part is mapped to the right part.

lines which are all unstable manifolds. We denote by $\mathcal{W}^s(\mathcal{W}^u)$ the set of full horizontal (vertical) stable (unstable) manifolds of length 1 just constructed. We point out that a stable horizontal manifold W_s will originate two disjoint full stable manifold when iterate backward by T^{-1}, not for the presence of singularity, but because the map T^{-1} will only be defined on the two images of $T(X)$ as illustrated in Fig. 1.

In order to obtain useful spectral information from the transfer operator $\mathcal{L}$, its action is restricted to a Banach space $\mathcal{B}$. We now give the construction of the norms on $\mathcal{B}$ and an associated "weak" space $\mathcal{B}_w$ in the case of the baker's map, following partly the exposition in [10]. In this case, those spaces are easier to define and the norms will be constructed directly on the horizontal stable manifolds instead of admissible leaves, which are smooth curves in approximately the stable direction, see [11]. As we anticipated above, we follow [10], but we slightly change the definition of the stable norms by adapting ourselves to that originally introduced in [11]. Let us explain why. First of all we will consider the collection Σ of all the intervals W of length less or equal to 1 that are contained in the stable manifolds $W \subset W_s \in \mathcal{W}^s$. Instead in [11], Σ was the set of full horizontal line segments of length 1 in X. The reason of our choice is that we will introduce small sets B_n, which could be identified as (fake) holes, and the preimages of such sets will cut the W_s. The smaller pieces generated in this way will enter the three norms given below and therefore it will be useful to count such pieces in Σ.

Then we denote $C^\kappa(W, \mathbb{C})$ the set of continuous complex-valued functions on W with Hölder exponent $\kappa \leq 1$ and define the norm

$$|\varphi|_{W,\kappa} := |W|^\kappa \cdot |\varphi|_{C^\kappa(W,\mathbb{C})}, \tag{2}$$

where $|W|$ denotes the length of W and

$$|\varphi|_{C^\kappa(W,\mathbb{C})} = |\varphi|_{C^0} + H^\kappa(\varphi), \ \ H^\kappa(\varphi) = \sup_{\substack{x,y\in W\\ x\neq y}} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^\kappa}.$$

For $h \in C^1(X,\mathbb{C})$ we define the *weak norm* of h by

$$|h|_w = \sup_{W\in\Sigma} \sup_{\substack{\varphi\in C^1(W,\mathbb{C}) \\ |\varphi|_{C^1(W,\mathbb{C})}\le 1}} \left| \int_W h\varphi \, dm \right|$$

where dm is the unnormalized Lebesgue measure along W, instead with m_L we will denote the Lebesgue measure over X. We now take[1] $(\kappa,\beta) \in (0,1)$ with $0 < \beta \le 1-\kappa$. The strong stable norm is defined as:

$$\|h\|_s = \sup_{W\in\Sigma} \sup_{\substack{\varphi\in C^1(W,\mathbb{C}) \\ |\varphi|_{W,\kappa}\le 1}} \left| \int_W h\varphi \, dm \right|. \tag{3}$$

We then need to define the strong unstable norm which allows us to compare expectations along different stable manifolds. If W_1 is a subset of the stable manifold W_s we could parameterize it as (t, s_{W_1}) where s_{W_1} is the common ordinate of the points in W_1 and $t \in [a_1,b_1] \subset [0,1]$. If W_2 is a subset of another stable manifold, parametrized as (t, s_{W_2}) with $t \in [a_2,b_2]$, we pose

$$d(W_1,W_2) = |s_{W_1} - s_{W_2}| + |[a_1,b_1]\Delta[a_2,b_2]| + |[a_1,b_1]\cap[a_2,b_2]|,$$

where Δ means the symetric difference, and for test functions $\varphi_i \in C^1(W_i,\mathcal{C}), i = 1,2:$

$$d_0(\varphi_1,\varphi_2) = \sup_{t\in[a_1,b_1]\cap[a_2,b_2]} |\varphi_1(s_{W_1},t) - \varphi_2(s_{W_1},t)|.$$

The strong unstable norm of h is defined as

$$\|h\|_u = \sup_{\epsilon\le 1} \sup_{\substack{W_1,W_2\in\mathcal{W}_s \\ d(W_1,W_2)\le\epsilon}} \sup_{\substack{\varphi_i\in C^1(W_i,\mathbb{C}) \\ |\varphi_i|_{C^1(W,\mathbb{C})}\le 1 \\ d_0(\varphi_1,\varphi_2)\le\epsilon}} \frac{1}{\epsilon^\beta} \left| \int_{W_1} h\varphi_1 dm - \int_{W_2} h\varphi_2 dm \right|, \tag{4}$$

Finally we can define the *strong norm* of h by

$$\|h\| = \|h\|_s + b\|h\|_u,$$

where b is a small constant to be fixed later on. We set $\mathcal{B}$ to be the completion of $C^1(X,\mathbb{C})$ with respect to the norm $\|\cdot\|$, and, similarly, we define $\mathcal{B}_w$ to the completion of $C^1(X,\mathbb{C})$ with respect to the norm $|\cdot|_w$.

Let us note that $\mathcal{B}$ lies in the dual of $C^1(X,\mathbb{C})$ and its elements are distributions. More precisely, any $h \in \mathcal{B}$ induces the linear functional $\varphi \to h(\varphi)$ with the property that

$$|h(\varphi)| \le |h|_w |\varphi|_{C^1}, \quad \text{for } \varphi \in C^1(X,\mathbb{C}), \tag{5}$$

see [11, Remark 3.4] for details[2]. In particular, for $h \in C^1(X,\mathbb{C})$ we have that (see [11, Remark 2.5])

$$h(\varphi) = \int_X h\varphi \, dm_L, \quad \text{for } \varphi \in C^1(X,\mathbb{C}). \tag{6}$$

The transfer operator $\mathcal{L}$ associated to the map T is defined as

$$(\mathcal{L}h)(\varphi) = h(\varphi\circ T), \quad \text{for } h \in C^1(X,\mathbb{C}) \text{ and } \varphi \in C^1(X,\mathbb{C}),$$

which, by completeness, can be extended to any $h \in \mathcal{B}$.

[1] The bound $\beta \le 1-\kappa$ is needed in the proof of Lemma 3.1 in [9].

[2] The proof of this fact will follow from similar statements shown in section 3.

For $h \in L^1(X, \mathbb{C})$, the space of m_L summable functions with complex values, we have, see [11, Section 2.1]:

$$\mathcal{L}h = \left(\frac{h}{|\det DT|}\right) \circ T^{-1} = \frac{h \circ T^{-1}}{\alpha^{-1}\gamma_a}, \tag{7}$$

where the last equality on the r.h.s. uses the particular choices for the parameters defining the map T.

3. The spectral approach for EVT

3.1. **Formulation of the problem.** We now take a ball $B(z, r)$ of center $z \in X$ and radius r and denote with $B(z, r)^c$ its complement, where $d(\cdot, \cdot)$ is the Euclidean metric.

Let us consider for $x \in X$ the observable

$$\phi(x) = -\log d(x, z) \tag{8}$$

and the function

$$M_n(x) := \max\{\phi(x), \cdots, \phi(T^{n-1}x)\}. \tag{9}$$

For $u \in \mathbb{R}_+$, we are interested in the distribution of $M_n \leq u$, where M_n is now seen as a random variable on the probability space (X, μ), with μ being the Sinai-Bowen-Ruelle (SRB) measure. Notice that the event $\{M_n \leq u\}$ is equivalent to the set $\{\phi \leq u, \ldots, \phi \circ T^{n-1} \leq u\}$ which in turn coincides with the set

$$E_n := B(z, e^{-u})^c \cap T^{-1}B(z, e^{-u})^c \cap \cdots \cap T^{-(n-1)}B(z, e^{-u})^c.$$

We are therefore following points which will enter the ball $B(z, e^{-u})$ for the first time after at least n steps, and $u \to \mu(E_n)$ is the distribution of the maximum of the observable $\phi \circ T^j, j = 0, \ldots, n-1$. It is well known from elementary probability that the distribution of the maximum of a sequence of i.i.d. random variables is degenerate. One way to overcome this is to make the *boundary level* u depend upon the time n in such a way the sequence u_n grows to infinity and gives, hopefully, a non-degenerate limit for $\mu(M_n \leq u_n)$.

From now on we set: $B_n = B(z, e^{-u_n})$ and B_n^c the complement of B_n.

We easily have

$$\mu(M_n \leq u_n) = \int \mathbf{1}_{B_n^c}(x)\mathbf{1}_{B_n^c}(Tx) \cdots \mathbf{1}_{B_n^c}(T^{n-1}x)\, d\mu. \tag{10}$$

By introducing the perturbed operator, for $h \in \mathcal{B}$:

$$\mathcal{L}_n h := \mathcal{L}(\mathbf{1}_{B_n^c} h), \tag{11}$$

we can write (10) as

$$\mu(M_n \leq u_n) = \mathcal{L}_n^n \mu(1). \tag{12}$$

We explicitly used here two facts which deserve justification.

- $\mathbf{1}_{B_n^c}$ and $\mathbf{1}_{B_n^c} h$ are in the Banach space, whenever $h \in \mathcal{B}$. If we prove it for $\mathbf{1}_{B_n^c}$, the same will hold for $\mathbf{1}_{B_n^c} h$ since both $\mathbf{1}_{B_n^c}$ and h will be the limit, in the $\mathcal{B}$ norm, of a sequence of functions in $C^1(X, \mathbb{C})$. Let us sketch the argument for $\mathbf{1}_{B_n^c}$. Take a sequence of C^∞ real functions $0 \leq \theta_k \leq 1$ defined on X, which are equal to 1 on B_n^c and equal to 0 on the complement of an open set U containing B_n^c and at distance $|U \setminus B_n^c| \leq 1/k$. Then for the weak norm of $\mathbf{1}_{B_n^c} - \theta_k$ we have to compute the integral

$$\left|\int_W (\mathbf{1}_{B_n^c} - \theta_k)\varphi dm\right|$$

where W is stable interval of length at most 1. We have $\left|\int_W (\mathbf{1}_{B_n^c} - \theta_k)\varphi dm\right| \leq 4\left|\int_{W\cap U\setminus B_n^c} \varphi dm\right|$. The set $W \cap U \setminus B_n^c$ will consist in fact of at most four connected pieces of stable manifold, therefore

$$|\mathbf{1}_{B_n^c} - \theta_k|_w \leq \sup_{W\in\Sigma} \sup_{\substack{\varphi\in C^1(W,\mathbb{C}) \\ |\varphi|_{C^1(W,\mathbb{C})}\leq 1}} \leq 4|W \cap U/B_n^c| \|\varphi\|_{C^0(W,\mathbb{C})} \leq \frac{4}{k}\|\varphi\|_{C^0(W,\mathbb{C})} \leq \frac{4}{k},$$

which goes to 0 when $k \to \infty$. Similar argument hold for the strong stable and unstable norms; this follows easily by using, for instance, the computations presented for such norms in item **A2** below.
- $\mathbf{1}_A h(\phi) = h(\mathbf{1}_A \phi)$, when h is a Borel measure. The proof in the preceding item holds for any compact set A. If we approximate, by density, h with $C^1(X, \mathbb{C})$ functions, we see that the equality we want to prove follows from the representation (6).

It has been proved in [10] that the operator $\mathcal{L}$ is quasi-compact, in the sense that it can be written as[3]

$$\mathcal{L} = \mu \otimes Z + Q, \tag{13}$$

where $\mu = \mathcal{L}\mu$ is the SRB measure normalized in such a way that $\mu(1) = 1$ and spanning the one-dimensional eigenspace corresponding to the eigenvalue 1; Z is the generator of the one-dimensional eigenspace of $\mathcal{L}^*$ in the dual space $\mathcal{B}^*$ and corresponding to the eigenvalue 1 and normalized in such a way that $Z(\mu) = 1$; finally Q is a linear operator on $\mathcal{B}$ with spectral radius $sp(Q)$ strictly less than one.

3.2. **The perturbative approach.** We now introduce the assumptions which allow us to apply the perturbative technique of Keller and Liverani [23]. They are split in two blocks: **A0**, **A2** and **A3** are needed to get the quasi-compact decomposition (16), which extends to the perturbed operators $\mathcal{L}_n$ the same decomposition for $\mathcal{L}$ required by **A1**. The assumptions **A4** and **A5** together with (16) are finally needed to apply the perturbative technique in [23] we referred to at the beginning of this section.

- **A0** $\mathcal{B}$ is continuously embedded into $\mathcal{B}_w$.
- **A1** The unperturbed operator $\mathcal{L}$ is quasi-compact in the sense expressed by (13).
- **A2** There are constants $0 < \rho < 1, D_1, D_2, D_3 > 0, M > 0, \rho < M$, such that $\forall n$ sufficiently large, $\forall h \in \mathcal{B}$ and $\forall k \in \mathbb{N}$ we have

$$|\mathcal{L}_n^k h|_w \leq D_1 M^k |h|_w, \tag{14}$$

$$||\mathcal{L}_n^k h|| \leq D_2 \rho^k ||h|| + D_3 M^k |h|_w. \tag{15}$$

  This will be proved below.
- **A3** We can bound the weak norm of $(\mathcal{L} - \mathcal{L}_n)h$, with $h \in \mathcal{B}$, in terms of the norm of h as:

$$|(\mathcal{L} - \mathcal{L}_n)h|_w \leq \chi_n ||h||$$

  where χ_n is a sequence converging to zero. We give immediately the proof of this fact since it is achieved by a simple adaptation of the computation of the strong stable norm in the proof of item **A2** below. Looking in fact at the notations and at the steps of such a demonstration, we have to control the term: $\int_W (\mathcal{L} - \mathcal{L}_n) h dm = \int_W \mathcal{L}(\mathbf{1}_{B_n} h)\varphi dm = \sum_{i=1,2} \int_{W_i \cap B_n} h(y)\varphi(Ty)\alpha\, dm(y) \leq \|h\|_s |B_n|^\kappa$. Then $\chi_n = |B_n|^\kappa$.

[3] If φ is a test function, eq. (13) means that $(\mathcal{L}h)(\varphi) = Z(h)\mu(\varphi) + Q(h)(\varphi)$.

Thanks to the assumptions **A2** (*uniform Lasota-Yorke inequalities*) and **A3** (*closeness of the operators in the triple norm*), we can apply the spectral theory in [24],[4] and get that the decomposition (13) holds for n large enough, namely

$$\lambda_n^{-1}\mathcal{L}_n = \mu_n \otimes Z_n + Q_n, \tag{16}$$

$$\mathcal{L}_n\mu_n = \lambda_n\mu_n, \tag{17}$$

$$Z_n\mathcal{L}_n = \lambda_n Z_n, \tag{18}$$

$$Q_n(\mu_n) = 0, \quad Z_nQ_n = 0, \tag{19}$$

where $\lambda_n \in \mathbb{C}$, $\mu_n \in \mathcal{B}$, $Z_n \in \mathcal{B}^*$, $Q_n \in \mathcal{B}$, and $\sup_n sp(Q_n) < sp(Q)$. We observe that the previous assumptions (16)–(19) imply that $Z_n(\mu_n) = 1, \forall n$; moreover μ_n can be normalized in such a way that $\mu_n(1) = 1$ and $Z(\mu_n) = 1$, see [23].

We now state assumption **A4** deferring **A5** to the next section.

- **A4** If we define

$$\Delta_n = Z(\mathcal{L} - \mathcal{L}_n)(\mu), \tag{20}$$

and for $h \in \mathcal{B}$

$$\eta_n := \sup_{||h||\leq 1} |Z(\mathcal{L}(h\mathbf{1}_{B_n}))|, \tag{21}$$

we must assume that

$$\lim_{n\to\infty} \eta_n = 0, \tag{22}$$

$$\eta_n||\mathcal{L}(\mathbf{1}_{B_n}\mu)|| \leq \text{const } \Delta_n. \tag{23}$$

It remains to prove **A2** and **A4**.

Let us start with the former, **A2**; notice that the proof we present is also valid for the unperturbed operator, and this will be explicitly used in the following. The proof is basically the same as the proof of Proposition 4.2 in [10], with the difference that we allow subsets of the stable manifolds of length less than one. By density of $C^1(X,\mathbb{C})$ in both $\mathcal{B}$ and $\mathcal{B}_w$, it will be enough to take $h \in C^1(X,\mathbb{C})$. We have to control integrals of type: $\int_W \mathcal{L}_n h\varphi\, dm$, where $W \in \Sigma$ and $\varphi \in C^1(W,\mathbb{C})$ (resp. $C^\kappa(W,\mathbb{C})$), according to the estimate of the weak (resp. strong) norm. Let us start for the weak norm and consider for instance $\mathcal{L}_n^2$, we have

$$\int_W \mathcal{L}_n^2 h\varphi dm = \int_W \frac{\mathbf{1}_{B_n^c}(T^{-1}x)\mathcal{L}(\mathbf{1}_{B_n^c}h)(T^{-1}x)\varphi(x)}{\alpha^{-1}\gamma_a}\, dm(x) = \tag{24}$$

$$\sum_{i=1,2}\int_{W_i} \frac{\mathbf{1}_{B_n^c}(y)\mathcal{L}(\mathbf{1}_{B_n^c}h)(y)\varphi(Ty)}{\alpha^{-1}}\, dm(y), \tag{25}$$

where $W_i, i = 1,2$ are the two preimages of W and we performed a change of variable along the stable manifold with Jacobian γ_a. The measure m along W_i is again the unnormalized Lebesgue measure. Iterating one more time we will produce at most two new pieces of stable manifolds, and we get:

$$\sum_{j=1,\cdots,4}\int_{W_j} \alpha^2 h(y)\varphi(T^2y)\mathbf{1}_{B_n^c}(y)\mathbf{1}_{B_n^c}(Ty)\, dm(y). \tag{26}$$

[4] This spectral theory also requires that if z is in the spectrum of $\mathcal{L}_n$ and $|z| > \rho$, then z is not in the residual spectrum of $\mathcal{L}_n$. This last fact is guaranteed by **A0** which implies that the spectral radius of $\mathcal{L}_n$ is bounded by ρ.

In the integral we replace each W_j with $(W_j \cap B_n^c \cap T^{-1}B_n^c)$ getting again at most two small pieces $W_j^{(n)}$ of stable manifolds, since $B_n^c \cap T^{-1}B_n^c$ could have only one connected component by the (linear) structure of the inverse of the map[5]. In order to compute the weak norm of $\mathcal{L}_n^2$ we must take a test function φ verifying $|\varphi|_{C^1(W,\mathbb{C})} \leq 1$. If we now take two points $y_1, y_2 \in W_j^{(n)}$ we have

$$|\varphi(T^2(y_1)) - \varphi(T^2(y_2))| \leq H^1(\varphi)|T^2(y_1) - T^2(y_2)| \leq H^1(\varphi)\gamma_a^2|y_1 - y_2|,$$

and therefore $|\varphi \circ T^2|_{C^1(W_j^{(n)},\mathbb{C})} \leq 1$. By multiplying and dividing (26) by $|\varphi \circ T^2|_{C^1(W_j^{(n)},\mathbb{C})}$ we finally get: (26) $\leq 2\sum_{j=1,\cdots,4} \alpha^2|h|_w \leq 2|h|_w$, where the last bound comes from our choice of $\alpha = \frac{1}{2}$. The proof generalizes immediately to any power $\mathcal{L}_n^k, k \geq 2$, by replacing the factor 2 in front of the sum with k, see the previous footnote:

$$|\mathcal{L}_n^k h|_w \leq k|h|_w.$$

To compute the strong stable norm, we closely follow the same calculations of section 4.1 in [10] and we write, still for the second iterate of the perturbed operator and using the notations above:

$$\int_W \mathcal{L}_n^2 h\varphi dm = 2\sum_{j=1,\cdots,4} \int_{W_j^{(n)}} \alpha^2 h(y)[\varphi(T^2y) - \overline{\varphi_{j,n}}]dm(y) + \int_{W_j^{(n)}} \alpha^2 h(y)\overline{\varphi_{j,n}}dm(y), \quad (27)$$

where

$$\overline{\varphi_{j,n}} = \frac{1}{|W_j^{(n)}|}\int_{W_j^{(n)}} \varphi(T^2y)dm(y).$$

Since $|\overline{\varphi_{j,n}}|_{C^1(W_j^{(n)})} \leq \sup_W |\varphi|$, we have immediately that the rightmost term in (27) is bounded by $2|h|_w$. Instead the first piece on the right hand side is bounded by

$$\sum_{j=1,\cdots,4} \alpha^2\|h\|_s|\varphi \circ T^2 - \overline{\varphi}_{j,n}|_{(W_j^{(n)}),\kappa}. \quad (28)$$

But $|\varphi\circ T^2 - \overline{\varphi}_{j,n}|_{C^\kappa(W_j^{(n)})} \leq |\varphi\circ T^2 - \overline{\varphi}_{j,n}|_{C^0} + \sup_{x\neq y}\frac{|\varphi(T^2x)-\varphi(T^2y)|}{|x-y|^\kappa} \leq |\varphi(T^2x) - \varphi(T^2x^*)| + H(\varphi)\gamma_a^{2\kappa} \leq 2H(\varphi)\gamma_a^{2\kappa}$, being x^* some point in $W_j^{(n)}$ by the mean value theorem. Therefore $|\varphi \circ T^2 - \overline{\varphi}_{j,n}|_{W_j^{(n)},\kappa} \leq 2\gamma_a^{2\kappa}|\varphi|_{W,\kappa} \leq 2\gamma_a^{2\kappa}$ and (28) $\leq 4\gamma_a^{2\kappa}\|h\|_s$. Generalizing to any k we finally get

$$\|\mathcal{L}_n^k h\|_s \leq k|h|_w + 2k\gamma_a^{\kappa k}\|h\|_s.$$

In order to treat the strong unstable norm, we follow section 4.3 in [11] adapted to our case, which is considerably much easier. Therefore, take two stable manifolds $W_{1,2}$ at distance at most ϵ, and φ_i on $W_i, i = 1, 2$ with $|\varphi_i|_{C^1(W_i,\mathbb{C})} \leq 1$. Call $U_1 \subset W_1$ and $U_2 \subset W_2$ the connected intervals parametrized respectively by $(s_{W_1}, t), (s_{W_2}, t)$, with t belonging to the same interval. We call *matched* these two pieces. We call $V_{1,2}$ the two *unmatched* pieces in $W_{1,2}$; notice that the length of these two pieces is less than ϵ. Define now by $U_{1,k}^{(j)}, U_{2,k}^{(j)}, j = 1, \ldots 2^k$ two preimages of order k respectively of U_1 and U_2 with the same history, which means that if $s_{U_{1,k}^{(j)}}, s_{U_{2,k}^{(j)}}$ are the common ordinates of the points in respectively $U_{1,k}^{(j)}$ and $U_{2,k}^{(j)}$, then $s_{U_{1,k}^{(j)}}$ and $s_{U_{2,k}^{(j)}}$ belong to the same inverse branch of

[5]If we consider higher iterates of $\mathcal{L}$, for instance of order k, we should control terms like $W \cap B_n^c \cap T^{-1}B_n^c \cap \cdots \cap T^{-(k-1)}B_n^c$, where W is a piece of stable manifold. Notice that each preimage $T^{-l}B_n, l = 1, \ldots, k-1$, is contained in 2^l disjoint horizontal rectangles. Therefore W could meet at most $k-1$ of such rectangles of different generation and hence at most $k-1$ preimages of B_n. This implies that the complement in W of such intersection is at most composed by k connected intervals

the map T_Y^k given in (1). Due to the linearity of the map, the sets $U_{1,k}^{(j)}$ and $U_{2,k}^{(j)}$ will be again matched and $d(U_{1,k}^{(j)}, U_{2,k}^{(j)}) = |s_{U_{1,k}^{(j)}} - s_{U_{2,k}^{(j)}}| \leq \alpha^k d(U_1, U_2) \leq \alpha^k \epsilon$. Since $U_{1,k}^{(j)}$ and $U_{2,k}^{(j)}$ could contain each at most k preimages of the ball B_n, we could have at most k matched intervals inside $U_{1,k}^{(j)}$ and $U_{2,k}^{(j)}$. Call $U_{1,k}^{(j,l)}$ and $U_{2,k}^{(j,l)}, l = 1, \ldots, k$ those smaller matched pieces. So their contribution to the $\mathcal{L}_n^k$ in (4) is

$$\sum_{j=1,\ldots,2^k} \sum_{l=1}^{k} \alpha^k \frac{1}{\epsilon^\beta} \left| \int_{U_{1,k}^{(j,l)}} h(y)\varphi_1(T^k y) dm(y) - \int_{U_{2,k}^{(j,l)}} h(y)\varphi_2(T^k y) dm(y) \right|. \tag{29}$$

Since $d_0(\varphi_1 \circ T^2, \varphi_2 \circ T^2) \leq \gamma_a^2 d_0(\varphi_1, \varphi_2) \leq \gamma_a^2 \epsilon \leq \epsilon$, and $d(U_{1,k}^{(j,l)}, U_{2,k}^{(j,l)}) = |s_{U_{1,k}^{(j)}} - s_{U_{2,k}^{(j)}}| \leq \alpha^k d(W_1, W_2) \leq \alpha^k \epsilon$, we have that, since $C^1(U_{m,k}^{(j,l)}) \leq 1, m = 1, 2$

$$(29) \leq k\alpha^{k\beta} \|h\|_u$$

For the unmatched pieces, we have to take into account those generated by the 2^k preimages of $V_{1,2}$, but also the unmatched pieces in the $U_{m,k}^{(j)}, m = 1, 2, j = 1, \ldots, 2^k$. By overcounting, the number of those unmatched pieces will be bounded by $4k2^k$. If we call V_k one of them and supposing it belongs to the backward images of W_1, we must estimate the strong stable norm of the quantity $\frac{1}{\epsilon^\beta} \left| \int_{V_k} h(y)\varphi(T^k y) dm(y) \right|$. We multiply it by $|V_k|^\kappa |\phi \circ T^k|_{C^\kappa(V_k, \mathbb{C})}$. But $|\phi \circ T^2|_{C^\kappa(V_k, \mathbb{C})} \leq |\phi|_{C^0(W_1, \mathbb{C})} + H(\phi)\gamma_a^2 \leq 1$, and $|V_k|^\kappa \leq \epsilon \gamma_a^{-k\kappa}$. Therefore all the unmatched pieces at the k-th generation in the estimate of the strong unstable norm will be bounded by $4k2^k \gamma_a^{-k\kappa} \|h\|_s$, since $\beta \leq 1$, and

$$\|\mathcal{L}_n^k h\|_u \leq k\alpha^{k\beta} \|h\|_u + 4k\gamma_a^{-k\kappa} \|h\|_s.$$

In conclusion we get for $k \geq 1$:

$$\|\mathcal{L}_n^k h\| = \|\mathcal{L}_n^k h\|_s + b\|\mathcal{L}_n^k h\|_u \leq k|h|_w + 2k\gamma_a^{\kappa k} \|h\|_s + b(k\alpha^{k\beta} \|h\|_u + 4k\gamma_a^{-k\kappa} \|h\|_s). \tag{30}$$

We now fix a value of k, say k_0, such that

$$4\sigma^{k_0} < 1/2; \ \rho := (4k_0 \sigma^{k_0})^{\frac{1}{k_0}} < 1,$$

where

$$\sigma := \max\{\gamma_a^\kappa, \alpha^\beta\} < 1$$

and we finally choose b such that

$$2b \leq \gamma_a^{2k_0\kappa}.$$

With these positions and by using blocks of length k_0, it is immediate to rewrite (30) as, for any $k > 0$:

$$\|\mathcal{L}_n^k h\| \leq \rho^k |h|_w + 2M^k \|h\|,$$

where $M := (k_0^{\frac{1}{k_0}})$, and this proves (15).

We now pass to justify **A4.** We remind that Z is the unique solution of the eigenvalue equation $\mathcal{L}^* Z = Z$, where $\mathcal{L}^*$ is the dual of the transfer operator. By setting

$$Z(h) := h(1), \ h \in \mathcal{B}, \tag{31}$$

we have for $h \in \mathcal{B}$:

$$\mathcal{L}^* Z(h) = Z(\mathcal{L}h) = (\mathcal{L}h)(1) = h(1 \circ T) = h(1) = Z(h).$$

Coming back to Δ_n we see immediately that

$$\Delta_n = Z(\mathcal{L}(\mathbf{1}_{B_n}\mu)) = \mathcal{L}(\mathbf{1}_{B_n}\mu)(1) = \int \mathbf{1}_{B_n}\, d\mu = \mu(B_n). \tag{32}$$

The term $||\mathcal{L}(\mathbf{1}_{B_n}\mu)||$ can be handled very easily using the Lasota-Yorke inequality which we proved in item **A2** above. It follows in fact from (15) that there are two constants C_1, C_2 depending only on the map such that

$$||\mathcal{L}(\mathbf{1}_{B_n}\mu)|| \leq C_1||\mathbf{1}_{B_n}\mu|| + C_2|\mathbf{1}_{B_n}\mu|_w.$$

Moreover it is easy to show that

$$||\mathbf{1}_{B_n}\mu|| \leq ||\mu|| \quad \text{and} \quad |\mathbf{1}_{B_n}\mu|_w \leq |\mu|_w.$$ [6]

By setting

$$C_3 := C_1||\mu|| + C_2|\mu|_w,$$

we are led to prove that (see (23)), $\eta_n C_3 \leq \text{const } \Delta_n$, namely

$$\eta_n \leq \text{ const } \Delta_n = \text{const } \mu(B_n). \tag{33}$$

Before continuing, we have to focus on $\mu(B_n) = \mu(B(z, e^{-u_n}))$. It is well known that for μ-almost all z and by taking the radius sufficiently small, depending on the value ι, $e^{-u_n(d+\iota)} \leq \mu(B(z, e^{-u_n}) \leq e^{-u_n(d-\iota)}$, where $\iota > 0$ is arbitrarily small. This follows from the existence of the limit

$$\lim_{r\to 0^+} \frac{\log \mu(B(x,r))}{\log r} = d, \text{ for } x \text{ chosen } \mu\text{-a.e.}, \tag{34}$$

and quantity d is the Hausdorff dimension of the measure μ which in our case reads [26], eq. (3.24):

$$d = 1 + d_s, \text{ where } d_s := \frac{\alpha \log \alpha^{-1} + (1-\alpha)\log(1-\alpha)^{-1}}{\log \gamma_a^{-1}}.$$

Notice that d_s is strictly smaller than 1; for instance, with the choices $\alpha = 0.5, \gamma_a = 0.25$, we get $d_s = 0.5$. We now have:

Lemma 3.1. *Assume $\kappa > d_s$.*
Then

$$\eta_n \leq 2\mu(B_n).$$

Proof. We have

$$Z(\mathcal{L}(h\ \mathbf{1}_{B_n})) = \int h\ \mathbf{1}_{B_n} dm.$$

Put $\tilde{W}_\xi = W_\xi \cap B_n$; by disintegrating along the stable partition $\mathcal{W}^s$ we get:

$$\begin{aligned}
\int h\ \mathbf{1}_{B_n}\, dm_L &= \int_\xi d\lambda(\xi) \left[\int_{W_\xi} (\mathbf{1}_{B_n} h)(x)\, dm(x)\right] \\
&\leq \int_\xi d\lambda(\xi) \left[|\tilde{W}_\xi|^\kappa \|h\|_s\right] \\
&\leq e^{-u_n\kappa}||h||_s \lambda(\xi; B_n \cap W_\xi \neq \emptyset),
\end{aligned}$$

[6] We give the proof for the weak stable norms, the others follows anagously. We approximate by density μ with functions $h \in C^1(X, \mathbb{C})$, as we did above when we proved that $\mathbf{1}_{B_n^c} h \in \mathcal{B}$. Since $\int_W \mathbf{1}_{B_n} h\varphi dm \leq \int_{B_n \cap W} h\varphi dm \leq \int_W h\varphi dm$ we have that $|\mathbf{1}_{B_n} h|_w \leq |h|_w$.

where λ is the quotient measure on the space of stable leaves W_ξ belonging to $\mathcal{W}^s$; and indexed by ξ, see for instance [27], Appendix A. By definition of disintegration we have that

$$\lambda(\xi; B_n \cap W_\xi \neq \emptyset) = m_L(\bigcup W_\xi, B_n \cap W_\xi \neq \emptyset) = 2e^{-u_n},$$

and therefore

$$\eta_n \leq 2e^{-u_n(\kappa+1)}.$$

We finally have

$$\eta_n \leq 2e^{-u_n(\kappa+1)} \leq 2e^{-u_n(d+\iota)} \leq 2\mu(B_n),$$

provided we choose

$$\kappa > d + \iota - 1 \tag{35}$$

which can be satisfied by assumption. □

Remark 3.2. *The local comparison between the Lebesgue and the SRB measure of a ball of center z obliged us to choose z μ-almost everywhere because in this way we have a precise value for the locally constant dimension d. We are therefore discarding several points, possibly periodic, where the limiting distribution for the Gumbel's law (see next section) could exhibit extremal indices different from* 1.

Remark 3.3. *For invertible, piecewise differentiable hyperbolic maps in dimension* 2*, the construction of the Banach space imposes that $\kappa < 1$; for billiard maps associated with Lorentz gases,* [12]*, it even verifies $\kappa \leq 1/6$. This could make difficult to check condition (35) for invariant sets with large d, like Anosov diffeomorphisms for instance. In some sense this difficulty was already raised in section 4.5 in the Keller's paper* [22]*, where an estimate like ours in terms of the Hölder exponent κ was given and the subsequent question of the comparison with the SRB measure was addressed.*

4. The limiting law

4.1. **The Gumbel law.** We have now all the tools to compute the asymptotic behavior of $\mathcal{L}_n$. We need one more ingredient which will constitute our last assumption:

- **A5** Let us suppose that the following limit exist for any $k \geq 0$:

$$q_k = \lim_{n\to\infty} q_{k,n} := \lim_{n\to\infty} \frac{Z\left([(\mathcal{L}-\mathcal{L}_n)\mathcal{L}_n^k(\mathcal{L}-\mathcal{L}_n)]\mu\right)}{\Delta_n} \tag{36}$$

Notice that

$$q_{k,n} = \frac{\mu(B_n \cap T^{-1}B_n^c \cap \cdots \cap T^{-k}B_n^c \cap T^{-(k+1)}B_n)}{\mu(B_n)}$$

and therefore by the Poincaré recurrence theorem

$$\sum_{k=0}^{\infty} q_{k,n} = 1.$$

Therefore if the limits (36) exist, the quantity

$$\theta = 1 - \sum_{k=0}^{\infty} q_k, \tag{37}$$

is well defined and verifies

$$0 \leq \theta \leq 1.$$

It is called the *extremal index* and it modulates the exponent of the Gumbel's law as we will see in a moment. We have in fact by Theorem 2.1 of [23]:

$$\lambda_n = 1 - \theta\Delta_n = \exp(-\theta\Delta_n + o(\Delta_n)),$$

or equivalently

$$\lambda_n^n = \exp(-\theta n\Delta_n + no(\Delta_n)).$$

Therefore we have

$$\mu(M_n \leq u_n) = \mathcal{L}_n^n\mu(1) = \lambda_n^n[\mu_n(1)Z_n(\mu) + Q_n^n(\mu)(1)]$$

and consequently

$$\mu(M_n \leq u_n) = \exp(-\theta n\Delta_n + no(\Delta_n))[O(1) + Q_n^n(\mu)(1)],$$

since $\mu_n(1) = 1$ and it has been proved in [23], Lemma 6.1, $Z_n(\mu) \to 1$ for $n \to \infty$. At this point we need an important assumption, which basically reduces to fix the sequence u_n and allow us to get a non-degenerate limit for the distribution of M_n. We in fact ask that

$$n\ \Delta_n \to \tau,\ \ n \to \infty, \tag{38}$$

where τ is a positive real number. With this assumption, using (5) and the fact that $|h|_w \leq \|h\|_s$, we have

$$|Q_n^n(\mu)(1)| \leq \text{const}\ sp(Q)^n||\mu|| \to 0.$$

In conclusion we get the Gumbel's law

$$\lim_{n\to\infty} \mu(M_n \leq u_n) = e^{-\theta\tau}.$$

4.2. **The extremal index.** We are now ready to compute the $q_{k,n}$, which will determine the extremal index. Let us first suppose that the center of the ball B_n is not a periodic point; then the points $T^j(z), j = 1, \cdots, k$ will be disjoint from z. Let us take the ball so small that is does not cross the set $T^j\Gamma, j = 1, \cdots, k$, where Γ is the discontinuity line $(y = \alpha)$. In this way the images of B_n will be ellipses with the long axis along the unstable manifold and the short axis stretched by a factor γ^k. By continuity and taking n large enough, we can manage that all the iterates of B_n up to T^k will be disjoint from B_n and for such n the numerator of $q_{k,n}$ will be zero. At this point we can state the following result:

Proposition 4.1. *Let T be the baker's transformation and consider the function $M_n(x) := \max\{\phi(x), \ldots, \phi(T^{n-1}x)\}$, where $\phi(x) = -\log d(x, z)$, and z is chosen μ-almost everywhere with respect to the SRB measure μ. Then, if z is not periodic, we have*

$$\lim_{n\to\infty} \mu(M_n \leq u_n) = e^{-\tau},$$

where the boundary level u_n is chosen to satisfy $n\mu(B(z, e^{-u_n})) \to \tau$.

Suppose now z is a periodic point of minimal period p. By doing as above we could stay away from the discontinuity lines up to p iterates and look simply to $T^{-p}(B_n) \cap B_n$. Since the map acts linearly, the p preimage of B_n would be an ellipse with center z and symmetric w.r.t. the unstable manifold passing trough z. So we have to compute the SRB measure of the intersection of the ellipse with the ball shown in Fig. 2.

It turns out that this computation is not easy. The natural idea would be to disintegrate the SRB measure along the unstable manifolds belonging to the unstable partition $\mathcal{W}^u$. We index such fibers as W_ν and we put $\zeta(\nu)$ the associated quotient measure. Let us recall that the conditional measures along leaves W_ν are normalized Lebesgue measures:

we denote them with l_ν. If we call $\mathcal{E}_{in}$ the region of the ellipse inside the ball B_n, we have to compute

$$\frac{\int l_\nu(\mathcal{E}_{in} \cap W_\nu)\, d\zeta(\nu)}{\int l_\nu(B_n \cap W_\nu)\, d\zeta(\nu)}. \tag{39}$$

Although simple geometry allows us to compute easily the length of $\mathcal{E}_{in} \cap W_\nu$ and $B_n \cap W_\nu$, and since they vary with W_ν, it is not at the end clear how to perform the integral with respect to the counting measure, especially because we need asymptotic estimates, not bounds. We therefore proceed by introducing a different metric, a nice trick which was already used in [8]. We use the l^∞ norm on $\mathbb{R}^2$ for which $|(x,y)|_\infty = \max\{|x|,|y|\}$. In this way the ball B_n will become a square with sides of length $r_n := e^{-u_n}$ and $T^{-p}(B_n)$ will be a rectangle with the long side of length $\gamma_a^{-p} r_n$ and the short side of length $\alpha^p r_n$. This rectangle will be placed symmetrically with respect to the square as indicated in Fig. 3. A quick inspection shows that the proof demonstrating that $\mathbf{1}_{B_n^c} \in \mathcal{B}$ remains valid whenever those balls are "squares". The ratio (39) can now be computed easily since the length in the integrals are constant and we get α^p. In conclusion:

Proposition 4.2. *Let T be the baker's transformation and consider the function $M_n(x) := \max\{\phi(x), \ldots, \phi(T^{n-1}x)\}$, where $\phi(x) = -\log d_\infty(x,z)$, and z is chosen μ-almost everywhere with respect to the SRB measure μ. Then, if z is a periodic point of minimal period p, we have*

$$\lim_{n\to\infty} \mu(M_n \le u_n) = e^{-\theta\tau},$$

where $n\mu(B(z, e^{-u_n})) \to \tau$ and

$$\theta = 1 - \alpha^p.$$

Remark 4.3. *Propositions 4.1 and 4.2 show that for a typical (non-periodic) point z the limiting distribution of the maximum is purely exponential. The baker's map is probably the easiest example of a singular attractor. It is annoying that we could not compute analytically the extremal index with respect to the Euclidean metric, which is the metric usually accessible in simulations and physical observations. Moreover, when $p \to \infty$, Fig. 2 tends to Fig. 3, with a very horizontally long and vertically thin green rectangle, so the extremal index for the Euclidean holes tends to that for the square holes.*[7]

5. Poisson statistics

5.1. **The spectral approach.** As mentioned in the introduction, the spectral technique has been recently generalised to study the statistics of the number of visits in balls shrinking around a point, [3]. We briefly introduce such an approach and the reader will see that we can easily adapt it to the baker's map. The starting point is to consider the following counting function

$$N^\tau_{B_n}(x) = \sum_{i=0}^{\lfloor \tau/\mu(B_n) \rfloor} \mathbf{1}_{B_n} \circ T^i(x),$$

where τ is a positive parameter and $x \in X$. The goal is to study the distribution of this discrete random variable in the limit $n \to \infty$; with the spectral approach will rather look at the characteristic function of such a variable.

We begin to define $S_{n,k} := \sum_{i=0}^{k} \mathbf{1}_{B_n} \circ T^i$ and put $S_{n,n} = N^\tau_{B_n}$. We then define the perturbed operator

$$\mathcal{L}_{n,s}(h) = \mathcal{L}(e^{is\mathbf{1}_{B_n}} h), \ s \in \mathbb{R}, \ h \in \mathcal{B}.$$

[7] We thank the anonymous referee for this observation.

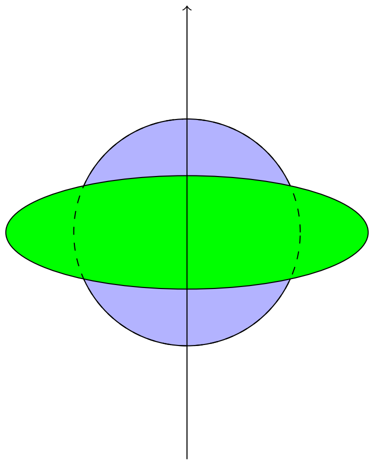

FIGURE 2. Computation of the extremal index around periodic point with the Euclidean metric. The vertical line is an unstable manifold. We should compute the green area inside the circle.

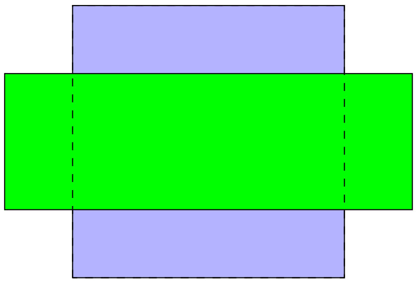

FIGURE 3. Computation of the extremal index around periodic point with the l^∞ metric. We should compute the green area inside the square.

A simple computation shows that

$$\mathcal{L}_{n,s}^k(\mu)(1) = \int e^{isS_{n,k}} d\mu,$$

which suggests to get information on the characteristic function of $S_{n,k}$ by the behavior of the top eigenvalue $\lambda_{n,s}$ of the perturbed operator $\mathcal{L}_{n,s}$. At this point the analysis proceeds in the same manner as for the perturbed operator $\mathcal{L}_n$ and we sketch here the main steps. The *difference* between the two operators is now quantified by

$$\Delta_{n,s} := Z(\mathcal{L} - \mathcal{L}_{n,s})(\mu) = (1 - e^{is})\mu(B_n),$$

and

$$\lambda_{n,s} = 1 - \theta(s)(1 - e^{is})\mu(B_n) + o(\mu(B_n)). \tag{40}$$

The quantity $\theta(s)$ plays the role of the extremal index and is defined according formula (36), which in the present case reduces to $\theta(s) = 1 - \sum_{k=0}^{\infty} q_k(s)$, where

$$q_k(s) = \lim_{n\to\infty} \frac{1}{1 - e^{is}} \sum_{\ell=0}^{k} (1 - e^{is})^2 e^{i\ell s} \beta_n^{(k)}(\ell) = (1 - e^{is}) \sum_{\ell=0}^{k} e^{i\ell s} \beta_k(\ell), \tag{41}$$

$$\beta_n^{(k)}(\ell) := \frac{\mu(x; x \in B_n, T^{k+1}(x) \in B_n, \sum_{j=1}^{k} 1_{B_n}(T^j x) = \ell)}{\mu(B_n)}. \tag{42}$$

and we suppose that the limit $\beta_k(\ell) := \lim_{n\to\infty} \beta_n^{(k)}(\ell)$ exists. Then we have

$$\theta(s) = 1 - (1 - e^{is}) \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} e^{i\ell s} \beta_k(\ell),$$

and the exponential decay of correlation of the measure μ allows us to show that the series $\sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \beta_k(\ell)$ converges absolutely[8] and therefore $\theta(s)$ is C^∞ in the neighborhod of 0. If now return to the eigenvalue (40), we exponentiate it at the power n and we use again the threshold condition (38), $n\mu(B_n) \to \tau$, we finally get

$$\lim_{n\to\infty} \int e^{isS_{n,n}} d\mu = e^{-\theta(s)(1-e^{is})} := \varphi(s).$$

Since $\varphi(s)$ is continuous in $s = 0$, it is the characteristic function of some random variable Z, eventually defined on a different probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The variable Z is clearly non-negative and integer valued and it is also infinite divisible since $e^{-\theta(s)(1-e^{is})t} = (e^{-\theta(s)(1-e^{is})t/N})^N$, for any N. This implies that Z has a compound Poisson (CP) distribution, see [15] or [3] for more references, namely it may be written as $Z := \sum_{j=1}^{N} X_j$, where the X_j are iid random variables defined on same probability space, and N is Poisson distributed with *intensity* $\varkappa$ and X_j has distribution $\mathbb{P}(X_j = l) = \rho_l$. We call the sequence $(\rho)_{l\geq 1}$ the *cluster size distribution* of Z. Among the CP distributions, two are particularly important, the standard Poisson distribution and the Pòlya-Aeppli distribution. For the standard Poisson $\rho_1 = 1$; for Pòlya-Aeppli the distribution of X_j is geometrical, namely $\rho_l = \eta(1-\eta)^l, \eta \in (0,1)$. For such distributions the associated characteristic functions are perfectly known. To determine them for our baker's system one should prove the existence and compute the quantities (42), which are of geometric and dynamical nature. This will be done in the next section in the context of a more probabilistic approach to Poisson-like statistics. Actually the quantities computed in the next section are not exactly those in (42), but it is not difficult to modify their derivation to get (42) and therefore reprove Proposition 5.2 with the spectral approach. As we said in the Introduction, we will present the alternative probabilistic approach since it will allow us to cover the example 5.3 which shows a CP distribution different from the standard Poisson and the Pòlya-Aeppli. The probabilistic approach gives also an alternative way to prove EVT for the baker's map which is recovered as the limiting distribution of $\mu(N^{\tau}_{B_n} = 0)$.

5.2. **The probabilistic approach.** We now use a recent technique developed in [20] and apply it to our baker's map. We will recover the usual dichotomy and get a pure Poisson distribution when the points are not periodic, and a Pólya-Aeppli distribution around periodic points with the parameter giving the geometric distribution of the size the clusters which coincide with the extremal index computed in the preceding section. This last result is achieved in particular if we use the l^∞ metric. This result is not surprising; what is interesting is the great flexibility of the technique of the proof which allows us to get easily the expected properties. In order to apply the theory in [20], we need to verify a certain number of assumptions, but otherwise defer to the aforementioned paper for precise definitions. Here we recall the most important requirements and prove in detail one of them.
Warning: the next considerations are carried over with the Euclidean metric which is more natural for applications. In order to cover visits to periodic points we will use the

[8]See section 3 in [3] for the proof of this convergence which applies to our case as well.

l^∞ metric and the following computations are even easier.

Decay of correlation. There exists a decay function $\mathcal{C}(k)$ so that

$$\left|\int_M G(H\circ T^k)\,d\mu - \mu(G)\mu(H)\right| \le \mathcal{C}(k)\|G\|_{Lip}\|H\|_\infty \qquad \forall k\in\mathbb{N},$$

for functions H which are constant on local stable leaves W_s of T and the functions $G: M\to\mathbb{R}$ being Lipschitz continuous. This is ensured by Theorem 2.5 in [10], where the role of H is taken by the test functions in $C^\kappa(W,\mathbb{C})$ and $G\in\mathcal{B}$, which is the completion of Lipschitz functions on X. The decay is exponential.

Cylinder sets. The proof requires the existence, for each $n\ge 1$, of a partition of each unstable leaf in subsets $\xi_n^{(k)}$, called n-cylinders (or cylinders of rank n), and indexed with k, where T^n is defined and the image $T^n\xi_n^{(k)}$ is an unstable leaf of full length for each k. These cylinders are obtained by taking the 2^n preimages of $\Gamma=\{y=\alpha\}$ by the map T_Y restricted to each leaf. In the following we will take $\alpha=1/2$ to simplify the exposition.

Exact dimensionality of the SRB measure. This quotes the existence of the limit (34). We shall need the following result.

Lemma 5.1. *(*Annulus type condition*) Let $w>1$. If x is a point for which the dimension limit* (34) *exists for a positive d, then there exists a $\delta>0$ so that*

$$\frac{\mu(B(x,r+r^w)\setminus B(x,r))}{\mu(B(x,r))} = O(r^\delta),$$

for all $r>0$ small enough.

Now we can apply the results of Section 7.4 in [20] to prove the following result which tracks the number of visits a trajectory of the point $x\in X$ makes to the set U on a suitable normalized orbit segment:

Proposition 5.2. *Consider the counting function*

$$N_{B_n}^\tau(z) = \sum_{i=0}^{\lfloor \tau/\mu(B_n)\rfloor} \mathbf{1}_{B_n}\circ T^i(x),$$

where τ is a positive parameter and z is a point for which the limit (34) exists and $n\mu(B(z,e^{-u_n}))\to\tau$.

- *If z is not a periodic point and using the Euclidean metric, then we get a pure Poisson distribution:*

$$\mu(N_{B_n}^\tau = k)\to \frac{e^{-\tau}\tau^k}{k!},\ n\to\infty.$$

- *If z is a periodic point of minimal period p and using the l^∞ metric, we get a compound Poisson distribution (Pólya-Aeppli):*

$$\mu(N_{B_n}^\tau = k)\to e^{-\theta\tau}\sum_{j=1}^k (1-\theta)^{k-j}\theta^{2j}\frac{s^j}{j!}\binom{k-1}{j-1},\ n\to\infty,$$

where θ is given as above by $\theta = 1-\lim_{n\to\infty}\frac{\mu(T^{-p}B_n\cap B_n)}{\mu(B_n)}$.

Proof of Lemma 5.1. We have to prove the lemma in the two cases when (I) the norm is ℓ^2 and (II) the norm is ℓ^∞ and the ball is geometrically a square.

(I) We now use the Euclidean metric and denote with $\mathcal{A}$ the annulus $\mathcal{A} = B(x, r + r^w) \setminus B(x, r)$ where $w > 1$. By disintegrating the SRB measure along the unstable manifolds we have:

$$\mu(\mathcal{A}) = \int l_\nu(\mathcal{A} \cap W_\nu)\, d\zeta(\nu).$$

We now split the subsets on each unstable manifold on the cylinders of rank n and condition with respect to the Lebesgue measure on them:

$$l_\nu(\mathcal{A} \cap W_\nu) = \sum_{\xi_n; \xi_n \cap \mathcal{A} \neq \emptyset} \frac{l_\nu(\mathcal{A} \cap W_\nu \cap \xi_n)}{l_\nu(\xi_n)} l_\nu(\xi_n). \tag{43}$$

We then iterate forward each cylinder with T^n; they will become of full length equal to 1 and subsequently we get $l_\nu(T^n \xi_n) = 1$. Since the action of T is locally linear and expanding by a factor 2^n (with the given choice of $\alpha = \frac{1}{2}$) on the unstable leaves and therefore has zero distortion, we have

$$\frac{l_\nu(\mathcal{A} \cap W_\nu \cap \xi_n)}{l_\nu(\xi_n)} = \frac{l_{\nu'}(T^n(\mathcal{A} \cap W_\nu \cap \xi_n))}{l_{\nu'}(T^n \xi_n)} = l_{\nu'}(T^n(\mathcal{A}) \cap W_{\nu'})$$

for some $W_{\nu'}$ so that $T^n(\mathcal{A} \cap W_\nu \cap \xi_n) \subset W_{\nu'}$. Therefore,

$$l_\nu(\mathcal{A} \cap W_\nu) = \sum_{\xi_n; \xi_n \cap \mathcal{A} \neq \emptyset} l_{\nu'}(T^n(\mathcal{A} \cap W_\nu \cap \xi_n)) l_\nu(\xi_n).$$

By elementary geometry we see that the largest intersection of $\mathcal{A}$ with the unstable leaves will produce a piece of length $O(r^{\frac{w+1}{2}})$; therefore $l_{\nu'}(T^n(\mathcal{A} \cap W_\nu \cap \xi_n)) = O(2^n r^{\frac{w+1}{2}})$, and:

$$\mu(\mathcal{A}) = O(2^n r^{\frac{w+1}{2}}) \int \sum_{\xi_n; \xi_n \cap \mathcal{A} \neq \emptyset} l_\nu(\xi_n)\, d\zeta(\nu)).$$

We now observe that in order to have our result, it will be enough to get it with a decreasing sequence r_n, $n \to \infty$, of exponential type, $r_n = b^{-t(n)}$, $b > 1$, and $t(n)$ increasing to infinity. We put $r = 2^{-n}$. With this choice and remembering that 2^{-n} is also the length of the n-cylinders, we have

$$\bigcup_{\xi_n; \xi_n \cap \mathcal{A} \neq \emptyset} \xi_n \subset B(x, r + r^w + 2^{-n}) \subset B(x, 2r + r^w) \subset B(x, 3r),$$

which, as the cylinders ξ_n are disjoint, yields the estimate for the integral above:

$$\mu(\mathcal{A}) = O(2^n r^{\frac{w+1}{2}} r^{d-\epsilon}).$$

Now by the exact dimensionality of the SRB measure one has for any $\varepsilon > 0$

$$(2r + r^w)^{d+\varepsilon} \leq \mu(B(x, 2r + r^w)) \leq (2r + r^w)^{d-\varepsilon}$$

for all r small enough i.e. n large enough. With this we can divide $\mu(\mathcal{A})$ by the measure of the ball of radius r and obtain the estimate

$$\frac{\mu(\mathcal{A})}{\mu(B(x, r))} = O(r^{\frac{w-1}{2} + d - \varepsilon - d - \varepsilon}) = O(r^{\frac{w-1}{2} - 2\varepsilon}) = O(r^{\frac{w-1}{4}}),$$

since $w > 1$, and provided ε is small enough.

(II) Now we shall use the ℓ^∞-distance and again denote by $\mathcal{A}$ the annulus $B(x, r + r^w) \setminus B(x, r)$. Since we are in two dimensions, we can cover the annulus by balls $B(y_j, 2r^w)$ of radii $2r^w$, with centers y_j for $j = 1, \ldots, N$. The number N of balls needed is bounded by

$8\frac{r}{r^w}$. For any $\varepsilon > 0$ there exists a constant c_1 so that $\mu(B(y_j, 2r^w)) \leq c_1 r^{w(d-\varepsilon)}$ for all r small enough. Thus

$$\mu(\mathcal{A}) \leq 8c_1 r^{1+w(d-1-\varepsilon)}$$

and since $\mu(B(x,r)) \geq c_3 r^{d+\varepsilon}$ for some $c_3 > 0$ we obtain

$$\frac{\mu(\mathcal{A})}{\mu(B(x,r))} \leq c_4 r^{(d-1)(w-1)-\varepsilon(w+1)}.$$

The exponent $\delta = (d-1)(w-1) - \varepsilon(w+1)$ is positive as $d, w > 1$ and $\varepsilon > 0$ can be chosen sufficiently small. □

Proof of Proposition 5.2. We can now prove the proposition by applying Theorem 1 from [20] to which we now refer for the following assumptions. Assumption (I) on the overlap of cylinders (pullbacks of local unstable leaves) follows from the product structure of the baker map. Since the decay of correlations is exponential, Assumption (II) is satisfied. Furthermore, distortion is bounded uniformly and the contraction of cylinders is uniformly exponential, thus implying Assumption (III) is satisfied with $\mathcal{G}_n$ being the full set. Moreover, since the dimension of the invariant measure is equal to $d = 1 + d_s$, where $d_s < 1$ is given above, we can choose $d_0 > 0$ and $d_1 < \infty$ so that $d_0 < d < d_1$. Since the decay of correlations and the decay rate of the diameters of the cylinders are both exponential, due to the uniform rates of expansion, the associated condition of Theorem 1 of [20] is satisfied. In addition the dimension of the restricted measure on the unstable leaves equals $u_0 = 1$ as it is Lebesgue. The annulus condition, Assumption (VI), was verified in Lemma 5.1.

If x is an aperiodic point then $\min\{j \geq 1 : B_\rho(x) \cap T^j B_\rho(x) \neq \varnothing\}$ goes to infinity as $\rho = e^{-u_n} \to 0$. Thus for the coefficients

$$\lambda_\ell(L) = \lim_{\rho \to 0} \frac{\mathbb{P}(Z^L = \ell)}{\mathbb{P}(Z^L \geq 1)}$$

we obtain that for every L: $\lambda_1 = 1$ and $\lambda_\ell = 0$ for all $\ell = 2, 3, \dots$, where $Z^L = \sum_{j=1}^{L} \chi_{B_\rho(x)}$ is the hit counter on the finite orbit segment of length L. This implies that $N_{B_n}^\tau$ converges in distribution to a standard Poisson random variable with parameter τ.

Let x be a periodic point with minimal period p and let $\tilde{B}_\rho$ be a square of size ρ centered at x and whose sides are aligned with the stable and unstable directions respectively. Then for $\ell = 2, 3, \dots$

$$\hat{\alpha}_\ell = \lim_{L\to\infty} \lim_{\rho\to 0} \mathbb{P}(\tilde{Z}^L \geq \ell | \tilde{B}_\rho) = \lim_{\rho\to 0} \frac{\mu(\tilde{B}_\rho \cap T^{-(\ell-1)p}\tilde{B}_\rho)}{\mu(\tilde{B}_\rho)} = \left(\lim_{\rho\to 0} \frac{\mu(\tilde{B}_\rho \cap T^{-p}\tilde{B}_\rho)}{\mu(\tilde{B}_\rho)} \right)^{\ell-1}$$

which implies that $\hat{\alpha}_\ell = \hat{\alpha}_2^{\ell-1}$, where $\tilde{Z}^L = \sum_{j=1}^{L} \chi_{\tilde{B}_\rho(x)}$. Then for $\alpha_\ell = \hat{\alpha}_\ell - \hat{\alpha}_{\ell+1}$ we thus obtain by [20] that $\lambda_\ell = \frac{\alpha_\ell - \alpha_{\ell+1}}{\alpha_1} = (1-\theta)\theta^{\ell-1}$, where $1 - \theta = \alpha_1 = 1 - \hat{\alpha}_2$ is the extremal index. Hence $N_{B_n}^\tau$ converges in distribution to a Pólya-Aeppli distributed random variable. □

Example 5.3. *The second statement of Proposition 5.2 about periodic points requires the neighborhoods B_n to be chosen in a dynamically relevant way. Here they turn out to be squares (or rectangles). If the measure has some mixing properties with respect to a partition then the sets B_n can be taken to be cylinder sets as it was done in* [19] *for periodic points and in* [18] *Corollary 1 for non-periodic points. Here we show that for Euclidean balls one cannot in general expect the limiting distribution at periodic points* *to be Pólya-Aeppli and therefore cannot be described by the single value of the extremal index.*

We assume that all parameters are equal, that is $\gamma_a = \gamma_b = \alpha = \beta = \frac{1}{2}$. This is the fat *baker's map for which the Lebesgue measure on $[0,1]^2$ is the SRB measure μ. Let x be a periodic point with minimal period p. Then $\mu(B(x,r)) = r^2\pi$ and*

$$\mu\left(\bigcap_{i=0}^{k} T^{-ip}B(x,r)\right) = 4r^2 2^{-kp}(1+\mathcal{O}(2^{-2kp})).$$

This yields

$$\hat{\alpha}_{k+1} = \lim_{r\to 0} \frac{\mu\left(\bigcap_{i=0}^{k} T^{-ip}B(x,r)\right)}{\mu(B(x,r))} = \frac{4}{\pi}\arctan 2^{-kp} = \frac{4}{\pi}2^{-kp}(1+\mathcal{O}(2^{-2kp}))$$

for $k = 1,2,\ldots$. According to [20] *Theorem 2 we then define the values $\alpha_k = \hat{\alpha}_k - \hat{\alpha}_{k+1}$ where the value α_1 is the extremal index, i.e. $\theta = \alpha_1$. If the limiting distribution is Pólya-Aeppli then the probabilities $\lambda_k = \frac{\alpha_k - \alpha_{k+1}}{\alpha_1}$, $k = 1,2,\ldots,$ are geometrically distributed and must satisfy $\lambda_k = \theta(1-\theta)^{k-1}$ which is equivalent to saying that $\hat{\alpha}_{k+1} = (1-\theta)^k$ for $k = 0,1,2,\ldots$ (see* [20] *Theorem 2). Evidently this condition is violated in the present case and we conclude that the limiting distribution given by the values $\hat{\alpha}_k$ is not Pólya-Aeppli and in fact obeys another compound Poisson distribution.*

5.3. **Compound point processes.** The compound Poisson distribution could be enriched by defining the *rare event point process* (REPP). Let us first introduce a few objects. Put $I_l = [a_l, b_l), l = 1,\ldots,k, a_l, b_l \in \mathbb{R}_0^+$ a finite number of semi-open intervals of the non-negative real axis; call $J = \cup_{l=1}^{k} I_l$ their disjoint union. If r is a positive real number, we write $rJ = \cup_{l=1}^{k} rI_l = \cup_{l=1}^{k}[ra_l, rb_l)$. We denote with $|I_l|$ the length of the interval I_l, which we also design with its Lebesgue measure $\mathrm{Leb}(I_l)$. The REPP counts the number of visits to the set B_n during the rescaled time period $v_n J$:

$$N_n(\cdot)(J) = \sum_{l\in v_n J\cap\mathbb{N}_0} 1_{B_n}(T^l\cdot), \tag{44}$$

where v_n is taken as

$$v_n = \left\lfloor \frac{\tau}{\mu(B_n)} \right\rfloor,\ \tau > 0.$$

Our REPP belongs to the class of the point processes on $\mathbb{R}_0^+$, see [21] for all the properties of point processes quoted below. They are given by any measurable map $N : (M, \mathcal{B}_M, \mu) \to \mathcal{N}_p([0,\infty))$, where $(X, \mathcal{F}_X, \mu)$ is the probability space of our original dynamical system with the invariant measure μ and the Borel σ-algebra $\mathcal{F}_X$, and $\mathcal{N}_p([0,\infty))$ denotes the set of counting measures $\mathfrak{c}$ on $\mathbb{R}_0^+$ endowed with the σ-algebra $\mathcal{M}_p(\mathbb{R}_0^+)$, which is the smallest σ-algebra making all evaluation maps $\mathfrak{c} \to \mathfrak{c}(B)$, from $\mathcal{N}_p([0,\infty)) \to [0,\infty]$ measurable for all $B \in \mathcal{B}_M$. Any counting measure $\mathfrak{c}$ has the form $\mathfrak{c} = \sum_{i=1}^{\infty}\delta_{x_i},\ x_i \in [0,\infty)$. The distribution of N, denoted μ_N, is the measure $\mu\circ N^{-1} = \mu[N\in\cdot]$, on $\mathcal{M}_p(\mathbb{R}_0^+)$. The set $\mathcal{N}_p([0,\infty))$ becomes a topological space with the vague topology, i.e. the sequence $\mathfrak{c}_n$ converges to $\mathfrak{c}$ whenever $\mathfrak{c}_n(\phi) \to \mathfrak{c}(\phi)$ for any continuous function $\phi : \mathbb{R}_0^+ \to \mathbb{R}$ with compact support. We also say that the sequence of point processes N_n converges in distribution to the point process N, eventually defined on another probability space $(X', \mathcal{F}'_{X'}, \mu')$, if μ_{N_n} converges weakly to μ'_N, that is for every continuous function φ defined on $\mathcal{N}_p([0,\infty))$ we have $\lim_{n\to\infty}\int\varphi d\mu\circ N_n^{-1} = \int\varphi d\mu'\circ N^{-1}$. In this case we will write $N_n \xrightarrow{\mu} N$.

If we now return to our REPP (44), we will see that a very common result is to get $N_n \xrightarrow{\mu} \tilde{N}$, where

$$\mu(x, \tilde{N}(x)(I_l) = k_l, 1 \leq l \leq n) = \prod_{l=1}^{n} e^{-\tau \mathrm{Leb}(I_l)} \frac{\tau^{k_l} \mathrm{Leb}(I_l)^{k_l}}{k_l!}, \tag{45}$$

for any disjoint bounded sets $I_1, \ldots, I_n$ and non-negative integers $k_1, \ldots, k_n$, which is called the *standard Poisson point process.* In general our REPP processes converges in distribution to a *compound point process* (CPP). We say that the point process $N : (X', \mathcal{F}'_{X'}, \mu') \to \mathcal{N}_p([0, \infty))$ is a CPP with intensity parameter t and cluster size distribution $(\lambda_l)_{l \geq 1}$ if it satisfies:

- For any finite sequence of measurable sets $B_1, \ldots, B_k$ in $\mathcal{F}'_{X'}$ and mutually disjoint, the random variables $N(\cdot)(B_i), i = 1, \ldots, k$, are independent.
- For any measurable set $B \in \mathcal{F}'_{X'}$, the random variable $N(\cdot)(B)$ is a CP random variable with intensity $t\mathrm{Leb}(B), t > 0$ and cluster size distribution $(\rho_l)_{l \geq 1}$, see the definition in section 5.

From now on we will simply write $N(\cdot)$ instead of $N(x)(\cdot)$ and we consider it as a CPP. In order to study the convergence of our REPP N_n to the CPP N two equivalent criteria are available. Before stating them we should remind the definition of the Laplace transform for a general point process $R : (X', \mathcal{F}'_{M'}, \mu') \to \mathcal{N}_p([0, \infty))$:

$$\psi_R(y_1, \ldots, y_k) = \mathbb{E}_{\mu'} \left(e^{-\sum_{l=1}^{k} y_l R(I_l)} \right), \tag{46}$$

for every non negative values $y_1, \ldots, y_k$, each choice of k disjoint intervals $I_i = [a_i, b_i), i = 1, \ldots, k$. In the case of a CPP N with intensity parameter t and cluster size distribution $(\rho_l)_{l \geq 1}$, we get

$$\psi_N(y_1, \ldots, y_k) = e^{-t \sum_{l=1}^{k} (1 - \varphi(y_l)) \mathrm{Leb}(I_l)}, \tag{47}$$

where $\varphi(y) = \sum_{i=0}^{\infty} e^{-yi} \rho_i$ is the Laplace transform of the cluster size distribution $(\rho_l)_{l \geq 1}$. Therefore in order to establish the convergence in distribution of the REPP N_n toward the CPP N it will be sufficient [21]:
- (C1): showing that for any k disjoint intervals $I_i = [a_i, b_i), i = 1, \ldots, k$ the joint distribution of N_n converges to the joint distribution of N, namely

$$(N_n(I_1), \ldots N_n(I_k)) \to (N(I_1), \ldots N(I_k)) .$$

-C(2): showing the convergence of the Laplace transforms:

$$\psi_{N_n}(y_1, \ldots, y_\zeta) = \mathbb{E} \left(e^{-\sum_{l=1}^{k} y_l N_n(I_l)} \right) \to \psi_N(y_1, \ldots, y_k) = e^{-t \sum_{l=1}^{k} (1 - \varphi(y_l)) \mathrm{Leb}(I_l)},$$

as $n \to \infty$.
The criterion C(1) lends itself to being studied with the probabilistic approach of [20] as two of us recently shown in ([1], Theorem 3), see also [16] for a different method. The criterion C(2) is *naturally* adapted to the spectral approach (just replacing characteristic functions with Laplace transforms), and the complete treatment, involving two of us, will appear soon [4]. Both criteria allow to extend immediately Proposition 5.2 to the point process framework giving

Proposition 5.4. *Consider the counting measure*

$$N_n(\cdot)(J) = \sum_{l \in v_n J \cap \mathbb{N}_0} 1_{B_n}(T^l \cdot)$$

where τ is a positive parameter, $v_n = \left\lfloor \frac{\tau}{\mu(B_n)} \right\rfloor$, and z is a point for which the limit (34) exists and $n\mu(B(z, e^{-u_n})) \to \tau$.

- *If z is not a periodic point and using the Euclidean metric, then N_n converges in distribution to a standard Poisson point process of intensity τ, see (45) for the finite size distributions.*
- *If z is a periodic point of minimal period p and using the l^∞ metric, we get a compound point process of Pólya-Aeppli type, namely a CPP with intensity $\tau\theta$ and cluster size distribution $\theta(1-\theta)^l, l \geq 1$, where θ is given as above by $\theta = 1 - \lim_{n\to\infty} \frac{\mu(T^{-p}B_n \cap B_n)}{\mu(B_n)}$.*

6. Acknowledgments

S.V. thanks the Laboratoire International Associé LIA LYSM, the INdAM (Italy), the UMI-CNRS 3483, Laboratoire Fibonacci (Pisa) where this work has been completed under a CNRS delegation and the *Centro de Giorgi* in Pisa for various supports. SV thanks M. Demers and D. Dragičević for enlightening discussions and Th. Caby for help on some matter of this paper.

N.H. was supported by a Simons Foundation Collaboration Grant (ID 526571).

J.A. was supported by an ARC Discovery Project and thanks the Centro de Giorgi in Pisa and CIRM in Luminy for their support and hospitality.

We finally thank the anonymous referees whose very careful and accurate reading of the paper, helped us to considerably improve it by adding several new proofs and explications.

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