Central limit theorems for sequential and random intermittent dynamical systems.

Matthew Nicol [*] Andrew Török [†] Sandro Vaienti [‡]

June 27, 2016

Contents

Abstract

We establish self-norming central limit theorems for non-stationary time series arising as observations on sequential maps possessing an indifferent fixed point. These

[*] Department of Mathematics, University of Houston, Houston Texas, USA. e-mail: <nicol@math.uh.edu>.

[†] Department of Mathematics, University of Houston, Houston Texas, USA and Institute of Mathematics of the Romanian Academy, P.O. Box 1–764, RO-70700 Bucharest, Romania. e-mail: <torok@math.uh.edu>.

[‡] Aix Marseille Université, CNRS, CPT, UMR 7332, 13288 Marseille, France and Université de Toulon, CNRS, CPT, UMR 7332, 83957 La Garde, France. e-mail:<vaienti@cpt.univ-mrs.fr>.

transformations are obtained by perturbing the slope in the Pomeau-Manneville map. We also obtain quenched central limit theorems for random compositions of these maps.

1 Introduction

In a preceding series of two papers [14], [3], we considered a few statistical properties of non-stationary dynamical systems arising by the sequential composition of (possibly) different maps. The first article [14] dealt with the Almost Sure Invariance Principle (ASIP) for the non-stationary process given by the observation along the orbit obtained by concatenating maps chosen in a given set. We choose maps in one and more dimensions which were piecewise expanding, more precisely their transfer operator (Perron-Frobenius, "PF") with respect to the Lebesgue measure was quasi-compact on a suitable Banach space. This allows to approximate the original process with a reverse martingale plus an error. By a recent result by Cuny and Merlevède [8], the reverse martingale satisfies the ASIP. The error is shown to be essentially bounded due to the presence of a spectral gap in the PF operator on a Banach space continuously injected in L^∞ (from now on all the L^p spaces will be with respect to the ambient Lebesgue measure m and they will be denoted with L^p or $L^p(m)$.). Moreover, the same spectral property allowed us to show that for expanding maps chosen close enough, the variance σ_n^2 grows linearly, which permits to approximate the original process almost everywhere with a finite sum of i.i.d. Gaussian variables with the same variance.

The second paper [3] considered composition of Pomeau-Manneville like maps, obtained by perturbing the slope at the indifferent fixed point 0. We got polynomial decay of correlations for particular classes of centered observables, which could also be interpreted as the decay of the iterates of the PF operator on functions of zero (Lebesgue) average; this fact is also known as *loss of memory.* In this situation the PF operator is not quasi-compact and although the process given by the observation along a sequential orbit can be decomposed again as the sum of a reverse martingale difference plus an error, apriori the latter turns out to be bounded only in L^1 and this was an obstacle to obtain an almost sure result like the ASIP by *only* looking at the almost sure convergence of the reverse martingale difference. Instead one could hope to get a (distributional) Central Limit Theorem (CLT); in this regard a general approach to CLT for sequential dynamical systems has been proposed and developed in [7]. It basically applies to systems with a quasi-compact PF operator and it is not immediately transposable to maps with do not admit a spectral gap. The main goal of our paper is to prove the CLT for the sequential composition of Pomeau-Manneville maps with varying slopes. A fundamental tool in obtaining such a result will be the polynomial loss of memory bound obtained in [3]; we are now going to recall it also because it will determine the regularity of the observables to which our CLT will apply; see Theorem 1.2.

We consider the family of Pomeau-Manneville maps

$$T_\alpha(x) = \begin{cases} x + 2^\alpha x^{1+\alpha}, \ 0 \le x \le 1/2 \\ 2x-1, \ 1/2 \le x \le 1 \end{cases} \qquad 0 < \alpha < 1. \tag{1.1}$$

Actually in [3] we considered a slightly different family of this type, but pointed out that both versions could be worked out with the same techniques (see [1]), and lead to the same result; here we prefer to use the *classical* version (1.1). As in [19], we identify the unit interval $[0,1]$ with the circle S^1, so that the maps become continuous. Given $0 < \beta_k \le \alpha < 1$, denote by P_{β_k} or P_k the *Perron-Frobenius* operator associated with the map $T_k = T_{\beta_k}$ w.r.t. the measure m. For concatenations we use equivalently the notations

$$\mathcal{T}_m^{n-m+1} := T_{\beta_n} \circ T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_m} = T_n \circ T_{n-1} \circ \cdots \circ T_m.$$

$$\mathcal{P}_m^{n-m+1} := P_{\beta_n} \circ P_{\beta_{n-1}} \circ \cdots \circ P_{\beta_m} = P_n \circ P_{n-1} \circ \cdots \circ P_m.$$

$$\mathcal{P}^n := \mathcal{P}_1^n \qquad \mathcal{T}^n := \mathcal{T}_1^n$$

where *the exponent denotes the number of maps in the concatenation.* For simplicity we use $\mathcal{T}^\infty := \cdots T_n \circ \cdots \circ T_1$ for a given sequence of transformations.

The Perron-Frobenius operator P_k associated to T_k satisfies the duality relation

$$\int_M P_k f \ g \ dm \ = \ \int_M f \ g \circ T_k \ dm, \quad \text{for all } f \in L^1, \ g \in L^\infty$$

and this is preserved under concatenation.

We next consider [19, 3] the cone $\mathcal{C}_2$ of functions given by (here $X(x) = x$ is the identity function):

$$\mathcal{C}_2 := \{f \in C^0((0,1]) \cap L^1(m) \mid \ f \ge 0, \ f \text{ decreasing}, \ X^{\alpha+1} f \text{ increasing}, \ f(x) \le a x^{-\alpha}\, m(f)\}^1$$

Remark 1.1 Some coefficients that appear later depend on the value a that defines the cone $\mathcal{C}_2$; however, we will not write explicitly this dependence.

[1] By "decreasing" we mean "nonincreasing".

Fix $0 < \alpha < 1$; as proven in [3], provided a is large enough, the cone $\mathcal{C}_2$ is preserved by all operators $P_\beta,\ 0 < \beta \le \alpha < 1$. The following polynomial decay result holds:

Theorem 1.2 ([3]) *Fix $0 < \alpha < 1$ and consider a cone $\mathcal{C}_2$ as above. Suppose ψ, φ in $\mathcal{C}_2$ have equal expectation, $\int \varphi dm = \int \psi dm$. Then for any sequence $T_{\beta_1}, \cdots, T_{\beta_n},\ n \ge 1$, of maps of Pomeau-Manneville type* (1.1) *with $0 < \beta_k \le \alpha < 1$, $k \in [1, n]$, we have*

$$\int |P_{\beta_n} \circ \cdots \circ P_{\beta_1}(\varphi) - P_{\beta_n} \circ \cdots \circ P_{\beta_1}(\psi)| dm \le C_\alpha (\|\varphi\|_1 + \|\psi\|_1) n^{-\frac{1}{\alpha}+1} (\log n)^{\frac{1}{\alpha}}, \quad (1.2)$$

where the constant C_α depends only on the map T_α, and $\|\cdot\|_1$ denotes the L^1 norm.

A similar rate of decay holds for observables φ and ψ that are C^1 on $[0,1]$; in this case the rate of decay has an upper bound given by

$$C_\alpha\ \mathcal{F}(\|\varphi\|_{C^1} + \|\psi\|_{C^1}) n^{-\frac{1}{\alpha}+1} (\log n)^{\frac{1}{\alpha}}$$

where the function $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ is affine.

For the proof of the CLT Theorem 3.1 we need better decay than in L^1. In this paper we improve the above result to decay in L^p, provided α is small enough.

Note that $\mathcal{P}^n \varphi \in \mathcal{C}_2$ if $\varphi \in \mathcal{C}_2$ and $m(\mathcal{P}^n \varphi) = m(\varphi)$, so

$$|[\mathcal{P}^n(\varphi) - \mathcal{P}^n(\psi)] \lfloor_x| \le |\mathcal{P}^n(\varphi) \lfloor_x| + |\mathcal{P}^n(\psi) \lfloor_x| \le a m(\varphi) x^{-\alpha} + a m(\psi) x^{-\alpha}$$

Proposition 1.3 *Under the assumptions on Theorem 1.2, if $1 \le p < 1/\alpha$ then*

$$\|P_{\beta_n} \circ \cdots \circ P_{\beta_1}(\varphi) - P_{\beta_n} \circ \cdots \circ P_{\beta_1}(\psi)\|_{L^p(m)} \le C_{\alpha,p} (\|\varphi\|_1 + \|\psi\|_1) n^{1-\frac{1}{p\alpha}} (\log n)^{\frac{1}{\alpha}\frac{1-\alpha p}{p-\alpha p}} \quad (1.3)$$

where the constant $C_{\alpha,p}$ depends only on the map T_α and p.

As in Theorem 1.2, a similar L^p-decay result also holds for observables $\varphi, \psi \in C^1([0,1])$.

Proof. For functions in the cone $\mathcal{C}_2$, Theorem 1.2 gives L^1-decay; then Lemma 2.7 together with the preceding discussion implies L^p-decay for α small enough. Note that we use this Lemma with $K = 2a(\|\varphi\|_1 + \|\psi\|_1)$ and the L^1-bound given by the Theorem, and then the coefficient in the L^p-bound is proportional to $(\|\varphi\|_1 + \|\psi\|_1)$ as well.

To prove the decay for C^1 observables, we use Lemma 2.4 (same approach as in the proof of Theorem 1.2). ∎

Note that the convergence of the quantity (1.2) implies the decay of the non-stationary correlations with respect to m:

$$\left| \int \psi\varphi \circ T_{\beta_n} \circ \cdots \circ T_{\beta_1} dm - \int \psi dm \int \varphi \circ T_{\beta_n} \circ \cdots \circ T_{\beta_1} dm \right|$$
$$\leq \|\varphi\|_\infty \left\| P_{\beta_n} \circ \cdots \circ P_{\beta_1}(\psi) - P_{\beta_n} \circ \cdots \circ P_{\beta_1} \left(\mathbf{1} \left(\int \psi dm \right) \right) \right\|_1$$

provided φ is essentially bounded and $(\int \psi dm)\mathbf{1}$ is in the functional space where the convergence of (1.2) takes place. In particular, this holds for C^1 observables, by Theorem 1.2.

As it is suggested by the preceding loss of memory result, centering the observable is the good way to define the process when it is not stationary, in order to consider limit theorems. To simplify the exposition, we introduce the following notation:

Definition 1.4 For $\varphi : [0,1] \to \mathbb{R}$ sufficiently regular (often C^1) introduce the following normalization *along a sequential orbit*:

$$[\varphi]_k := \varphi - \int \varphi(T_k \circ \cdots \circ T_1) dm. \tag{1.4}$$

However, to simplify notation, it is convenient to set $[\varphi]_0 = 0$.

Conze and Raugi [7] defined the sequence of transformations $\{T_1, T_2, T_3, \dots\}$ to be *pointwise ergodic* whenever the law of large numbers is satisfied, namely

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[\varphi(T_k \circ \cdots \circ T_1 x) - \int \varphi(T_k \circ \cdots \circ T_1) dm \right] = 0 \text{ for Lebesgue-a.e. } x.$$

We will prove in Theorem 2.10 that such a law of large numbers holds for our observations provided $0 < \alpha < 1$. It is therefore natural to ask about a non-stationary Central Limit Theorem for the sums

$$S_n := \sum_{k=1}^{n} [\varphi]_k \circ T_k \circ \cdots \circ T_1 \tag{1.5}$$

for a given sequence $\mathcal{T}^\infty := \cdots \circ T_n \circ \cdots \circ T_1$: this will be the content of the next sections.

To be more specific we will prove in Theorem 3.1 a non-stationary central limit theorem similar to that proved by Conze and Raugi [7] for (piecewise expanding) sequential systems:

$$\frac{S_n}{\sqrt{\mathrm{Var}(S_n)}} \to^d \mathcal{N}(0,1). \tag{1.6}$$

At this point, we would like to make a few comments about our result compared to that of Conze and Raugi. Theorem 5.1 in [7] shows that, when applied to the quantities defined above and for classes of maps enjoying a quasi-compact transfer operator:

(1) If the norms $||S_n||_2$ are bounded, then the sequence $S_n, n \geq 1$ is bounded.

(2) If $||S_n||_2 \to \infty$, then (1.6) holds.

We are not able to prove item (1) for the intermittent map following the same approach as in [7], since it uses the uniform boundedness of the sequence $\mathbf{H}_n \circ \mathcal{T}^k$, where the function $\mathbf{H}_n$ is defined in (2.1) and is just the error in the martingale approximation as we discussed above. We can only prove that $\mathbf{H}_n$ is bounded uniformly in n on each set of the form $[a, 1), a > 0$, and do not expect it to be bounded near 0 (look at the stationary case).

Instead, our central limit theorem will satisfy item (2) under the assumption that the variance $||S_n||_2^2$ grows at a certain rate and for some limitation on the range of values of α. It seems difficult to get such a result in full generality for the intermittent map considered here. Conze and Raugi proved the linear growth of the variance in their Theorem 5.3 under a certain number of assumptions, including the presence of a spectral gap for the transfer operator. We showed in our paper [14] that those assumptions apply to several classes of expanding maps even in higher dimensions.

However, for concatenations given by the *same* intermittent map T_α with $\alpha < 1/2$, the variance is linear in n, provided the observable is not a coboundary for T_α. In section 4 we prove that the linear growth of the variance still holds if we take maps T_{β_n} with β_n arbitrary but close to a fixed β, and an observable is not a coboundary for T_β; therefore, the CLT holds. See Theorem 4.1. Our proof of Theorem 4.1 uses an estimate of interesting related work of Leppänen and Stenlund [17], which we learnt about after a first version of this paper was completed. Their result allowed us to give another example where variance grows linearly for a sequential dynamical system of intermittent type maps, and hence the non-stationary CLT holds. The focus of [17] is however more on the strong law of large numbers and convergence in probability rather than the CLT. They also consider quasi static systems, introduced in [18].

In section 5 we show that the variance grows linearly for almost all sequences when we compose intermittent maps chosen from a finite set and we take them according to a fixed probability distribution. This means that for almost all sequences (with respect to the induced Bernoulli measure) of maps, the central limit theorem holds (a *quenched* CLT). See Theorem 5.3.

Remark 1.5 For simplicity, in many of the following statements we will use as rate of decay $n^{-\frac{1}{\alpha}+1}$, ignoring the $\log n$-factor. This is correct if we take for α a slightly larger value (and is actually the correct rate of decay for the stationary case).

Notation 1.6 For any sequences of numbers $\{a_n\}$ and $\{b_n\}$, we will write $a_n \approx b_n$ if $c_1 b_n \leq a_n \leq c_2 b_n$ for some constants $c_2 \geq c_1 > 0$ and $n \gg 1$; similarly, use $a_n \gtrsim b_n$ for a one sided asymptotic relation.

2 Cones and Martingales

In order to get the right martingale representation, we begin by recalling a few formulas concerning the transfer operator; the conditional expectation is considered with respect to the measure m, and $\mathcal{B}$ denotes the Borel σ-algebra on $[0,1]$. We have:

$$\mathbb{E}[\varphi \mid \mathcal{T}^{-k}\mathcal{B}] = \frac{\mathcal{P}^k(\varphi)}{\mathcal{P}^k(\mathbf{1})} \circ \mathcal{T}^k$$

$$P(\varphi \circ T \cdot \psi) = \varphi \cdot P(\psi)$$

and therefore, for $0 \leq \ell \leq k$

$$\mathbb{E}[\varphi \circ \mathcal{T}^\ell \mid \mathcal{T}^{-k}\mathcal{B}] = \frac{\mathcal{P}_{\ell+1}^{k-\ell}(\varphi \cdot \mathcal{P}^\ell(\mathbf{1}))}{\mathcal{P}^k(\mathbf{1})} \circ \mathcal{T}^k.$$

Recall that for $L^2(m)$-functions these conditional expectations are the orthogonal projections in $L^2(m)$.

We denote, as in Definition 1.4, $\varphi - m(\varphi \circ \mathcal{T}^j)$ by $[\varphi]_j$, with the convention that $[\varphi]_0 = 0$. Therefore we have for the centered sum (1.5): $S_n = \sum_{k=1}^n [\varphi]_k \circ \mathcal{T}^k = \sum_{k=0}^n [\varphi]_k \circ \mathcal{T}^k$.

Introduce

$$\mathbf{H}_n \circ \mathcal{T}^n := \mathbb{E}(S_{n-1} \mid \mathcal{T}^{-n}\mathcal{B}).$$

Hence $\mathbf{H}_1 = 0$, and the explicit formula for $\mathbf{H}_n$ is

$$\mathbf{H}_n = \frac{1}{\mathcal{P}^n\mathbf{1}} \left[P_n([\varphi]_{n-1} \mathcal{P}^{n-1}\mathbf{1}) + P_n P_{n-1}([\varphi]_{n-2} \mathcal{P}^{n-2}\mathbf{1}) + \cdots + P_n P_{n-1} \dots P_1([\varphi]_0 \mathcal{P}^0\mathbf{1}) \right]. \tag{2.1}$$

It is not hard to check that setting

$$S_n = M_n + \mathbf{H}_{n+1} \circ \mathcal{T}^{n+1}$$

the sequence $\{M_n\}$ is a reverse martingale for the decreasing filtration $\{\mathcal{B}_n := \mathcal{T}^{-n}\mathcal{B}\}$:

$$\mathbb{E}(M_n \mid \mathcal{B}_{n+1}) = 0.$$

In particular,

$$M_n - M_{n-1} = \psi_n \circ \mathcal{T}^n \quad \text{with} \quad \psi_n := [\varphi]_n + \mathbf{H}_n - \mathbf{H}_{n+1} \circ T_{n+1}. \tag{2.2}$$

We recall three lemmas from [15], stated in the current context:

Lemma 2.1 ([15, Lemma 2.6])

$$\sigma_n^2 := \mathbb{E}[(\sum_{i=1}^{n} [\varphi]_i \circ \mathcal{T}^i)^2] = \sum_{i=1}^{n} \mathbb{E}[\psi_i^2 \circ \mathcal{T}^i] - \int \mathbf{H}_1^2 + \int \mathbf{H}_{n+1}^2 \circ \mathcal{T}^{n+1}$$

(and $\mathbf{H}_1 = 0$*).*

To prove this Lemma we replace our $\mathbf{H}_n$ with ω_n in [15].

Lemma 2.2 ([15, proof of Lemma 3.3]) *Let* $\mathbf{H}_j^\varepsilon = \mathbf{H}_j \mathbf{1}_{\{|\mathbf{H}_j| \leq \varepsilon\sigma_n\}}$, *where for simplicity of notation we have left out the dependence on* n. *Then*

$$\int \left(\sum_{j=1}^{n} \psi_j \circ \mathcal{T}^j \cdot \mathbf{H}_{j+1}^\varepsilon \circ \mathcal{T}^{j+1} \right)^2 = \sum_{j=1}^{n} \int \left(\psi_j \circ \mathcal{T}^j \cdot \mathbf{H}_{j+1}^\varepsilon \circ \mathcal{T}^{j+1}\right)^2$$

The last formula in the proof of [15, Lemma 2.6] gives:

Lemma 2.3

$$\sigma_n^2 = \sum_{i=1}^{n} \mathbb{E}[[\varphi]_i^2 \circ \mathcal{T}^i] + 2 \sum_{i=1}^{n} \mathbb{E}[(\mathbf{H}_i \, [\varphi]_i) \circ \mathcal{T}^i]$$

The following Lemma plays a crucial role all along this paper. In a slightly different form it was introduced and used in [19, Sect. 4], without a proof, and subsequently in [3]. We now give a detailed proof in a more general setting.

Lemma 2.4 *Assume given a* C^1*-function* $\varphi : [0,1] \to \mathbb{R}$ *and* $h \in \mathcal{C}_2$*. where the cone* $\mathcal{C}_2$ *is defined with* $a > 1$.

Denote by X the function $X(x) = x$. If

$$\begin{aligned}\lambda &\leq -|\varphi'|_\infty \\ \nu &\geq -|\varphi + \lambda X|_\infty \\ \delta &\geq \frac{a}{\alpha+1}\big(|\varphi'|_\infty + |\lambda|\big)m(h) \\ \delta &\geq \frac{a}{a-1}|\varphi + \lambda X + \nu|_\infty m(h)\end{aligned}$$

then

$$(\varphi + \lambda X + \nu)h + \delta \in \mathcal{C}_2.$$

Remark 2.5 It follows immediately that if $\varphi \in C^1([0,1])$ and $h \in \mathcal{C}_2$ then we can use Theorem 1.2 and Proposition 1.3 to obtain decay of $\mathcal{P}^\ell(\varphi h - m(\varphi h))$: consider $\Phi := (\varphi + \lambda X + \nu)h + \delta$, $\Psi := (\lambda X + \nu)h + \delta + m(\varphi h)$, with constants chosen according to Lemma 2.4 so that $\Phi, \Psi \in \mathcal{C}_2$ (by definition, $m(\Phi) = m(\Psi)$), and write

$$\mathcal{P}^\ell\big(\varphi \cdot h - m(\varphi \cdot h)\big) = \mathcal{P}^\ell(\Phi - \Psi).$$

Corollary 2.6 *In particular, for a sequence $\omega_k \in C^1([0,1])$ with $\|\omega_k\|_{C^1} \leq K$ and $h_k \in \mathcal{C}_2$ with $m(h_k) \leq M$ (e.g, $h_k := \mathcal{P}^k(\mathbf{1})$), one can choose constants λ, ν and δ so that*

$$(\omega_k + \lambda X + \nu)h_k + \delta, (\lambda X + \nu)h_k + \delta + m(\omega_k h_k) \in \mathcal{C}_2 \quad \text{for all } k \geq 1$$

and therefore

$$||\mathcal{P}^n\big(\omega_k h_k - m(\omega_k h_k)\big)||_1 \leq C_{\alpha,K,M}\ n^{-\frac{1}{\alpha}+1}(\log n)^{\frac{1}{\alpha}} \quad \text{for all } n \geq 1,\ k \geq 1,$$

where the constant $C_{\alpha,K,M}$ has an explicit expression in terms of α, K and M. Decay in L^p now follows from Lemma 2.7: if $1 \leq p < 1/\alpha$ then

$$||\mathcal{P}^n\big(\omega_k h_k - m(\omega_k h_k)\big)||_p \leq C_{\alpha,K,M,p}\ n^{-\frac{1}{p\alpha}+1} \quad \text{for all } n \geq 1,\ k \geq 1$$

(ignoring the log*-correction, see Remark 1.5) where the constant on the right hand side depends now upon p too.*

Proof of Lemma 2.4. Denote $\Phi := (\varphi + \lambda X + \nu)h + \delta$. There are three conditions for Φ to be in $\mathcal{C}_2$.

<u>Φ nonnegative and decreasing.</u> If $\lambda \le -\sup \varphi'$ and $\nu \ge -\inf(\varphi + \lambda X)$ then $\varphi + \lambda X + \nu$ is decreasing and nonnegative. Therefore Φ is also decreasing (because $h \in \mathcal{C}_2$) and nonnegative provided $\delta \ge 0$.

<u>$\Phi X^{1+\alpha}$ increasing.</u> For $0 < x < y \le 1$, need

$$\begin{aligned} & \big[(\varphi(x) + \lambda x + \nu)h(x) + \delta\big]x^{1+\alpha} \le \big[(\varphi(y) + \lambda y + \nu)h(y) + \delta\big]y^{1+\alpha} \\ \iff \quad & [\varphi(x) + \lambda x + \nu] \le [\varphi(y) + \lambda y + \nu]\frac{h(y)}{h(x)}\frac{y^{\alpha+1}}{x^{\alpha+1}} + \delta\left[\frac{y^{\alpha+1}}{x^{\alpha+1}} - 1\right]\frac{1}{h(x)} \end{aligned}$$

Since $hX^{\alpha+1} \ge 0$ is increasing, $1 \le \frac{h(y)}{h(x)}\frac{y^{\alpha+1}}{x^{\alpha+1}}$, so it suffices to have

$$\begin{aligned} & \varphi(x) + \lambda x + \nu \le [\varphi(y) + \lambda y + \nu] + \delta\left[\frac{y^{\alpha+1}}{x^{\alpha+1}} - 1\right]\frac{1}{h(x)} \\ \iff \quad & \delta \ge -\big[(\varphi(y) + \lambda y + \nu) - (\varphi(x) + \lambda x + \nu)\big]\frac{h(x)}{\frac{y^{\alpha+1}}{x^{\alpha+1}} - 1}. \end{aligned}$$

By the mean value theorem and using that $\alpha \le 1$, $y^{\alpha+1} - x^{\alpha+1} = (\alpha+1)\xi^\alpha(y-x) \ge (\alpha+1)x^\alpha(y-x) \ge (\alpha+1)x(y-x)$; therefore

$$0 \le \frac{h(x)}{\frac{y^{\alpha+1}}{x^{\alpha+1}} - 1} = \frac{h(x)x^{\alpha+1}}{y^{\alpha+1} - x^{\alpha+1}} \le \frac{h(x)x^\alpha}{(\alpha+1)(y-x)} \le \frac{am(h)}{(\alpha+1)(y-x)}.$$

Meanwhile,

$$-\big[(\varphi(y) + \lambda y + \nu) - (\varphi(x) + \lambda x + \nu)\big] \le (|\varphi'|_\infty + |\lambda|)(y-x).$$

Using these in the above lower bound for δ, we conclude that it suffices to have

$$\delta \ge \frac{a}{\alpha+1}\big(|\varphi'|_\infty + |\lambda|\big)m(h)$$

<u>$\Phi X^\alpha \le am(\Phi)$.</u> Using that $hX^\alpha \le am(h)$,

$$[(\varphi + \lambda X + \nu)h + \delta]X^\alpha \le (\varphi + \lambda X + \nu)hX^\alpha + \delta \le \sup(\varphi + \lambda X + \nu)am(h) + \delta.$$

On the other hand, $am((\varphi + \lambda X + \nu)h + \delta) \ge a\inf(\varphi + \lambda X + \nu)m(h) + a\delta$, so it suffices to have

$$\begin{aligned} & \sup(\varphi + \lambda X + \nu)am(h) + \delta \le a\inf(\varphi + \lambda X + \nu)m(h) + a\delta \\ \iff \quad & \delta \ge \frac{a}{a-1}\big[\sup(\varphi + \lambda X + \nu) - \inf(\varphi + \lambda X + \nu)\big]m(h). \end{aligned}$$

∎

Note that, since the transfer operators are monotone,

$$\left| P_n \dots P_{k+1}[\varphi \mathcal{P}^k \mathbf{1}] \downharpoonright_x \right| \le P_n \dots P_{k+1}[|\varphi|_\infty \mathcal{P}^k \mathbf{1}] \downharpoonright_x = |\varphi|_\infty P_n \dots P_{k+1}[\mathcal{P}^k \mathbf{1}] \downharpoonright_x .$$

Since $|\varphi|_\infty P_n \dots P_{k+1}[\mathcal{P}^k \mathbf{1}]$ lies in the cone $\mathcal{C}_2$ this implies that

$$|P_n \dots P_{k+1}[\varphi \mathcal{P}^k \mathbf{1}]| \downharpoonright_x \le a |\varphi|_\infty x^{-\alpha}.$$

The following Lemma gives control over the L^p-norm of functions with such a bound.

Lemma 2.7 *Suppose that $f \in L^1(m)$ and $|f(x)| \le K x^{-\alpha}$. Then, provided $p \ge 1$ and $\alpha p < 1$,*

$$||f||_p \le C_{\alpha,p} ||f||_1^{\frac{1-\alpha p}{p-p\alpha}} K^{\frac{p-1}{p-p\alpha}}$$

In particular, if $|f(x)| \le K x^{-\alpha}$ and $||f||_1 \le M n^{1-\frac{1}{\alpha}}$, then

$$||f||_p \le C_{K,M,\alpha,p} n^{1-\frac{1}{p\alpha}} \text{ for } 1 \le p < 1/\alpha.$$

Therefore, for $1 \le p < 1/(2\alpha)$, there is $\delta > 0$ such that $||f||_p \le C_{K,M,\alpha,p} n^{-1-\delta}$.

Proof. The case $p = 1$ is obviously true, so we assume from now on that $p > 1$. Denote $C_1 := ||f||_1$. Compute, for $0 < x_* \le 1$, and $\alpha p < 1$: $\int_{x_*}^1 |f|^p dx \le \sup\{|f(x)|^{p-1} \mid x_* \le x \le 1\} \int_0^1 |f| dx \le K^{p-1} x_*^{-\alpha(p-1)} C_1$, and $\int_0^{x_*} |f|^p dx \le K^p \int_0^{x_*} x^{-\alpha p} dx = \frac{K^p}{1-\alpha p} x_*^{1-\alpha p}$. We want to minimize over x_* the quantity

$$G(x_*) := K^{p-1} C_1 x_*^{-\alpha(p-1)} + K^p \frac{1}{1-\alpha p} x_*^{1-\alpha p} = A x_*^{-\alpha(p-1)} + B x_*^{1-\alpha p}.$$

It reaches its minimum value for $x_*^{\alpha-1} = \frac{B(1-\alpha p)}{A\alpha(p-1)}$, which gives for the minimum of $G^{1/p}$ the value

$$C_{\alpha,p} C_1^{\frac{1-\alpha p}{1-\alpha}\frac{1}{p}} K^{\frac{p-1}{p}\frac{1}{1-\alpha}}.$$

For the last statement notice that $\frac{1-p\alpha}{p\alpha} > 1 \iff 0 < \alpha p < 1/2$. ∎

Corollary 2.8 *We have:*

1. *$||\mathbf{H}_n||_q$ is uniformly bounded in n for $1 \le q < \frac{1}{2\alpha}$.*

2. $||\mathbf{H}_n \circ \mathcal{T}^n||_r$ *is uniformly bounded in* n *for* $1 \le r < \frac{1}{2\alpha} - \frac{1}{2}$.

Proof. Recall that $\mathbf{H}_n$ is given in (2.1). By [3, Remark 1.3], $\mathcal{P}^n(\mathbf{1}) \ge D_\alpha > 0$ on $(0,1]$. We now apply Minkowski's inequality in the sum defining $\mathbf{H}_n$. Thanks to Lemma 2.7 each term of the form $P_n P_{n-1} \dots P_{n-\ell}([\varphi]_{n-\ell-1}\, \mathcal{P}^{n-\ell-1}\mathbf{1})$, $\ell \in [0, n-1]$ will be bounded in L^p by $\frac{2}{D_\alpha}\, C_{\alpha,K,p}\, \ell^{1-\frac{1}{p\alpha}}$, where K is the C^1 norm of φ. The role of h_k in Lemma 2.6 is now played by $\mathcal{P}^{n-\ell-1}\mathbf{1}$ and therefore $M = 1$. By summing over ℓ from 1 to infinity, we get a convergent series whenever $p\alpha < 1/2$. We now write $\int |\mathbf{H}_n \circ \mathcal{T}^n|^r dx = \int |\mathbf{H}_n|^r \mathcal{P}^n\mathbf{1}\, dx$. Since $\mathcal{P}^n\mathbf{1}$ belongs to $L^p(m)$ for $1 \le p < \frac{1}{\alpha}$ by the definition of $\mathcal{C}_2$ and its invariance property, it suffices that the function $|\mathbf{H}_n|^{r\frac{p}{p-1}}$ be uniformly in $L^1(m)$, and therefore, by the previous item, that $r\frac{p}{p-1} < \frac{1}{2\alpha}$. Thus it suffices to have $1 \le r < \frac{p-1}{2p\alpha}$ for some $1 \le p < \frac{1}{\alpha}$, which means $1 \le r < \frac{1}{2\alpha} - \frac{1}{2}$. ∎

As we said in the Introduction, we will also have a pointwise bound on the $\mathbf{H}_n$'s.

Lemma 2.9 *For* $0 < \alpha < 1/2$, *there is a constant* C *depending on* α *and* $K = ||\varphi||_{C^1}$, *such that*

$$|\mathbf{H}_n(x)| \le C x^{-\alpha-1} \quad \text{for all } x \in (0,1],\ n \ge 1. \tag{2.3}$$

Proof. By using again formula (2.1) for $\mathbf{H}_n$ (where $\varphi_0 = 0$) and the bound $\mathcal{P}^n(\mathbf{1}) \ge D_\alpha > 0$ we are left with the pointwise estimate of

$$P_n([\varphi]_{n-1}\, \mathcal{P}^{n-1}\mathbf{1}) + P_n P_{n-1}([\varphi]_{n-2}\, \mathcal{P}^{n-2}\mathbf{1}) + \cdots + P_n P_{n-1} \dots P_1([\varphi]_0\, \mathcal{P}^0\mathbf{1}).$$

By Corollary 2.6, for each $k \ge 1$ one can write $[\varphi]_k\, \mathcal{P}^k\mathbf{1} = \big(\varphi - m(\varphi \circ \mathcal{T}^k)\big)\mathcal{P}^k\mathbf{1} = A_k - B_k$ where $A_k, B_k \in \mathcal{C}_2$ with $m(A_k), m(B_k)$ uniformly bounded by some constant $C_{\alpha,K} < \infty$. Therefore, by the decay Theorem 1.2 (and ignoring the log-correction), there is a new constant C' depending only on α and K such that

$$\|\mathcal{P}_{k+1}^{n-k}(A_k - B_k)\|_1 \le C'(n-k)^{-\frac{1}{\alpha}+1}. \tag{2.4}$$

We now recall the footnote to the proof of [19, Lemma 2.3]: if $f \in \mathcal{C}_2$ with $m(f) \le M$ then

$$|x^{\alpha+1} f(x) - y^{\alpha+1} f(y)| \le a(1+\alpha)M|x-y| \quad \text{for } 0 < x, y \le 1. \tag{2.5}$$

But a bound $|g(x) - g(y)| \le L|x-y|$ for the Lipschitz-seminorm $|g|_{\mathrm{Lip}}$ implies

$$\|g\|_1 \ge C_L \|g\|_\infty. \tag{2.6}$$

Combining the above observations and since $m(\mathcal{P}_{k+1}^{n-k}(f)) = m(f)$, we obtain that $|X^{\alpha+1}\mathcal{P}_{k+1}^{n-k}(A_k - B_k)|_{\text{Lip}} \le |X^{\alpha+1}\mathcal{P}_{k+1}^{n-k}(A_k)|_{\text{Lip}} + |X^{\alpha+1}\mathcal{P}_{k+1}^{n-k}(B_k)|_{\text{Lip}} \le L$ uniformly for $n \ge 1$, $1 \le k < n$, and then

$$\|X^{\alpha+1}\mathcal{P}_{k+1}^{n-k}(A_k - B_k)\|_\infty \le 1/C_L \|X^{\alpha+1}\mathcal{P}_{k+1}^{n-k}(A_k - B_k)\|_1 \le C''(n-k)^{-\frac{1}{\alpha}+1}$$

for a new constant C'' depending only on α, K, L, which implies that

$$|\mathcal{P}_{k+1}^{n-k}(A_k - B_k)(x)| \le x^{-\alpha-1}C''(n-k)^{-\frac{1}{\alpha}+1}$$

and therefore, for $0 < \alpha < 1/2$,

$$\left|\sum_{k=1}^{n-1}\mathcal{P}_{k+1}^{n-k}(A_k - B_k)(x)\right| \le x^{-\alpha-1}C''\sum_{k=1}^{n-1}(n-k)^{-\frac{1}{\alpha}+1} \le Cx^{-\alpha-1}$$

as desired. ∎

We finish this Section by proving a type of Borel-Cantelli Lemma which is an unavoidable tool in proving non-stationary limit theorems.

Theorem 2.10 (Strong Borel-Cantelli) *Suppose that for $j \ge 1$, $\psi_j \in C^1([0,1])$ with uniformly bounded C^1-norms.*

(a) If $0 < \alpha < 1/2$ then

$$\sum_{j=1}^n \psi_j(\mathcal{T}^j) - \sum_{j=1}^n m(\psi_j(\mathcal{T}^j)) = O(n^{1/2}(\log\log n)^{3/2}) \quad m\text{-a.e.}$$

and therefore, if $\liminf_j m(\psi_j \circ \mathcal{T}^j) > 0$ then

$$\frac{\sum_{j=1}^n \psi_j(\mathcal{T}^j x)}{\sum_{j=1}^n m(\psi_j \circ \mathcal{T}^j)} \to 1 \quad m\text{-a.e. } x.$$

(b) If $0 < \alpha < 1$ then

$$\frac{1}{n}\left[\sum_{j=1}^n \psi_j(\mathcal{T}^j x) - \sum_{j=1}^n m(\psi_j \circ \mathcal{T}^j)\right] \to 0 \quad m\text{-a.e. } x.$$

Proof. To prove the first statement in part (a) we will use the Sprindzuk's Theorem 6.1 in the Appendix. By adding the same constant to all the ψ_j's and rescaling, we can assume without loss of generality that $\inf_j m(\psi_j \circ \mathcal{T}^j) > 0$ and $\sup_j m(\psi_j \circ \mathcal{T}^j) \le 1$. We take

$g_k = m(\psi_k \circ \mathcal{T}^k)$ and $h_k = 1$ in Theorem 6.1, thus it suffices to give a linear upper bound for $\mathbb{E}[(\sum_{j=1}^n \psi_j \circ \mathcal{T}^j - b_n)^2]$, where $b_n := \sum_{j=1}^n m(\psi_j \circ \mathcal{T}^j)$; note that the same estimate can be derived for sums over $m \le j \le n$. Expand

$$\mathbb{E}[(\sum_{j=1}^n \psi_j \circ \mathcal{T}^j - b_n)^2] = \sum_{j=1}^n \mathbb{E}[\psi_j \circ \mathcal{T}^j - m(\psi_j \circ \mathcal{T}^j)]^2 + 2\sum_{i=1}^n \sum_{j>i} \mathbb{E}[(\psi_j \circ \mathcal{T}^j - m(\psi_j \circ \mathcal{T}^j)(\psi_i \circ \mathcal{T}^i - m(\psi_i \circ \mathcal{T}^i))]$$

and use the decay to estimate the mixed terms. Denote, following Definition 1.4, $[g]_j := g - m(g \circ \mathcal{T}^j)$. Then, for $j > i \ge 1$,

$$\begin{aligned} |\mathbb{E}[(\psi_j \circ \mathcal{T}^j - m(\psi_j \circ \mathcal{T}^j)(\psi_j \circ \mathcal{T}^j - m(\psi_j \circ \mathcal{T}^j)]| &= |\mathbb{E}[[\psi_j]_j \circ \mathcal{T}^j \cdot [\psi_i]_i \circ \mathcal{T}^i]| \\ = |\mathbb{E}[([\psi_j]_j \circ \mathcal{T}_{i+1}^{j-i} \cdot [\psi_i]_i \cdot \mathcal{P}^i(\mathbf{1})]| &= |\mathbb{E}[([\psi_j]_j \cdot \mathcal{P}_{i+1}^{j-i}([\psi_i]_i \mathcal{P}^i(\mathbf{1}))]| \\ &\le \| [\psi_j]_j \|_\infty \| \mathcal{P}_{i+1}^{j-i}([\psi_i]_i \mathcal{P}^i(\mathbf{1}))\|_1 \le C(j-i)^{1-\frac{1}{\alpha}} \end{aligned}$$

where in the last inequality we used Corollary 2.6. Therefore

$$\begin{aligned} &\mathbb{E}[(\sum_{j=1}^n \psi_j \circ \mathcal{T}^j - b_n)^2] \\ &\le 2\sum_{i=1}^n |(\psi_j \circ \mathcal{T}^j - m(\psi_j \circ \mathcal{T}^j)|_\infty m(\psi_j \circ \mathcal{T}^j) + 2C \sum_{i=1}^n \sum_{j>i} (j-i)^{1-\frac{1}{\alpha}} \le nC', \end{aligned}$$

where the constants C, C' are independent of j and n.. The conclusion now follows from the Sprindzuk's Theorem 6.1.

For (b), note that for $1/2 \le \alpha < 1$ the above computation still gives

$$\mathbb{E}[(\sum_{j=1}^n \psi_j \circ \mathcal{T}^j - b_n)^2] \le C n^{3-\frac{1}{\alpha}}$$

which implies that

$$\sum_{j=1}^n \psi_j \circ \mathcal{T}^j - b_n = O(n^{1-\eta}) \text{ a.s.}$$

for some $\eta > 0$, see the standard Lemma 2.11. ∎

Lemma 2.11 *Assume the random variables X_n have mean zero, and there are $M < \infty$, $\gamma < 2$ such that*

$$\|X_n\|_\infty \le M, \quad \operatorname{Var}\Big(\sum_{k=1}^n X_k\Big) \le Cn^\gamma \qquad \textit{for all } n.$$

Then

$$\sum_{k=1}^n X_k = O(n^\eta) \ \textit{a.s. for } \eta > \frac{\gamma+1}{3}.$$

Proof. Denote $S_n := \sum_{k=1}^n X_k$. From Tchebycheff's inequality,

$$P(|S_n| > n^{1-\delta}) \le \frac{\operatorname{Var}(S_n)}{(n^{1-\delta})^2} \le Cn^{\gamma-2\delta-2}.$$

Pick $\delta > 0$ so that $\gamma - 2\delta - 2 < 0$ and $\omega > 0$ such that $\omega(2 - \gamma + 2\delta) > 1$. Then, for the subsequence $n_k := k^\omega$,

$$\sum_k P(|S_{n_k}| > n_k^{1-\delta}) < \infty$$

so, by Borel-Cantelli,

$$|S_{n_k}| = O(n_k^{1-\delta}) \text{ a.s.} \tag{2.7}$$

Using (2.7), one has a.s.: if $n_k \le n < n_{k+1}$ for some k, then

$$|S_n| \le |S_{n_k}| + [n_{k+1} - n_k] \sup \|X_\ell\|_\infty \le O(n_k^{1-\delta}) + Ck^{\omega-1}M \le O(n^{1-\delta}) + C(n^{1/\omega})^{\omega-1}M$$

therefore $|S_n| = O(n^\eta)$ a.s. with

$$\eta = \max\left\{1 - \delta, \frac{\omega - 1}{\omega}\right\}.$$

Optimize over δ and ω to get the claimed lower bound on η. ∎

3 Central Limit Theorem

We assume in this section that $0 < \alpha < 1/2$ (note that in the stationary case the CLT holds only in this range). With our approach we can only prove the non-stationary CLT for a lower upper bound on α, which will be stated later.

We define scaling constants $\sigma_n^2 = \mathbb{E}[(\sum_{j=1}^n [\varphi]_j \circ \mathcal{T}^j)^2]$. This sequence of constants play the role of non-stationary variance. As we pointed out in the Introduction, giving estimates on the growth and non-degeneracy of σ_n in this non-stationary setting is more difficult than in the usual stationary case.

Theorem 3.1 (CLT for C^1 functions) *Let φ be a $C^1([0,1])$ function, and define S_n as in* (1.5),

$$S_n := \sum_{k=1}^{n} [\varphi]_k \circ T_{\beta_k} \circ \cdots \circ T_{\beta_1}.$$

Assume that

$$\sigma_n^2 := \mathrm{Var}(S_n) = \mathbb{E}[(\sum_{i=1}^{n} [\varphi]_i \circ \mathcal{T}^i)^2] \gtrsim n^{\beta}.$$

Then

$$0 < \alpha < \frac{1}{9} \text{ and } \beta > \frac{1}{2(1-2\alpha)} \quad \Longrightarrow \quad \frac{S_n}{\sigma_n} \to^d \mathcal{N}(0,1).$$

In particular, $\beta > 9/14 = 0.643$ suffices for any $0 < \alpha < \frac{1}{9}$, and the lower bound on β approaches $\frac{1}{2}$ as α approaches zero.

Remark 3.2 The above Theorem holds, with the same proof, if we allow φ to vary but stay bounded in C^1 (as in our Strong Borel-Cantelli Theorem 2.10). That is, consider

$$S_n := \sum_{k=1}^{n} [\varphi_k]_k \circ T_{\beta_k} \circ \cdots \circ T_{\beta_1}$$

where $\varphi_k \in C^1([0,1])$ have uniformly bounded C^1-norms.

To keep the notation simpler, we will not prove this more general case.

Following the approach of Gordin we will express $S_n = \sum_{j=1}^{n} [\varphi]_j \circ \mathcal{T}^j$ as the sum of a (non-stationary) martingale difference array and a controllable error term and then use the following Theorem from Conze and Raugi [7, Theorem 5.8], which is a modification of a result of B. M. Brown [6] from martingale differences to reverse martingale differences.

Theorem 3.3 ([7, Theorem 5.8]) *Let $(X_i, \mathcal{B}_i)$ be a sequence of differences of square integrable reversed martingales, defined on a probability space $(\Omega, \mathcal{B}, \mathcal{P})$. For $n \geq 0$ let*

$$S_n = X_0 + \ldots + X_{n-1},\ \sigma_n^2 = \sum_{k=0}^{n-1} \mathbb{E}[X_k^2],\ V_n = \sum_{k=0}^{n-1} \mathbb{E}[X_k^2 | \mathcal{B}_{k+1}].$$

Assume the following two conditions hold:

(i) the sequence of random variables $(\sigma_n^{-2} V_n)_{n \geq 1}$ converges in probability to 1.

(ii) For each $\varepsilon > 0$, $\lim_{n\to\infty} \sigma_n^{-2} \sum_{k=0}^{n-1} \mathbb{E}[X_k^2 \mathbf{1}_{\{|X_k| > \varepsilon \sigma_n\}}] = 0$.

Then

$$\lim_{n\to\infty}\sup_{\alpha\in\mathbb{R}}\left|P\left[\frac{S_n}{\sigma_n}<a\right]-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha}e^{-\frac{x^2}{2}}\ dx\right|=0.$$

Proof of Theorem 3.1.

We will apply Theorem 3.3 with the following identifications:

- $X_n = \psi_n \circ \mathcal{T}^n$.
- $\mathcal{B}_n = \mathcal{T}^{-n}\mathcal{B}$.
- $\sigma_n^2 = \mathbb{E}[(\sum_{i=1}^n [\varphi]_i \circ \mathcal{T}^i)^2]$ as defined earlier, but if $\alpha < \frac{1}{5}$ then $\sigma_n^2 = \mathbb{E}[(\sum_{i=1}^n \psi_i \circ \mathcal{T}^i)^2] + O(1)$ by Lemma 2.1 and Corollary 2.8.

Let us take $\mathbf{H}_n$ defined in (2.1) and ψ_n given in (2.2)

$$\psi_n := [\varphi]_n + \mathbf{H}_n - \mathbf{H}_{n+1} \circ T_{n+1}.$$

Recall that $\psi_n \circ \mathcal{T}^n$ is a reverse martingale difference scheme, uniformly bounded in $L^{r_1}(m)$ provided $1 \le r_1 < \frac{1}{2\alpha} - \frac{1}{2}$ (because so is $\mathbf{H}_k \circ \mathcal{T}^k$, see the second item in Corollary 2.8). Once we establish (i) and (ii) it follows that $\lim_{n\to\infty} \frac{1}{\sigma_n}\sum_{j=1}^n \psi_j \circ \mathcal{T}^j \to \mathcal{N}(0,1)$ in distribution. Finally, since $[\sum_{j=1}^n [\varphi]_j \circ \mathcal{T}^j] - [\sum_{j=1}^n \psi_j \circ \mathcal{T}^j] = \mathbf{H}_{n+1} \circ \mathcal{T}^{n+1}$ is uniformly bounded in L^2 if $\alpha < 1/5$, we conclude that $\lim_{n\to\infty} \frac{1}{\sigma_n}\sum_{j=1}^n [\varphi]_j \circ \mathcal{T}^j \to \mathcal{N}(0,1)$ in distribution as well.

We will now verify conditions (i) and (ii) of Theorem 3.3. We defer to the end of this proof the discussion about the possible choices for α and β, see (3.8).

For condition (ii) we begin by noticing that the functions $(\psi_n \circ \mathcal{T}^n)^2$ have a uniformly bounded L^p-norm if the same is true for $(\mathbf{H}_{n+1} \circ \mathcal{T}_{n+1})^2$; this holds provided $1 \le 2p < \frac{1}{2\alpha} - \frac{1}{2}$, and we also need $p > 1$ (for a Hölder inequality, see below). By Minkowski's inequality, $\|(\psi_n \circ \mathcal{T}^n)^2\|_{L^p(m)}$ will therefore be bounded uniformly in n by some constant $\hat{C}_p$. Then we have by Hölder's and Tchebycheff's inequality, where $1/p + 1/q = 1$:

$$\begin{aligned}\sigma_n^{-2}\sum_{k=0}^{n-1}\mathbb{E}[(\psi_k\circ\mathcal{T}^k)^2\mathbf{1}_{\{|\psi_k\circ\mathcal{T}^k)|>\varepsilon\sigma_n\}}] &\le \sigma_n^{-2}\ \sum_{k=0}^{n-1}\|(\psi_k\circ\mathcal{T}^k)^2\|_p m(|\psi_k\circ\mathcal{T}^k|>\varepsilon\sigma_n)^{\frac{1}{q}}\\ &\le \sigma_n^{-2}\sum_{k=0}^{n-1}\|(\psi_k\circ\mathcal{T}^k)^2\|_p\left[\frac{1}{(\varepsilon\sigma_n)^s}\mathbb{E}(|\psi_k\circ\mathcal{T}^k|^s)\right]^{\frac{1}{q}}\\ &\le \sup_k\|\psi_k\circ\mathcal{T}^k\|_{2p}^2\sup_k\|\psi_k\circ\mathcal{T}^k\|_s^{\frac{s}{q}}\frac{n}{\varepsilon^{\frac{s}{q}}\sigma_n^{2+\frac{s}{q}}}\le C\frac{n}{\varepsilon^{\frac{s}{q}}\sigma_n^{2+\frac{s}{q}}}\end{aligned}\tag{3.1}$$

if $1 \le s < \frac{1}{2\alpha} - \frac{1}{2} := \widetilde{s}(\alpha)$. Since $q = (1-1/p)^{-1} > \frac{1-\alpha}{1-5\alpha} := \widetilde{q}(\alpha)$ provided $\alpha < \frac{1}{5}$, the largest value we can use for the exponent of σ_n is $2 + \frac{s}{q} = 2 + \frac{\widetilde{s}(\alpha)-\iota}{\widetilde{q}(\alpha)+\iota}$, for $0 < \iota$ small. If we now assume that the variance grows as $\sigma_n^2 \gtrsim n^\beta$, then we need $\beta > \frac{2(\widetilde{q}(\alpha)+\iota)}{2\widetilde{q}(\alpha)+\widetilde{s}(\alpha)+\iota} := b_\alpha(\iota)$, in order for the upper bound (3.1) to vanish as n tends to infinity. It is easy to check that when $\widetilde{q}(\alpha)$ and $\widetilde{s}(\alpha)$ are positive then the function $\iota \mapsto b_\alpha(\iota)$ is decreasing for $\iota > 0$, so suffices to require that $\beta > b_\alpha(0) = \frac{4\alpha}{1-\alpha}$.

The hard part lies in establishing (i). This is in contrast with the stationary setting where condition (i) is usually a straightforward consequence of the ergodic theorem.

For (i), we first prove that

$$\frac{1}{\sigma_n^2}\sum_{j=1}^n \psi_j^2 \circ \mathcal{T}^j \to 1 \quad \text{in probability as } n \to \infty. \tag{3.2}$$

That (3.2) implies (i) follows from Theorem 3.6.

We follow [15, Lemma 3.3 and proof of Theorem 3.1 (II)], which uses an argument of Peligrad [20]. Since $\psi_j = [\varphi]_j + \mathbf{H}_j - \mathbf{H}_{j+1} \circ T_{j+1}$,

$$\begin{aligned} \psi_j^2 &= [\varphi]_j^2 + 2[\varphi]_j \mathbf{H}_j + \mathbf{H}_j^2 + \mathbf{H}_{j+1}^2 \circ T_{j+1} - 2\mathbf{H}_{j+1} \circ T_{j+1}([\varphi]_j + \mathbf{H}_j) \\ &= [\varphi]_j^2 + 2[\varphi]_j \mathbf{H}_j + \mathbf{H}_j^2 + \mathbf{H}_{j+1}^2 \circ T_{j+1} - 2\mathbf{H}_{j+1} \circ T_{j+1}(\psi_j + \mathbf{H}_{j+1} \circ T_{j+1}) \\ &= [\varphi]_j^2 + (\mathbf{H}_j^2 - \mathbf{H}_{j+1}^2 \circ T_{j+1}) - 2\psi_j \cdot \mathbf{H}_{j+1} \circ T_{j+1} + 2[\varphi]_j \mathbf{H}_j. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{j=1}^n \psi_j^2 \circ \mathcal{T}^j &= \left(\mathbf{H}_1^2 \circ \mathcal{T}_1 - \mathbf{H}_{n+1}^2 \circ \mathcal{T}_{n+1}\right) - \left[\sum_{j=1}^n \psi_j \circ \mathcal{T}^j \cdot \mathbf{H}_{j+1} \circ \mathcal{T}^{j+1}\right] \\ &\quad + \left[\sum_{j=1}^n [\varphi]_j^2 \circ \mathcal{T}^j\right] + 2\left[\sum_{j=1}^n ([\varphi]_j \cdot \mathbf{H}_j) \circ \mathcal{T}^j\right]. \end{aligned}$$

By Corollary 2.8, $\mathbf{H}_n \circ \mathcal{T}^n$ is uniformly bounded in L^2 for $\alpha < \frac{1}{5}$, so $\frac{1}{\sigma_n^2}\mathbf{H}_{n+1}^2 \circ \mathcal{T}^{n+1} \to 0$ in probability.

Next we show that

$$\frac{1}{\sigma_n^2}\left[\sum_{j=1}^n \psi_j \circ \mathcal{T}^j \cdot \mathbf{H}_{j+1} \circ \mathcal{T}^{j+1}\right] \to 0 \qquad \text{in probability.} \tag{3.3}$$

Define

$$\mathbf{H}_j^\varepsilon := \mathbf{H}_j \mathbf{1}_{\{|\mathbf{H}_j| \le \varepsilon\sigma_n\}}.$$

By Lemma 2.2,

$$U_n^2 := \int \left(\sum_{j=1}^{n} [\psi_j \circ \mathcal{T}^j \cdot \mathbf{H}^{\varepsilon}_{j+1} \circ \mathcal{T}^{j+1}] \right)^2 = \int \sum_{j=1}^{n} [\psi_j \circ \mathcal{T}^j \cdot \mathbf{H}^{\varepsilon}_{j+1} \circ \mathcal{T}^{j+1}]^2.$$

Hence, using Lemma 2.1 for the equality in the next computation (note that $\mathbf{H}_k \circ \mathcal{T}^k \in L^2$ if $\alpha < \frac{1}{5}$),

$$\begin{aligned} U_n^2 &\le \varepsilon^2 \sigma_n^2 \sum_{j=1}^{n} \int \psi_j^2 \circ \mathcal{T}^j \\ &= \varepsilon^2 \sigma_n^2 \left[\int (\sum_{j=1}^{n} [\varphi]_j \circ \mathcal{T}^j)^2 + \int \mathbf{H}_1^2 \circ \mathcal{T}^1 - \int \mathbf{H}_{n+1}^2 \circ \mathcal{T}^{n+1} \right] \le \varepsilon^2 \sigma_n^4. \end{aligned} \tag{3.4}$$

For any $a > \varepsilon$ we obtain, using Tchebycheff's inequality in the third and fourth lines below, the inequality (3.4), and that $\mathbf{H}_j \circ \mathcal{T}^j$ is uniformly bounded in L^r by some constant $\hat{D}$ (Corollary 2.8)

$$\begin{aligned} & m \left(\left| \frac{1}{\sigma_n^2} \sum_{j=1}^{n} \psi_j \circ \mathcal{T}^j \cdot \mathbf{H}_{j+1} \circ \mathcal{T}^{j+1} \right| > a \right) \\ \le \;& m \left(\max_{1 \le j \le n} \left| \mathbf{H}_{j+1} \circ \mathcal{T}^{j+1} \right| > \varepsilon \sigma_n \right) + m \left(\left| \frac{1}{\sigma_n^2} \sum_{j=1}^{n} \psi_j \circ \mathcal{T}^j \cdot \mathbf{H}^{\varepsilon}_{j+1} \circ \mathcal{T}^{j+1} \right| > a \right) \\ \le \;& \sum_{j=1}^{n} m(|\mathbf{H}_{j+1} \circ \mathcal{T}^{j+1}| > \varepsilon \sigma_n) + \frac{1}{a^2 \sigma_n^4} U_n^2 \\ \le \;& \frac{n}{(\varepsilon \sigma_n)^r} \left(\max_{1 \le j \le n} \int |\mathbf{H}_{j+1} \circ \mathcal{T}^{j+1}|^r \right) + \frac{\varepsilon^2}{a^2} \le \frac{n \hat{D}^r}{(\varepsilon \sigma_n)^r} + \frac{\varepsilon^2}{a^2}. \end{aligned}$$

Take $a = \sqrt{\varepsilon}$; if we use that $\sigma_n^2 \gtrsim n^{\beta}$, then $\beta > \frac{2}{r}$ with $1 \le r < \frac{1}{2\alpha} - \frac{1}{2}$, that is $\beta > \frac{4\alpha}{1-\alpha}$, allows us to obtain (3.3).

Finally, we show that

$$\frac{1}{\sigma_n^2} \sum_{j=1}^{n} ([\varphi]_j^2 + 2\,[\varphi]_j\, \mathbf{H}_j) \circ \mathcal{T}^j \to 1 \quad \text{in probability.} \tag{3.5}$$

We know from our Strong Borel-Cantelli Theorem 2.10 that

$$\sum_{j=1}^{n} [\varphi]_j^2 \circ \mathcal{T}^j = \sum_{j=1}^{n} \mathbb{E}[[\varphi]_j^2 \circ \mathcal{T}^j] + o(n^{\frac{1}{2}+\varepsilon}) \quad m\text{-a.e.} \tag{3.6}$$

We will show in Lemma 3.4 that

$$\frac{1}{\sigma_n^2}\left(\sum_{j=1}^{n}([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j-\sum_{j=1}^{n}\mathbb{E}[([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j]\right)\to 0 \text{ in probability.} \tag{3.7}$$

In view of Lemma 2.3, equations (3.5) and (3.7) imply $\frac{1}{\sigma_n^2}[\sum_{j=1}^n[\varphi]_j^2\circ\mathcal{T}^j + 2\sum_{j=1}^n([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j]\to 1$ in probability. ∎

Lemma 3.4 *For* $0<\alpha<1/5$ *and the variance growing as* $\sigma_n^2\gtrsim n^\beta$ *with* $\beta>\frac{1}{2(1-2\alpha)}$, *we have*

$$\frac{1}{\sigma_n^2}\left(\sum_{j=1}^{n}([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j-\sum_{j=1}^{n}\mathbb{E}[([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j]\right)\to 0 \text{ in probability.}$$

Proof. Write $S_n=\sum_{j=1}^n([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j$ and $E_n=\sum_{j=1}^n\mathbb{E}[([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j]$ and estimate

$$\begin{aligned}\mathbb{E}(|S_n-E_n|>\sigma_n^2\varepsilon) &= \mathbb{E}(|S_n-E_n|^2>\sigma_n^4\varepsilon^2)\\ &\le \frac{1}{\sigma_n^4\varepsilon^2}\mathbb{E}(|S_n-E_n|^2).\end{aligned}$$

When we expand $\mathbb{E}(|S_n-E_n|^2)$ we have, as usual, the diagonal terms and a double summation of off-diagonal terms:

$$\mathbb{E}(|S_n-E_n|^2)=\sum_{j=1}^{n}\mathbb{E}(([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j-m[([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j)]^2)$$
$$+\,2\sum_{j=1}^{n}\sum_{i=1}^{j-1}\int[([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j-m(([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j)][([\varphi]_i\,\mathbf{H}_i)\circ\mathcal{T}^i-m(([\varphi]_i\,\mathbf{H}_i)\circ\mathcal{T}^i)]dx.$$

The sum of diagonal terms is $O(n)$ as $([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j\in L^2(m)$ with uniformly bounded norm if $\alpha<1/5$. Therefore, if $\sigma_n^2\approx n^\beta$, then the exponent β must verify $\beta>1/2$.

We now consider

$$\sum_{j=1}^{n}\sum_{i=1}^{j-1}\int[([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j - m(([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j)][([\varphi]_i\,\mathbf{H}_i)\circ\mathcal{T}^i - m(([\varphi]_i\,\mathbf{H}_i)\circ\mathcal{T}^i)]dx$$

$$=\sum_{j=1}^{n}\sum_{i=1}^{j-1}\int[[\varphi]_j\,\mathbf{H}_j - m(([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j)]\circ\mathcal{T}^j\cdot[[\varphi]_i\,\mathbf{H}_i - m(([\varphi]_i\,\mathbf{H}_i)\circ\mathcal{T}^i)]\circ\mathcal{T}^i dx$$

$$=\sum_{j=1}^{n}\sum_{i=1}^{j-1}\int[[\varphi]_j\,\mathbf{H}_j - m(([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j)]\circ\mathcal{T}_{i+1}^{j-i}\cdot[[\varphi]_i\,\mathbf{H}_i - m(([\varphi]_i\,\mathbf{H}_i)\circ\mathcal{T}^i)]\cdot\mathcal{P}^i\mathbf{1}\;dx$$

$$=\sum_{j=1}^{n}\sum_{i=1}^{j-1}\int[[\varphi]_j\,\mathbf{H}_j - m(([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j)]\cdot\mathcal{P}_{i+1}^{j-i}[\mathbf{H}_i\,[\varphi]_i\,\mathcal{P}^i\mathbf{1} - m(([\varphi]_i\,\mathbf{H}_i)\circ\mathcal{T}^i)\mathcal{P}^i\mathbf{1}]\;dx.$$

We will prove in Lemma 3.5 below that $\alpha < 1/2$ implies $||\mathcal{P}_{i+1}^{j-i}[\mathcal{P}^i\mathbf{1}\mathbf{H}_i\,[\varphi]_i - \mathcal{P}^i\mathbf{1}m(([\varphi]_i\,\mathbf{H}_i)\mathcal{T}^i)]||_2 \le \frac{C^* i}{(j-i)^{\alpha^*}}$, where C^* is a constant depending only on α and the C^1 norm of φ (and uniform in i and j). Here the numerator i comes about as $1 \le i \le j-1$ and $\alpha^* = \frac{1-2\alpha}{2\alpha}$ follows from the decay Theorem 1.2 and Lemma 2.7. Note also that $||([\varphi]_j\,\mathbf{H}_j) - m(([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j)||_2$ is uniformly bounded in j provided $\alpha < \frac{1}{4}$, see Corollary 2.8.

We have to show that each row summation satisfies

$$|\sum_{i=1}^{j-1}\int[([\varphi]_j\,\mathbf{H}_j) - m(([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j)]\mathcal{P}_{i+1}^{j-i}[\mathcal{P}^i\mathbf{1}\mathbf{H}_i\,[\varphi]_i - \mathcal{P}^i\mathbf{1}m(([\varphi]_i\,\mathbf{H}_i)\circ\mathcal{T}^i)]\;dx| \le j^{\chi}$$

where $n^{1+\chi} = o(\sigma_n^4)$ otherwise the double summation contributes a term which is too large.

So we divide the sum into two parts, with $0 < \delta < 1$

$$\sum_{i=j-j^{\delta}}^{j-1}\int[([\varphi]_j\,\mathbf{H}_j) - m(([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j)]\mathcal{P}_{i+1}^{j-i}[\mathcal{P}^i\mathbf{1}\mathbf{H}_i\,[\varphi]_i - \mathcal{P}^i\mathbf{1}m(([\varphi]_i\,\mathbf{H}_i)\circ\mathcal{T}^i)]\;dx$$

$$+\sum_{i=1}^{j-j^{\delta}}\int[([\varphi]_j\,\mathbf{H}_j) - m(([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j)]\mathcal{P}_{i+1}^{j-i}[\mathcal{P}^i\mathbf{1}\mathbf{H}_i\,[\varphi]_i - \mathcal{P}^i\mathbf{1}m(([\varphi]_i\,\mathbf{H}_i)\circ\mathcal{T}^i)]\;dx.$$

We bound the first sum by $C^* j^{\delta}$ using L^2 bounds without decay. The second sum uses our decay estimate (see Lemma 3.5) and we get $\sum_{i=1}^{j-j^{\delta}}\frac{C^* i}{(j-i)^{\alpha^*}} \le C^* j^{1-(\alpha^*-1)\delta} = C^*\; j^{1+\delta-\alpha^*\delta}$ provided $\alpha^* > 1$ ($\iff$ $0 < \alpha < 1/2$). Then $|\sum_{i=1}^{j-1}\int[([\varphi]_j\,\mathbf{H}_j) - m(([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j)]\mathcal{P}_{i+1}^{j-i}[\mathcal{P}^i\mathbf{1}\mathbf{H}_i\,[\varphi]_i - \mathcal{P}^i\mathbf{1}m(([\varphi]_i\,\mathbf{H}_i)\circ\mathcal{T}^i)]\;dx| \le C(j^{\delta} + j^{1+\delta-\alpha^*\delta})$ which is lowest for

$\delta = 1/\alpha_*$. We obtain

$$|\sum_{j=1}^{n}\sum_{i=1}^{j-1}\int[([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j - m(([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j)][([\varphi]_i\,\mathbf{H}_i)\circ\mathcal{T}^i - m(([\varphi]_i\,\mathbf{H}_i)\circ\mathcal{T}^i)]dx|$$

$$\leq C^*\; n^{1+1/\alpha_*} = C^*\; n^{1/(1-2\alpha)}$$

so

$$E(|S_n - E_n|^2) \leq Cn^{1/(1-2\alpha)}.$$

By dividing for σ_n^4 and asking again for a growth like $\sigma_n^2 \gtrsim n^\beta$ we have now that $\beta > \frac{1}{2(1-2\alpha)}$. This estimate allows us to show that $\frac{1}{\sigma_n^2}\left(\sum_{j=1}^n([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j - \sum_{j=1}^n E[([\varphi]_j\,\mathbf{H}_j)\circ\mathcal{T}^j]\right) \to 0$ in probability. ∎

We now collect the various inequalities involving α and β, which is the scaling of $\sigma_n^2 \gtrsim n^\beta$:

- for our proof of condition (ii) in Brown's Theorem 3.3 we need $\alpha < \frac{1}{5}$ and $\beta > \frac{4\alpha}{1-\alpha}$, $\beta > \frac{1}{2}$;
- in Peligrad's argument we needed $\alpha < \frac{1}{5}$ and $\beta > \frac{4\alpha}{1-\alpha}$;
- in Lemma 3.4, using that $\alpha < \frac{1}{5}$, we have $\beta > \frac{1}{2(1-2\alpha)}$.
- for Theorem 3.6 we use $\beta > \frac{1}{2}$, and $\alpha < \frac{1}{9}$ to obtain a uniform L^4-bound for $\psi_n \circ \mathcal{T}^n = [\varphi]_n \circ \mathcal{T}^n + \mathbf{H}_n \circ \mathcal{T}^n - \mathbf{H}_{n+1} \circ \mathcal{T}^{n+1}$ (see Corollary 2.8).

Therefore, it is sufficient to take

$$0 < \alpha < \frac{1}{9} \text{ and } \beta > \max\left\{\frac{1}{2}, \frac{4\alpha}{1-\alpha}, \frac{1}{2(1-2\alpha)}\right\} = \frac{1}{2(1-2\alpha)}. \tag{3.8}$$

To conclude the proof we need Theorem 3.6 to show that (3.2) implies condition (i) of Brown's Theorem 3.3, and the statement of Lemma 3.5.

Lemma 3.5 *For* $1 \leq p < 1/\alpha$

$$\|\mathcal{P}_k^n\left([\mathcal{P}^i\mathbf{1}\mathbf{H}_i\,[\varphi]_i - \mathcal{P}^i\mathbf{1}m(([\varphi]_i\,\mathbf{H}_i)\circ\mathcal{T}^i)]\right)\|_p \leq i\;C_{\alpha,p}\;C_\varphi\;n^{-\frac{1}{p\alpha}+1}\,(\log n)^{\frac{1}{\alpha}\frac{1-\alpha p}{p-\alpha p}}$$

Proof. See Section 6.2 in the Appendix. ∎

Theorem 3.6 *Assume $\psi_j \circ \mathcal{T}^j$ is uniformly bounded in L^4 and $\sigma_n^2 = \mathbb{E}(\sum_{j=1}^n \psi_j^2 \circ \mathcal{T}^j) + O(1) \gtrsim n^\beta$ with $\beta > \frac{1}{2}$. Then*

$$\frac{1}{\sigma_n^2} \sum_{j=1}^n (\psi_j^2 \circ \mathcal{T}^j - \mathbb{E}[\psi_j^2 \circ \mathcal{T}^j | \mathcal{B}_{j+1}]) \to 0 \quad \textit{in probability.}$$

Proof. Define

$$V_k := \psi_k^2 \circ \mathcal{T}^k - \mathbb{E}[\psi_k^2 \circ \mathcal{T}^k | \mathcal{B}_{k+1}], \qquad T_n := \sum_{j=1}^n V_j.$$

Note that $\mathbb{E}[V_k | \mathcal{B}_{k+1}] = 0$, so V_k is a reverse martingale difference; by Pythagoras, $\mathbb{E}(V_k^2) \leq \mathbb{E}((\psi_k^2 \circ \mathcal{T}^k)^2)$. Applying Pythagoras again,

$$\mathbb{E}\Big[\big(\sum_{j=1}^n V_j\big)^2\Big] = \sum_{j=1}^n \mathbb{E}(V_j^2) \leq \sum_{j=1}^n \mathbb{E}(\psi_j^4 \circ \mathcal{T}^j) \lesssim n$$

therefore, by Tchebycheff

$$P(|T_n| > \sigma_n^2 \varepsilon) = P(|T_n|^2 > \sigma_n^4 \varepsilon^2) \lesssim \frac{n}{\varepsilon^2 \sigma_n^4}$$

Since we assumed that $\sigma_n^2 \gg n^{1/2}$, it follows that

$$\frac{1}{\sigma_n^2} T_n \to 0 \text{ in probability}$$

as claimed. ∎

4 Central Limit Theorem for nearby maps

Theorem 4.1 *Given $\beta \in (0, \frac{1}{9})$ and $\varphi \in C^1([0,1])$, if φ is not a coboundary (up to a constant) for T_β there exists $\varepsilon > 0$ such that for all parameters $\beta_k \in (\beta - \varepsilon, \beta + \varepsilon)$ the variance grows linearly for any sequential system formed from concatenation of the maps T_{β_k}.*

Therefore, by Theorem 3.1, the CLT holds.

Proof.

Recall the quantities defined by a concatenation of different maps:

$$\mathbf{H}_n = \frac{1}{\mathcal{P}^n \mathbf{1}} \left[P_n([\varphi]_{n-1} \mathcal{P}^{n-1} \mathbf{1}) + P_n P_{n-1}([\varphi]_{n-2} \mathcal{P}^{n-2} 1) + \cdots + P_n P_{n-1} \ldots P_1([\varphi]_0 \mathcal{P}^0 \mathbf{1}) \right]$$

and

$$\psi_n := [\varphi]_n + \mathbf{H}_n - \mathbf{H}_{n+1} \circ T_{n+1}.$$

First assume that the maps all coincide with T_β so that $P_\beta^n 1 \to h_\beta$ (at a polynomial rate in L^2), $P_n P_{n-1}...P_{n-k} = P_\beta^k$, where h_β is the invariant density for T_β and P_β is the transfer operator for T_β with respect to Lebesgue measure. Furthermore $[\varphi]_n = \varphi - m(\varphi(T_\beta^n)) \to \varphi - \int \varphi h_\beta dx$. Denote the $\mathbf{H}_n$ corresponding to this situation by $\mathbf{H}_{\beta,n}$.

Note the terms $P_n P_{n-1}...P_{n-j}([\varphi]_{n-j-1} \mathcal{P}^{n-j-1}\mathbf{1})$ decay at a polynomial rate in L^2, $\|P_n P_{n-1}...P_{n-j}([\varphi]_{n-j-1} \mathcal{P}^{n-j-1}\mathbf{1})\|_2 \le \frac{C}{j^\tau}$ for some $\tau > 1$ for $\beta < 1/4$, by Proposition 1.3 and Lemma 2.4. Note that C and τ may be taken as uniform over all T_{β_k} if β_k is close to β.

Combining this with the fact that $P_\beta^n \mathbf{1} \to h_\beta$ in L^2 (and hence $\frac{1}{P_\beta^n \mathbf{1}} \to \frac{1}{h_\beta}$ in L^2 as both h_β and $P_\beta^n \mathbf{1}$ are bounded below by a positive constant[2]), we see that given $\varepsilon > 0$ there exists an N such that for all $n > N$, $\mathbf{H}_{\beta,n} = \frac{1}{h_\beta}[P_\beta(h_\beta \varphi - \int \varphi h_\beta dx) + P_\beta^2(h_\beta \varphi - \int \varphi h_\beta dx) + ... + P_\beta^N(h_\beta \varphi - \int \varphi h_\beta dx)] + \gamma(\beta, n)$ where $\|\gamma(\beta, n)\|_2 < \varepsilon$. We define $G_{\beta,N} = \frac{1}{h_\beta}[P_\beta(h_\beta \varphi - \int \varphi h_\beta dx) + P_\beta^2(h_\beta \varphi - \int \varphi h_\beta dx) + ... + P_\beta^N(h_\beta \varphi - \int \varphi h_\beta dx)]$ so that $\mathbf{H}_{\beta,n} = G_{\beta,N} + \gamma(\beta, n)$.

Now suppose φ is not a coboundary for T_β. Denote by $\widetilde{P}_\beta$ the transfer operator for T_β with respect to the invariant measure $d\mu_\beta = h_\beta dx$. Then $\widetilde{P}_\beta^n(\varphi) = \frac{1}{h_\beta} P_\beta^n(h_\beta \varphi)$ where P_β is the transfer operator for T_β with respect to Lebesgue measure.

Hence $\frac{1}{h_\beta}[P_\beta(h_\beta \varphi - \int \varphi h_\beta dx) + P_\beta^2(h_\beta \varphi - \int \varphi h_\beta dx) + ... + P_\beta^N(h_\beta \varphi - \int \varphi h_\beta dx)] = \sum_{k=1}^N \widetilde{P}_\beta^k[\varphi - \int \varphi d\mu_\beta]$. If φ is not a coboundary then $\sum_{k=1}^\infty \widetilde{P}_\beta^k[\varphi - \int \varphi d\mu_\beta]$ converges to a coboundary $\widetilde{H}_\beta$ so that

$$\varphi = \widetilde{\psi}_\beta + \widetilde{H}_\beta \circ T_\beta - \widetilde{H}_\beta$$

defines a martingale difference sequence $\{\widetilde{\psi}_\beta \circ T_\beta^n\}$, where $\widetilde{\psi}_\beta \neq 0$ in L^2 (as φ is not a coboundary for T_β). Suppose $\|\widetilde{\psi}_\beta\|_2 > \eta$.

Choose N large enough that for all $n > N$, $\|[H_{\beta,n} - H_{\beta,n+1} \circ T_\beta] - [\widetilde{H}_\beta - \widetilde{H}_\beta \circ T_\beta]\|_2 < \frac{\eta}{20}$ and $\|\widetilde{H}_\beta - \sum_{k=1}^N \widetilde{P}_\beta^k[\varphi - \int \varphi d\mu_\beta]\|_2 < \frac{\eta}{20}$. Then $\|\psi(\beta, n)\|_2 > \frac{\eta}{2}$ for all $n > N$.

Now we consider a concatenation of maps T_{β_k} where β_k is close to β. The idea is to break $\mathbf{H}_n$ into a sum of N terms uniformly close to $G(\beta, N)$ (no matter what the sequence of maps) and a small error.

Choose all β_k's sufficiently close to β that when we form a concatenation of the maps

[2] These facts, in particular that $P_\beta^n \mathbf{1}$ are uniformly in n bounded from below by a strictly positive constant, are proved in [19].

T_{β_k} we have

$$\|G_{\beta,N} - \frac{1}{\mathcal{P}^n\mathbf{1}}\big[P_n([\varphi]_{n-1}\,\mathcal{P}^{n-1}\mathbf{1}) + P_nP_{n-1}([\varphi]_{n-2}\,\mathcal{P}^{n-2}\mathbf{1}) + \dots + P_nP_{n-1}\cdots P_{n-N}([\varphi]_{n-N-1}\,\mathcal{P}^{n-N-1}\mathbf{1})\big]\|_2 < \frac{\eta}{20}.$$

We can do this as we have fixed N and the finite terms are continuous in L^2 as $\beta_k \to \beta$, see [17, Theorem 5.1] and Lemmas 2.4, 2.7.

Recall we also have $\|\gamma(\beta, n)\|_2 < \frac{\eta}{20}$ for all $n \geq N$.

Using the uniform contraction (τ and C are uniform for T_β where β is in a small neighborhood of β) we have

$$\|\mathbf{H}_n - \frac{1}{\mathcal{P}^n\mathbf{1}}\big[P_n([\varphi]_{n-1}\,\mathcal{P}^{n-1}\mathbf{1}) + P_nP_{n-1}([\varphi]_{n-2}\,\mathcal{P}^{n-2}\mathbf{1}) + \dots + P_nP_{n-1}\cdots P_{n-N}([\varphi]_{n-N-1}\,\mathcal{P}^{n-N-1}1)\big]\|_2 < \frac{\eta}{20}$$

for all $n > N$. Then $\|\psi_n\|_2 > \frac{\eta}{10}$ for all $n > N$ and we have linear growth of variance for the concatenation of maps as $\sigma_n^2 = \sum_{k=1}^n E[\psi_n \circ \mathcal{T}^k]^2$. ∎

5 Random compositions of intermittent maps

Suppose $S = \{T_{\alpha_1}, \dots, T_{\alpha_\ell}\}$ is a finite number of intermittent type maps as in Section 1, with $\alpha_i < \frac{1}{9}$. We will take an iid selection of maps from S according to a probability vector $p = (p_1, \dots, p_\ell)$ where the probability of choosing map T_{α_i} is p_i. This induces a Bernoulli measure ν on the shift space $\Omega := \{1, \dots, l\}^{\mathbb{N}}$, where $(i_1, i_2, \dots, i_n, \dots)$ corresponds to the sequence of maps: first apply $T_{\alpha_{i_1}}$, then $T_{\alpha_{i_2}}$ and so on. Writing elements of $\omega \in \Omega$ as sequences $\omega := (\omega_0, \omega_1, \dots, \omega_n, \dots)$ the shift operator $S : \Omega \to \Omega$, $(S\omega)_i = \omega_{i+1}$ preserves the measure ν.

This random system also induces a Markov process on $[0, 1]$ with the transition probability function $P(x, A) = \sum_{i=1}^{\ell} p_{\alpha_i} 1_A(T_{\alpha_i}(x))$. A measure μ is invariant for the Markov process if $P^*\mu = \mu$. In this setting Bahsoun and Bose [5] have shown (among other results) that there is a unique absolutely continuous invariant measure μ and that if $\varphi : [0, 1] \to \mathbb{R}$ is a Hölder function then φ satisfies an *annealed* CLT for this random dynamical system in the sense that if $\int \varphi d\ \mu = 0$ then

$$(\nu \times \mu)\{(\omega, x) : \frac{1}{n}\sum_{j=1}^{n} \varphi(T_{(S^j\omega)_0} \dots T_{(\omega)_0}x) \in A\} \to \frac{1}{\sqrt{2\pi\sigma^2}} \int_A e^{-\frac{x^2}{2\sigma^2}} dx$$

for some $\sigma^2 \geq 0$.

In fact the result of Bahsoun and Bose [5] also shows that this convergence is with respect to $(\nu \times m)$ where m is Lebesgue measure on $[0,1]$. This follows from a well known result by Eagleson [10] which states the equivalence of the convergence in distribution for measures which are absolutely continuous with respect to each other.

We will strengthen this to a quenched result: almost every realization of choices of concatenations of maps (with respect to the product measure ν), satisfies a self-norming CLT provided φ is not a coboundary – up to a constant – for all maps (see the precise statement below). First we show that, in this situation, ν almost surely a random composition of a finite number of intermittent type maps has linear growth of the variance. Therefore, we can apply the CLT proven earlier.

Lemma 5.1 *Assume $\alpha_i < \frac{1}{4}$ for all $1 \leq i \leq \ell$ and let*

$$\sigma_n^2(\omega) := \int \left(\sum_{j=1}^{n} \varphi \circ \mathcal{T}_\omega^j - m(\varphi \circ \mathcal{T}_\omega^j) \right)^2 dx$$

where $\mathcal{T}_\omega^j$ stands for $T_{(S^{j-1}\omega)_0} \circ \ldots \circ T_{(\omega)_0}$.

If φ is not a coboundary (up to a constant) for one of the maps, i.e.

> *there exists an i such that $\varphi \neq c + \psi \circ T_{\alpha_i} - \psi$ for any measurable ψ and any constant c*

then for ν-almost every ω there exists a $C > 0$ (independent of ω) and an integer $N(\omega)$ such that $\sigma_n^2(\omega) \geq Cn$ for all $n \geq N(\omega)$.

Remark 5.2 A similar cohomological condition was presented in [4] in the setting of two random commuting toral automorphisms, and conditions on the maps are given under which all $\varphi \neq 0$ have a linear rate of growth of variance.

Proof.

We will assume that φ is not a coboundary (up to constants) for one of the maps, suppose, without loss of generality, that this map is T_{α_1}.

Given any k, for ν-a.e. ω, the sequence of m consecutive applications of the map T_{α_1} will occur in the sequence of composed maps prescribed by ω at a fixed asymptotic frequency of p_1^m.

Now we consider T_{α_1} as a fixed map. T_{α_1} has an absolutely continuous invariant probability measure μ_{α_1} whose density h_{α_1} is in the cone $\mathcal{C}_2$. We let Q_{α_1} denote the transfer operator of T_{α_1} with respect to the invariant measure μ_{α_1}. Then $Q_{\alpha_1}\mathbf{1} = \mathbf{1}$, $P_{\alpha_1}h_{\alpha_1} = h_{\alpha_1}$.

First we construct a martingale decomposition for φ using the transfer operator Q_{α_1} corresponding to the invariant measure μ_{α_1} for T_{α_1}. Note that P_{α_1} is the transfer operator of T_{α_1} with respect to Lebesgue measure m, so the relation between P_{α_1} and Q_{α_1} is $Q_{\alpha_1}(\varphi) = \frac{1}{h_{\alpha_1}}P_{\alpha_1}(h_{\alpha_1}\varphi)$, so $Q^n_{\alpha_1}(\varphi) = \frac{1}{h_{\alpha_1}}P^n_{\alpha_1}(h_{\alpha_1}\varphi)$ for all $n > 0$. Q_{α_1} has the same decay rate as P_{α_1}.

Define

$$H_{\alpha_1} = \sum_{j=1}^{\infty} Q^j_{\alpha_1}[\varphi - \int \varphi d\mu_{\alpha_1}] \qquad \text{and} \qquad \varphi - \int \varphi d\mu_{\alpha_1} = \psi_{\alpha_1} + H_{\alpha_1} - H_{\alpha_1} \circ T_{\alpha_1}.$$

Although it will not be used, but note that $\{\psi_{\alpha_1} \circ T^n_{\alpha_1}\}$ is a reverse martingale difference scheme with respect to μ_{α_1} and the decreasing filtration $\mathcal{F}_n := T^{-n}_{\alpha_1}\mathcal{B}$, where $\mathcal{B}$ is the σ-algebra of Borel sets on $[0, 1]$.

Since $\varphi - \int \varphi d\mu_{\alpha_1} = \psi_{\alpha_1} + H_{\alpha_1} - H_{\alpha_1} \circ T_{\alpha_1}$ and there are no measurable solutions to $\varphi = c + H_{\alpha_1} - H_{\alpha_1} \circ T_{\alpha_1}$ with c constant, the martingale difference function ψ_{α_1} is not zero, so $\|\psi_{\alpha_1}\|_2 > \rho > 0$.[3]

Now we consider the analogous quantities defined by a concatenation of different maps, not just iterates of T_{α_1}. We will use the notation from previous sections, so that $\mathcal{P}^n := P_{\alpha_{i_n}} \circ P_{\alpha_{i_{n-1}}} \circ \cdots \circ P_{\alpha_{i_1}}$ for some sequence $\mathcal{T}^n := T_{\alpha_{i_n}} \circ T_{\alpha_{i_{n-1}}} \circ \cdots \circ T_{\alpha_{i_1}}$ (leaving out the dependence on ω for convenience).

Defining as before

$$\mathbf{H}_n = \frac{1}{\mathcal{P}^n\mathbf{1}}\left[P_n([\varphi]_{n-1}\,\mathcal{P}^{n-1}\mathbf{1}) + P_nP_{n-1}([\varphi]_{n-2}\,\mathcal{P}^{n-2}\mathbf{1}) + \cdots + P_nP_{n-1}\ldots P_1([\varphi]_0\,\mathcal{P}^0\mathbf{1})\right]$$

and $\psi_n := [\varphi]_n + \mathbf{H}_n - \mathbf{H}_{n+1} \circ T_{n+1}$, the sequence $\{\psi_n \circ \mathcal{T}^n\}$ is a reverse martingale difference scheme for m and the decreasing filtration $\{\mathcal{T}^{-n}\mathcal{B}\}$.

Our strategy is to show that if k is sufficiently large (independent of n) then $\|\psi_{n+2k} - \psi_{\alpha_1}\|_2 < \frac{\rho}{2}$ and $\|\psi_{n+2k+1} - \psi_{\alpha_1}\|_2 < \frac{\rho}{2}$ every time that ψ_{n+2k} corresponds to the reverse martingale difference produced by following any sequence of n maps chosen from S by $2k+1$ applications of T_{α_1} (i.e., the last $2k + 1$ maps applied were T_{α_1}).

[3]Unless explicitly stated, L^2 stands for $L^2(m)$, and conditional expectations are with respect to m.

More precisely we will show that $\|\mathbf{H}_{n+2k} - H_{\alpha_1}\|_2 < \frac{\rho}{10}$ and $\|\mathbf{H}_{n+2k+1} - H_{\alpha_1}\|_2 < \frac{\rho}{10}$, which will imply $\|\psi_{n+2k} - \psi_{\alpha_1}\|_2 < \frac{\rho}{2}$ since $\psi_{\alpha_1} - \psi_{n+2k} = [H_{\alpha_1} \circ T_{\alpha_1} - H_{\alpha_1}] - [\mathbf{H}_{n+2k+1} \circ T_{\alpha_1} - \mathbf{H}_{n+2k}]$. The proof that $\|\mathbf{H}_{n+2k+1} - H_{\alpha_1}\|_2 < \frac{\rho}{10}$ is exactly the same as the proof that $\|\mathbf{H}_{n+2k} - H_{\alpha_1}\|_2 < \frac{\rho}{10}$, so we only give details in the latter case. In fact, to simplify notation we consider $2k$ applications of the T_{α_1} after n applications of any sequence of maps from S.

Once we have established this, by Lemma 2.1,

$$\sigma_m^2 \approx \sum_{j=1}^{m} \mathbb{E}[\psi_j^2 \circ \mathcal{T}^j] = \sum_{j=1}^{m} \int \psi_j^2 \cdot \mathcal{P}^j(\mathbf{1}) dm,$$

and hence (since $\mathcal{P}^j(\mathbf{1})$ is bounded away from zero) there is linear growth as for any integer r, r consecutive applications of T_{α_1} will occur with an asymptotic frequency of p_1^r for ν a.e. ω.

To set the stage for our estimates we make the assumption that n maps have been applied followed by $2k$ applications of T_{α_1} and write

$$\mathbf{H}_{n+2k} = \frac{1}{\mathcal{P}^{n+2k}\mathbf{1}}[P_{n+2k}([\varphi]_{n+2k-1}\,\mathcal{P}^{n+2k-1}\mathbf{1}) + P_{n+2k}P_{n+2k-1}([\varphi]_{n+2k-2}\,\mathcal{P}^{n+2k-2}\mathbf{1})$$

$$+ \cdots + P_{n+2k}P_{n+2k-1}\dots P_1([\varphi]_0\,\mathcal{P}^0\mathbf{1})]$$

as

$$H_{n+2k} = A(k,n)[B(k,n) + C(k,n)]$$

where $A(k,n) := \frac{1}{\mathcal{P}^{n+2k}\mathbf{1}}$, $B(k,n) := \sum_{j=0}^{k} P_{n+2k}P_{n+2k-1}\dots P_{n+2k-j}([\varphi]_{n+2k-j-1}\,\mathcal{P}^{n+2k-j-1}\mathbf{1})$ and $C(k,n) := \sum_{j=k+1}^{n+2k-1} P_{n+2k}P_{n+2k-1}\dots P_{n+2k-j}([\varphi]_{n+2k-j-1}\,\mathcal{P}^{n+2k-j-1}\mathbf{1})$.

Recall that $\sum_{j=1}^{k} Q_{\alpha_1}^j[\varphi - \int \varphi d\mu_{\alpha_1}] = \frac{1}{h_{\alpha_1}}[P_{\alpha_1}(h_{\alpha_1}\varphi - h_{\alpha_1}\int \varphi h_{\alpha_1} dx) + P_{\alpha_1}^2(h_{\alpha_1}\varphi - h_{\alpha_1}\int \varphi h_{\alpha_1} dx) + ... + P_{\alpha_1}^k(h_{\alpha_1}\varphi - h_{\alpha_1}\int \varphi h_{\alpha_1} dx)]$ converges to H_{α_1} at a polynomial rate in L^2.

We define $\alpha(k) := \frac{1}{h_{\alpha_1}}$ (which does not actually depend on k), $\beta(k) := P_{\alpha_1}(h_{\alpha_1}\varphi - h_{\alpha_1}\int \varphi h_{\alpha_1} dx) + P_{\alpha_1}^2(h_{\alpha_1}\varphi - h_{\alpha_1}\int \varphi h_{\alpha_1} dx) + ... + P_{\alpha_1}^{k+1}(h_{\alpha_1}\varphi - h_{\alpha_1}\int \varphi h_{\alpha_1} dx)$ and $\gamma(k) := \sum_{j=k+2}^{\infty} P_{\alpha_1}^j[h_{\alpha_1}\varphi - h_{\alpha_1}\int \varphi d\mu_{\alpha_1}]$.

We will show that as k increases, uniformly in n, $\|A(k,n) - \alpha(k)\|_2 \to 0$, $\|B(k,n) - \beta(k)\|_2 \to 0$, $\|C(k,n)\|_2 \to 0$ and $\|\gamma(k)\|_2 \to 0$. As $H_{\alpha_1} = \alpha(k)[\beta(k) + \gamma(k)]$ and $\mathbf{H}_{n+2k} = A(k,n)[B(k,n) + C(k,n)]$ this implies (because $A(k,n)$ and $\alpha(k)$ are uniformly bounded in L^∞) that $\|\mathbf{H}_{n+2k} - H_{\alpha_1}\|_2 \le \frac{\rho}{10}$ for sufficiently large k.

We first consider the terms $A(k,n)$ and $\alpha(k)$.

For any j and any sequence of j maps $T_{\alpha_{i_j}} \circ T_{\alpha_{j-1}} \circ \cdots \circ T_{\alpha_{i_1}}$ chosen from S the corresponding transfer operator with respect to Lebesgue measure $\mathcal{P}^j = P_{\alpha_{i_j}} \circ P_{\alpha_{i_{j-1}}} \circ \cdots \circ P_{\alpha_{i_1}}$ (again, we leave out the dependence on ω for notational convenience) has the property that $\mathcal{P}^j \mathbf{1}$ lies in the cone $\mathcal{C}_2$ and $\int \mathcal{P}^j \mathbf{1} dx = 1$.

Furthermore for any n

$$P_{\alpha_1}^{2k}[h_{\alpha_1} - \mathcal{P}^n \mathbf{1}] \to 0 \text{ as } k \to \infty$$

in L^2 at a uniform polynomial rate, in fact $\|P_{\alpha_1}^{2k}[h_{\alpha_1} - \mathcal{P}^n \mathbf{1}]\|_2 \leq C\frac{1}{(2k)^{1+\eta}}$ where C and η are uniform over $\mathcal{P}^n 1$.

Hence $\frac{1}{P_{\alpha_1}^{2k}\mathcal{P}^n \mathbf{1}} \to \frac{1}{h_{\alpha_1}}$ in L^2 at a polynomial rate as both h_{α_1} and $P_{\alpha_1}^{2k}\mathcal{P}^n \mathbf{1}$ are uniformly bounded below by a positive constant. Thus there exists $C_1 > 0$ such that for all k and n

$$\|\frac{1}{P_{\alpha_1}^{2k}\mathcal{P}^n \mathbf{1}} - \frac{1}{h_{\alpha_1}}\|_2 \leq C_1 \frac{1}{(2k)^{1+\eta}}$$

This is the same as

$$\|A(k,n) - \alpha(k)\|_2 \leq C_1 \frac{1}{(2k)^{1+\eta}}$$

Now we consider $C(k,n)$ and $\gamma(k)$.

The terms $P_n P_{n-1} ... P_{n-j}([\varphi]_{n-j-1}\, \mathcal{P}^{n-j-1}\mathbf{1})$ decay at a polynomial rate in L^2, in fact $\|P_n P_{n-1} ... P_{n-j}([\varphi]_{n-j-1}\, \mathcal{P}^{n-j-1}\mathbf{1})\|_2 \leq \frac{C}{j^{1+\eta}}$. Note that C and η may be taken as uniform over all choices of T_{α_i} in the concatenation. Hence

$$\|C(k,n)\|_2 \leq \frac{C_2}{(2k)^{\delta}}$$

Similarly

$$\|\gamma(k)\|_2 \leq \frac{C_3}{(2k)^{\delta}}$$

Finally we consider the terms $B(k,n)$ and $\beta(k)$. Observe that

$$B(k,n) - \beta(k) = \sum_{j=0}^{k} P_{\alpha_1}^{j+1} \left[([\varphi]_{n+2k-j-1}\, \mathcal{P}^{n+2k-j-1}\mathbf{1}) - (h_{\alpha_1}\varphi - h_{\alpha_1} \int \varphi h_{\alpha_1} dx) \right]$$

where the terms in square brackets have Lebesgue integral zero and are "uniformly" differences of functions in the cone $\mathcal{C}_2$ (see Corollary 2.6). Therefore

$$\left\| P_{\alpha_1}^{j+1} \left[([\varphi]_{n+2k-j-1}\, \mathcal{P}^{n+2k-j-1}\mathbf{1}) - (h_{\alpha_1}\varphi - h_{\alpha_1} \int \varphi h_{\alpha_1} dx) \right] \right\|_2 \leq \frac{C}{j^{1+\delta}}.$$

uniformly over n, k and j.

Hence uniformly over n,

$$\|B(k,n)-\beta(k)]\|_2 \leq \frac{C_2}{k^{1+\delta}}$$

To summarize: we have shown that if we choose k large enough then $\|\mathbf{H}_{n+2k}-H_{\alpha_1}\|_2 < \frac{\rho}{10}$ and $\|\mathbf{H}_{n+2k_1}-H_{\alpha_1}\|_2 < \frac{\rho}{10}$, hence $\|\psi_{n+2k}\|_2 > \frac{\rho}{2}$, whenever, independently of n, the last $2k+1$ maps in the sequence are all T_{α_1}. This implies linear growth in the random composition setting as almost all choices of maps will have $2k+1$ long sequences of the map T_{α_1} at a fixed frequency p_1^{2k+1}. ∎

The next theorem is an immediate consequence of the previous Lemma and Theorem 3.1.

Theorem 5.3 *If $\alpha_i < \frac{1}{9}$ for all $1 \leq i \leq \ell$ and φ is not a coboundary (up to constants) for one of the maps T_{α_i} then $\sigma_n^2(\omega) \geq Cn$ for some $C > 0$ and $n > N(\omega)$, and hence φ satisfies a CLT, for ν almost every sequence ω of maps.*

6 Appendices

6.1 Sprindzuk's Theorem.

We recall the following result, as formulated by W. Schmidt [21, 22] and stated by Sprindzuk [23][4]:

Theorem 6.1 ([23, page 45, Lemma 10]) *Let $(\Omega, \mathcal{B}, \mu)$ be a probability space and let $f_k(\omega)$, $(k = 1, 2, \ldots)$ be a sequence of non-negative μ measurable functions and g_k, h_k be sequences of real numbers such that $0 \leq g_k \leq h_k \leq 1$, $(k = 1, 2, \ldots,)$. Suppose there exists $C > 0$ such that*

$$\int \left(\sum_{m<k\leq n} (f_k(\omega) - g_k) \right)^2 d\mu \leq C \sum_{m<k\leq n} h_k$$

for arbitrary integers $m < n$. Then for any $\varepsilon > 0$

$$\sum_{1\leq k\leq n} f_k(\omega) = \sum_{1\leq k\leq n} g_k + O(\Theta^{1/2}(n)\log^{3/2+\varepsilon}\Theta(n))$$

for μ-a.e. $\omega \in \Omega$, where $\Theta(n) = \sum_{1\leq k\leq n} h_k$.

[4] Quoting Sprindzuk [23]: "The Lemma is abstracted from the work of W. Schmidt, and is based on the idea of the well-known method of Rademacher in the theory of orthogonal series."

6.2 Proof of Lemma 3.5

Proof. For simplicity of notation we discuss only the case $k = 1$; the general case is the same, since we use the n Perron-Frobenius maps in $\mathcal{P}_k^n$ only for the decay given by Theorem 1.2.

The idea is to write $[\mathcal{P}^i \mathbf{1}\mathbf{H}_i [\varphi]_i - \mathcal{P}^i \mathbf{1} m(([\varphi]_i \mathbf{H}_i) \circ \mathcal{T}^i)]$ as a difference of $2i$ functions in the cone of the same integral. By writing explicitely $\mathbf{H}_i$ we get

$$[\mathcal{P}^i \mathbf{1}\mathbf{H}_i [\varphi]_i - \mathcal{P}^i \mathbf{1} m(([\varphi]_i \mathbf{H}_i) \circ \mathcal{T}^i)] = \left[\sum_{k=1}^{i} \prod_{j=0}^{k-1} P_{i-j}([\varphi]_{i-k} \mathcal{P}^{i-k}\mathbf{1}) [\varphi]_i - \mathcal{P}^i \mathbf{1} m(([\varphi]_i \mathbf{H}_i) \circ \mathcal{T}^i)\right]$$

$$= \left[\sum_{k=1}^{i} \prod_{j=0}^{k-1} P_{i-j}([\varphi]_{i-k} \mathcal{P}^{i-k}\mathbf{1}) [\varphi]_i - \mathcal{P}^i \mathbf{1} \sum_{k=1}^{i} m(([\varphi]_i \frac{1}{\mathcal{P}^i \mathbf{1}} \prod_{j=0}^{k-1} P_{i-j}([\varphi]_{i-k} \mathcal{P}^{i-1}\mathbf{1}) \circ \mathcal{T}^i)\right]$$

$$= \sum_{k=1}^{i} \left[[\varphi]_i \mathcal{P}_{i-k+1}^k([\varphi]_{i-k} \mathcal{P}^{i-k}\mathbf{1}) - \mathcal{P}^i \mathbf{1} m(([\varphi]_i \frac{1}{\mathcal{P}^i \mathbf{1}} \mathcal{P}_{i-k+1}^k([\varphi]_{i-k} \mathcal{P}^{i-1}\mathbf{1}) \circ \mathcal{T}^i)\right]$$

Call $C_{k,i} := m(([\varphi]_i \frac{1}{\mathcal{P}^i \mathbf{1}} \mathcal{P}_{i-k+1}^k([\varphi]_{i-k} \mathcal{P}^{i-1} 1) \circ \mathcal{T}^i)$; then consider the quantity

$$(*) := [\varphi]_i \mathcal{P}_{i-k+1}^k([\varphi]_{i-k} \mathcal{P}^{i-k}\mathbf{1}) - \mathcal{P}^i \mathbf{1} C_{k,i}.$$

Since $[\varphi]_{i-k} \in C^1$ and $\mathcal{P}^{i-k}\mathbf{1} \in \mathcal{C}_2$ we can write by Lemma 2.4

$$[\varphi]_{i-k} \mathcal{P}^{i-k}\mathbf{1} = F_{i-k} - G_{i-k}$$

with $F_{i-k}, G_{i-k} \in \mathcal{C}_2$. By the invariance of the cone, the functions $h_{i-k}^{(1)} := \mathcal{P}_{i-k+1}^k F_{i-k}$; $h_{i-k}^{(2)} := \mathcal{P}_{i-k+1}^k G_{i-k}$ are still in the cone, and we rewrite (*) as

$$(*) = [\varphi]_i h_{i-k}^{(1)} - [\varphi]_i h_{i-k}^{(2)} - C_{i,k} \mathcal{P}^i \mathbf{1}.$$

Although the functions (in the cone), F_{i-k}, G_{i-k} are not of zero mean, we can still apply Lemma 2.4 and split the product of $[\varphi]_i$ with them into the differences of two new functions belonging to the cone, namely

$$[\varphi]_i h_{i-k}^{(1)} = M_{i-k}^{(1)} - M_{i-k}^{(2)};\ [\varphi]_i h_{i-k}^{(2)} = N_{i-k}^{(1)} - N_{i-k}^{(2)}$$

with $M_{i-k}^{(1,2)}, N_{i-k}^{(1,2)} \in \mathcal{C}_2$. We finally have

$$(*) = [M_{i-k}^{(1)} + N_{i-k}^{(2)}] - [M_{i-k}^{(2)} + N_{i-k}^{(1)} + C_{i,k} \mathcal{P}^i \mathbf{1}] := R_{i,k} - S_{i,k}$$

where the functions $R_{i,k}, S_{i,k}$ are in the cone and have the same expectation. Before continuing, let us summarize what we got

$$[\mathcal{P}^i \mathbf{1} \mathbf{H}_i [\varphi]_i - \mathcal{P}^i \mathbf{1} m(([\varphi]_i \mathbf{H}_i) \circ \mathcal{T}^i)] = \sum_{k=1}^{i} (R_{i,k} - S_{i,k}).$$

By taking the power $\mathcal{P}^n$ on both sides we have by our Theorem 1.2 on the loss of memory and Proposition 1.3

$$\|\mathcal{P}^n \left([\mathcal{P}^i \mathbf{1} \mathbf{H}_i [\varphi]_i - \mathcal{P}^i \mathbf{1} m(([\varphi]_i \mathbf{H}_i) \circ \mathcal{T}^i)] \right) \|_p \leq \sum_{k=1}^{i} C_{\alpha,p} (\|R_{i,k}\|_1 + \|S_{i,k}\|_1) n^{-\frac{1}{p\alpha}+1} (\log n)^{\frac{1}{\alpha} \frac{1-\alpha p}{p-\alpha p}}.$$

From Lemma 2.4, one observes that if we have $\varphi \in C^1([0,1])$ and $H \in \mathcal{C}_2$ the splitting $\varphi H = A - B$, with $A, B \in \mathcal{C}_2$ is such that the functions A, B depend only on the C^1 norm of φ and the integrals $m(H), m(\varphi H)$. In our case since $[\varphi]_i(x) = \varphi(x) - m(\varphi \circ \mathcal{T}^i)$, we have that $\| [\varphi]_i \|_{C^1} \leq \|\varphi\|_{C^1}$; moreover, at each application of Lemma 2.4, the function H is either $\mathcal{P}^i \mathbf{1}$ or obtained by applying $\mathcal{P}^\ell$ to a function obtained in the previous step and which only depends upon $\|\varphi\|_{C^1}$; in conclusion the norms $\|R_{i,k}\|_1, \|S_{i,k}\|_1$ are bounded by a function C_φ which only depends on the choice of the observable φ. We finally get

$$\|\mathcal{P}^n \left(\mathcal{P}^i \mathbf{1} [\mathbf{H}_i [\varphi]_i - m(([\varphi]_i \mathbf{H}_i) \circ \mathcal{T}^i)] \right) \|_p \leq i \; C_{\alpha,p} \; C_\varphi \; n^{-\frac{1}{p\alpha}+1} (\log n)^{\frac{1}{\alpha} \frac{1-\alpha p}{p-\alpha p}}.$$

∎

Acknowledgments

SV was supported by the ANR-Project *Perturbations* and by the PICS (Projet International de Coopération Scientifique), *Propriétés statistiques des systèmes dynamiques déterministes et aléatoires*, with the University of Houston, n. PICS05968. SV thanks the University of Houston for supporting his visits during the preparation of this work. SV thanks the Leverhulme Trust for support thorough the Network Grant IN-2014-021, and for useful conversations with S. Galatolo and W. Bahsoun. MN was supported by NSF grant DMS 1101315 and Simons Foundation Collaboration Grant Number 349664. AT was partially supported by the Simons Foundation grant 239583. The authors warmly thank R. Aimino for useful suggestions concerning Theorem 2.10. All three authors are grateful for the support received from the Erwin Schrödinger Institute in Vienna during their May 2016 visit.

References

[1] R. Aimino, *Vitesse de mélange et théorèmes limites pour les systèmes dynamiques aléatoires et non-autonomes*, Ph. D. Thesis, Université de Toulon, (2014)

[2] R. Aimino, M. Nicol and S. Vaienti. *Annealed and quenched limit theorems for random expanding dynamical systems*, Probability Theory and Related Fields, 162 (1), 233-274, (2015).

[3] R. Aimino, H. Hu, M. Nicol, A. Török and S. Vaienti. *Polynomial loss of memory for maps of the interval with a neutral fixed point*, Discrete and Continuous Dynamical Systems, A, **35**, 3, 793-806 (2015)

[4] A. Ayyer, C. Liverani and M. Stenlund. *Quenched CLT for random toral automorphism*, Discrete Contin. Dyn. Syst. 24 , no. 2, 331?348, (2009).

[5] W. Bahsoun and C. Bose. *Mixing Rates and Limit Theorems for Random Intermittent Maps.* Preprint.

[6] B. M. Brown. *Martingale central limit theorems*, Ann. Math. Statist. **42** (1971), 59–66.

[7] J-P Conze and A. Raugi. *Limit theorems for sequential expanding dynamical systems on [0,1]* Ergodic theory and related fields, **89-121**, Contemp. Math., 430, Amer. Math. Soc., Providence, RI, 2007.

[8] C. Cuny and F. Merlevède. *Strong invariance principles with rate for "reverse" martingales and applications* J. Theoret. Probab., **28** 2015, 137–183.

[9] P. Hall and C. C. Heyde, *Martingale limit theory and its application*, Probability and Mathematical Statistics, Academic Press, 1980, New York.

[10] G. K. Eagleson, *Some simple conditions for limit theorems to be mixing*, Teor.Verojatnost. i Primenen. 21 (1976) 653-660. (English translation: Theor. Prob. Appl. 21 (1976) 637-642, 1977.

[11] N. Haydn, *Convergence of the transfer operator for rational maps*, Erg. Th. Dyn. Sys **19(3)** (1999), 657–669.

[12] N. Haydn, Y. Lacroix and S. Vaienti, *Hitting and return time statistics in ergodic dynamical systems*, Ann. Probab., **33** (2005), 2043–2050.

[13] N. Haydn, M. Nicol, T. Persson and S. Vaienti. *A note on Borel–Cantelli lemmas for non-uniformly hyperbolic dynamical systems*, Erg. Th. Dyn. Syst., to appear

[14] N. Haydn, M. Nicol, A. Török, S. Vaienti, *Almost sure invariance principle for sequential and non-stationary dynamical systems*, submitted, to appear on Trans. Amer Math. Soc., arXiv:1406.4266

[15] N. Haydn, M. Nicol, S. Vaienti, L. Zhang, *Central limit theorems for the shrinking target problem*, Journal of Statistical Physics, **153**, (2013), p. 864-887.

[16] H. Hennion, L. Hervé, *Limit theorems for Markov chains and stochastic properties of dynamical systems by quasicompactness*, Lect. Notes in Math., **1766**, (2001), Springer-Verlag

[17] J. Leppänen, M. Stenlund, *Quasistatic dynamics with intermittency*, arXiv:1510.02748v1

[18] N. Dobbs and M. Stenlund. *Quasistatic dynamical systems*, arXiv:1504.01926

[19] C. Liverani, B. Saussol and S. Vaienti, *A probabilistic approach to intermittency*, Ergodic Theory Dynam. Systems **19(3)** (1999), 671–685.

[20] M. Peligrad, *Central limit theorem for triangular arrays of non-homogeneous Markov chains* To appear in Prob. Theory and Related Fields.

[21] W. Schmidt, *A metrical theory in diophantine approximation*, Canad. J. Math, **12** (1960), 619–631.

[22] W. Schmidt, *Metrical theorems on fractional parts of sequences*, Trans. Amer. Math. Soc., **110** (1964), 493–518.

[23] Vladimir G. Sprindzuk, *Metric theory of Diophantine approximations*, V. H. Winston and Sons, Washington, D.C., 1979, Translated from the Russian and edited by Richard A. Silverman, With a foreword by Donald J. Newman, Scripta Series in Mathematics.