A SPECTRAL APPROACH FOR QUENCHED LIMIT THEOREMS FOR RANDOM HYPERBOLIC DYNAMICAL SYSTEMS

ABSTRACT. We extend the recent spectral approach for quenched limit theorems developed for piecewise expanding dynamics under general random driving [11] to quenched random piecewise hyperbolic dynamics.

For general ergodic sequences of maps in a neighbourhood of a hyperbolic map we prove a quenched large deviations principle (LDP), central limit theorem (CLT), and local central limit theorem (LCLT).

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1. Introduction

In our previous paper [11] we extended the Nagaev-Guivarc'h spectral method to obtain limit theorems, such as the Central Limit Theorem (CLT), the Large Deviation Principle (LDP) and the Local Central Limit Theorem (LCLT), for random dynamical systems governed by a *cocycle* of maps $T_\omega^{(n)} := T_{\sigma^{n-1}\omega} \circ \cdots \circ T_{\sigma\omega} \circ T_\omega$, assuming uniform-in-$\omega$ eventual expansivity conditions on the maps T_ω. The random driving was a general ergodic, invertible transformation $\sigma : \Omega \circlearrowleft$ on a probability space $(\Omega, \mathbb{P})$, and the real observable g was defined on the product space $\Omega \times X \to \mathbb{R}$.

Before introducing our new results, we briefly recap the essence of the Nagaev-Guivarc'h spectral method in the deterministic setting, where one deals with a single map T, deferring to the original articles by Nagaev [27, 28] and Guivarc'h [30, 18] and to the excellent survey [16] for more details. The spectral method uses the transfer operator $\mathcal{L} : \mathcal{B} \circlearrowleft$ acting on a Banach space $\mathcal{B}$, and in particular, the *twisted* transfer operator $\mathcal{L}^\theta f := \mathcal{L}(e^{\theta g} f)$, for f and $g \in \mathcal{B}$. In the situation where $\mathcal{L}^\theta$ is quasi-compact for θ near zero, regularity of the leading eigenvalues and eigenprojectors have been used to prove limit theorems [21, 20, 29, 30, 21, 3, 30, 26, 20, 31, 14] and more, namely Berry-Esseen theorems [18, 14] and almost-sure invariance principles [15]. The key equality was $\mathbb{E}(e^{\theta S_n g} f) = \mathbb{E}((\mathcal{L}^\theta)^n f)$, where $S_n g$ denotes the Birhkoff sum of the observable g and the expectation is taken with respect to the unique eigenmeasure m of the adjoint of $\mathcal{L}$. Since the map $\theta \mapsto \mathcal{L}^\theta$ is holomorphic, classical perturbation theory allows one to obtain $\mathbb{E}(e^{\theta S_n g}) = c(\theta)\lambda(\theta)^n + d_n(\theta)$, where $\lambda(\theta)$ is the leading eigenvalue of $\mathcal{L}^\theta$, with c and λ analytic in θ, and $\sup_\theta |d_n(\theta)| \to 0$. We can therefore easily compute the characteristic function and the log generating function of the process $g \circ T^n$ with respect to the invariant probability measure of T which can be identified as the unique eigenvector of $\mathcal{L}$ corresponding to the leading eigenvalue 1.

In the quenched random setting we must replace the n-th power of the twisted operator with the twisted transfer operator cocycle $\mathcal{L}_\omega^{\theta,(n)} := \mathcal{L}^\theta_{\sigma^{n-1}\omega} \circ \cdots \circ \mathcal{L}^\theta_{\sigma\omega} \circ \mathcal{L}^\theta_\omega$. By using the multiplicative ergodic theorem adapted to the study of such cocycles and generalizing a theorem of Hennion and Hérve [20] to the random setting, we were able in our previous paper [11] to show that the cocycle $\mathcal{L}_\omega^{\theta,(n)}$ is quasi-compact for θ near to 0. We therefore thus obtained that for such values of θ and for $\mathbb{P}$-a.e. $\omega \in \Omega$, the top Lyapunov exponent $\Lambda(\theta)$ (analogous to the logarithm of $\lambda(\theta)$ in the deterministic setting) of the cocycle is analytic and given by

$$\lim_{n\to\infty} \frac{1}{n} \log |\mathbb{E}_{\mu_\omega}(e^{\theta S_n g(\omega,\cdot)})| = \Lambda(\theta),$$

where μ_ω is the equivariant probability measure on the ω-fiber (see below). This result together with the exponential decay of the norm of the elements in the complement of the top Oseledets space, which handled the error corresponding to quantity d_n above, allowed us to achieve the desired limit theorems.

In the present paper we move from cocycles of piecewise expanding maps to cocycles of hyperbolic maps both smooth and piecewise smooth. To our knowledge, this is the

first time that this setting has been investigated with multiplicative ergodic theory tools. One of the primary differences with [11] is the use of anisotropic Banach spaces here in place of the space of functions of bounded variation in [11]. Specifically, in the smooth hyperbolic setting and in any dimension, we use the functional analytic setup of Gouëzel and Liverani [17], and in the piecewise hyperbolic case in dimension two we use the spaces from Demers and Liverani [7] (as well as Demers and Zhang [8, 9]). This increased technicality in the underlying spaces necessitates a certain amount of checking of relevant conditions, however, we wish to highlight the fact that a wholesale change of the theory of [11] is not required, which demonstrates the power and flexibility of our approach. The use of transfer operators in the study of statistical properties and limit theorems for hyperbolic dynamical systems has flourished in the last years, and [2] presents a thorough discussion of the various spaces that have been used in the literature. Our intention in this work has not been to find the most general version of the results, but rather to illustrate the applicability of the methods. In fact, we expect the methods presented here to remain applicable in some (or all) of these functional analytic scenarios.

We first consider cocycles $T_\omega^{(n)}$ where the family of maps $\{T_\omega\}_{\omega\in\Omega}$ are selected from a C^{r+1}-neighbourhood of a topologically transitive Anosov map T of class C^{r+1} (in Section 10 we consider piecewise hyperbolic maps also describing periodic Lorentz gas). The random driving $\sigma : \Omega \circlearrowleft$ is a general (ergodic, invertible) automorphism preserving a probability measure $\mathbb{P}$. If $d_{C^{r+1}}(T_\omega, T) < \Delta$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ and Δ is sufficiently small [5] the random dynamical system generated by the cocycle $T_\omega^{(n)}$ supports a measure μ, invariant under the skew product $\tau(\omega, x) = (\sigma\omega, T_\omega x)$. We obtain this measure by explicitly constructing the family μ_ω along the marginal $\mathbb{P}$, namely $\mu = \int_\Omega \mu_\omega \, d\mathbb{P}(\omega)$, and satisfying the usual equivariance condition $\mu_\omega \circ T_\omega^{-1} = \mu_{\sigma\omega}$. Our observable g satisfies $g(\omega, \cdot) \in C^r$ for $\mathbb{P}$-a.e. ω, ess $\sup_{\omega\in\Omega} \|g(\omega, \cdot)\|_{C^r} < \infty$, and is fiberwise centred: $\int_X g(\omega, x)\, d\mu_\omega(x) = 0$ for $\mathbb{P}$-a.e. ω. Our limit theorems concern random Birkhoff sums

$$S_n g(\omega, x) := \sum_{i=0}^{n-1} g(\tau^i(\omega, x)) = \sum_{i=0}^{n-1} g(\sigma^i, T_\omega^{(i)} x), \quad (\omega, x) \in \Omega \times X, n \in \mathbb{N}. \tag{1}$$

We now summarize our main results for the perturbed Anosov systems described above; we defer to the main body for more precise statements.

- **A: Quenched large deviations theorem** We can find $\epsilon_0 > 0$ and a non-random function $c\colon (-\epsilon_0, \epsilon_0) \to \mathbb{R}$ which is nonnegative, continuous, strictly convex, vanishing only at 0 and such that

  $$\lim_{n\to\infty} \frac{1}{n} \log \mu_\omega(S_n g(\omega, \cdot) > n\epsilon) = -c(\epsilon), \quad \text{for } 0 < \epsilon < \epsilon_0 \text{ and } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

- **B: Quenched central limit theorem** There exists a positive variance Σ^2 such that for every bounded and continuous function $\phi\colon \mathbb{R} \to \mathbb{R}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, we have

  $$\lim_{n\to\infty} \int \phi\left(\frac{S_n g(\omega, x)}{\sqrt{n}}\right) d\mu_\omega(x) = \int \phi \, d\mathcal{N}(0, \Sigma^2).$$

One of the main achievements of our previous paper was the proof of the local central limit theorem (LCLT) in the non-arithmetic and arithmetic cases. Our basic assumption, which for convenience we simply call (L), expresses the exponential decay of the strong norm of the twisted operator when the parameter $\theta = it$ has $t \neq 0$. Moreover we showed

[5] The neighborhood of a given map T will be precisely quantified, and therefore the value of Δ as well, as $\mathcal{O}_{\delta_0}(T, \mathcal{B}^{1,1})$ in section 3 for the Anosov case and as $\mathcal{O}_{\delta_0}(T, \mathcal{B})$ in section 10.2 for the piecewise hyperbolic case.

under additional assumptions that we will recall in section 8, that hypothesis (L) was equivalent to a co-boundary condition which is better known as the *aperiodicity condition*. In the present paper we prove the LCLT in the non-arithmetic case by assuming (L). Recently Hafouta and Kifer [19] proposed a new set of assumptions which allow us to check condition (L). We will see that some of these assumptions can be verified easily for our systems, provided we restrict the class of the driving maps and observables (see Corollary 9.5).

- **C: Quenched local central limit theorem** Let us suppose condition (L) holds; then , for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every bounded interval $J \subset \mathbb{R}$, we have

$$\lim_{n\to\infty} \sup_{s\in\mathbb{R}} \left| \Sigma\sqrt{n}\mu_\omega(s + S_n g(\omega,\cdot) \in J) - \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2n\Sigma^2}} |J| \right| = 0.$$

In Section 10 we consider random cocycles of *piecewise* hyperbolic maps of the type considered in [7] on two-dimensional compact Riemannian manifolds. As we will explain later on and in order to apply the multiplicative ergodic theorem, we have now less choice for the random distribution of the maps, but for instance we can deal with countably many maps. All the preceding theorems **A, B, C** still hold.

Apart from [11] there are some quenched limit theorems (LDP and CLT) that have been obtained using different methods. Kifer derives a large deviation principle [22, 23, 24] for occupational measures using theory of equilibrium states, and a central limit theorem via martingale methods; in both cases, he treats random subshifts of finite type and random smooth expanding maps. Recently, Hafouta and Kifer [19] proved limit theorems for these systems in the more general "nonconventional setting". They used (complex) cone techniques, where the cones were defined in the functional space upon which the transfer operator acts. We emphasize that they don't consider the case of hyperbolic dynamics studied in the present paper. In fact, is not clear if their cone techniques can be adapted to the present setting. Bakhtin [1] is probably the closest to our work; he proves a central limit theorem and large deviation estimates for mixing sequences of smooth hyperbolic maps with common expanding and contracting directions, under a variance growth condition on the Birkhoff sums. He also used cones, but living on the tangent space of the manifold. In comparison to Bakhtin, we can additionally treat the case of random piecewise hyperbolic maps, and moreover we exhibit explicitly the rate function which produces asymptotic large deviation bounds; the local CLT is also new in this setting.

2. Preliminaries

Let X be a d-dimensional C^∞ compact connected Riemannian manifold and let T be a topologically transitive Anosov map of class C^{r+1}, where $r > 2$. We follow the setup of [17]. Replacing the Riemannian metric by an adapted metric [25], we use hyperbolicity constants $0 < \nu < 1 < \lambda$, where λ is less than the minimal expansion along the unstable directions, ν is greater than the minimal contraction along the stable directions, and the angles between the stable and unstable spaces (of dimensions d_s, d_u, respectively) are close to $\pi/2$. A collection of C^∞ coordinate charts $\psi_i : (-r_i, r_i)^d \to X$, $i = 1, \ldots, N$ are defined so that $\bigcup_{i=1}^N \psi_i((-r_i/2, r_i/2)^d)$ cover X, with the r_i small enough that $D\psi_i(0) \cdot (\mathbb{R}^{d_s} \times \{0\}) = E^s(\psi_i(0))$, $|\psi_i|_{C^{r+1}}, |\psi_i^{-1}|_{C^{r+1}} \leq 1 + \kappa$, and κ small enough in such a way that the stable cone at x in $\mathbb{R}^d$ is compatibly mapped to the stable cone at $\psi_i(x)$ in X. For such values of κ, the stable cone at $x \in X$ is defined as $\mathcal{C}(x) = \{u + v \in T_x X | u \in E^s(x), v \perp E^s(x), ||v|| \leq \kappa ||u||\}$, where $T_x X$ denotes the tangent space at x and $||\cdot||$ is

the (Mather) adapted metric on X. With this norm $DT^{-1}(x)$ expands the vectors in $\mathcal{C}(x)$ by ν^{-1}, (see [17], section 3 for details).
Let $G_i(K)$ denote the set of graphs of C^{r+1} functions $\chi : (-r_i, r_i)^{d_s} \to (-r_i, r_i)^{d_u}$ with $|\chi|_{C^{r+1}} \leq K$ (and with $|D\chi| \leq c_i$ so that the tangent space of the graph belongs to the stable cone in $\mathbb{R}^d$ mentioned above). For large enough K, the coordinate map $\psi_j^{-1} \circ T^{-1} \circ \psi_i$ maps $G_i(K)$ into $G_j(K')$ for some $K' < K$. For A sufficiently large, (depending on κ and ν) and δ small enough that $A\delta < \min_i r_i/6$, an admissible graph is a map $\chi : \bar{B}(x, A\delta) \to (-2r_i/3, 2r_i/3)^{d_u}$, with range $(\mathbf{Id}, \chi) \in G_i(K)$, where $\bar{B}(x, A\delta)$ denotes some ball included in $(-2r_i/3, 2r_i/3)^{d_s}$; the collection of admissible graphs is denoted Ξ_i.

For $p \in \mathbb{N}$, $p \leq r$, $q \geq 0$ and $h \in C^r(X, \mathbb{C})$, $\varphi \in C^q(X, \mathbb{C})$ we define (using the notation in [17])

$$\|h\|_{p,q}^{\sim} := \sup_{\substack{|\alpha|=p \\ 1\leq i\leq N}} \sup_{\substack{\chi\colon \overline{B}(x,A\delta)\to\mathbb{R}^{d_u} \\ \chi\in\Xi_i}} \sup_{\substack{\varphi\in C_0^q(\overline{B}(x,\delta),\mathbb{C}) \\ |\varphi|_{C^q}\leq 1}} \left| \int_{B(x,\delta)} \left[\partial^\alpha (h \circ \psi_i)\right] \circ (\mathrm{Id}, \chi) \cdot \varphi \right|. \tag{2}$$

Finally, for p and q as above satisfying $p + q < r$, we set

$$\|h\|_{p,q} := \sup_{0\leq k\leq p} \|h\|_{k,q+k}^{\sim} = \sup_{p'\leq p, q'\geq q+p'} \|h\|_{p',q'}^{\sim}. \tag{3}$$

The space $\mathcal{B}^{p,q}$ is defined to be the completion of $C^r(X, \mathbb{C})$ with respect to the norm $\|\cdot\|_{p,q}$.
The following proposition will be useful when applying the multiplicative ergodic theorem.

Proposition 2.1. *The space $\mathcal{B}^{p,q}$ is separable.*

Proof. The desired conclusion follows directly from [17, Remark 4.3] after we note that $C^\infty(X, \mathbb{C})$ has a countable subset which is dense with respect to the C^r norm. □

We recall from [17, Section 4] that the elements of $\mathcal{B}^{p,q}$ are distributions of order at most q. More precisely, there exists $C > 0$ such that any $h \in \mathcal{B}^{p,q}$ induces a linear functional $\varphi \to h(\varphi)$ with the property that

$$|h(\varphi)| \leq C\|h\|_{p,q}\|\varphi\|_{C^q}, \quad \text{for } \varphi \in C^q(X, \mathbb{C}). \tag{4}$$

In particular, for $h \in C^r$ we have that

$$h(\varphi) = \int_X h\varphi \, dm, \quad \text{for } \varphi \in C^q(X, \mathbb{C}), \tag{5}$$

where m denotes the Lebesgue measure on X. We say that $h \in \mathcal{B}^{p,q}$ is nonnegative and write $h \geq 0$ if $h(\varphi) \geq 0$ for any $\varphi \in C^q(X, \mathbb{R})$ such that $\varphi \geq 0$.

Let $\mathcal{L}_T \colon \mathcal{B}^{p,q} \to \mathcal{B}^{p,q}$ be the transfer operator associated to T defined by

$$(\mathcal{L}_T h)(\varphi) = h(\varphi \circ T), \quad \text{for } h \in \mathcal{B}^{p,q} \text{ and } \varphi \in C^q(X, \mathbb{C}). \tag{6}$$

We recall that for $h \in C^r(X, \mathbb{C})$, $\mathcal{L}_T$ is the function given by

$$\mathcal{L}_T h = \left(\frac{h}{|\det T|}\right) \circ T^{-1}. \tag{7}$$

Take $g \in C^r(X, \mathbb{C})$ and $h \in \mathcal{B}^{p,q}$. Then, there exists a sequence $(h_n)_n \subset C^r(X, \mathbb{C})$ that converges to h in $\mathcal{B}^{p,q}$. It follows that $(gh_n)_n \subset C^r(X, \mathbb{C})$ is a Cauchy sequence in $\mathcal{B}^{p,q}$ and therefore it converges to some element of $\mathcal{B}^{p,q}$ which we denote by $g \cdot h$. It is straightforward to verify that the above construction does not depend on the particular choice of the sequence $(h_n)_n$. Moreover, the action of $g \cdot h$ as a distribution is given by

$$(g \cdot h)(\varphi) = h(g\varphi), \quad \varphi \in C^q(X, \mathbb{C}). \tag{8}$$

Moreover, one can easily verify that there exists $C > 0$ such that

$$\|g \cdot h\|_{p,q} \leq C\|g\|_{C^r} \cdot \|h\|_{p,q} \quad \text{for } g \in C^r(X, \mathbb{C}) \text{ and } h \in \mathcal{B}^{p,q}. \tag{9}$$

We will need the following result.

Lemma 2.2. *For $h \in \mathcal{B}^{p,q}, g \in C^r(X, \mathbb{C})$ one has $\mathcal{L}_T(g \circ T \cdot h) = g \cdot \mathcal{L}_T h$.*

Proof. Let $\varphi \in C^q(X, \mathbb{C})$. It follows from (6) and (8) that $[\mathcal{L}_T(g \circ T \cdot h)](\varphi) = (g \circ T \cdot h)(\varphi \circ T) = h(g \circ T \cdot \varphi \circ T) = \mathcal{L}_T h(g \cdot \varphi) = (g \cdot \mathcal{L}_T h)(\varphi)$, which yields the desired result. □

3. Building the cocycle $\mathcal{L}$

In the sequel we will consider the case $p = q = 1$ and $r > 2$, but we will also require T to be C^{r+1}, to be in a suitable framework for perturbations. Using the fact that the unit ball in $\mathcal{B}^{1,1}$ is relatively compact in $\mathcal{B}^{0,2}$ [17, Lemma 2.1], it follows from [17, Theorem 2.3] that the associated transfer operator $\mathcal{L}_T$ is quasicompact on $\mathcal{B}^{1,1}$, 1 is a simple eigenvalue and there are no other eigenvalues of modulus 1. This in particular implies (using the terminology as in [6, Definition 2.6]) that $\mathcal{L}_T$ is exact in $\{h \in \mathcal{B}^{1,1} : h(1) = 0\}$. Let

$$\mathcal{M}_\epsilon(T) = \{S : \ S \text{ is an Anosov map of class } C^{r+1} \text{ satisfying } d_{C^{r+1}}(S, T) < \epsilon\}.$$

We also recall (see [17, Lemmas 2.1. and 2.2] and the discussion at the beginning of §7 [17]) that there exist $\epsilon, B > 0$ and $a \in (0, 1)$ such that for any $T_1, \ldots, T_n \in \mathcal{M}_\epsilon(T)$, we have

- for each $n \in \mathbb{N}$ and $h \in \mathcal{B}^{1,1}$,

$$\|\mathcal{L}_{T_n} \circ \cdots \circ \mathcal{L}_{T_1} h\|_{0,2} \leq B\|h\|_{0,2}; \tag{10}$$

- for each $n \in \mathbb{N}$ and $h \in \mathcal{B}^{1,1}$,

$$\|\mathcal{L}_{T_n} \circ \cdots \circ \mathcal{L}_{T_1} h\|_{1,1} \leq Ba^n\|h\|_{1,1} + B\|h\|_{0,2}. \tag{11}$$

For $\delta > 0$, set

$$\mathcal{O}_\delta(T, \mathcal{B}^{1,1}) = \left\{\mathcal{L}_S : \mathcal{B}^{1,1} \to \mathcal{B}^{1,1} : S \in \mathcal{M}_\epsilon(T) \text{ and } \sup_{\|h\|_{1,1} \leq 1} \|\mathcal{L}_S h - \mathcal{L}_T h\|_{0,2} \leq \delta\right\}.$$

It follows from [6, Proposition 2.10] (applied to the case where $\|\cdot\| = \|\cdot\|_{0,2}$ and $|\cdot|_v = \|\cdot\|_{1,1}$) that there exist $\delta_0 > 0$, $D, \lambda > 0$ such that for any $\mathcal{L}_{T_1}, \ldots, \mathcal{L}_{T_n} \in \mathcal{O}_{\delta_0}(T, \mathcal{B}^{1,1})$, we have that

$$\|\mathcal{L}_{T_n} \circ \cdots \circ \mathcal{L}_{T_2} \circ \mathcal{L}_{T_1} h\|_{1,1} \leq De^{-\lambda n}\|h\|_{1,1} \quad \text{for } h \in \mathcal{B}^{1,1} \text{ satisfying } h(1) = 0. \tag{12}$$

On the other hand, [17, Lemma 7.1] implies that there exist $0 < \epsilon_0 \leq \epsilon$ such that

$$\{\mathcal{L}_S : S \in \mathcal{M}_{\epsilon_0}(T)\} \subset \mathcal{O}_{\delta_0}(T, \mathcal{B}^{1,1}).$$

Remark 3.1. *In order to apply* [6, Proposition 2.10] *instead of* (11) *we need to have that*

$$\|\mathcal{L}_{T_n} \circ \cdots \circ \mathcal{L}_{T_1} h\|_{1,1} \leq a^n\|h\|_{1,1} + B\|h\|_{0,2}.$$

However, as pointed out to us by J-P. Conze [5]*, the arguments from* [6] *can be easily modified in a way that the conclusion given by* [6, Proposition 2.10] *still holds true in our case* [6]

[6]The proof of Proposition 2.7 in [6] on which the result of Proposition 2.10 in [6] is based, was done by taking our $B = 1$. In order to make comparison with that proof we establish the correspondences among our quantities and those in [6]. Therefore our a is identified with ρ and our $\|h\|_{1,1}$ with $|f|_v$. Then for $B > 1$ it will be enough to multiply by B: (i) the exponential factors a^n in the quantities named $\beta(n)$ and $\beta_q(n)$ in [6] and (ii) the norm $\|h\|_{1,1}$, for $n = 0$. Then the proofs goes exactly in the same way and at the end our factor D will depend on B as well.

We now build the cocycle $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{B}^{1,1}, \mathcal{L})$, simply referred to as $\mathcal{L}$, as follows:

(1) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω is a Borel subset of a separable, complete metric space and $\sigma : \Omega \to \Omega$ an ergodic, invertible $\mathbb{P}$-preserving transformation.

(2) Let $\mathcal{T} : \Omega \to \mathcal{M}_{\epsilon_0}(T)$ be a measurable map given by $\omega \mapsto T_\omega$.

3.1. **Strong measurability of $\omega \mapsto \mathcal{L}_\omega$.** In this section we demonstrate strong measurability of the map $\mathcal{L} : \Omega \to \mathcal{O}_{\delta_0}(T, \mathcal{B}^{1,1})$ given by $\omega \mapsto \mathcal{L}_\omega := \mathcal{L}_{T_\omega}$; this is required to establish the existence of measurable Oseledets spaces for the cocycle. To prove strong measurability of $\omega \mapsto \mathcal{L}_\omega := \mathcal{L}_{T_\omega}$, we will show that the map from $\mathcal{M}_{\epsilon_0}(T)$ to the space of all bounded linear operators on $\mathcal{B}^{1,1}$ defined by $S \mapsto \mathcal{L}_S$ is strongly continuous. For this, let $S \in \mathcal{M}_{\epsilon_0}(T)$ and $h \in \mathcal{B}^{1,1}$. We must show that $\|\mathcal{L}_{\tilde{S}}h - \mathcal{L}_S h\|_{1,1} \to 0$ as $d_{C^{r+1}}(\tilde{S}, S) \to 0$. First, assume $h \in C^r$. Then, we need to estimate differences of the form

$$\Big| \int_{B(x,\delta)} \big[\partial^\alpha (\mathcal{L}_S h \circ \psi_i)\big] \circ (\mathrm{Id}, \chi) \cdot \varphi - \int_{B(x,\delta)} \big[\partial^\alpha (\mathcal{L}_{\tilde{S}} h \circ \psi_i)\big] \circ (\mathrm{Id}, \chi) \cdot \varphi \Big|,$$

where α, χ and φ vary as in the definition in (2), with $p = q = 1$. Arguing as in [17, Lemma 7.1], and employing the corresponding notation, we write

$$\int_{B(x,\delta)} \big[\partial^\alpha (\mathcal{L}_S h \circ \psi_i)\big] \circ (\mathrm{Id}, \chi) \cdot \varphi = \sum_{|\beta| \le |\alpha|} \sum_{j=1}^{l} \int_{B(x_j,\delta)} \partial^\beta \tilde{h}_j \circ (\mathrm{Id}, \chi_j) \cdot F_{\alpha,\beta,S,j} \cdot \rho_j, \quad (13)$$

where $\chi_1, \ldots, \chi_l$ are γ-admissible graphs whose corresponding γ-admissible leaves cover $S^{-1}(W)$, with W an admissible leaf corresponding to the graph of χ; $\tilde{h}_j = h \circ \psi_{i(j)}$; $\{\rho_j\}_{j=1,\ldots,l}$ is a partition of unity subordinated to the γ-admissible leaves of χ_j; and $F_{\alpha,\beta,S,j}$ are functions bounded in $C^{q+|\beta|}$. A similar expression holds for $\int_{B(x,\delta)} \big[\partial^\alpha (\mathcal{L}_{\tilde{S}} h \circ \psi_i)\big] \circ (\mathrm{Id}, \chi) \cdot \varphi$, with $F_{\alpha,\beta,S,j}$ replaced by $F_{\alpha,\beta,\tilde{S},j}$ and χ_j replaced by $\tilde{\chi}_j$, the graph corresponding to $\psi_{i(j)}^{-1} \circ \tilde{S}^{-1} \circ S \circ \psi_{i(j)} \circ (Id, \chi_j)(B(x_j, \gamma A\delta))$. Furthermore, if $d_{C^{r+1}}(S, \tilde{S})$ is small enough, each $\tilde{\chi}_j$ is a graph in $\Xi_{i(j)}$, and $|\chi_j - \tilde{\chi}_j|_{C^2(\bar{B}(x_j, A\delta))} < C d_{C^{r+1}}(S, \tilde{S})$. Also, $\|F_{\alpha,\beta,S,j}\|_{C^{q+|\beta|}}, \|F_{\alpha,\beta,\tilde{S},j}\|_{C^{q+|\beta|}}$ are uniformly bounded for $S, \tilde{S} \in \mathcal{M}_\epsilon(T)$ and $\|F_{\alpha,\beta,S,j} - F_{\alpha,\beta,\tilde{S},j}\|_{C^{q+|\beta|}} \to 0$ as $d_{C^{r+1}}(\tilde{S}, S) \to 0$, uniformly over φ as in (2). Hence, as $d_{C^{r+1}}(\tilde{S}, S) \to 0$, we get

$$\Big| \int_{B(x_j,\delta)} \partial^\beta (\tilde{h}_j) \circ (\mathrm{Id}, \tilde{\chi}_j) \cdot F_{\alpha,\beta,\tilde{S},j} \cdot \rho_j - \partial^\beta (\tilde{h}_j) \circ (\mathrm{Id}, \chi_j) \cdot F_{\alpha,\beta,S,j} \cdot \rho_j \Big| \to 0,$$

uniformly over χ (and so χ_j) and φ as in (2). It then follows from (13) that $\|\mathcal{L}_{\tilde{S}}h - \mathcal{L}_S h\|_{1,1} \to 0$ as $d_{C^{r+1}}(\tilde{S}, S) \to 0$, as claimed.

The result for general $h \in \mathcal{B}^{1,1}$ follows from an approximation argument by C^r functions, because if $d_{C^{r+1}}(S, \tilde{S})$ is sufficiently small, then $\|\mathcal{L}_{\tilde{S}}\|_{1,1} \le 1 + \|\mathcal{L}_S\|_{1,1} =: M$. Indeed, let $\{h_j\}_{j \in \mathbb{N}}$ be a sequence of C^r functions such that $\lim_{j\to\infty} h_j = h$ in $\mathcal{B}^{1,1}$, and let $t > 0$. Then, there exists $n \in \mathbb{N}$ such that $\|h - h_n\|_{1,1} < \frac{t}{3M}$. Hence, $\|\mathcal{L}_{\tilde{S}}h - \mathcal{L}_S h\|_{1,1} \le \|\mathcal{L}_{\tilde{S}}h - \mathcal{L}_{\tilde{S}}h_n\|_{1,1} + \|\mathcal{L}_{\tilde{S}}h_n - \mathcal{L}_S h_n\|_{1,1} + \|\mathcal{L}_S h_n - \mathcal{L}_S h\|_{1,1} \le \frac{2t}{3} + \|\mathcal{L}_{\tilde{S}}h_n - \mathcal{L}_S h_n\|_{1,1}$. Since $h_n \in C^r$, we have that $\limsup_{d_{C^{r+1}}(\tilde{S},S)\to 0} \|\mathcal{L}_{\tilde{S}}h - \mathcal{L}_S h\|_{1,1} \le \frac{2t}{3}$. Since the choice of $t > 0$ is arbitrary, the result follows.

3.2. **Quasi-compactness of the cocycle $\mathcal{L}$ and existence of Oseledets splitting.** For each $\omega \in \Omega, n \in \mathbb{N}$, let $\mathcal{L}_\omega^{(n)} := \mathcal{L}_{\sigma^{n-1}\omega} \circ \cdots \circ \mathcal{L}_{\sigma\omega} \circ \mathcal{L}_\omega$. It follows readily from (12) that

$$\|\mathcal{L}_\omega^{(n)} h\|_{1,1} \le D e^{-\lambda n} \|h\|_{1,1} \quad \text{for any } \omega \in \Omega,\ n \in \mathbb{N} \text{ and } h \in \mathcal{B}^{1,1},\ h(1) = 0. \quad (14)$$

Moreover, observe that (10) and (11) imply that

$$\|\mathcal{L}_\omega^{(n)} h\|_{0,2} \le B\|h\|_{0,2}, \quad \|\mathcal{L}_\omega^{(n)} h\|_{1,1} \le Ba^n\|h\|_{1,1} + B\|h\|_{0,2}, \tag{15}$$

which in particular implies that

$$\|\mathcal{L}_\omega^{(n)} h\|_{1,1} \le K\|h\|_{1,1} \tag{16}$$

where $K := Ba + B > 0$.

Proposition 3.2. *There exists a unique family $(h_\omega^0)_{\omega\in\Omega} \subset \mathcal{B}^{1,1}$ such that:*

(1) $\mathcal{L}_\omega h_\omega^0 = h_{\sigma\omega}^0$ for $\mathbb{P}$-a.e. $\omega \in \Omega$;
(2) h_ω^0 is nonnegative and $h_\omega^0(1) = 1$ for $\mathbb{P}$-a.e. $\omega \in \Omega$;
(3) $\omega \to h_\omega^0$ is a measurable map from Ω to $\mathcal{B}^{1,1}$;
(4)

$$\text{ess sup}_{\omega\in\Omega}\|h_\omega^0\|_{1,1} < \infty. \tag{17}$$

Proof. Let

$$Y = \{v\colon \Omega \to \mathcal{B}^{1,1} :\ v \text{ is measurable and } \|v\|_\infty := \text{ess sup}_{\omega\in\Omega}\|v(\omega)\|_{1,1} < \infty\}.$$

Then, $Y = (Y, \|\cdot\|_\infty)$ is a Banach space. Furthermore, let Z be the subset of Y that consists of $v \in Y$ with the property that $v(\omega)$ is nonnegative and $v(\omega)(1) = 1$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. It is easy to verify that Z is a closed subset of Y. Indeed, assume that $(v_n)_{n\in\mathbb{Z}}$ is a sequence in Z converging to $v \in Y$. It follows from (4) that

$$|v(\omega)(\varphi) - v_n(\omega)(\varphi)| \le C\|v_n(\omega) - v(\omega)\|_{1,1}|\varphi|_{C^1} \le C\|v_n - v\|_\infty|\varphi|_{C^1},$$

and thus $v_n(\omega)(\varphi) \to v(\omega)(\varphi)$ for $\varphi \in C^1$ and $\mathbb{P}$-a.e. $\omega \in \Omega$. Thus, $v(\omega)(\varphi) \ge 0$ for $\varphi \ge 0$ and $v(\omega)(1) = 1$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ and we conclude that $v \in Y$.

We define $\mathbb{L}\colon Z \to Z$ by

$$(\mathbb{L}v)(\omega) = \mathcal{L}_{\sigma^{-1}\omega} v(\sigma^{-1}\omega) \quad \omega \in \Omega,\ v \in Z.$$

It follows from (6) and (16) that $\mathbb{L}$ is a well-defined and continuous map on Z. Using (14), one can easily verify (see [10, Proposition 1]) that there exists $n_0 \in \mathbb{N}$ such that $\mathbb{L}^{n_0}$ is a contraction on Z. Thus, $\mathbb{L}$ has a unique fixed point $\bar{v} \in Z$. It is easy to verify that the family h_ω^0, $\omega \in \Omega$ defined $h_\omega^0 = \bar{v}(\omega)$, $\omega \in \Omega$ satisfies the desired properties. Conversely, each family satisfying properties (1)-(4) induces a fixed point of $\mathbb{L}$ which then must coincide with $\bar{v}$. □

Proposition 3.3. *Let $(h_\omega^0)_{\omega\in\Omega}$ be as in Proposition 3.2. Then h_ω^0 is a probability measure on $\mathcal{B}^{1,1}$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.*

Proof. Using Lemma 2.2, we have that

$$h_\omega^0(\varphi) = \mathcal{L}_{\sigma^{-n}\omega}^{(n)} h_{\sigma^{-n}\omega}^0(\varphi) = h_{\sigma^{-n}\omega}^0(\varphi \circ T_{\sigma^{-n}\omega}^{(n)}), \quad \text{for } \omega \in \Omega \text{ and } \varphi \in C^1.$$

Hence, using the arguments as in [7, Lemma 5.3], and equations (4) and (17), we find that there exists a constant $D > 0$ such that

$$|h_\omega^0(\varphi)| \le D|\varphi|_\infty \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and } \varphi \in C^1.$$

Since C^1 is dense in C^0, we conclude that for $\mathbb{P}$-a.e. $\omega \in \Omega$, h_ω^0 can be extended to a bounded linear functional on C^0. By the Riesz representation theorem, h_ω^0 is a signed measure. By invoking the nonnegativity of h_ω^0 together with $h_\omega^0(1) = 1$, we conclude that h_ω^0 is a probability measure for $\mathbb{P}$-a.e. $\omega \in \Omega$. □

We may apply Kingman's subadditive ergodic theorem to form the following limits, which are constant for $\mathbb{P}$-a.e. $\omega \in \Omega$:

$$\Lambda(\mathcal{L}) := \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)}\|_{1,1}, \text{ and}$$

$$\kappa(\mathcal{L}) := \lim_{n\to\infty} \frac{1}{n} \log ic(\mathcal{L}_\omega^{(n)}), \quad \text{where}$$

$ic(A) := \inf\{r > 0 : A(B_{\mathcal{B}_{1,1}})$ can be covered with finitely many balls of radius $r\}$, and $B_{\mathcal{B}^{1,1}}$ is the unit ball in $\mathcal{B}^{1,1}$. The cocycle $\mathcal{L}$ is called *quasi-compact* if $\Lambda(\mathcal{L}) > \kappa(\mathcal{L})$. Observe that (16) implies that

$$\Lambda(\mathcal{L}) \leq 0. \tag{18}$$

Remark 3.4. *The notion of quasicompactness for cocycles of linear operators was introduced by Thieullen* [32]. *Its relevance stems from the fact that all known versions of the multiplicative ergodic theorem for linear cocycles acting on Banach spaces require that a cocycle is quasicompact (see* [12, 13, 32] *and references therein). As pointed out by Buzzi* [4, Section 0.2]*, this requirement is not equivalent to the usual (deterministic) quasicompactness of the global (or annealed) transfer operator associated to the skew-product transformation.*

Lemma 3.5. *The cocycle $\mathcal{L}$ is quasi-compact.*

Proof. Observe that it follows from (4) and Proposition 3.2 that

$$\limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{L}_\omega^{(n)} h_\omega^0\|_{1,1} = \limsup_{n\to\infty} \frac{1}{n} \log\|h_{\sigma^n\omega}^0\|_{1,1} \geq \limsup_{n\to\infty} \frac{1}{n} \log C^{-1} = 0,$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. The above inequality together with (18) implies that $\Lambda(\mathcal{L}) = 0$.

Now we argue as in the proof of [11, Theorem 3.12.]. More precisely, we choose $N \in \mathbb{N}$ such that $Ba^N < 1$ and we consider the cocycle $\mathcal{L}^N$ over $(\Omega, \mathcal{F}, \mathbb{P}, \sigma^N)$ whose generator is the map $\omega \mapsto \mathcal{L}_\omega^{(N)}$. Then, it is easy to verify that $\Lambda(\mathcal{L}^N) = N\Lambda(\mathcal{L}) = 0$ and $\kappa(\mathcal{L}^N) = N\kappa(\mathcal{L})$. On the other hand, it follows from the inequalities (15) and (16) (applied for $n = N$) together with [11, Lemma 2.1] that $\kappa(\mathcal{L}^N) < \Lambda(\mathcal{L}^N)$ which immediately yields that $\kappa(\mathcal{L}) < \Lambda(\mathcal{L}) = 0$. We conclude that $\mathcal{L}$ is quasi-compact.

□

By separability of $\mathcal{B}^{1,1}$, and quasi-compactness and strong measurability of $\mathcal{L}$, the multiplicative ergodic theorem (Theorem A, [13]) yields: (i) $1 \leq l \leq \infty$ and a sequence of exceptional Lyapunov exponents $0 = \lambda_1 > \lambda_2 > \ldots > \lambda_l > \kappa(\mathcal{L})$ (or in the case $l = \infty$, $0 = \lambda_1 > \lambda_2 > \ldots$; $\lim_{n\to\infty} \lambda_n = \kappa(\mathcal{L})$) and (ii) a unique measurable Oseledets splitting

$$\mathcal{B}^{1,1} = \left(\bigoplus_{j=1}^{l} Y_j(\omega)\right) \oplus V(\omega),$$

where each component of the splitting is equivariant under $\mathcal{L}_\omega$, that is, $\mathcal{L}(\omega)(Y_j(\omega)) = Y_j(\sigma\omega)$ and $\mathcal{L}(\omega)(V(\omega)) \subset V(\sigma\omega)$. The $Y_j(\omega)$ are finite-dimensional and for each $y \in Y_j(\omega) \setminus \{0\}$, $\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} y\| = \lambda_j$. For $y \in V(\omega)$, $\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} y\| \leq \kappa(\mathcal{L})$.

Proposition 3.6. *The top Oseledets space $Y_1(\omega)$ of the cocycle $\mathcal{L}$ is one-dimensional, and spanned by h_ω^0.*

Proof. In the proof of Lemma 3.5 we have showed that $h_\omega^0 \in Y_1(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. We now claim that h_ω^0 spans $Y_1(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Indeed, assume that there exists $g_\omega \notin$

$span\{h^0_\omega\}$, $g_\omega \in Y_1(\omega)$ and choose α, β scalars (that depend on ω) such that $|\alpha| + |\beta| > 0$ and $(\alpha h^0_\omega + \beta g_\omega)(1) = 0$. Then, it follows from (14) that

$$\lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{L}^{(n)}_\omega(\alpha h^0_\omega + \beta g_\omega)\|_{1,1} \leq -\lambda < 0.$$

On the other hand, since $\alpha h^0_\omega + \beta g_\omega \in Y_1(\omega) \setminus \{0\}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{L}^{(n)}_\omega(\alpha h^0_\omega + \beta g_\omega)\|_{1,1} = 0,$$

which yields a contradiction. We conclude that $Y_1(\omega) = span\{h^0_\omega\}$ and thus $Y_1(\omega)$ is one-dimensional for $\mathbb{P}$-a.e. $\omega \in \Omega$.

□

4. Quasi-compactness of the twisted cocycle $\mathcal{L}^\theta$

We build a twisted cocycle $\mathcal{L}^\theta$, by setting

$$\mathcal{L}^\theta_\omega(h) = \mathcal{L}_\omega(e^{\theta g(\omega,\cdot)} \cdot h), \quad \text{for } \omega \in \Omega,\ \theta \in \mathbb{C}, \text{ and } h \in \mathcal{B}^{1,1}.$$

We will from now write $e^{\theta g(\omega,\cdot)}h$ instead of $e^{\theta g(\omega,\cdot)} \cdot h$. Our (centered) observable g will be a map $g\colon \Omega \times X \to \mathbb{R}$ such that $g(\omega,\cdot) \in C^r$ for $\omega \in \Omega$,

$$\text{ess sup}_{\omega\in\Omega}\|g(\omega,\cdot)\|_{C^r} < \infty, \tag{19}$$

and for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$h^0_\omega(g(\omega,\cdot)) = 0. \tag{20}$$

This twisted cocycle gives us access to an ω-wise moment-generating function for Birkhoff sums of g.

Lemma 4.1. *For $\mathbb{P}$-a.e. $\omega \in \Omega$, $h \in \mathcal{B}^{1,1}$ and $\varphi \in C^1(X, \mathbb{C})$ one has*

$$(\mathcal{L}^{\theta,(n)}_\omega h)(\varphi) = h(e^{\theta S_n g(\omega,\cdot)}(\varphi \circ T^n_\omega)). \tag{21}$$

Proof. One can follow the proof of Lemma 3.3 (part 2) [11], using the definition of the untwisted transfer operator (6) and Lemma 2.2. □

The following lemma is required as an auxiliary result in the proof of quasi-compactness of the twisted cocycle (Proposition 4.4).

Lemma 4.2. *There exists $C > 0$ such that for $\theta_1, \theta_2 \in B_{\mathbb{C}}(0,1) := \{\theta \in \mathbb{C} : |\theta| < 1\}$, we have that*

$$\text{ess sup}_{\omega\in\Omega}\|e^{\theta_1 g(\sigma^{-1}\omega,\cdot)} - e^{\theta_2 g(\sigma^{-1}\omega,\cdot)}\|_{C^2} \leq C|\theta_1 - \theta_2|.$$

Proof. By applying the mean value theorem for the map $g(z) = e^{zg(\sigma^{-1}\omega,x)}$, where $x \in X$ is fixed and using (19), we find that

$$\text{ess sup}_{\omega\in\Omega}\|e^{\theta_1 g(\sigma^{-1}\omega,\cdot)} - e^{\theta_2 g(\sigma^{-1}\omega,\cdot)}\|_{C^0} \leq C|\theta_1 - \theta_2|. \tag{22}$$

Furthermore, for $j = 1, \ldots, d$

$$\begin{aligned}\|\partial^j(e^{\theta_1 g(\sigma^{-1}\omega,\cdot)} - e^{\theta_2 g(\sigma^{-1}\omega,\cdot)})\|_{C^0} &= \|e^{\theta_1 g(\sigma^{-1}\omega,\cdot)}\theta_1\partial^j g(\sigma^{-1}\omega,\cdot) - e^{\theta_2 g(\sigma^{-1}\omega,\cdot)}\theta_2\partial^j g(\sigma^{-1}\omega,\cdot)\|_{C^0}\\ &\leq |\theta_1 - \theta_2| \cdot \|e^{\theta_1 g(\sigma^{-1}\omega,\cdot)}\partial^j g(\sigma^{-1}\omega,\cdot)\|_{C^0}\\ &\quad + |\theta_2| \cdot \|e^{\theta_1 g(\sigma^{-1}\omega,\cdot)} - e^{\theta_2 g(\sigma^{-1}\omega,\cdot)}\|_{C^0} \cdot \|\partial^j g(\sigma^{-1}\omega,\cdot)\|_{C^0}.\end{aligned}$$

It now follows from (19) and (22) that

$$\text{ess sup}_{\omega\in\Omega}\|\partial^j(e^{\theta_1 g(\sigma^{-1}\omega,\cdot)} - e^{\theta_2 g(\sigma^{-1}\omega,\cdot)})\|_{C^0} \leq C|\theta_1 - \theta_2|.$$

One can now proceed and obtain the same estimates for the second derivatives of the map $e^{\theta_1 g(\sigma^{-1}\omega,\cdot)} - e^{\theta_2 g(\sigma^{-1}\omega,\cdot)}$ which implies the desired conclusion. □

We need the following basic regularity result for the operators $\mathcal{L}_\omega^\theta$.

Proposition 4.3. *There exists a continuous function $K\colon \mathbb{C} \to (0,\infty)$ such that*

$$\|\mathcal{L}_\omega^\theta h\|_{1,1} \le K(\theta)\|h\|_{1,1}, \quad \text{for } h \in \mathcal{B}^{1,1},\ \theta \in \mathbb{C} \text{ and } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{23}$$

Proof. We first note that it follows from (16) that

$$\|\mathcal{L}_\omega^\theta h\|_{1,1} = \|\mathcal{L}_\omega(e^{\theta g(\omega,\cdot)}h)\|_{1,1} \le K\|e^{\theta g(\omega,\cdot)}h\|_{1,1}, \quad \text{for } h \in \mathcal{B}^{1,1},\ \theta \in \mathbb{C} \text{ and } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Hence, we need to estimate $\|e^{\theta g(\omega,\cdot)}h\|_{1,1}$. Note that by (3),

$$\|e^{\theta g(\omega,\cdot)}h\|_{1,1} = \max\{\|e^{\theta g(\omega,\cdot)}h\|_{0,1}^{\sim}, \|e^{\theta g(\omega,\cdot)}h\|_{1,2}^{\sim}\}.$$

It follows easily from (2) that

$$\|e^{\theta g(\omega,\cdot)}h\|_{0,1}^{\sim} \le \Big(\max_{1\le i\le N} \sup_{\substack{\chi\colon \overline{B}(x,A\delta)\to\mathbb{R}^{d_u}\\ \chi\in\Xi_i}} \|(e^{\theta g(\omega,\cdot)}\circ\psi_i)\circ(\mathrm{Id},\chi)\|_{C^1}\Big)\cdot\|h\|_{0,1}^{\sim}$$

and

$$\begin{aligned}\|e^{\theta g(\omega,\cdot)}h\|_{1,2}^{\sim} \le{}& \Big(\max_{1\le i\le N} \sup_{\substack{\chi\colon \overline{B}(x,A\delta)\to\mathbb{R}^{d_u}\\ \chi\in\Xi_i}} \|(e^{\theta g(\omega,\cdot)}\circ\psi_i)\circ(\mathrm{Id},\chi)\|_{C^2}\Big)\cdot\|h\|_{1,2}^{\sim}\\ &+\left(\max_{\substack{1\le j\le d\\ 1\le i\le N}} \sup_{\substack{\chi\colon \overline{B}(x,A\delta)\to\mathbb{R}^{d_u}\\ \chi\in\Xi_i}} \|\big[\partial^j(e^{\theta g(\omega,\cdot)}\circ\psi_i)\big]\circ(\mathrm{Id},\chi)\|_{C^1}\right)\cdot\|h\|_{0,1}^{\sim},\end{aligned}$$

which together with (19) implies the desired conclusion. □

We can now state the main result of this section on quasi-compactness.

Proposition 4.4. *For θ close to 0, the cocycle $(\mathcal{L}_\omega^\theta)_{\omega\in\Omega}$ is quasi-compact.*

Proof. We follow closely [11, Lemma 3.13]. Observe (15) and choose $N \in \mathbb{N}$ such that $\gamma := Ba^N < 1$. Hence,

$$\begin{aligned}\|\mathcal{L}_\omega^{\theta,(N)}h\|_{1,1} &\le \|\mathcal{L}_\omega^{(N)}h\|_{1,1} + \|\mathcal{L}_\omega^{\theta,(N)} - \mathcal{L}_\omega^{(N)}\|_{1,1}\cdot\|h\|_{1,1}\\ &\le \gamma\|h\|_{1,1} + B\|h\|_{0,2} + \|\mathcal{L}_\omega^{\theta,(N)} - \mathcal{L}_\omega^{(N)}\|_{1,1}\cdot\|h\|_{1,1}.\end{aligned}$$

On the other hand, we have that

$$\mathcal{L}_\omega^{\theta,(N)} - \mathcal{L}_\omega^{(N)} = \sum_{j=0}^{N-1} \mathcal{L}_{\sigma^{N-j}\omega}^{\theta,(j)}(\mathcal{L}_{\sigma^{N-1-j}\omega}^{\theta} - \mathcal{L}_{\sigma^{N-1-j}\omega})\mathcal{L}_\omega^{(N-1-j)}.$$

It follows from (16) and (23) that

$$\|\mathcal{L}_\omega^{(N-1-j)}\|_{1,1} \le K^{N-1-j} \quad \text{and} \quad \|\mathcal{L}_{\sigma^{N-j}\omega}^{\theta,(j)}\|_{1,1} \le K(\theta)^j.$$

Furthermore, using (16), we have that for any $h \in \mathcal{B}^{1,1}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\|(\mathcal{L}_\omega^\theta - \mathcal{L}_\omega)(h)\|_{1,1} = \|\mathcal{L}_\omega(e^{\theta g(\omega,\cdot)}h - h)\|_{1,1} \le K\|(e^{\theta g(\omega,\cdot)} - 1)h\|_{1,1}.$$

Moreover,

$$\|(e^{\theta g(\omega,\cdot)} - 1)h\|_{1,1} = \max\{\|(e^{\theta g(\omega,\cdot)} - 1)h\|_{0,1}^{\sim}, \|(e^{\theta g(\omega,\cdot)} - 1)h\|_{1,2}^{\sim}\}.$$

Now Lemma 4.2 (applied for $\theta_1 = \theta$ and $\theta_2 = 0$) implies that there exists $C > 0$ such for $\theta \in B_{\mathbb{C}}(0,1)$,

$$\|(e^{\theta g(\omega,\cdot)} - 1)h\|_{1,1} \le C|\theta|\|h\|_{1,1} \quad \text{for } h \in \mathcal{B}^{1,1}.$$

We conclude that

$$\|\mathcal{L}_\omega^{\theta,(N)} - \mathcal{L}_\omega^{(N)}\|_{1,1} \le C|\theta| \sum_{j=0}^{N-1} K^{N-1-j} K(\theta)^j,$$

and therefore there exists $\tilde{\gamma} \in (0,1)$ such that for any θ sufficiently close to 0 and $h \in \mathcal{B}^{1,1}$,

$$\|\mathcal{L}_\omega^{\theta,(N)} h\|_{1,1} \le \tilde{\gamma}\|h\|_{1,1} + B\|h\|_{0,2}. \tag{24}$$

Similarly, one can show that there exists $\tilde{B} > 0$ such that for any θ sufficiently close to 0 and $h \in \mathcal{B}^{1,1}$,

$$\|\mathcal{L}_\omega^{\theta} h\|_{0,2} \le \tilde{B}\|h\|_{0,2}. \tag{25}$$

The conclusion of the proposition follows from (24) and (25) by arguing as in the quasi-compactness part of the proof of [11, Theorem 3.12]. □

5. Regularity of the top Oseledets space of the twisted cocycle

Let $\mathcal{S}'$ be the space of measurable maps $\mathcal{W}\colon \Omega \to \mathcal{B}^{1,1}$ with the property that

$$\|\mathcal{W}\|_\infty := \operatorname{ess\,sup}_{\omega\in\Omega} \|\mathcal{W}(\omega)\|_{1,1} < \infty.$$

Then, $(\mathcal{S}', \|\cdot\|_\infty)$ is a Banach space. Furthermore, let $\mathcal{S}$ be the set of all $\mathcal{W} \in \mathcal{S}'$ such that $\mathcal{W}(\omega)(1) = 0$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Arguing as in the proof of Proposition 3.2, it is easy to verify that $\mathcal{S}$ is a closed subspace of $\mathcal{S}'$. For $\mathcal{W} \in \mathcal{S}'$ and $\omega \in \Omega$ we will often write $\mathcal{W}_\omega$ instead of $\mathcal{W}(\omega)$.

5.1. **Regularity of the cocycles.**

Lemma 5.1.

(1) For $\mathbb{P}$-a.e. $\omega \in \Omega$, the map $\theta \mapsto \mathcal{L}_\omega^\theta$ is analytic in the norm topology of $\mathcal{B}^{1,1}$.
(2) The map $\mathcal{P} : B_{\mathbb{C}}(0,1) \times \mathcal{S} \to \mathcal{S}$, given by $\mathcal{P}(\theta,\mathcal{W})_\omega = \mathcal{L}^\theta_{\sigma^{-1}\omega}(\mathcal{W}_{\sigma^{-1}\omega})$ is analytic in θ and bounded, linear in $\mathcal{W}$. In particular, $\mathcal{P}$ is C^∞.
(3) The map $\mathcal{P}_1 : B_{\mathbb{C}}(0,1) \times \mathcal{S} \to \mathcal{L}^\infty(\Omega)$, given by $\mathcal{P}_1(\theta,\mathcal{W})_\omega = (\mathcal{L}^\theta_{\sigma^{-1}\omega}(\mathcal{W}_{\sigma^{-1}\omega}))(1)$ is analytic in θ and bounded, linear in $\mathcal{W}$. In particular, $\mathcal{P}_1$ is C^∞.

Proof. We claim that for every $h \in \mathcal{B}^{1,1}$, the following holds:

$$\mathcal{L}_\omega^\theta(h) = \sum_{k=0}^\infty \frac{\theta^k}{k!} \mathcal{L}_\omega(g(\omega,\cdot)^k h), \quad \text{in } \mathcal{B}^{1,1}. \tag{26}$$

To verify this, note that [17, Lemma 3.2] implies that

$$\|\mathcal{L}_\omega(g(\omega,\cdot)^k h)\|_{1,1} \le C\|g(\omega,\cdot)^k\|_{C^2}\|h\|_{1,1} \le C\|g(\omega,\cdot)\|_{C^2}^k \|h\|_{1,1},$$

so by (19), the RHS of (26) is a well defined element of $\mathcal{B}^{1,1}$. The fact that it coincides with $\mathcal{L}_\omega^\theta(h)$ is straightforward to check, using linearity of $\mathcal{L}_\omega$, the power series expansion of $e^{\theta g(\omega,\cdot)}$, and testing against functions $\varphi \in \mathcal{C}^1$. This concludes the proof of Lemma 5.1(1).

Let us prove Lemma 5.1(2). For each $k \ge 0$ and $\mathcal{W} \in \mathcal{S}$, let $(g^k \cdot \mathcal{W})(\omega,\cdot) := g(\omega,\cdot)^k \mathcal{W}(\omega,\cdot)$. Then, $g^k \cdot \mathcal{W} \in \mathcal{S}$, because of (19) and [17, Lemma 3.2]. We claim that

$$\mathcal{P}(\theta,\mathcal{W}) = \sum_{k=0}^\infty \frac{\theta^k}{k!} \mathcal{P}(0, g^k \cdot \mathcal{W}) \quad \text{in } \mathcal{S}. \tag{27}$$

Indeed, (26) implies that

$$\mathcal{P}(\theta,\mathcal{W})_\omega = \sum_{k=0}^\infty \frac{\theta^k}{k!} \mathcal{P}(0, g^k \cdot \mathcal{W})_\omega \quad \text{in } \mathcal{B}^{1,1}. \tag{28}$$

Furthermore, using once again [17, Lemma 3.2], in combination with the uniform over ω bounds (16) and (19), we have that there exists $C > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\|\mathcal{P}(\theta, \mathcal{W})_\omega\|_{1,1} \leq \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \|\mathcal{P}(0, g^k \cdot \mathcal{W})_\omega\|_{1,1} \leq C \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \operatorname{ess\,sup}_{\omega\in\Omega} \|g(\omega, \cdot)\|_{C^2}^k \|\mathcal{W}\|_\infty. \quad (29)$$

Hence, the series in (27) indeed converges in $\mathcal{S}$ and yields analyticity as required. The fact that $\mathcal{W} \mapsto \mathcal{P}(\theta, \mathcal{W})$, and also $\mathcal{W} \mapsto \mathcal{P}(0, g^k \cdot \mathcal{W})$ is linear and bounded is straightforward to check. Hence, the C^∞ claim follows immediately.

The proof of Lemma 5.1(3) is similar to that of Lemma 5.1(2). Indeed,

$$\mathcal{P}_1(\theta, \mathcal{W})_\omega = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \langle \mathcal{P}(0, g^k \cdot \mathcal{W})_\omega, 1 \rangle, \quad (30)$$

and (4) implies that $|\langle \mathcal{P}(0, g^k \cdot \mathcal{W})_\omega, 1\rangle| \leq C\|\mathcal{P}(0, g^k \cdot \mathcal{W})_\omega\|_{1,1}$, which was bounded uniformly over ω in (29). Hence, the series (30) converges to $\mathcal{P}_1(\theta, \mathcal{W})$ in $L^\infty(\Omega)$. □

5.2. **An auxiliary function F and its regularity.** For $\theta \in \mathbb{C}$ and $\mathcal{W} \in \mathcal{S}$, set

$$F(\theta, \mathcal{W})(\omega) = \frac{\mathcal{L}^\theta_{\sigma^{-1}\omega}(\mathcal{W}(\sigma^{-1}\omega) + h^0_{\sigma^{-1}\omega})}{\mathcal{L}^\theta_{\sigma^{-1}\omega}(\mathcal{W}(\sigma^{-1}\omega) + h^0_{\sigma^{-1}\omega})(1)} - \mathcal{W}(\omega) - h^0_\omega, \quad \omega \in \Omega. \quad (31)$$

We define two further auxiliary functions, which will be used in the sequel. Let $G\colon \mathbb{C} \times \mathcal{S} \to \mathcal{S}'$ and $H\colon \mathbb{C} \times \mathcal{S} \to L^\infty(\Omega)$ be given by

$$G(\theta, \mathcal{W})(\omega) := \mathcal{P}(\theta, \mathcal{W} + h^0)(\omega) = \mathcal{L}^\theta_{\sigma^{-1}\omega}(\mathcal{W}_{\sigma^{-1}\omega} + h^0_{\sigma^{-1}\omega}), \quad \omega \in \Omega, \quad (32)$$

$$H(\theta, \mathcal{W})(\omega) := \mathcal{P}_1(\theta, \mathcal{W} + h^0)(\omega) = \mathcal{L}^\theta_{\sigma^{-1}\omega}(\mathcal{W}_{\sigma^{-1}\omega} + h^0_{\sigma^{-1}\omega})(1), \quad \omega \in \Omega. \quad (33)$$

It follows readily from (17) and Lemma 5.1 that G and H are well defined, and in fact C^∞ functions. Direct calculations, analogous to those of [11, Appendix B], yield the following:

Lemma 5.2. *For $\omega \in \Omega; \theta, z \in \mathbb{C}$; $\mathcal{W}, \mathcal{H} \in \mathcal{S}$, the following identities hold:*[7]

$$D_1 G(\theta, \mathcal{W})(z)_\omega = z\mathcal{L}_{\sigma^{-1}\omega}(g(\sigma^{-1}\omega, \cdot) e^{\theta g(\sigma^{-1}\omega, \cdot)}(\mathcal{W}_{\sigma^{-1}\omega} + h^0_{\sigma^{-1}\omega})), \quad (34)$$

$$D_2 G(\theta, \mathcal{W})(\mathcal{H})_\omega = \mathcal{L}^\theta_{\sigma^{-1}\omega}(\mathcal{H}_{\sigma^{-1}\omega}), \quad (35)$$

$$D_{11} G(\theta, \mathcal{W})(z_1, z_2)_\omega = z_1 z_2 \mathcal{L}_{\sigma^{-1}\omega}(g(\sigma^{-1}\omega, \cdot)^2 e^{\theta g(\sigma^{-1}\omega, \cdot)}(\mathcal{W}_{\sigma^{-1}\omega} + h^0_{\sigma^{-1}\omega})), \quad (36)$$

$$D_{12} G(\theta, \mathcal{W})(z, \mathcal{H})_\omega = D_{21} G(\theta, \mathcal{W})(\mathcal{H}, z)_\omega = z\mathcal{L}_{\sigma^{-1}\omega}(g(\sigma^{-1}\omega, \cdot) e^{\theta g(\sigma^{-1}\omega, \cdot)} \mathcal{H}_{\sigma^{-1}\omega}), \quad (37)$$

$$D_{22} G = 0. \quad (38)$$

Moreover, the expressions for the derivatives of H are equal to the corresponding expression for G applied to the constant function 1.

Lemma 5.3. *There exist $\epsilon, R > 0$ such that $F\colon \mathcal{D} \to \mathcal{S}$ is a well-defined map on*

$$\mathcal{D} := \{\theta \in \mathbb{C} : |\theta| < \epsilon\} \times B_{\mathcal{S}}(0, R),$$

where $B_{\mathcal{S}}(0, R)$ denotes the ball of radius R in $\mathcal{S}$ centered at 0.

[7] Here $D_1 G(\theta, \mathcal{W})$ is a linear operator from $\mathbb{C}$ to $\mathcal{S}'$ whose argument is denoted by z. Similar considerations apply to the other differentiability operators used in the Lemma.

Proof. Let G and H be defined as in (32) and (33). The function H is continuous on a neighborhood of $(0,0)$ in $\mathbb{C}\times\mathcal{S}$ and obviously $H(0,0)(\omega)=h^0_\omega(1)=1$ for $\mathbb{P}$-a.e. $\omega\in\Omega$. Hence,

$$|H(\theta,\mathcal{W})(\omega)|\geq 1-|H(0,0)(\omega)-H(\theta,\mathcal{W})(\omega)|\geq 1-\|H(0,0)-H(\theta,\mathcal{W})\|_{L^\infty},$$

for $\mathbb{P}$-a.e. $\omega\in\Omega$. Continuity of H implies that $\|H(0,0)-H(\theta,\mathcal{W})\|_{L^\infty}\leq\frac{1}{2}$ for all $(\theta,\mathcal{W})$ in a neighborhood of $(0,0)$ in $\mathbb{C}\times\mathcal{S}$ and hence, in such a neighborhood,

$$\text{ess inf}_{\omega\in\Omega}|H(\theta,\mathcal{W})(\omega)|\geq\frac{1}{2}.$$

This together with (17) and a simple observation that $F(\theta,\mathcal{W})(1)=0$ immediately yields the desired conclusion.

□

Notice that map F defined by (31) satisfies $F(\theta,\mathcal{W})(\omega)=G(\theta,\mathcal{W})(\omega)/H(\theta,\mathcal{W})(\omega)-\mathcal{W}(\omega)-h^0_\omega$. The proof of Lemma 5.3 ensures that for $(\theta,\mathcal{W})$ in a neighbourhood $\mathcal{D}$ of $(0,0)\in\mathbb{C}\times\mathcal{S}$, ess $\inf_{\omega\in\Omega}|H(\theta,\mathcal{W})(\omega)|\geq\frac{1}{2}$. Thus, the following result is a direct consequence of Lemma 5.2.

Proposition 5.4. *The map F defined by* (31) *is of class C^∞ on the neighborhood $\mathcal{D}$ of $(0,0)\in\mathbb{C}\times\mathcal{S}$ from Lemma 5.3. Moreover, for $\omega\in\Omega,(\theta,\mathcal{W})\in\mathcal{D}$ and $\mathcal{H}\in\mathcal{S}$,*

$$D_2F(\theta,\mathcal{W})(\mathcal{H})_\omega=\frac{1}{H(\theta,\mathcal{W})(\omega)}\mathcal{L}^\theta_{\sigma^{-1}\omega}\mathcal{H}_{\sigma^{-1}\omega}-\frac{\mathcal{L}^\theta_{\sigma^{-1}\omega}\mathcal{H}_{\sigma^{-1}\omega}(1)}{[H(\theta,\mathcal{W})(\omega)]^2}G(\theta,\mathcal{W})_\omega-\mathcal{H}_\omega,$$

$$\begin{aligned}D_1F(\theta,\mathcal{W})_\omega=&\frac{1}{H(\theta,\mathcal{W})(\omega)}\mathcal{L}_{\sigma^{-1}\omega}(g(\sigma^{-1}\omega,\cdot)e^{\theta g(\sigma^{-1}\omega,\cdot)}(\mathcal{W}_{\sigma^{-1}\omega}+h^0_{\sigma^{-1}\omega}))\\&-\frac{\mathcal{L}_{\sigma^{-1}\omega}(g(\sigma^{-1}\omega,\cdot)e^{\theta g(\sigma^{-1}\omega,\cdot)}(\mathcal{W}_{\sigma^{-1}\omega}+h^0_{\sigma^{-1}\omega}))(1)}{[H(\theta,\mathcal{W})(\omega)]^2}\mathcal{L}^\theta_{\sigma^{-1}\omega}(\mathcal{W}_{\sigma^{-1}\omega}+h^0_{\sigma^{-1}\omega}),\end{aligned}$$

where we have identified $D_1F(\theta,\mathcal{W})$ with its value at 1.

Lemma 5.5. *Let $\mathcal{D}=\{\theta\in\mathbb{C}:|\theta|<\epsilon\}\times B_{\mathcal{S}}(0,R)$ be as in Lemma 5.3. Then, $F:\mathcal{D}\to\mathcal{S}$ is C^∞ and the equation*

$$F(\theta,\mathcal{W})=0 \tag{39}$$

has a unique solution $O(\theta)\in\mathcal{S}$, for every θ in a neighborhood of 0. Furthermore, $O(\theta)$ is a C^∞ function of θ.

Proof. Note that $F(0,0)=0$. Furthermore, Proposition 5.4 implies that F is of class C^∞ on a neighborhood of $(0,0)$. In addition, Lemma 5.2 implies that

$$(D_2F(0,0)\mathcal{X})(\omega)=\mathcal{L}_{\sigma^{-1}\omega}\mathcal{X}(\sigma^{-1}\omega)-\mathcal{X}(\omega),\quad\omega\in\Omega,\ \mathcal{X}\in\mathcal{S}.$$

Using (14) and proceeding as in [11, Lemma 3.5], one can show that $D_2F(0,0)$ is invertible and that

$$(D_2F(0,0)^{-1}\mathcal{X})(\omega)=-\sum_{j=0}^{\infty}\mathcal{L}^{(j)}_{\sigma^{-j}\omega}\mathcal{X}(\sigma^{-j}\omega)\quad\omega\in\Omega,\ \mathcal{X}\in S. \tag{40}$$

The conclusion of the lemma now follows directly from the implicit function theorem. □

6. Properties of $\Lambda(\theta)$

Let $0 < \epsilon < 1$ be as in Lemma 5.3 and $O(\theta)$ be as in Lemma 5.5. Let

$$h^\theta_\omega := h^0_\omega + O(\theta)(\omega) \in \mathcal{B}^{1,1}, \quad \omega \in \Omega. \tag{41}$$

We notice that $h^\theta_\omega(1) = 1$ and by Lemma 5.5, $\theta \mapsto h^\theta$ is continuously differentiable.

Let us define

$$\hat{\Lambda}(\theta) := \int \log \left| h^\theta_\omega(e^{\theta g(\omega,\cdot)}) \right| d\mathbb{P}(\omega), \tag{42}$$

and

$$\lambda^\theta_\omega := h^\theta_\omega(e^{\theta g(\omega,\cdot)}) = \mathcal{L}^\theta_\omega h^\theta_\omega(1). \tag{43}$$

6.1. **A differentiable lower bound for $\Lambda(\theta)$.** Lemma 6.1 deals with differentiability properties of $\hat{\Lambda}(\theta)$.

Lemma 6.1.

(1) For every $\theta \in B_{\mathbb{C}}(0,\epsilon)$, $\hat{\Lambda}(\theta) \le \Lambda(\theta)$.

(2) $\hat{\Lambda}$ is differentiable on a neighborhood of 0, and

$$\hat{\Lambda}'(\theta) = \Re\left(\int \frac{\overline{\lambda^\theta_\omega}((O(\theta)(\omega) + h^0_\omega)(g(\omega,\cdot)e^{\theta g(\omega,\cdot)}) + O'(\theta)(\omega)(e^{\theta g(\omega,\cdot)}))}{|\lambda^\theta_\omega|^2} \, d\mathbb{P}(\omega) \right),$$

where $\Re(z)$ denotes the real part of z and $\overline{z}$ the complex conjugate of z.

(3) For $\mathbb{P}$-a.e. $\omega \in \Omega$, and θ in a neighborhood of 0, the map $\theta \mapsto Z_\omega(\theta) := \log|\lambda^\theta_\omega|$ is differentiable. Moreover,

$$Z'_\omega(\theta) = \frac{\Re\Big(\overline{\lambda^\theta_\omega}((O(\theta)(\omega) + h^0_\omega)(g(\omega,\cdot)e^{\theta g(\omega,\cdot)}) + O'(\theta)(\omega)(e^{\theta g(\omega,\cdot)}))\Big)}{|\lambda^\theta_\omega|^2}.$$

(4) $\hat{\Lambda}'(0) = 0$.

Proof. The proof of part 1 is identical to the proof of Lemma 3.8 [11] replacing $\|\cdot\|_{\mathcal{B}}$ with $\|\cdot\|_{1,1}$ and $\|\mathcal{L}^{\theta,(n)}_\omega v^\theta_\omega\|_1$ with $|\mathcal{L}^{\theta,(n)}_\omega h^\theta_\omega(1)|$.

The proof of part 2 is identical to the proof of Lemma 3.9 [11], using Lemma 5.5 in place of Lemma 3.5 [11] and replacing the final two equation blocks with:

$$\begin{aligned} |(O(\theta)(\omega) + h^0_\omega)(g(\omega,\cdot)e^{\theta g(\omega,\cdot)})| &\le C\|O(\theta)(\omega) + h^0_\omega\|_{1,1} \cdot \|g(\omega,\cdot)e^{\theta g(\omega,\cdot)}\|_{C^1} \\ &\le C\|O(\theta)\|_\infty + C, \end{aligned}$$

and

$$|O'(\theta)(\omega)(e^{\theta g(\omega,\cdot)})| \le C\|O'(\theta)(\omega)\|_{1,1} \cdot \|e^{\theta g(\omega,\cdot)}\|_{C^1} \le C\|O'(\theta)\|_\infty.$$

The proof of part 3 is identical to the proof of Lemma 3.10 [11], using differentiability of H and O in Lemmas 5.2 and 5.5.

The proof of part 4 is identical to proof of Lemma 3.11 [11]. □

6.2. **One-dimensionality of $Y^\theta_1(\omega)$ and differentiability of Λ.** Let $Y^\theta_1(\omega)$ denote the top Oseledets subspace of the cocycle $(\mathcal{L}^\theta_\omega)_{\omega\in\Omega}$. The proof of part 1 of the following result can be obtained by repeating the argument as in [11, Theorem 3.12], using Proposition 4.4. Part 2 follows by arguing as in [11, Corollary 3.14].

Proposition 6.2. *For $\theta \in \mathbb{C}$ near 0*

(1) $\dim Y^\theta_1(\omega) = 1$.

(2) $\Lambda(\theta) = \hat{\Lambda}(\theta)$. In particular, $\Lambda(\theta)$ is differentiable near 0 and $\Lambda'(0) = 0$.

6.3. **Convexity of $\Lambda(\theta)$.** By Proposition 3.3, we can regard h^0_ω as Borel probability measure on X which we will denote by μ_ω. The family $(\mu_\omega)_{\omega\in\Omega}$ induces a probability measure on $\Omega\times X$ given by

$$\mu(A\times B)=\int_A \mu_\omega(B)\,d\mathbb{P}(\omega),\quad \text{for measurable sets } A\subset\Omega \text{ and } B\subset X.$$

Then, μ is invariant for the skew-product transformation $\tau\colon \Omega\times X\to\Omega\times X$ defined by

$$\tau(\omega,x)=(\sigma\omega,T_\omega(x)),\quad \omega\in\Omega,\ x\in X.$$

Let us now establish the appropriate decay of correlations result.

Proposition 6.3. *There exists $D'>0$ such that*

$$\left|\int_X \varphi(\psi\circ T^n_\omega)\,d\mu_\omega\right|\le D'e^{-\lambda n}\|\varphi\|_{C^r}\cdot\|\psi\|_{C^1},$$

for $\mathbb{P}$-a.e. $\omega\in\Omega$, $n\in\mathbb{N}$, $\varphi\in C^r(X,\mathbb{C})$ such that $\int_X\varphi\,d\mu_\omega=0$ and $\psi\in C^1(X,\mathbb{C})$.

Proof. We have that

$$\int_X \varphi(\psi\circ T^n_\omega)\,d\mu_\omega=\mu_\omega(\varphi(\psi\circ T^n_\omega))=(\mu_\omega\cdot\varphi)(\psi\circ T^n_\omega)=\mathcal{L}^{(n)}_\omega(\mu_\omega\cdot\varphi)(\psi).$$

Furthermore, observe that $(\mu_\omega\cdot\varphi)(1)=\int_X\varphi\,d\mu_\omega=0$. Now the desired conclusion follows readily from (4), (9), (14) and (17). □

The following result can be obtained by repeating the arguments in [10, Lemma 12.] (and by using Proposition 6.3).

Proposition 6.4. *We have that*

$$\Sigma^2:=\int_{\Omega\times X} g(\omega,x)^2\,d\mu(\omega,x)+2\sum_{n=1}^{\infty}\int_{\Omega\times X} g(\omega,x)g(\tau^n(\omega,x))\,d\mu(\omega,x)\tag{44}$$

exists and that $\Sigma^2\ge 0$.

Proposition 6.5. *We have that $\Sigma^2=0$ if and only if there exists $r\in L^2_\mu(\Omega\times X)$ such that $g=r-r\circ\tau$.*

Proof. Assume first that $\Sigma^2=0$. For $n\in\mathbb{N}$, set

$$X_n(\omega,x)=\sum_{k=0}^{n-1} g(\tau^k(\omega,x)),\quad (\omega,x)\in\Omega\times X.$$

By arguing as in [10, Proposition 3], one can show that the sequence $(X_n)_{n\in\mathbb{N}}$ is bounded in $L^2_\mu(\Omega\times X)$ and thus there exists a subsequence $(X_{n_k})_k$ that converges weakly to some $r\in L^2_\mu(\Omega\times X)$. We claim that $g=r-r\circ\tau$. Take $w=\mathbf{1}_A\cdot\varphi\in L^2_\mu(\Omega\times X)$, where $A\in\mathcal{F}$ and $\varphi\in C^1(X,\mathbb{C})$. Proceeding again as in the proof of [10, Proposition 3], we conclude that

$$\begin{aligned}\int_{\Omega\times X} w(g-r+r\circ\tau)\,d\mu&=\lim_{k\to\infty}\int_{\Omega\times X} w(g\circ\tau^{n_k})\,d\mu\\&=\lim_{k\to\infty}\int_\Omega \mathbf{1}_A\left(\int_X\varphi(g_{\sigma^{n_k}\omega}\circ T^{n_k}_\omega)\,d\mu_\omega\right)d\mathbb{P}(\omega)\\&=0,\end{aligned}$$

where in the last step we have used Proposition 6.3. Hence, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $\varphi \in C^1(X, \mathbb{C})$, we have that

$$\int_X (g_\omega - r_\omega + r_{\sigma\omega} \circ T_\omega)\varphi \, d\mu_\omega = 0. \tag{45}$$

Since $C^1(X, \mathbb{C})$ is dense in $C(X, \mathbb{C})$, one can easily conclude that (45) holds also for $\varphi \in C(X, \mathbb{C})$. Finally, $C(X, \mathbb{C})$ is dense in $L^2(\mu_\omega)$ (since μ_ω is a Radon measure) and thus (45) is valid for $\varphi \in L^2(\mu_\omega)$. Consequently, for $\mathbb{P}$-a.e. $\omega \in \Omega$, we have that $g_\omega - r_\omega + r_{\sigma\omega} \circ T_\omega = 0$, μ_ω-a.e. Therefore, $g = r - r \circ \tau$.

The converse statement can be obtained by arguing exactly as in [10, Proposition 3]. □

Proposition 6.6. *Suppose that $\Sigma^2 > 0$. Then, on a neighbourhood of 0,*

(1) Λ is of class C^2 and $\Lambda''(0) = \Sigma^2$.

(2) Λ is strictly convex.

Proof. The proof of part (1) is identical to the proof of [11, Lemma 3.15] and part (2) is a direct consequence of part (1) and our assumption that $\Sigma^2 > 0$. □

From now on we will assume that $\Sigma^2 > 0$.

7. Large deviation principle and central limit theorem

For $\theta \in \mathbb{C}$ sufficiently close to 0, we have that $\dim Y^\theta(\omega) = 1$, where $Y^\theta(\omega)$ now denotes $Y_1^\theta(\omega)$ for the sake of simplicity. Choose $h_\omega^\theta \in Y_1^\theta(\omega)$ such that $h_\omega^\theta(1) = 1$. We note that h_ω^θ is actually given by (41). Furthermore, let $\lambda_\omega^\theta \in \mathbb{C}$ be such that

$$\mathcal{L}_\omega^\theta h_\omega^\theta = \lambda_\omega^\theta h_{\sigma\omega}^\theta. \tag{46}$$

Note that

$$\lambda_\omega^\theta = h_\omega^\theta(e^{\theta g(\omega,\cdot)}), \tag{47}$$

which coincides with (43). Let $Y_\omega^{*\theta}$ denotes the top-Oseledets space of the adjoint cocycle $\mathcal{L}^*$ over $(\Omega, \mathcal{F}, \mathbb{P}, \sigma^{-1})$ whose generator is the map $\omega \mapsto (\mathcal{L}_{\sigma^{-1}\omega})^*$. Next, let us fix $\phi_\omega^\theta \in Y_\omega^{*\theta}$ so that $\phi_\omega^\theta(h_\omega^\theta) = 1$. We recall that one can indeed apply MET for the adjoint cocycle and that $\dim Y_\omega^{*\theta} = \dim Y^\theta(\omega) = 1$ (see [11, Corollary 2.5.]).

Furthermore, one can show (see [11, p. 30]) that

$$(\mathcal{L}_\omega^\theta)^* \phi_{\sigma\omega}^\theta = \lambda_\omega^\theta \phi_\omega^\theta. \tag{48}$$

Remark 7.1. *The differentiability of $\theta \mapsto \phi^\theta$ follows similarly to the presentation in* [11, Appendix C]. *The proofs of Lemmas C.4 and C.6 make use of regularity estimates (90) and (99) in terms of variation; in the present work, these estimates may be replaced with C^1 estimates. In the proof of Lemma C.2, the expression $\|v_\omega^0\|_1$ may be replaced with $|h_\omega^0(1)|$ and bounded by (4) in the present work.*

In addition, let

$$\mathcal{B}^{1,1} = Y_\omega^\theta \oplus H_\omega^\theta \quad \text{and} \quad (\mathcal{B}^{1,1})^* = Y_\omega^{*\theta} \oplus H_\omega^{*\theta}$$

be the Oseledets splitting of cocycles $(\mathcal{L}_\omega^\theta)_{\omega\in\Omega}$ and $((\mathcal{L}_\omega^\theta)^*)_{\omega\in\Omega}$ respectively into a direct sum of the top space and the sum of all other Oseledets subspaces.

We are now going to state and prove the first two main results of this paper; we now remind the general assumption on the choice of the cocycle and the regularity of the maps:

- **General assumptions for the limit theorems**: Having fixed a topological transitive Anosov map T of class C^{r+1}, with $r > 2$, we will consider the neighborhood $\mathcal{O}_{\delta_0}(T, \mathcal{B}^{1,1})$ and choose maps into it in order to build the cocycle $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{B}^{1,1}, \mathcal{L})$. This cocycle will verify the assumptions of Section 3 and the observable $g\colon \Omega \times X \to \mathbb{R}$ will satisfy (19) and (20). Finally, suppose that $\Sigma^2 > 0$.

7.1. **Large deviation principle.** The following lemmas link the limits of characteristic functions of Birkhoff sums to the function Λ.

Lemma 7.2. *Let $\theta \in \mathbb{C}$ be sufficiently close to 0 and $h \in \mathcal{B}^{1,1}$ be such that $h \notin H^\theta_\omega$, i.e. $\phi^\theta_\omega(h) \neq 0$. Then,*

$$\lim_{n\to\infty} \frac{1}{n} \log \left| h(e^{\theta S_n g(\omega,\cdot)}) \right| = \Lambda(\theta).$$

Proof. Identical to the proof of Lemma 4.2 [11] with $h \in \mathcal{B}^{1,1}$ in the present paper playing the role of $\int f \cdot \, dm$ in the proof of Lemma 4.2 [11], and (21) replacing [11, (43)]. □

Lemma 7.3. *For all complex θ in a neighborhood of 0, and $\mathbb{P}$-a.e. $\omega \in \Omega$, we have that*

$$\lim_{n\to\infty} \frac{1}{n} \log \left| \int e^{\theta S_n g(\omega,x)} \, d\mu_\omega(x) \right| = \Lambda(\theta).$$

Proof. We follow the proof of Lemma 4.3 [11], observing that

$$\int e^{\theta S_n g(\omega,x)} \, d\mu_\omega(x) = h^0_\omega(e^{\theta S_n g(\omega,\cdot)}),$$

and recalling the differentiability of the map $\theta \mapsto \phi^\theta$ in Remark 7.1. □

We are now ready to state our Theorem A.

Theorem A (Quenched large deviations theorem)**.** *There exists $\epsilon_0 > 0$ and a non-random function $c\colon (-\epsilon_0, \epsilon_0) \to \mathbb{R}$ which is nonnegative, continuous, strictly convex, vanishing only at 0 and such that*

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_\omega(S_n g(\omega,\cdot) > n\epsilon) = -c(\epsilon), \quad \text{for } 0 < \epsilon < \epsilon_0 \text{ and } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Proof. Following the proof of Theorem A [11], by applying Proposition 6.6 and Lemma 7.3, together with the Gärtner-Ellis theorem (see [20] or Theorem 4.1 [11]), we obtain the large deviation principle. □

7.2. **Central limit theorem.** The proof of the following result is completely analogous to the proof of [11, Lemma 4.4].

Lemma 7.4. *There exist $C > 0, 0 < r < 1$ such that for every $\theta \in \mathbb{C}$ sufficiently close to 0, every $n \in \mathbb{N}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, we have*

$$\left| \mathcal{L}^{\theta,(n)}_\omega (h^0_\omega - \phi^\theta_\omega(h^0_\omega) h^\theta_\omega)(1) \right| \leq C r^n. \tag{49}$$

Theorem B (Quenched central limit theorem)**.** *Let us assume that the non-random variance $\Sigma^2 > 0$. Then, for every bounded and continuous function $\phi\colon \mathbb{R} \to \mathbb{R}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, we have*

$$\lim_{n\to\infty} \int \phi\left(\frac{S_n g(\omega,x)}{\sqrt{n}} \right) d\mu_\omega(x) = \int \phi \, d\mathcal{N}(0, \Sigma^2).$$

Proof. The proof is identical to the proof of Theorem B [11], with the same modifications as those listed in the proof of Lemma 7.2. Differentiability of $\theta \mapsto \phi^\theta$ is used (see Remark 7.1) as well as Lemma 4.1 to obtain the coding of the Birkhoff sums via the twisted transfer operator. Lemma 4.5 [11] is proved is proved in an identical way. □

8. Local central limit theorem

We begin by recalling the concept of $\mathbb{P}$-continuity which we will also use in section 9.2.1. We say that our cocycle is $\mathbb{P}$-continuous (a concept introduced in [32]) if the map $\omega \mapsto \mathcal{L}_\omega$ is continuous on each of countably many Borel subsets of Ω, whose union has full $\mathbb{P}$ measure. This for example happens if the map $\omega \mapsto T_\omega$ has $\mathbb{P}$-a.e. a countable range (besides being measurable). We refer to [12] for details.

In our earlier paper [11] we proved the local central limit theorem in the non-arithmetic case (we also separately treated the arithmetic case) under the condition that we called (L) in the introduction, namely

- (L) For $\mathbb{P}$-a.e. $\omega \in \Omega$ and for every compact interval $J \subset \mathbb{R}\backslash\{0\}$ there exists $C = C(\omega) > 0$ and $\rho \in (0,1)$ such that

$$||\mathcal{L}^{it,(n)}||_{\mathcal{B}} \leq C\rho^n, \text{ for } t \in J \text{ and } n \geq 0. \tag{50}$$

Moreover under the assumption that the cocycle is $\mathbb{P}$-continuous, we proved [11, Lemma 4.7.] that (L) is equivalent to the following *aperiodicity* condition

- For every $t \in \mathbb{R}$, either (i) $\Lambda(it) < 0$ or (ii) the cocycle $\mathcal{L}_\omega^{it}$ is quasicompact and the equation

$$e^{itg(\omega,x)}\mathcal{L}_\omega^*\psi_{\sigma\omega} = \gamma_\omega^{it}\psi_\omega,$$

  where $\gamma_\omega^{it} \in S^1$, $\mathcal{L}_\omega^*$ denotes the adjoint of $\mathcal{L}_\omega$ and $\psi_\omega \in \mathcal{B}^*$, only has a measurable non-zero solution $\psi := \{\psi_\omega\}_{\omega\in\Omega}$ when $t = 0$. In this case $\gamma_\omega^0 = 1$ and $\psi_\omega(f) = \int f dm$ (up to a scalar multiplicative factor) for $\mathbb{P}$-a.e. $\omega \in \Omega$.

In our present Anosov setting the bound (50) will be replaced with the following:

$$\|\mathcal{L}_\omega^{it,(n)}\|_{1,1} \leq C\rho^n, \quad \text{for } t \in J \text{ and } n \geq 0. \tag{51}$$

Still in the present setting, we can not prove at the moment the equivalence between (L) and the aperiodicity condition although several of the technical steps which formed the skeleton of our proof of [11, Lemma 4.7.] for expanding maps and functions of bounded variation can be transferred to Anosov maps and the anisotropic Banach spaces used in this work.

We now state and prove our Theorem C.

Theorem C (Quenched local central limit theorem)**.** *Let us suppose condition (L) holds. Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every bounded interval $J \subset \mathbb{R}$, we have*

$$\lim_{n\to\infty} \sup_{s\in\mathbb{R}} \left| \Sigma\sqrt{n}\mu_\omega(s + S_n g(\omega,\cdot) \in J) - \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2n\Sigma^2}} |J| \right| = 0.$$

Proof. The proof assuming (L) follows now exactly as in the proof of Theorem C [11], with the following minor modifications. We use Lemma 4.1 to obtain the coding of the Birkhoff sums through powers of the twisted transfer operator. The control of term (III) in the proof of Theorem C [11] uses Lemma 7.4 in place of Lemma 4.4 [11]. The control of term (IV) in the proof of Theorem C [11] uses (L1) in place of the analogous condition (C5) in [11]. □

9. Sufficient conditions under which (L) holds

In this section we formulate sufficient conditions under which the condition (L) holds. This is of central importance when dealing with Theorem C, since the requirement (L) is obviously quite difficult to verify directly in concrete situations. For this, we will rely heavily on the work of Hafouta and Kifer [19]. More precisely, we will formulate three

conditions (HK A1, HK A2 and HK A3 below) that in conjunction imply (L) and then we will formulate sufficient conditions under which each of those three conditions hold.

Let us now explicity introduce these conditions:

- HK A1: The probability measure $\mathbb{P}$ assigns positive measure to open sets, σ is a homeomorphism and there exist $\omega_0 \in \Omega$ and $m_0 \in \mathbb{N}$ so that $\sigma^{m_0}\omega_0 = \omega_0$. Moreover for each $i \in \{0, 1, \dots, m_0 - 1\}$, there exists a neighborhood of $\sigma^i\omega_0$ on which the map $\omega \mapsto T_\omega$ is constant (note that on this neighborhood we also have that the map $\omega \mapsto \mathcal{L}_\omega$ is constant).
- HK A2: For each compact interval $J \subset \mathbb{R}$, the family of maps $\omega \mapsto \mathcal{L}_\omega^{it}$, where $t \in J$, is equicontinuous at the points $\omega = \sigma^i\omega_0, 0 \leq i \leq m_0$ with respect to the operator norm, and there exists a constant $B = B(J) \geq 1$, such that $\mathbb{P}$-a.e.,
$$\|\mathcal{L}_\omega^{it,(n)}\|_{1,1} \leq B, \tag{52}$$
for any $n \in \mathbb{N}$ and $t \in J$.
- HK A3: For any compact interval $J \subset \mathbb{R}$ that does not contain the origin, there exists constants $c = c(J) > 0$ and $b = b(J) \in (0, 1)$ such that
$$\|\mathbf{L}_{it}^s\|_{1,1} \leq cb^s, \tag{53}$$
for any $s \in \mathbb{N}$ and $t \in J$, where the (deterministic) operator $\mathbf{L}_{it}$ is defined as $\mathbf{L}_{it} := \mathcal{L}_{\omega_0}^{it,(m_0)}$.

Under these three assumptions, it was proved in Lemma 2.10.4 [19] that condition (L) holds.

Observe that HK A1 represents a mild requirement. Indeed, it is easily satisfied by requiring that σ is a homeomorphism that has at least one periodic point ω_0 and by building the cocycle in a way that $\omega \mapsto T_\omega$ is locally constant at all points that belong to the orbit of ω_0. Of course, we also need to work with $\mathbb{P}$ that assigns positive measure to all open nonempty subsets of Ω.

9.1. **Discussion regarding HK A2.** We now turn to the condition HK A2. We begin by stating the following auxiliary result.

Lemma 9.1. *Let us suppose $(\Omega, \mathcal{F}, \mathbb{P}, \sigma)$ is an invertible and ergodic measure-preserving dynamical system satisfying HK A1 and that for each $i \in \{0, 1, \dots, m_0 - 1\}$, the observable g satisfies*
$$\lim_{\omega \to \sigma^i\omega_0} \|g(\omega, \cdot) - g(\sigma^i\omega_0, \cdot)\|_{C^2} = 0. \tag{54}$$
Furthermore, let $J \subset \mathbb{R}$ be a compact interval. Then, the family of maps $\{\omega \mapsto \mathcal{L}_\omega^{it} : t \in J\}$ is equicontinuous in all points ω that belong to the orbit of ω_0.

Remark 9.2. *Observe that it is not enough to simply prescribe that* (54) *holds since we also need to make sure that this requirement is not spoiled when we center our observable (see* (20)*). It turns out that under the condition HK A1,* (54) *is preserved under centering. Let us first recall that* (14) *and* (15) *hold for every $\omega \in \Omega$. Let us now modify slightly the proof of Proposition 3.2 to ensure that under HK A1, we can say more about the top Oseledets space of our cocycle.*

Set
$$\mathcal{Y} = \{v \colon \Omega \to \mathcal{B}^{1,1} : v \text{ measurable and } \|v\|_\infty := \sup_{\omega \in \Omega} \|v(\omega)\|_{1,1} < \infty\}.$$
Then, $(\mathcal{Y}, \|\cdot\|_\infty)$ is a Banach space. Let Y be a set of all $v \in \mathcal{Y}$ that are continuous at points $\sigma^i\omega_0$, $i = 0, 1, \dots, m_0 - 1$. We claim that Y is a closed subset of $\mathcal{Y}$. Indeed, take a

sequence $(v_n)_n \subset Y$ *such that* $v_n \to v$ *in* $\mathcal{Y}$ *and fix* $i \in \{0, 1, \ldots, m_0 - 1\}$. *Then, we have that*

$$\begin{aligned}\|v(\omega) - v(\sigma^i\omega_0)\|_{1,1} &\leq \|v(\omega) - v_n(\omega)\|_{1,1} + \|v_n(\omega) - v_n(\sigma^i\omega_0)\|_{1,1} \\ &\quad + \|v_n(\sigma^i\omega_0) - v(\sigma^i\omega_0)\|_{1,1} \\ &\leq 2\|v - v_n\|_\infty + \|v_n(\omega) - v_n(\sigma^i\omega_0)\|_{1,1}.\end{aligned}$$

Take $\varepsilon > 0$ *and choose* n *such that*

$$\|v - v_n\|_\infty < \frac{\varepsilon}{3}.$$

Since $v_n \in Y$, *we have that*

$$\|v_n(\omega) - v_n(\sigma^i\omega_0)\|_{1,1} < \frac{\varepsilon}{3},$$

whenever ω *is sufficiently close to* $\sigma^i\omega_0$. *Hence,*

$$\|v(\omega) - v(\sigma^i\omega_0)\|_{1,1} < \varepsilon,$$

whenever ω *is sufficiently close to* $\sigma^i\omega_0$. *Therefore,* $v \in Y$ *and* Y *is closed.*

Set

$$Z := \{v \in Y; v(\omega) \geq 0 \text{ and } v(\omega)(1) = 1 \text{ for } \omega \in \Omega\}.$$

Then, Z *is a closed subset of* Y *(see the argument in the proof of Proposition 3.2) and hence it is a complete metric space. We consider* $\mathbb{L}\colon Z \to Z$ *defined by*

$$(\mathbb{L}v)(\omega) = \mathcal{L}_{\sigma^{-1}\omega} v(\sigma^{-1}\omega), \quad \omega \in \Omega,\ v \in Z.$$

In order to show that $\mathbb{L}$ *is well-defined, we only need to note that*

$$\omega \mapsto \mathcal{L}_{\sigma^{-1}\omega} v(\sigma^{-1}\omega)$$

is continuous at $\sigma^i\omega_0$, $i \in \{0, \ldots, m_0 - 1\}$. *However, this follows from the fact that* $v \in Z$ *(and thus* $v \in Y$*) and our assumption that* $\omega \mapsto T_\omega$ *(and thus also* $\omega \mapsto \mathcal{L}_\omega$*) is locally constant along the orbit of* ω_0. *It follows from* (14) *that* $\mathbb{L}$ *has the unique fixed point* $h^0 \in Z$. *This easily implies that*

$$\omega \mapsto h^0_\omega(g(\omega, \cdot)), \quad h^0_\omega := h^0(\omega)$$

is continuous at $\sigma^i\omega_0$, $i = 0, \ldots, m_0 - 1$ *and therefore* (54) *will remain valid even after centering.*

Proof of Lemma 9.1. We will prove the desired equicontinuity property in ω_0. The argument for all other points in the orbit of ω_0 is completely analogous. Observe that for all $\omega \in \Omega$ sufficiently close to ω_0, we have that $\mathcal{L}_\omega = \mathcal{L}_{\omega_0}$. Therefore, for all ω close to ω_0, we have that

$$(\mathcal{L}^{it}_\omega - \mathcal{L}^{it}_{\omega_0})(h) = \mathcal{L}_{\omega_0}((e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)})h),$$

and thus

$$\|(\mathcal{L}^{it}_\omega - \mathcal{L}^{it}_{\omega_0})(h)\|_{1,1} \leq \|\mathcal{L}_{\omega_0}\|_{1,1} \cdot \|(e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)})h\|_{1,1},$$

for each $h \in \mathcal{B}^{1,1}$. Observe that

$$\begin{aligned}&\|(e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)})h\|_{1,1} = \\ &= \max\{\|(e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)})h\|^\sim_{0,1}, \|(e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)})h\|^\sim_{1,2}\}.\end{aligned}$$

As in the proofs of Propositions 4.3 and 4.4, we need to estimate

$$\|e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)}\|_{C^2}.$$

Take $x \in X$. By applying the mean-value theorem for the map $z \mapsto e^{itz}$, we see that

$$|e^{itg(\omega,x)} - e^{itg(\omega_0,x)}| \leq |t| \cdot |g(\omega, x) - g(\omega_0, x)|.$$

Thus,

$$\|e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)}\|_{C^0} \leq |t| \cdot \|g(\omega,\cdot) - g(\omega_0,\cdot)\|_{C^0},$$

which implies that

$$\|e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)}\|_{C^0} \leq \max\{|t| : t \in J\}\|g(\omega,\cdot) - g(\omega_0,\cdot)\|_{C^2}. \tag{55}$$

Moreover, we have that

$$\begin{aligned}\partial^j(e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)}) &= ite^{itg(\omega,\cdot)}\partial^j(g(\omega,\cdot)) - ite^{itg(\omega_0,\cdot)}\partial^j(g(\omega_0,\cdot)) \\ &= ite^{itg(\omega,\cdot)}\partial^j(g(\omega,\cdot)) - ite^{itg(\omega_0,\cdot)}\partial^j(g(\omega,\cdot)) \\ &\quad + ite^{itg(\omega_0,\cdot)}\partial^j(g(\omega,\cdot)) - ite^{itg(\omega_0,\cdot)}\partial^j(g(\omega_0,\cdot)),\end{aligned}$$

for each $j \in \{1,\ldots,d\}$. By (55), we have that

$$\begin{aligned}\|\partial^j(e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)})\|_{C^0} &\leq \max\{|t|^2 : t \in J\}\|g(\omega,\cdot) - g(\omega_0,\cdot)\|_{C^2} \cdot \|g(\omega,\cdot)\|_{C^2} \\ &\quad + \max\{|t| : t \in J\}\|g(\omega,\cdot) - g(\omega_0,\cdot)\|_{C^2},\end{aligned}$$

for every $j \in \{1,\ldots,d\}$. Thus,

$$\max_{1\leq j\leq d}\|\partial^j(e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)})\|_{C^0} \leq C\|g(\omega,\cdot) - g(\omega_0,\cdot)\|_{C^2}, \tag{56}$$

for some $C > 0$ which is independent on t and ω. Finally, for each $k, j \in \{1,\ldots,d\}$, we have that

$$\begin{aligned}\partial^k\partial^j(e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)}) &= -t^2e^{itg(\omega,\cdot)}\partial^k(g(\omega,\cdot))\partial^j(g(\omega,\cdot)) \\ &\quad + ite^{itg(\omega,\cdot)}\partial^k\partial^j(g(\omega,\cdot)) \\ &\quad + t^2e^{itg(\omega_0,\cdot)}\partial^k(g(\omega_0,\cdot))\partial^j(g(\omega_0,\cdot)) \\ &\quad - ite^{itg(\omega_0,\cdot)}\partial^k\partial^j(g(\omega_0,\cdot)).\end{aligned}$$

Observe that

$$\begin{aligned}ite^{itg(\omega,\cdot)}\partial^k\partial^j(g(\omega,\cdot)) - ite^{itg(\omega_0,\cdot)}\partial^k\partial^j(g(\omega_0,\cdot)) &= ite^{itg(\omega,\cdot)}\partial^k\partial^j(g(\omega,\cdot)) \\ &\quad - ite^{itg(\omega,\cdot)}\partial^k\partial^j(g(\omega_0,\cdot)) \\ &\quad + ite^{itg(\omega,\cdot)}\partial^k\partial^j(g(\omega_0,\cdot)) \\ &\quad - ite^{itg(\omega_0,\cdot)}\partial^k\partial^j(g(\omega_0,\cdot)).\end{aligned}$$

Thus (using (55)),

$$\begin{aligned}&\|ite^{itg(\omega,\cdot)}\partial^k\partial^j(g(\omega,\cdot)) - ite^{itg(\omega_0,\cdot)}\partial^k\partial^j(g(\omega_0,\cdot))\|_{C^0} \\ &\leq \max\{|t| : t \in J\}\|g(\omega,\cdot) - g(\omega_0,\cdot)\|_{C^2} \\ &\quad + \max\{|t|^2 : t \in J\}\|g(\omega,\cdot) - g(\omega_0,\cdot)\|_{C^2}\|g(\omega_0,\cdot)\|_{C^2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&- t^2e^{itg(\omega,\cdot)}\partial^k(g(\omega,\cdot))\partial^j(g(\omega,\cdot)) + t^2e^{itg(\omega_0,\cdot)}\partial^k(g(\omega_0,\cdot))\partial^j(g(\omega_0,\cdot)) \\ &= -t^2e^{itg(\omega,\cdot)}\partial^k(g(\omega,\cdot))\partial^j(g(\omega,\cdot)) + t^2e^{itg(\omega_0,\cdot)}\partial^k(g(\omega,\cdot))\partial^j(g(\omega,\cdot)) \\ &\quad - t^2e^{itg(\omega_0,\cdot)}\partial^k(g(\omega,\cdot))\partial^j(g(\omega,\cdot)) + t^2e^{itg(\omega_0,\cdot)}\partial^k(g(\omega_0,\cdot))\partial^j(g(\omega,\cdot)) \\ &\quad - t^2e^{itg(\omega_0,\cdot)}\partial^k(g(\omega_0,\cdot))\partial^j(g(\omega,\cdot)) + t^2e^{itg(\omega_0,\cdot)}\partial^k(g(\omega_0,\cdot))\partial^j(g(\omega_0,\cdot))\end{aligned}$$

Hence, (55) implies that

$$\begin{aligned}
&\|-t^2 e^{itg(\omega,\cdot)}\partial^k(g(\omega,\cdot))\partial^j(g(\omega,\cdot)) + t^2 e^{itg(\omega_0,\cdot)}\partial^k(g(\omega_0,\cdot))\partial^j(g(\omega_0,\cdot))\|_{C^0}\\
&\leq \max\{|t|^3 : t \in J\}\|g(\omega,\cdot) - g(\omega_0,\cdot)\|_{C^2}\cdot\|g(\omega,\cdot)\|_{C^2}^2\\
&\quad + t^2\|g(\omega,\cdot) - g(\omega_0,\cdot)\|_{C^2}\cdot\|g(\omega,\cdot)\|_{C^2}\\
&\quad + t^2\|g(\omega,\cdot) - g(\omega_0,\cdot)\|_{C^2}\cdot\|g(\omega_0,\cdot)\|_{C^2}.
\end{aligned}$$

We conclude that (by increasing C) we have that

$$\begin{aligned}
&\sup_{1\leq k,j\leq d}\|-t^2 e^{itg(\omega,\cdot)}\partial^k(g(\omega,\cdot))\partial^j(g(\omega,\cdot)) + t^2 e^{itg(\omega_0,\cdot)}\partial^k(g(\omega_0,\cdot))\partial^j(g(\omega_0,\cdot))\|_{C^0}\\
&\leq C\|g(\omega,\cdot) - g(\omega_0,\cdot)\|_{C^2}.
\end{aligned}$$

Thus,

$$\|e^{itg(\omega,\cdot)} - e^{itg(\omega_0,\cdot)}\|_{C^2} \leq C\|g(\omega,\cdot) - g(\omega_0,\cdot)\|_{C^2},$$

and

$$\|\mathcal{L}_\omega^{it} - \mathcal{L}_{\omega_0}^{it}\| \leq C\|g(\omega,\cdot) - g(\omega_0,\cdot)\|_{C^2},$$

for $t \in J$ and ω in a neighborhood of ω_0. The conclusion of the lemma follows directly from (54). □

Hence, it follows from Lemma 9.1 (together with Remark 9.2) that HK A1 and (54) imply that the first requirement in HK A2 holds. We now consider the second requirement in HK A2. As a direct consequence of the following lemma, we show that (52) holds without any additional assumptions.

Lemma 9.3. *For each $t \in \mathbb{R}$, there exist $A_t, B_t > 0$, $0 < \gamma_t < 1$ such that for every $n \geq 0, h \in \mathcal{B}^{1,1}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,*

$$\|\mathcal{L}_\omega^{it,(n)}h\|_{1,1} \leq A_t\gamma_t^n\|h\|_{1,1} + B_t\|h\|_{0,2}.$$

Moreover, for each $J \subset \mathbb{R}$ compact interval, we have that

$$\sup_{t\in J}\max\{A_t, B_t\} < \infty.$$

Proof. (Sketch). The proof follows verbatim the proof of Lemma 6.3 in [17], with two differences. First, we work with composition of maps, but if they are close enough we can easily adapt the *deterministic* arguments (we recall that this was explicitly emphasized in [17, Section 7], in particular allowing for a *random version* of [17, Lemma 3.3] to be applied). Second, since we use the twisted operator instead of the usual one, in the various estimates in the proof of Lemma 6.3 [17] we find the extra multiplicative factor $e^{itS_n g(\omega,\cdot)}$. The proof of Lemma 6.3 [17] is done by induction on the index p and the first step is to get a weak version of the Lasota–Yorke, namely for each $t \in \mathbb{R}$, there exists $C_t \geq 0$ such that for every $n \geq 0$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\|\mathcal{L}_\omega^{it,(n)}\|_{0,q} \leq C_t. \tag{57}$$

We now prove (57) (with $q = 1$) to show how to handle the additional multiplicative factor. We use the notation as in [17, p.202]. Recall that

$$\mathcal{L}_\omega^{(n)}(h)(x) = \frac{h((T_\omega^{(n)})^{-1}(x))}{|\det DT_\omega^{(n)}((T_\omega^{(n)})^{-1}(x))|},$$

and therefore

$$\mathcal{L}_\omega^{it,(n)}(h)(x) = \frac{e^{itS_n g(\omega,(T_\omega^{(n)})^{-1}(x))}h((T_\omega^{(n)})^{-1}(x))}{|\det DT_\omega^{(n)}((T_\omega^{(n)})^{-1}(x))|},$$

for each $h \in C^r$ and $x \in X$. Thus,

$$\int_W \mathcal{L}_\omega^{it,(n)} h \cdot \varphi = \int_{(T_\omega^{(n)})^{-1}(W)} \bar{h}_n \cdot \varphi \circ T_\omega^{(n)} \cdot J_W T_\omega^{(n)},$$

where $J_W T_\omega^{(n)}$ is the Jacobian of $T_\omega^{(n)} \colon (T_\omega^{(n)})^{-1}(W) \to W$ and

$$\bar{h}_n := \frac{h e^{itS_n g(\omega,\cdot)}}{|\det DT_\omega^{(n)}|}.$$

Let $\varphi_j = \varphi \circ T_\omega^{(n)} \cdot \rho_j$, where $\rho_1, \dots, \rho_\ell$ is a partition of unity on $(T_\omega^{(n)})^{-1}W$, as provided by (the random analogue of) Lemma 3.3 [17] (using $\gamma = 1$), and $W_1, \dots, W_\ell$ the corresponding admissible leaves such that $(T_\omega^{(n)})^{-1}(W) \subset \cup_{j=1}^{\ell} W_j$. Hence, [17, (6.2)] becomes

$$\left| \int_{W_j} \bar{h}_n \cdot \varphi_j \cdot J_W T_\omega^{(n)} \right| \le C\|h\|_{0,1} \left| |\det DT_\omega^{(n)}|^{-1} \cdot e^{itS_n g(\omega,\cdot)} \cdot \varphi_j \cdot J_W T_\omega^{(n)} \right|_{C^1(W_j)}.$$

Note that

$$\left| |\det DT_\omega^{(n)}|^{-1} \cdot e^{itS_n g(\omega,\cdot)} \cdot \varphi_j \cdot J_W T_\omega^{(n)} \right|_{C^1(W_j)} \le \left| |\det DT_\omega^{(n)}|^{-1} \cdot J_W T_\omega^{(n)} \right|_{C^1(W_j)} \cdot |\varphi_j|_{C^1(W_j)} \cdot |e^{itS_n g(\omega,\cdot)}|_{C^1(W_j)}.$$

It follows from [17, Lemma 6.2] that

$$\sum_{j \le \ell} ||\det DT_\omega^{(n)}|^{-1} \cdot J_W T_\omega^{(n)}|_{C^1(W_j)} \le C.$$

In addition, from the argument at the bottom of [17, p. 203], it follows that

$$|\varphi_j|_{C^1(W_j)} \le |\varphi \circ T_\omega^{(n)}|_{C^1(W_j)} \cdot |\rho_j|_{C^1(W_j)} \le C.$$

Hence, in order to complete the proof of the weak Lasota–Yorke inequality, it is sufficient to show that

$$|e^{itS_n g(\omega,\cdot)}|_{C^1(W_j)} \le C. \tag{58}$$

Note that

$$|e^{itS_n g(\omega,\cdot)}|_{C^0(W_j)} = 1 \quad \text{and}$$

$$|\partial^\alpha (e^{itS_n g(\omega,\cdot)})|_{C^0(W_j)} = |t| \cdot |\partial^\alpha (S_n g(\omega,\cdot))|_{C^0(W_j)} \le |t| \sum_{i=0}^{n-1} |\partial^\alpha (g(\sigma^i\omega, T_\omega^{(i)}(\cdot)))|_{C^0(W_j)}.$$

In order to bound $|\partial^\alpha (g(\sigma^i\omega, T_\omega^{(i)}(\cdot)))|_{C^0(W_j)}$, we proceed as in [7, (4.3)]. For each i and $x, y \in W_j$, we have

$$\begin{aligned} \frac{|g(\sigma^i\omega, T_\omega^{(i)}x) - g(\sigma^i\omega, T_\omega^{(i)}y)|}{d(x,y)} &= \frac{|g(\sigma^i\omega, T_\omega^{(i)}x) - g(\sigma^i\omega, T_\omega^{(i)}y)|}{d(T_\omega^{(i)}x, T_\omega^{(i)}y)} \cdot \frac{d(T_\omega^{(i)}x, T_\omega^{(i)}y)}{d(x,y)} \\ &\le C\nu^i \operatorname{ess\,sup}_{\omega\in\Omega} \|g(\omega,\cdot)\|_{C^1}, \end{aligned}$$

and thus

$$|\partial^\alpha (g(\sigma^i\omega, T_\omega^{(i)}(\cdot)))|_{C^0(W_j)} \le C\nu^i \operatorname{ess\,sup}_{\omega\in\Omega} \|g(\omega,\cdot)\|_{C^1}.$$

In view of (19), (58) holds. Now one can repeat arguments in [17] to obtain the weak Lasota–Yorke inequality for the twisted cocycle, (57). The proof of the strong Lasota–Yorke inequality can be obtained in a similar manner. □

9.2. **Discussion regarding HK A3.** In this subsection we discuss how to ensure that HK A3 holds. We begin by emphasizing that this assumption is purely deterministic since it deals only with (deterministic) operators $\mathcal{L}_{\omega_0}^{it,(m_0)}$, $t \in \mathbb{R} \setminus \{0\}$.

Let us first assume for simplicity that ω_0 is a fixed point for σ and suppose that T_{ω_0} is transitive. By μ_{SRB} we will denote the unique SRB measure for T_{ω_0}. We will show that HK A3 holds if $g(\omega_0, \cdot)$ cannot be written in the form

$$g(\omega_0, \cdot) = \psi \circ T_{\omega_0} - \psi + \frac{2\pi}{t}k + r, \tag{59}$$

for $t \in \mathbb{R} \setminus \{0\}$, $r \in \mathbb{R}$, ψ measurable and k integer-valued maps.

In order to prove this, let us fix $t \in \mathbb{R} \setminus \{0\}$. It follows from Lemma 9.3 (together with the corresponding weak LY-inequality that can be established in analogous manner) that $\mathcal{L}_{\omega_0}^{it}$ is quasicompact and that its spectral radius satisfies $r(\mathcal{L}_{\omega_0}^{it}) \leq 1$. Let us assume that $r(\mathcal{L}_{\omega_0}^{it}) = 1$. Then, there exists $z \in \mathbb{C}$, $|z| = 1$ and $h \in \mathcal{B}^{1,1}$ such that

$$\mathcal{L}_{\omega_0}^{it} h = zh. \tag{60}$$

We need the following auxiliary result.

Lemma 9.4. *We have that h is a complex measure. In addition, h is absolutely continuous with respect to μ_{SRB}.*

Proof. By adapting the proofs of Lemmas 3.3 and 5.3 in [7], there exists a constant $\hat{C} > 0$ such that for $\varphi \in C^1(X, \mathbb{C})$, we have

$$|h(\varphi)| = |h(e^{itS_n g(\omega_0,\cdot)} \varphi \circ T_{\omega_0}^n)| \leq \hat{C}||h||_{0,2} \left(|\varphi|_\infty (1 + |D^s(e^{itS_n g(\omega_0,\cdot)})|_\infty) + |D^s(\varphi \circ T_{\omega_0}^n)|_\infty\right)$$

$$\leq \hat{C}||h||_{0,2} \left(|\varphi|_\infty + \nu_{\omega_0}^n |D^s(\varphi)|_\infty\right),$$

where $D^s\phi$ denotes derivative of ϕ along the stable direction. The first inequality follows from Lemma 3.3 [7]. The estimate for $|D^s(e^{itS_n g(\omega_0,\cdot)})|_\infty$, which follows from (19) and (58) has been absorbed into $\hat{C}$ in the second inequality. Finally the term $\nu_{\omega_0}^{-1}$ has been introduced in section 2 (and it was called ν^{-1} there since it referred to the map T), as the expansion rate of $DT_{\omega_0}^{-1}$ in the stable cone. By sending $n \to \infty$, we conclude that h is a measure.

We now prove that $h \ll \mu_{SRB}$. We will follow closely the arguments in the proof of [7, Lemma 5.5]. Let Π_z^{it} denote the projection onto the eigenspace corresponding to z. Then, we have that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} z^{-k} (\mathcal{L}_{\omega_0}^{it})^k = \Pi_z^{it}.$$

Choose $h' \in C^r(X,\mathbb{C})$ such that $h = \Pi_z^{it} h'$. Hence, for each $\varphi \in C^1(X,\mathbb{C})$ we have that

$$\begin{aligned}
|h(\varphi)| = |\Pi_z^{it} h'(\varphi)| &\leq \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\mathcal{L}_{\omega_0}^k (e^{itS_k g(\omega_0,\cdot)} h')(\varphi)| \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Big| \int_X e^{itS_k g(\omega_0,\cdot)} (\varphi \circ T_{\omega_0}^k) h' \, dm \Big| \\
&\leq \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_X (|\varphi| \circ T_{\omega_0}^k) |h'| \, dm \\
&\leq \|h'\|_{C^0} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} (\mathcal{L}_{\omega_0}^k 1)(|\varphi|) \\
&= \|h'\|_{C^0} \cdot \mu_{SRB}(|\varphi|).
\end{aligned}$$

This easily implies that $h \ll \mu_{SRB}$.

□

Set $\rho := \frac{dh}{d\mu_{SRB}}$. Observe that we can normalize h to ensure that $\int_X |\rho| \, d\mu_{SRB} = 1$. For each $\varphi \in C^1(X,\mathbb{C})$, it follows from (60) that

$$\mathcal{L}_{\omega_0}(e^{itg(\omega_0,\cdot)} h)(\varphi) = z h(\varphi),$$

and therefore

$$\int_X e^{itg(\omega_0,\cdot)} (\varphi \circ T_{\omega_0}) \rho \, d\mu_{SRB} = z \int_X (\varphi \circ T_{\omega_0})(\rho \circ T_{\omega_0}) \, d\mu_{SRB}.$$

This easily implies that

$$e^{itg(\omega_0,\cdot)} \rho = z(\rho \circ T_{\omega_0}), \quad \mu_{SRB}\text{-a.e.} \tag{61}$$

Hence,

$$|\rho| = |\rho| \circ T_{\omega_0}, \quad \mu_{SRB}\text{-a.e.}$$

Then, it follows from ergodicity of μ_{SRB} that $|\rho| = 1$, μ_{SRB}-a.e. Hence, by writing z in the form $z = e^{ir}$ for $r \in \mathbb{R}$, it follows from (61) that $g(\omega_0,\cdot)$ can be written in the form (59), which yields a contradiction. We conclude that $r(\mathcal{L}_{\omega_0}^{it}) < 1$ for $t \in \mathbb{R} \setminus \{0\}$. This then implies HK A3 by applying [20, Corollary III.13].

Assume now that ω_0 is a periodic point for σ of period m_0 and that T_{ω_0} is transitive (which also implies that $T_{\omega_0}^{m_0}$ is transitive). Moreover, suppose that $S_{m_0} g(\omega_0,\cdot)$ cannot be written in the form

$$S_{m_0} g(\omega_0,\cdot) = \psi \circ T_{\omega_0} - \psi + \frac{2\pi}{t} k + r, \tag{62}$$

for $t \in \mathbb{R} \setminus \{0\}$, $r \in \mathbb{R}$, ψ measurable and k integer-valued. By arguing as in the case $m_0 = 1$, one can show that HK A3 holds.

9.3. **Conclusion.** We now state the following simple consequence of Theorem C and the discussion in previous subsections that represents a more operable version of the local central limit theorem.

Corollary 9.5. *Let $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{B}^{1,1}, \mathcal{L})$ be as in Section 3 and suppose that the observable $g\colon \Omega \times X \to \mathbb{R}$ satisfies (19) and (20). Moreover, we assume that:*

- *$\mathbb{P}$ assigns positive measure to open sets, σ is a homeomorphism and there exist $\omega_0 \in \Omega$ and $m_0 \in \mathbb{N}$ so that $\sigma^{m_0}\omega_0 = \omega_0$. Moreover for each $i \in \{0, 1, \ldots, m_0 - 1\}$, there exists a neighborhood of $\sigma^i \omega_0$ on which the map $\omega \mapsto T_\omega$ is constant;*

- *for each* $i \in \{0, 1, \ldots, m_0 - 1\}$*, the observable* g *satisfies* (54)*;*
- $S_{m_0} g(\omega_0, \cdot)$ *cannot be written in the form* (62) *for* $t \in \mathbb{R} \setminus \{0\}$*,* $r \in \mathbb{R}$*,* ψ *measurable and* k *integer-valued.*

Then, for $\mathbb{P}$*-a.e.* $\omega \in \Omega$ *and every bounded interval* $J \subset \mathbb{R}$*, we have*

$$\lim_{n\to\infty} \sup_{s\in\mathbb{R}} \left| \Sigma\sqrt{n}\mu_\omega(s + S_n g(\omega, \cdot) \in J) - \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2n\Sigma^2}} |J| \right| = 0.$$

10. Piecewise hyperbolic dynamics

In this section, we apply the previous theory to obtain statistical laws for the random compositions $T_\omega^{(n)} = T_{\sigma^{n-1}\omega} \circ \cdots \circ T_{\sigma\omega} \circ T_\omega$ of piecewise uniformly hyperbolic maps T_ω of the type studied in [7]. The class of maps T_ω considered contains piecewise toral automorphisms and piecewise hyperbolic maps with bounded derivatives; see Remark 2.2 [7].

10.1. **Preliminaries.** We follow the construction of [7]. Let X be a two-dimensional compact Riemannian manifold, possibly with boundary and not necessarily connected and let $T\colon X \to X$ be a piecewise hyperbolic map in the sense of [7]. That is, the domain X is broken into a finite number of pairwise disjoint open regions $\{X_i^+\}$ with piecewise C^1 boundary curves of finite length, such that $\bigcup_i \overline{X_i^+} = X$. The image of each X_i^+ under T is denoted $X_i^- = T(X_i^+)$; we assume that $\bigcup \overline{X_i^-} = X$. The sets $\mathcal{S}^\pm := X \setminus \bigcup_i X_i^\pm$ are the "singularity sets" for T and T^{-1}, respectively. Assume that T is a C^2 diffeomorphism from the complement of $\mathcal{S}^+$ to the complement of $\mathcal{S}^-$, and that for each i, there is a C^2 extension of T to $\overline{X_i^+}$. On each X_i, the map T is uniformly hyperbolic: there are two continuous, strictly DT-invariant families of cones C^s and C^u defined on $X \setminus (\mathcal{S}^+ \cup \partial X)$ satisfying

$$\begin{aligned}
\lambda &:= \inf_{x\in X\setminus\mathcal{S}^+} \inf_{v\in C^u} \frac{\|DTv\|}{\|v\|} > 1, \\
\mu &:= \inf_{x\in X\setminus\mathcal{S}^+} \inf_{v\in C^s} \frac{\|DTv\|}{\|v\|} < 1, \\
\mu_+^{-1} &:= \inf_{x\in X\setminus\mathcal{S}^-} \inf_{v\in C^s} \frac{\|DT^{-1}v\|}{\|v\|} > 1.
\end{aligned}$$

Assume that vectors tangent to the singularity curves in $\mathcal{S}^-$ are bounded away from C^s. The singularity curves and their images and preimages should not intersect at too many points. Denote by $\mathcal{S}_n^-$ (resp. $\mathcal{S}_n^+$) the set of singularity curves for T^{-n} (resp. T^n), and let $M(n)$ denote the maximum number of singularity curves that meet at a single point. Assume that there is an α_0 and an integer $n_0 > 0$ such that $\lambda\mu^{\alpha_0} > 1$ and $(\lambda\mu^{\alpha_0})^{n_0} > M(n_0)$; this condition is satisfied if $M(n)$ has polynomial growth, for example.

For each $n \in \mathbb{N}$, let $\mathcal{K}_n$ be the set of connected components of $X \setminus \mathcal{S}_n^+$, and let $C^1(\overline{K}, \mathbb{R})$ be the set of functions $\varphi \in C^1(\mathring{K}, \mathbb{R})$ with C^1 extension in a neighbourhood of $\overline{K}$. Let $(C^1_{\mathcal{S}_n^+})' := \{\varphi \in L^\infty(X) : \varphi \in C^1(\overline{K}, \mathbb{R})\ \forall K \in \mathcal{K}_n\}$. If $h \in (C^1_{\mathcal{S}_n^+})'$ is an element of the dual of $C^1_{\mathcal{S}_n^+}$, then $\mathcal{L} : (C^1_{\mathcal{S}_n^+})' \to (C^1_{\mathcal{S}_{n-1}^+})'$ acts on h by

$$\mathcal{L}h(\varphi) = h(\varphi \circ T) \quad \forall \varphi \in C^1_{\mathcal{S}_{n-1}^+}.$$

In order to obtain useful spectral information from $\mathcal{L}$, its action is restricted to a Banach space $\mathcal{B}$, analogous to the space $\mathcal{B}^{p,q} = \mathcal{B}^{1,1}$ in Section 2. We now briefly outline the construction of the norms on $\mathcal{B}$ and an associated "weak" space $\mathcal{B}_w$; see [7] for details.

The norms are defined using "admissible leaves" W in a set of admissible leaves Σ. These leaves are smooth curves in approximately the stable direction, and are analogues of the $\psi_i \circ (\mathrm{Id}, \chi)$ defined in Section 2. Since we are going to recall several times estimates in [7], we intend to comply with the notation there. In particular the functions χ defined on the charts will now become F and the image of the graph of F, namely the *admissible leaves*, will be denoted with G_F. For $\alpha, \beta, q < 1$ such that $0 < \beta \leq \alpha \leq 1 - q \leq \alpha_0$ let $C^\alpha(W, \mathbb{C})$ denote the set of continuous complex-valued functions on W with Hölder exponent α and define the norm

$$|\varphi|_{W,\alpha,q} := |W|^\alpha \cdot |\varphi|_{C^q(W,\mathbb{C})}, \tag{63}$$

where $|W|$ denotes unnormalised induced Riemannian volume of W. For $h \in C^1(X, \mathbb{C})$ we define the weak norm of h by

$$|h|_w = \sup_{W \in \Sigma} \sup_{\substack{\varphi \in C^1(W,\mathbb{C}) \\ |\varphi|_{C^1(W,\mathbb{C})} \leq 1}} \left| \int_W h\varphi \, dm \right|$$

and the strong norm by

$$\|h\| = \|h\|_s + b\|h\|_u,$$

where the strong stable norm is

$$\|h\|_s = \sup_{W \in \Sigma} \sup_{\substack{\varphi \in C^1(W,\mathbb{C}) \\ |\varphi|_{W,\alpha,q} \leq 1}} \left| \int_W h\varphi \, dm \right| \tag{64}$$

and the strong unstable norm is

$$\|h\|_u = \sup_{\epsilon \leq \epsilon_0} \sup_{\substack{W_1, W_2 \in \Sigma \\ d_\Sigma(W_1, W_2) \leq \epsilon}} \sup_{\substack{|\varphi_i|_{C^1(W_i,\mathbb{C})} \leq 1 \\ d_q(\varphi_1, \varphi_2) \leq \epsilon}} \frac{1}{\epsilon^\beta} \left| \int_{W_1} h\varphi_1 \, dm - \int_{W_2} h\varphi_2 \, dm \right|, \tag{65}$$

where d_Σ and d_q are defined precisely in §3.1 [7]. In comparison to the setting in Section 2, the norm $|\cdot|_w$ plays the role of $\|\cdot\|_{p-1,q+1} = \|\cdot\|_{0,2}$, and the norm $\|\cdot\|$ plays the role of $\|\cdot\|_{p,q} = \|\cdot\|_{1,1}$.

Let $\mathcal{B}$ be the completion of $C^1(X, \mathbb{C})$ with respect to the norm $\|\cdot\|$. Similarly, we define $\mathcal{B}_w$ to the completion of $C^1(X, \mathbb{C})$ with respect to the norm $|\cdot|_w$.

We recall that the elements of $\mathcal{B}$ are distributions. More precisely, there exists $C > 0$ such that any $h \in \mathcal{B}$ induces a linear functional $\varphi \to h(\varphi)$ with the property that

$$|h(\varphi)| \leq C|h|_w|\varphi|_{C^1}, \quad \text{for } \varphi \in C^1(X, \mathbb{C}), \tag{66}$$

see [7, Remark 3.4] for details. In particular, for $h \in C^1(X, \mathbb{C})$ we have that (see [7, Remark 2.5])

$$h(\varphi) = \int_X h\varphi, \quad \text{for } \varphi \in C^1(X, \mathbb{C}). \tag{67}$$

We say that $h \in \mathcal{B}$ is nonnegative and write $h \geq 0$ if $h(\varphi) \geq 0$ for any $\varphi \in C^1(X, \mathbb{R})$ such that $\varphi \geq 0$. Finally, we recall (see [7, Section 2.1]) that for $h \in L^1(X, \mathbb{C})$,

$$\mathcal{L}h = \left(\frac{h}{|\det DT|} \right) \circ T^{-1}. \tag{68}$$

Proposition 10.1. *We have that*

$$(\mathcal{L}h)(\varphi) = h(\varphi \circ T), \quad \textit{for } h \in \mathcal{B} \textit{ and } \varphi \in C^1(X, \mathbb{C}).$$

Proof. For $h \in C^1(X, \mathbb{C})$ the desired conclusion can be easily obtained from (67) and (68) by using a change of variables. This immediately implies that the conclusion holds for any $h \in \mathcal{B}$. □

10.2. **Building the cocycle.** This section follows the material in Section 3, replacing $(\mathcal{B}^{1,1}, \|\cdot\|_{1,1})$ with $(\mathcal{B}, \|\cdot\|)$ and $(\mathcal{B}^{0,2}, \|\cdot\|_{0,2})$ with $(\mathcal{B}_w, \|\cdot\|_w)$. We have included this material to make the relevant references to [7] transparent.

Assume from now on that $(T, \bar{\mu})$ is ergodic, where $\bar{\mu}$ is given by [7, Theorem 2.8.]. Moreover, assume that T is topologically mixing. Then, [7, Theorem 2.8] implies that the associated transfer operator $\mathcal{L}_T$ is quasicompact on $\mathcal{B}$, 1 is a simple eigenvalue and there are no other eigenvalues of modulus 1. This in particular implies (using the terminology as in [6, Definition 2.6]) that $\mathcal{L}_T$ is exact in $\{h \in \mathcal{B} : h(1) = 0\}$.

Let Γ_{B_*} and X_ϵ be the sets of maps as defined in [7, Section 2.4]. It then follows from [7, Lemma 3.5] and the discussion on [7, Section 2.4] that there exist $\epsilon, B > 0$ and $c \in (0, 1)$ such that for any $T_1, \ldots, T_n \in X_\epsilon$, we have that

- the unit ball in $\mathcal{B}$ is relatively compact in $\mathcal{B}_w$;
- $|\mathcal{L}_{T_n} \circ \ldots \circ \mathcal{L}_{T_1} h|_w \leq B|h|_w$ for each $n \in \mathbb{N}$ and $h \in \mathcal{B}$;
- $\|\mathcal{L}_{T_n} \circ \ldots \circ \mathcal{L}_{T_1} h\| \leq Bc^n\|h\| + B|h|_w$ for each $n \in \mathbb{N}$ and $h \in \mathcal{B}$.

For $\delta > 0$, set

$$\mathcal{O}_\delta(T, \mathcal{B}) = \left\{ \mathcal{L}_S \colon \mathcal{B} \to \mathcal{B} : S \in X_\epsilon \text{ and } \sup_{\|h\| \leq 1} \|\mathcal{L}_S - \mathcal{L}_T\|_w \leq \delta \right\}.$$

It follows from [6, Proposition 2.10] (applied to the case where $\|\cdot\| = |\cdot|_w$ and $|\cdot|_v = \|\cdot\|$; see also Remark 3.1) that there exist $\delta_0 > 0$, $D, \lambda > 0$ such that for any $\mathcal{L}_{T_1}, \ldots, \mathcal{L}_{T_n} \in \mathcal{O}_{\delta_0}(T, \mathcal{B})$, we have that

$$\|\mathcal{L}_{T_n} \circ \ldots \circ \mathcal{L}_{T_1} h\| \leq De^{-\lambda n}\|h\| \quad \text{for } h \in \mathcal{B} \text{ satisfying } h(1) = 0. \tag{69}$$

On the other hand, [7, Lemma 6.1] implies that there exist $0 < \epsilon_0 \leq \epsilon$ such that

$$\{\mathcal{L}_S : S \in X_{\epsilon_0}\} \subset \mathcal{O}_{\delta_0}(T, \mathcal{B}).$$

We now build our cocycle by prescribing that for each $\omega \in \Omega$, $T_\omega \in X_{\epsilon_0}$ and we consider $\mathcal{L}_\omega$ which is the transfer operator associated to T_ω. Then, it follows readily from (69) that

$$\|\mathcal{L}_\omega^{(n)} h\| \leq De^{-\lambda n}\|h\| \quad \text{for any } \omega \in \Omega,\ n \in \mathbb{N} \text{ and } h \in \mathcal{B},\ h(1) = 0. \tag{70}$$

In addition, we have that

$$|\mathcal{L}_\omega^{(n)} h|_w \leq B|h|_w \quad \text{and} \quad \|\mathcal{L}_\omega^{(n)} h\| \leq Ba^n\|h\| + B|h|_w, \tag{71}$$

for every $\omega \in \Omega$, $n \in \mathbb{N}$ and $h \in \mathcal{B}$. In particular, there exists $K > 0$ such that

$$\|\mathcal{L}_\omega^{(n)} h\| \leq K\|h\| \quad \text{for } \omega \in \Omega,\ n \in \mathbb{N} \text{ and } h \in \mathcal{B}. \tag{72}$$

10.2.1. $\mathbb{P}$-*continuity of* $\omega \mapsto \mathcal{L}_\omega$. We assume Ω is a Borel subset of a complete separable metric space, $\mathcal{F}$ is the Borel sigma-algebra and σ is a homeomorphism. Unfortunately, in this (piecewise-hyperbolic) setting we are unable to establish strong measurability of the map $\omega \mapsto \mathcal{L}_\omega$ under the assumption that $\omega \mapsto T_\omega$ is measurable. In order to be able to apply the weaker version of MET from [12], we ask instead that $\omega \mapsto T_\omega$ is measurable and that it has a countable range.

10.2.2. *Quasi-compactness of the cocycle $\mathcal{L}$ and existence of Oseledets splitting.* By arguing as in the proofs of Propositions 3.2 and 3.3, one can construct a unique family of probability measures $(h^0_\omega)_{\omega\in\Omega} \subset \mathcal{B}$ such that ess $\sup_{\omega\in\Omega}\|h^0_\omega\|$ and for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\mathcal{L}_\omega h^0_\omega = h^0_{\sigma\omega}$ One can now repeat the arguments in the proof of Proposition 3.5 to show that $\mathcal{L}$ is quasi-compact and that $\Lambda(\mathcal{L}) = 0$. Consequently, the multiplicative ergodic theorem (Theorem 17, [12]) yields the existence of a unique $\mathbb{P}$-continuous Oseledets splitting

$$\mathcal{B}_{1,1} = \left(\bigoplus_{j=1}^{l} Y_j(\omega)\right) \oplus V(\omega),$$

where each component of the splitting is equivariant under $\mathcal{L}_\omega$. The $Y_j(\omega)$ are finite-dimensional and have corresponding (finite or infinite) sequence of Lyapunov exponents $0 = \lambda_1 > \lambda_2 > \ldots$. Moreover, $Y_1(\omega)$ is spanned by h^0_ω as in Proposition 3.6.

10.3. **The twisted cocycle.** Our observable will be a map $g\colon \Omega \times X \to \mathbb{R}$ such that $g(\omega, \cdot) \in C^1$ for $\omega \in \Omega$ and

$$M := \text{ess } \sup_{\omega\in\Omega}\|g(\omega, \cdot)\|_{C^1} < \infty. \tag{73}$$

We assume that g is ω-fibrewise centred: for $\mathbb{P}$-a.e. $\omega \in \Omega$, $h^0_\omega(g(\omega, \cdot)) = 0$.

For $g \in C^1(X, \mathbb{C})$ and $h \in \mathcal{B}$, we can introduce $g \cdot h \in \mathcal{B}$ as in Section 2. Furthermore, for $\omega \in \Omega$, $\theta \in \mathbb{C}$, and $h \in \mathcal{B}$ set $\mathcal{L}^\theta_\omega(h) = \mathcal{L}_\omega(e^{\theta g(\omega,\cdot)}h)$. We will need the following lemma (see [8, Lemma 6.1] or [9, Lemma 5.3]).

Lemma 10.2. *For $h \in \mathcal{B}$ and $g \in C^1(X, \mathbb{C})$, we have that*

$$\|gh\| \le C|g|_{C^1}\|h\|,$$

for some $C > 0$, independent of g and h.

The following proposition is analogous to Proposition 4.3.

Proposition 10.3. *There exists a continuous function $K\colon \mathbb{C} \to (0, \infty)$ such that*

$$\|\mathcal{L}^\theta_\omega h\| \le K(\theta)\|h\|, \quad \text{for } h \in \mathcal{B},\ \theta \in \mathbb{C} \text{ and } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{74}$$

Proof. Note that it follows from (72) and Lemma 10.2 that

$$\|\mathcal{L}^\theta_\omega h\| = \|\mathcal{L}_\omega(e^{\theta g(\omega,\cdot)}h)\| \le K\|e^{\theta g(\omega,\cdot)}h\| \le CK|e^{\theta g(\omega,\cdot)}|_{C^1}\|h\|,$$

for $h \in \mathcal{B}$, $\theta \in \mathbb{C}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$. Furthermore, observe that (73) implies that

$$|e^{\theta g(\omega,\cdot)}|_{C^0} \le e^{M|\theta|} \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Similarly, it follows from the mean-value theorem (applied for a map $z \mapsto e^{\theta z}$) and (73) that

$$\sup_{x\neq y} \frac{|e^{\theta g(\omega,x)} - e^{\theta g(\omega,y)}|}{|x - y|} \le |\theta|e^{2M|\theta|} \sup_{x\neq y} \frac{|g(\omega, x) - g(\omega, y)|}{|x - y|} \le M|\theta|e^{2M|\theta|}.$$

The desired conclusion follows directly from the above estimates.

□

Analogously to Proposition 4.4 we have:

Proposition 10.4. *For θ close to 0, the cocycle $(\mathcal{L}^\theta_\omega)_{\omega\in\Omega}$ is quasicompact.*

Proof. We follow closely [11, Lemma 3.13]. Observe (71) and choose $N \in \mathbb{N}$ such that $\gamma := Ba^N < 1$. Hence,

$$\begin{aligned}\|\mathcal{L}_\omega^{\theta,(N)} h\| &\le \|\mathcal{L}_\omega^{(N)} h\| + \|\mathcal{L}_\omega^{\theta,(N)} - \mathcal{L}_\omega^{(N)}\| \cdot \|h\| \\ &\le \gamma \|h\| + B|h|_w + \|\mathcal{L}_\omega^{\theta,(N)} - \mathcal{L}_\omega^{(N)}\| \cdot \|h\|.\end{aligned}$$

On the other hand, we have that

$$\mathcal{L}_\omega^{\theta,(N)} - \mathcal{L}_\omega^{(N)} = \sum_{j=0}^{N-1} \mathcal{L}_{\sigma^{N-j}\omega}^{\theta,(j)} (\mathcal{L}_{\sigma^{N-1-j}\omega}^{\theta} - \mathcal{L}_{\sigma^{N-1-j}\omega}) \mathcal{L}_\omega^{(N-1-j)}.$$

It follows from (72) and (74) that

$$\|\mathcal{L}_\omega^{(N-1-j)}\| \le K^{N-1-j} \quad \text{and} \quad \|\mathcal{L}_{\sigma^{N-j}\omega}^{\theta,(j)}\| \le K(\theta)^j.$$

Furthermore, using (72) and Lemma 10.2, we have that for any $h \in \mathcal{B}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\|(\mathcal{L}_\omega^\theta - \mathcal{L}_\omega)(h)\| = \|\mathcal{L}_\omega(e^{\theta g(\omega,\cdot)} h - h)\| \le K\|(e^{\theta g(\omega,\cdot)} - 1)h\| \le CK|e^{\theta g(\omega,\cdot)} - 1|_{C^1} \|h\|.$$

On the other hand, using (73) and applying the mean value theorem for the map $z \mapsto e^{\theta z}$, it is easy to verify that there exists $C' > 0$ such that for $\theta \in B_{\mathbb{C}}(0,1)$,

$$|e^{\theta g(\omega,\cdot)} - 1|_{C^1} \le C'|\theta| \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{75}$$

Hence, there exists $\tilde{C} > 0$ such that

$$\|\mathcal{L}_\omega^\theta - \mathcal{L}_\omega\| \le \tilde{C}|\theta|, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

We conclude that

$$\|\mathcal{L}_\omega^{\theta,(N)} - \mathcal{L}_\omega^{(N)}\| \le \tilde{C}|\theta| \sum_{j=0}^{N-1} K^{N-1-j} K(\theta)^j,$$

and therefore there exists $\tilde{\gamma} \in (0,1)$ such that for any θ sufficiently close to 0 and $h \in \mathcal{B}$,

$$\|\mathcal{L}_\omega^{\theta,(N)} h\| \le \tilde{\gamma}\|h\| + B|h|_w. \tag{76}$$

Similarly, one can show that there exists $\tilde{B} > 0$ such that for any θ sufficiently close to 0 and $h \in \mathcal{B}$,

$$|\mathcal{L}_\omega^\theta h|_w \le \tilde{B}|h|_w. \tag{77}$$

The conclusion of the proposition follows from (76) and (77) by arguing as in [11, Theorem 3.12]. □

10.4. **Regularity of the top Oseledets space, convexity of Λ.** The regularity of the top Oseledets space of the twisted cocycles follows identically as in Section 5, with Lemma 10.2 used in place of Lemma 3.2 [17] in the proof of Lemma 5.1. Moreover the family of probability measures h_ω^0 will allow us to define the fibred measure μ_ω as we did in Section 6.3.

10.5. **Large deviation principle and central limit theorem.** The results of Sections 6 and 7 follow verbatim with the obvious modifications. Of course, and in order to build the cocycle, we must restrict to the neighborhood $\mathcal{O}_{\delta_0}(T, \mathcal{B})$. We thus obtain our main results for piecewise hyperbolic dynamics.

Theorem D (Quenched large deviations theorem)**.** *In the setting of Section 10, there exists $\epsilon_0 > 0$ and a non-random function $c\colon (-\epsilon_0, \epsilon_0) \to \mathbb{R}$ which is nonnegative, continuous, strictly convex, vanishing only at 0 and such that*

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_\omega(S_n g(\omega,\cdot) > n\epsilon) = -c(\epsilon), \quad \textit{for } 0 < \epsilon < \epsilon_0 \textit{ and } \mathbb{P}\textit{-a.e. } \omega \in \Omega.$$

Theorem E (Quenched central limit theorem). *In the setting of Section 10, assume that the non-random variance Σ^2, defined in (44) satisfies $\Sigma^2 > 0$. Then, for every bounded and continuous function $\phi\colon \mathbb{R} \to \mathbb{R}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, we have*

$$\lim_{n\to\infty} \int \phi\left(\frac{S_n g(\omega, x)}{\sqrt{n}}\right) d\mu_\omega(x) = \int \phi \, d\mathcal{N}(0, \Sigma^2).$$

(The discussion in §6.3 deals with the degenerate case $\Sigma^2 = 0$).

10.6. **Local central limit theorem.**

Theorem F (Quenched Local central limit theorem). *In the setting of Section 10, suppose that condition (L) holds, where the functional norm in (L) is now $\mathcal{B}$.*

Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every bounded interval $J \subset \mathbb{R}$, we have

$$\lim_{n\to\infty} \sup_{s\in\mathbb{R}} \left| \Sigma\sqrt{n}\mu_\omega(s + S_n g(\omega, \cdot) \in J) - \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2n\Sigma^2}} |J| \right| = 0.$$

The LCLT can also be obtained under the assumptions HK A1, A2, A3 and the hypothesis of Lemma 9.1 with the obvious change of the functional space which is now $\mathcal{B}$. The Lasota–Yorke inequality for the twisted operator follows now by adapting the analogous proof in [7] for the usual operator. Moreover, all the discussion and results in Subsections 9.1 and 9.2 remain unchanged in this setting and consequently, we have the analogous statement as in Corollary 9.5. We refrain from formulating it explicitly since it is essentially the same as Corollary 9.5.

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