METASTABILITY OF CERTAIN INTERMITTENT MAPS

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ABSTRACT. We study an intermittent map which has exactly two ergodic invariant densities. The densities are supported on two subintervals with a common boundary point. Due to certain perturbations, leakage of mass through subsets, called *holes*, of the initially invariant subintervals occurs and forces the subsystems to merge into one system that has exactly one invariant density. We prove that the invariant density of the perturbed system converges in the L^1-norm to a *particular* convex combination of the invariant densities of the intermittent map. In particular, we show that the ratio of the weights in the combination equals to the limit of the ratio of the measures of the *holes*.

1. INTRODUCTION

Open and metastable dynamical systems are currently very active topics of research in ergodic theory and dynamical systems. A dynamical system is called open if there is a subset in the phase space, called a *hole*, such that whenever an orbit lands in it, the dynamics of this obit is terminated (see [9, 10] and references therein). A typical example of an open dynamical system is a billiard table with holes. Probabilistic and topological aspects of open dynamical systems have recently been of central interest to ergodic theorists [1, 6, 7, 8, 13, 12, 15].

A dynamical system is called metastable if it has two or more stable states. For example, a system which consists of two adjacent billiard tables that are linked via a small hole in their common boundary is a metastable dynamical system. Researchers have recognised that studying open dynamical systems can bring insights into the dynamics of metastable dynamical systems [11, 14, 15]. In particular, it has been recognised that closed systems that are metastable behave approximately like a collection of open systems: the infrequent transitions between stable states in a metastable system are similar to infrequent escapes from associated open systems [14, 15].

A particularly transparent description of this phenomenon is discussed in the recent work of González-Tokman, Hunt and Wright [14]. In [14], a metastable expanding system is described by a piecewise smooth and expanding interval map which has two invariant sub-intervals and exactly two ergodic invariant densities. Due to small perturbations, the system starts to allow for infrequent leakage through

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subsets (also called *holes*) of the initially invariant sub-intervals, forcing the two invariant sub-systems to merge into one perturbed system which has exactly one invariant density. The authors of [14] proved that the unique invariant density of the perturbed interval map can be approximated by a convex combination of the two invariant densities of the original interval map, with the weights in the combination depending on the sizes of the holes.

In this paper, we depart to the *non-uniformly* hyperbolic setting[1]. In particular, we study an *intermittent map* which has exactly two ergodic invariant densities. The densities are supported on two subintervals with a common boundary point. Due to certain perturbations, leakage of mass through *holes* of the initially invariant subintervals occurs and forces the subsystems to merge into one system that has exactly one invariant density. We prove that the invariant density of the perturbed system converges in the L^1-norm to a *particular* convex combination of the invariant densities of the intermittent map. In particular, we show that the ratio of the weights in the combination equals to the limit of the ratio of the measures of the *holes*.

We would like to comment on the relationship between our work and the issue of statistical stability. The latter is usually established in the context of systems which admit a unique SRB measure (in our case an absolutely continuous invariant measure, *a.c.i.m.*) and which are successively perturbed and the perturbed maps posses an SRB measure too. One way to formulate the statistical stability is by asking that the perturbed density converges to the unperturbed one in L^1, w.r.t. the Lebesgue measure and whenever the SRB measure is absolutely continuous. A general result of this kind has been established by Alves and Viana in the paper [3], and successively by Alves [2] where sufficient conditions are given to prove the statistical stability but still for the same class of maps. The latter is given by non-uniformly expanding maps which admit an induction structure with the first return map which is uniformly expanding, with bounded distortion and finally with *long branches* of the domains of local injectivity. The perturbed map is chosen in an open neighbourhood of the unperturbed one in the C^k topology with $k \geq 2$, and a few more conditions are given to insure that the subsets with the same return times in the induction set are close and moreover the structural parameters of the maps (especially those bounding the derivative and the distortion) could be chosen uniformly in a C^k neighbourhood of the unperturbed map. The main result is that when the perturbed maps converge to the unperturbed ones in the C^k topology then the corresponding densities of the a.c.i.m. converge to each other in the L^1 norm, w.r.t. the Lebesgue measure.

There are two main differences with our situation. First our unperturbed map admit more than one a.c.i.m.; second, the maps are only close in C^0, a better regularity being restored only locally on the open domain of injectivity of the branches. These two facts obliged us to find a completely different proof.

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[1]With the exceptions of [6, 13], most of the results in ergodic theory of open and metastable systems have been obtained for uniformly hyperbolic systems. See also [9, 13] for further details.

In section 2 we recall the result of [14] about metastable expanding maps in a slightly more general setting. In section 3 we introduce our metastable intermittent system and its corresponding induced system. We then show that the induced system satisfies the assumptions of section 2. Moreover, we prove a lemma that relates invariant densities of the induced system to those of the original one. In section 4 we setup the problem of the metastable intermittent. Further, we derive the formula of the *particular* invariant density which is needed to approximate in the L^1-norm the invariant density of the perturbed system. This section also includes the statement of our main result (Theorem 4.3) and the strategy of our proof. Section 5 contains proofs of some technical lemmas and the proof of Theorem 4.3.

Notation.
Δ is an interval subset of $[0,1]$. We denote by m the normalized Lebesgue measure over the unit interval and with $\|\cdot\|_1$ the associated L^1 norm. Given two sequences a_n and b_n, when writing $a_n \lesssim b_n$, or equivalently $a_n = O(b_n)$ with a_n and b_n non-negative, we mean that $\exists C \geq 1$, independent of n and such that $a_n \leq Cb_n$, $\forall n \geq 1$. By $a_n \approx b_n$ we mean that $\exists C \geq 1$, independent of n and such that $C^{-1}b_n \leq a_n \leq Cb_n$, $\forall n \geq 1$. With $a_n \sim b_n$ we mean that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. We will also use the symbols "O" in the usual sense. Finally, $|Z|$ denotes the length of the interval Z.

2. Invariant Densities of Metastable Expanding Maps

2.1. **The expanding system.** Let $\hat{T} : \Delta \to \Delta$ be a map which satisfies the following conditions:

(A1) There exists a countable partition of Δ, which consists of a sequence of intervals $\{I_i\}_{i=1}^{\infty}$, $I_i \cap I_j = \emptyset$ for $i \neq j$, $\bar{I}_i := [c_{i,0}, c_{i+1,0}]$ and there exists $\delta > 0$ such that $\hat{T}_{i,0} := \hat{T}|_{(c_{i,0},c_{i+1,0})}$ is C^2 which extends to a C^2 function $\bar{T}_{i,0}$ on a neighbourhood $[c_{i,0} - \delta, c_{i+1,0} + \delta]$ of $\bar{I}_i$;
(A2) $\inf_{x\in I\setminus\mathcal{C}_0} |\hat{T}'(x)| \geq \beta_0^{-1} > 2$, where $\mathcal{C}_0 = \{c_{i,0}\}_{i=1}^{\infty}$.
(A3) We assume that the collection $\hat{T}(I_i)_{i=1}^{\infty}$ consists only of finitely many different intervals.
(A4) $\exists$ b in the interior of Δ such that $\hat{T}|_{\Delta_*} \subseteq \Delta_*$, where $* \in \{l, r\}$, Δ_* is an interval such that $\Delta_l \cup \Delta_r = \Delta$ and $\Delta_l \cap \Delta_r = \{b\}$.
(A5) Let $H_0 := \hat{T}^{-1}\{b\} \setminus \{b\}$. We call H_0 the set of *infinitesimal holes* and we assume that for every $n \geq 1$, $(\hat{T}^n\mathcal{C}_0) \cap H_0 = \emptyset$.
(A6) $\hat{T}$ verifies the Adler condition, namely there exists a constant $D_A > 0$ such that $\sup_i \sup_{x\in I_i} \frac{|D^2\hat{T}(x)|}{(D\hat{T}(x))^2} \leq D_A$. In this case there will be an a.c.i.m. with a finite number of ergodic components [17]. We will make the assumption that $\hat{T}$ admits exactly two ergodic a.c.i.ms μ_*, such that each μ_* is supported on Δ_* and the corresponding density $\hat{h}_*$ is positive at each of the points of $H_0 \cap \Delta_*$.

2.2. **Perturbations of the expanding system.** Let $\hat{T}_\varepsilon : \Delta \to \Delta$ be a perturbation of $\hat{T}$ which satisfies the following conditions:

(B1) There exists a countable partition of Δ, which consists of a sequence of intervals $\{I_{i,\varepsilon}\}_{i=1}^{\infty}$, $I_{i,\varepsilon} \cap I_{j,\varepsilon} = \emptyset$ for $i \neq j$, $\bar{I}_{i,\varepsilon} := [c_{i,\varepsilon}, c_{i+1,\varepsilon}]$ such that

(i) for each i, $\varepsilon \to c_{i,\varepsilon}$ is a C^2 function for all $\varepsilon \geq 0$ and for ε sufficiently small we have that $[c_{i,\varepsilon}, c_{i+1,\varepsilon}] \subset [c_{i,0} - \delta, c_{i+1,0} + \delta]$;
(ii) $\hat{T}_\varepsilon|_{[c_{i,\varepsilon},c_{i+1,\varepsilon}]}$ has a C^2 extension $\bar{T}_{i,\varepsilon} : [c_{i,0} - \delta, c_{i+1,0} + \delta] \to \mathbb{R}$, and $\bar{T}_{i,\varepsilon} \to \bar{T}_{i,0}$ in the C^2 topology.
(B2) We assume that the collection $\hat{T}_\varepsilon(I_{i,\varepsilon})_{i=1}^{\infty}$ consists only of finitely many different intervals;
(B3) We assume that for each $\varepsilon > 0$, $\hat{T}_\varepsilon$ admits a unique a.c.i.m. with density $\hat{h}_\varepsilon$.
(B4) Boundary condition:
(i) if $b \notin \mathcal{C}_0$, then $\hat{T}(b) = b$ and for all $\varepsilon > 0$, $\hat{T}_\varepsilon(b) = b$;
(ii) if $b \in \mathcal{C}_0$, then $\hat{T}(b-) < b < \hat{T}(b+)$ and for all $\varepsilon > 0$, $b \in \mathcal{C}_\varepsilon$, where $\mathcal{C}_\varepsilon = \{c_{i,\varepsilon}\}_{i=1}^{\infty}$.

2.3. **Holes in the expanding system** $(\hat{T}_\varepsilon, \Delta)$**.** We are interested in perturbations of $\hat{T}$ which produce "leakage" of mass from Δ_l to Δ_r and vice versa. For this purpose we define the following sets:

$$\hat{H}_{l,\varepsilon} := \Delta_l \cap \hat{T}_\varepsilon^{-1}(\Delta_r)$$

and

$$\hat{H}_{r,\varepsilon} := \Delta_r \cap \hat{T}_\varepsilon^{-1}(\Delta_l).$$

The sets $\hat{H}_{l,\varepsilon}$ and $\hat{H}_{r,\varepsilon}$ are called the "left hole" and the "right hole", respectively, of the perturbed expanding system $(\hat{T}_\varepsilon, \Delta)$. Thus, when $\hat{T}_\varepsilon$ allows leakage of mass from Δ_l to Δ_r, this leakage occurs when orbits of $\hat{T}_\varepsilon$ fall in the set $\hat{H}_{l,\varepsilon}$. Similarly, when $\hat{T}_\varepsilon$ allows leakage of mass from Δ_r to Δ_l, this leakage occurs when orbits of $\hat{T}_\varepsilon$ fall in the set $\hat{H}_{r,\varepsilon}$.

Following [14] the *limiting hole ratio* $(l.h.r)$ is defined by

$$l.h.r = \lim_{\varepsilon \to 0} \frac{\hat{\mu}_r(\hat{H}_{r,\varepsilon})}{\hat{\mu}_l(\hat{H}_{l,\varepsilon})},$$

whenever the limit exists.

In the following we will denote by $BV([u, v])$ the space of functions of bounded variation defined on the closed interval $[u, v]$. We will equip this set with the complete norm given by the sum of the total variation plus the L^1 norm with respect to m. We denote this norm by $\|\cdot\|_{BV([u,v])}$ and the corresponding Banach space by $BV([u, v])$. By P_ε we denote the Perron-Frobenius operator associated with the map $\hat{T}_\varepsilon$ [4, 5] and acting on $BV(\Delta)$.

Proposition 2.1.

(1) *There exists a* $\beta \in (0, 1)$ *and a* $B \in (0, \infty)$*, such that for any* $\varepsilon \geq 0$ *and* $f \in BV(\Delta)$*, we have*

$$\|P_\varepsilon f\|_{BV(\Delta)} \leq \beta \|f\|_{BV(\Delta)} + B||f||_1.$$

(2) *Suppose that the l.h.r. exists. Then*

$$\lim_{\varepsilon \to 0} ||\hat{h}_\varepsilon - \hat{h}_p||_1 = 0,$$

where $\hat{h}_p = \hat{\lambda}_p \hat{h}_l + (1 - \hat{\lambda}_p)\hat{h}_r$ *and* $\frac{\hat{\lambda}_p}{1-\hat{\lambda}_p} = l.h.r.$.

Proof. The proof of the first statement, which is uniform Lasota-Yorke inequality, is standard for C^2 perturbations of $\hat{T}$ with $|\hat{T}'(x)| \geq \beta_0^{-1} > 2$ and satisfying Adler's condition. The proof of the second statement is exactly the same as the proof provided by [14] for Lasota-Yorke maps with finite number of branches[2]. □

Remark 2.2. It will be important in the following that β and B could be chosen independently of ε and ε small. This could be easily achieved by recalling that those quantities are in fact explicitly determined in terms of the map, we refer to [3] for the details. In particular they depend on: (i) the infimum of the absolute value of the derivative, which we denoted by β_0 for T and which persist larger than 2 by condition (B1); (ii) the constant D_A bounding the Adler's condition which by its definition (see above), could also be chosen uniformly in ε for ε small enough.

3. A metastable intermittent map

A main issue of our work will be to compare a map of the interval with a neutral fixed point (intermittent map), with a perturbation of it. Instead of studying a general class of maps, we prefer to work with a particular example which allows us to analyze in a precise manner the steps of our approach. By looking at the proofs in the following sections, it will be clear that our approach can be extended to other intermittent maps.

3.1. **The intermittent map and its perturbation.** Let $\alpha \in (0,1)$. For each $\varepsilon \geq 0$ define the continuous map $T_\varepsilon : [0,1] \to [0,1]$ by:

$$T_\varepsilon(x) = \begin{cases} T_{1,\varepsilon} := x + 4^\alpha(1+4\varepsilon)x^{1+\alpha} & \text{for } 0 \leq x < \frac{1}{4} \\ T_{2,\varepsilon} := -4(1+2\varepsilon)x + \frac{3}{2} + 3\varepsilon & \text{for } \frac{1}{4} \leq x < \frac{3}{8} \\ T_{3,\varepsilon} := 4x - \frac{3}{2} & \text{for } \frac{3}{8} \leq x < \frac{1}{2} \\ T_{4,\varepsilon}(x) & \text{for } \frac{1}{2} \leq x \leq 1, \end{cases} \quad . \tag{3.1}$$

The component $T_{4,\varepsilon}(x)$ continuously extends $T_{3,\varepsilon}$ on the right; it is piecewise expanding with the absolute value of the derivative bigger that 2, of class C^2 but in the points of relative minima and with a finite number of long branches. We will assume that it has only one spike emerging on the right side of 1/2 (see Figure 2) and this spike is located at the point of relative minimum s_r which does not move with ε. We finally suppose that the height of the spike is exactly ε like on the left side. Notice that for $\varepsilon = 0$, the intermittent map $T_0 := T$ has exactly two ergodic invariant densities, h_l supported on $[0, 1/2]$ and h_r supported on $[1/2, 1]$. Moreover, for any $\varepsilon > 0$, the perturbed map has a unique invariant density h_ε We will elaborate more on the uniqueness of h_ε in the Appendix. The graph of the map is shown in Figure 2. Let us point out that with our assumptions T and T_ε are C^0 close, namely $\lim_{\varepsilon\to 0} \|T_\varepsilon - T\|_0 = 0$. Since T and T_ε are also continuous (and hence uniformly continuos on the closed unit interval), this implies that for

[2]We impose the same conditions as the ones imposed by [14], except that we relax the assumption on the number of branches. Instead of requiring the map to have only finite number of branches, we allow maps with countable number of branches whose image set is finite. The proofs of [14] only depend on exploiting the locations and sizes of the jumps of the sets of discontinuities of the invariant densities h_ε which occur on the forward trajectories of the partition points of $\hat{T}_\varepsilon$. Thus their proof follows verbatim for the class of maps $\hat{T}_\varepsilon$ of this paper.

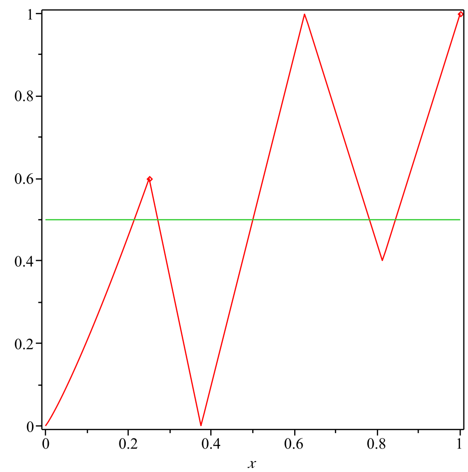

FIGURE 1. The graph of T_ε for the values $\alpha = 0.3$ and $\varepsilon = 0.1$

any $n > 0$ we have as well $\lim_{\varepsilon\to 0} \|T_\varepsilon^n - T^n\|_0 = 0$.

3.2. **Holes in the intermittent system** $(T_\varepsilon, [0,1])$**.** We are interested in perturbations of $\hat{T}$ which produce "leakage" of mass from $I_l := [0, b]$ to $I_r : [b, 1]$ and vice versa. For this purpose we define the following sets:

$$H_{l,\varepsilon} := I_l \cap T_\varepsilon^{-1}(I_r)$$

and

$$H_{r,\varepsilon} := I_r \cap T_\varepsilon^{-1}(I_l).$$

The sets $H_{l,\varepsilon}$ and $H_{r,\varepsilon}$ are called the "left hole" and the "right hole", respectively, of the perturbed intermittent system $(T_\varepsilon, [0, 1])$.

3.3. **The induced system.** For each $\varepsilon \geq 0$, we induce T_ε on the same set $\Delta := [a_0, 1]$, where $a_0 := 1/4$. We also set $b_0 := 1/4$. It is important to notice that a_0 and consequently Δ are independent of ε. Then for $n \geq 1$ we define

$$b_{n+1,\varepsilon} = T_{1,\varepsilon}^{-1}(b_{n,\varepsilon}),\ a_{n,\varepsilon} = T_{2,\varepsilon}^{-1}(b_{n,\varepsilon}),\ \text{and}\ a'_{n,\varepsilon} = T_{3,\varepsilon}^{-1}(b_{n,\varepsilon}).$$

Then for $\varepsilon \geq 0$ we define the induced map $\hat{T}_\varepsilon : \Delta \to \Delta$ by

$$\hat{T}_\varepsilon(x) = \begin{cases} T_\varepsilon(x) & \text{for } x \in Z_{1,\varepsilon} \\ T_\varepsilon^{n+1}(x) & \text{for } x \in Z_{n,\varepsilon} \end{cases}, \tag{3.2}$$

where $Z_{1,\varepsilon} := (a_0, a_{1,\varepsilon}) \cup (a'_{1,\varepsilon}, 1)$ and $Z_{n,\varepsilon} := (a_{n-1,\varepsilon}, a_{n,\varepsilon}) \cup (a'_{n,\varepsilon}, a'_{n-1,\varepsilon})$.

We now define the following sets:

$$W_{0,\varepsilon} := (a_0, 1) \text{ and } W_{n,\varepsilon} := (b_{n,\varepsilon}, b_{n-1,\varepsilon}),\ n \geq 1.$$

Observe that

$$T_\varepsilon(Z_{n,\varepsilon}) = W_{n-1,\varepsilon} \text{ and } \tau_{Z_{n,\varepsilon}} = n,$$

where $\tau_{Z_{n,\varepsilon}}$ is the first return time of $Z_{n,\varepsilon}$ to Δ.

Lemma 3.1.

(1) *For $\varepsilon = 0$, the invariant densities of $\hat{T}$, $\hat{h}_l$ and $\hat{h}_r$, are Lipschitz continuous and bounded away from 0 on $[a_0, b]$, $[b, 1]$ respectively.*
(2) *For $\varepsilon = 0$, the induced map $\hat{T} : \Delta \to \Delta$ satisfies assumptions (A1)-(A6).*
(3) *For $\varepsilon > 0$, the perturbed induced map $\hat{T}_\varepsilon$ satisfies conditions (B1)-(B4).*
(4) *The limiting hole ratio of the induced system*

$$l.h.r = \lim_{\varepsilon \to 0} \frac{\hat{\mu}_r(\hat{H}_{r,\varepsilon})}{\hat{\mu}_l(\hat{H}_{l,\varepsilon})}$$

exists and it is different from zero and infinity.

Proof. Statement (1) follows from the fact that $\hat{T}_{|[a_0,b]}$ is piecewise C^2, piecewise onto and expanding (see [5] for example). The same properties hold for $\hat{T}_{|[b,1]}$. To prove (2), observe that $\sup_{x\in\Delta} |\hat{T}'(x)| > 3$. Moreover, for all $n \geq 1$, $\hat{T}^n(\mathcal{C}_0) = \{b, 1\} \cap H_0 = \emptyset$. Statement (3) is satisfied, in particular, condition (B4). We now prove (4). We first observe that

$$\frac{\mu_r(\hat{H}_{r,\varepsilon})}{\mu_l(\hat{H}_{l,\varepsilon})} = \frac{\int_{\hat{H}_{r,\varepsilon}} \hat{h}_r dx}{\int_{\hat{H}_{l,\varepsilon}} \hat{h}_l dx} = \frac{\hat{h}_r(\xi_{r,\varepsilon})|\hat{H}_{r,\varepsilon}|}{\sum_{k=1}^{\infty} \hat{h}_l(\xi^k_{l,\varepsilon})|Q_{k,\varepsilon}|}$$

where we applied the mean value theorem: $\xi_{r,\varepsilon}$ is a point in $\hat{H}_{r,\varepsilon}$, $Q_{k,\varepsilon} = [a_{k-1,\varepsilon}, w_{k,\varepsilon}]$, where $w_{k,\varepsilon} = \hat{T}_\varepsilon^{-1}(b) \cap Z_{k,\varepsilon}$ and $\xi^k_{l,\varepsilon}$ is a point in $Q_{k,\varepsilon}$. We now treat the denominator; again by the mean value theorem there will be a point $\chi^k_{l,\varepsilon} \in Q_{k,\varepsilon}$ and such that $|Q_{k,\varepsilon}| = \frac{\varepsilon}{|D\hat{T}_\varepsilon(\chi^k_{l,\varepsilon})|}$. Since $\chi^k_{l,\varepsilon} \to a_{k-1}$ when $\varepsilon \to 0$, we write $D\hat{T}_\varepsilon(\chi^k_{l,\varepsilon}) - D\hat{T}(a_{k-1}) = D\hat{T}_\varepsilon(\chi^k_{l,\varepsilon}) - D\hat{T}(\chi^k_{l,\varepsilon}) + D\hat{T}(\chi^k_{l,\varepsilon}) - D\hat{T}(a_{k-1})$. When ε goes to zero the first couple on the right hand side goes to zero by the assumption (B1) (convergence in C^2 over $[a_{k-1}-\delta, a_k+\delta]$), while the second couple goes to zero by the continuity of $D\hat{T}$ still on $[a_{k-1}-\delta, a_k+\delta]$. Therefore, when $\varepsilon \to 0$, the denominator tends to ε times: $\sum_{k=1}^{\infty} \hat{h}_l(a_{k-1})|D\hat{T}(a_{k-1})|^{-1}$. We now show that this quantity is finite and different from 0. First of all the density is bounded away from zero and infinity in the preimages of b by assumption (6); this assumption is surely verified in our case since the density is Lipschitz continuous and bounded from below on $[b_0, b]$. Then we observe that the assumptions (A1, A2, A3, A6) imply that the first return map has bounded distortion. Therefore, there exists a constant C_d independent of k which allows us to bound $|D\hat{T}(a_{k-1})|^{-1} \leq C_d|D\hat{T}(v_k)|^{-1}$ where v_k is a point in $Z_{k,\varepsilon}$ for which the inverse of the derivative gives the length $|Z_{k,\varepsilon}|$ of $Z_{k,\varepsilon}$ times the inverse of the length of $[b_0, b]$; finally the sum over the lengths of the $Z_{k,\varepsilon}$ on $[b_0, b]$ gives of course $b - b_0$. We now bound the numerator; by the assumptions on the branch $\hat{T}_{4,\varepsilon}$ we get immediately that $|\hat{H}_{r,\varepsilon}| = \varepsilon\left[|D\hat{T}_{4,\varepsilon}(u_{l,\varepsilon})|^{-1} + |D\hat{T}_{4,\varepsilon}(u_{r,\varepsilon})|^{-1}\right]$, where $u_{l,\varepsilon}$ (resp. $u_{r,\varepsilon}$) is a point on the left hand side (resp. right hand side) of s_r. By an argument similar to that used above we have that the term in the square bracket converges, as $\varepsilon \to 0$ to $\left[|D_l\hat{T}_4(s_r)|^{-1} + |D_r\hat{T}_4(s_r)|^{-1}\right]$, where $D_l\hat{T}_4(s_r)$ (resp. $D_r\hat{T}_4(s_r)$) denotes the right (resp. left) derivative of $\hat{T}_4$ at the point s_r. □

Remark 3.2. Lemma 3.1 implies that results of Proposition 2.1 hold for the induced system. In particular,

$$\lim_{\varepsilon \to 0} ||\hat{h}_\varepsilon - \hat{h}_p||_1 = 0,$$

where

$$\hat{h}_p := \hat{\lambda}_p \hat{h}_l + (1 - \hat{\lambda}_p)\hat{h}_r. \tag{3.3}$$

and

$$l.h.r. = \frac{\hat{\lambda}_p}{1 - \hat{\lambda}_p}.$$

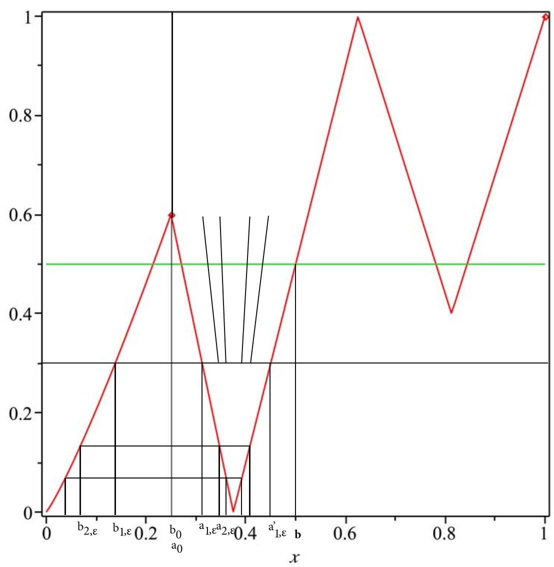

FIGURE 2. The graph of induced system $\hat{T}_\varepsilon$ for the values $\alpha = 0.3$ and $\varepsilon = 0.1$

3.4. **Pulling back the invariant density.** For all $\varepsilon \geq 0$, we can find an a.c.i.m., μ_ε, of T_ε using the a.c.i.m., $\hat{\mu}_\varepsilon$, of $\hat{T}_\varepsilon$ [17]. In particular, for any measurable set $B \subset [0, 1]$, we have

$$\mu_\varepsilon(B) = c_{\tau,\varepsilon} \sum_{n=1}^{\infty} \sum_{j=0}^{\tau_{Z_{n,\varepsilon}-1}} \hat{\mu}_\varepsilon(T_\varepsilon^{-j} B \cap Z_{n,\varepsilon}), \tag{3.4}$$

where $c_{\tau,\varepsilon}^{-1} = \sum_{k=1}^{\infty} \tau_{Z_{k,\varepsilon}} \hat{\mu}_\varepsilon(Z_{k,\varepsilon})$. In the following lemma we provide a lemma expressing the density of μ_ε in terms of that of $\hat{\mu}_\varepsilon$. This will play a crucial role in the proof of our main result.

Lemma 3.3. *Let μ_ε be a T_ε-acim, defined as in (3.4). Then, for $\varepsilon \geq 0$,*

$$h_\varepsilon(x) = \begin{cases} c_{\tau,\varepsilon} \hat{h}_\varepsilon(x) & \text{for } x \in \Delta \\ c_{\tau,\varepsilon} \sum_{n=k+1}^{\infty} \left(\sum_{i=2}^{3} \frac{\hat{h}_\varepsilon(T_{i,\varepsilon}^{-1} T_{1,\varepsilon}^{-(n-k-1)} x)}{|DT_\varepsilon^{(n-k)}(T_{i,\varepsilon}^{-1} T_{1,\varepsilon}^{-(n-k-1)} x)|} \right) & \text{for } x \in W_{k,\varepsilon} \end{cases}, \tag{3.5}$$

where h_ε and $\hat{h}_\varepsilon$ are the densities of μ_ε and $\hat{\mu}_\varepsilon$ respectively.

Proof. By (3.4), for any measurable set $B \subset \Delta$, we have

$$\mu_\varepsilon(B) = c_{\tau,\varepsilon}\hat{\mu}_\varepsilon(B).$$

Passing to the densities and for Lebesgue almost all $x \in \Delta$, we obtain

$$h_\varepsilon(x) = c_{\tau,\varepsilon}\hat{h}_\varepsilon(x).$$

We then extend h_ε to a bounded variation function as $\hat{h}_\varepsilon$. This proves formula (3.5) for $x \in \Delta$.
Now suppose $B = W_{k,\varepsilon}$, for some k. Then by (3.4), we have

$$\mu_\varepsilon(W_{k,\varepsilon}) = c_{\tau,\varepsilon}\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\hat{\mu}_\varepsilon(T_\varepsilon^{-j}W_{k,\varepsilon} \cap Z_{n,\varepsilon}) = c_{\tau,\varepsilon}\sum_{n=k+1}^{\infty}\hat{\mu}_\varepsilon(Z_{n,\varepsilon}).$$

Therefore, if $B \subseteq W_{k,\varepsilon}$, we obtain

$$\mu_\varepsilon(B) = c_{\tau,\varepsilon}\sum_{n=k+1}^{\infty}\hat{\mu}_\varepsilon(T_\varepsilon^{-(n-k)}B \cap Z_{n,\varepsilon}).$$

consequently,

$$\int_B h_\varepsilon dx = \sum_{n=k+1}^{\infty}\int_{T_\varepsilon^{-(n-k)}B\cap Z_{n,\varepsilon}}\hat{h}_{\varepsilon(x)}dx.$$

We now perform the change of variable $T^{n-k}y = x$ by observing that the set B is pushed backward $n-k-1$ times with $T_{1,\varepsilon}^{-1}$ and then it splits into three parts according to the actions of $T_{1,\varepsilon}^{-1}, T_{2,\varepsilon}^{-1}, T_{3,\varepsilon}^{-1}$. Therefore,

$$\int_B h_\varepsilon dx = c_{\tau,\varepsilon}\sum_{n=k+1}^{\infty}\int_B\left(\sum_{i=2}^{3}\frac{\hat{h}_\varepsilon(T_{i,\varepsilon}^{-1}T_{1,\varepsilon}^{-(n-k-1)}y)}{|DT_\varepsilon^{(n-k)}(T_{i,\varepsilon}^{-1}T_{1,\varepsilon}^{-(n-k-1)}y)|}\right)dy,$$

where $DT_\varepsilon^{(n-k)}(z)$ is the derivative of $T_\varepsilon^{(n-k)}$ evaluated at the point z. Thus, for Lebesgue almost all $x \in W_{k,\varepsilon}$ we obtain

$$\begin{aligned} h_\varepsilon(x) &= c_{\tau,\varepsilon}\sum_{n=k+1}^{\infty}\left(\sum_{i=2}^{3}\frac{\hat{h}_\varepsilon(T_{i,\varepsilon}^{-1}T_{1,\varepsilon}^{-(n-k-1)}x)}{|DT_\varepsilon^{(n-k)}(T_{i,\varepsilon}^{-1}T_{1,\varepsilon}^{-(n-k-1)}x)|}\right) \\ &= c_{\tau,\varepsilon}\sum_{n=1}^{\infty}\left(\sum_{i=2}^{3}\frac{\hat{h}_\varepsilon(T_{i,\varepsilon}^{-1}T_{1,\varepsilon}^{-(n-1)}x)}{|DT_\varepsilon^{(n)}(T_{i,\varepsilon}^{-1}T_{1,\varepsilon}^{-(n-1)}x)|}\right). \end{aligned}$$

The last expression shows that h_ε could be extended to a bounded variation function over all Δ^c and therefore over all the unit interval. □

4. The problem of the original intermittent system

4.1. **The problem.** In subsection 3.1 we noted that the intermittent map T has exactly two ergodic invariant densities, h_l supported on $[0, 1/2]$ and h_r supported on $[1/2, 1]$. Moreover, for any $\varepsilon > 0$, the perturbed map has a unique invariant density h_ε. The uniqueness of the invariant density h_ε is proved in the Appendix.

Our main goal is to prove that the invariant density of the perturbed system h_ε converges in the L^1-norm to a *particular* convex combination of the invariant densities, h_l and h_r, of the intermittent map. We define

$$h_p(x) := \begin{cases} c_{\tau,p}\hat{h}_p(x) & \text{for } x \in \Delta \\ c_{\tau,p}\sum_{n=k+1}^{\infty}\left(\sum_{i=2}^{3}\frac{\hat{h}_p(T_i^{-1}T_1^{-(n-k-1)}x)}{|DT^{(n-k)}(T_i^{-1}T_1^{-(n-k-1)}x)|}\right) & \text{for } x \in W_k \end{cases}, \tag{4.1}$$

where $c_{\tau,p}^{-1} = \sum_{k=1}^{\infty} k\hat{\mu}_p(Z_k)$, $\hat{\mu}_p = \hat{\lambda}_p\hat{\mu}_l + (1-\hat{\lambda}_p)\hat{\mu}_r$.

Remark 4.1. Note that, by Lemma 3.3, h_p is a T-invariant density. Moreover, since T has exactly two ergodic invariant densities h_l and h_r, h_p is a convex combination of h_l and h_r. In fact, h_p is a *particular* convex combination of h_l and h_r. In the following proposition, we give an explicit representation of h_p in terms of h_l and h_r.

Proposition 4.2. *The representation of h_p in terms of h_l and h_r is given by*

$$h_p(x) = \lambda_p h_l(x) + (1-\lambda_p)h_r(x),$$

where $\lambda_p = \frac{\hat{\lambda}_p c_{\tau,r}}{\hat{\lambda}_p c_{\tau,r} + (1-\hat{\lambda}_p)c_{\tau,l}}$, $c_{\tau,l}^{-1} = \sum_{k=1}^{\infty} k\hat{\mu}_l(Z_k)$ *and* $c_{\tau,r}^{-1} = \sum_{k=1}^{\infty} k\hat{\mu}_r(Z_k)$.

Proof. First, using Lemma 3.3, we have

$$h_l(x) = \begin{cases} c_{\tau,l}\hat{h}_l(x) & \text{for } x \in \Delta \\ c_{\tau,l}\sum_{n=k+1}^{\infty}\left(\sum_{i=2}^{3}\frac{\hat{h}_l(T_i^{-1}T_1^{-(n-k-1)}x)}{|DT^{(n-k)}(T_i^{-1}T_1^{-(n-k-1)}x)|}\right) & \text{for } x \in W_k \end{cases},$$

and for all $x \in [0,1]$

$$h_r(x) = c_{\tau,r}\hat{h}_r(x).$$

Moreover,

$$c_{\tau,p}^{-1} = \sum_{k=1}^{\infty} k\hat{\mu}_p(Z_k) = \hat{\lambda}_p\sum_{k=1}^{\infty} k\hat{\mu}_l(Z_k) + (1-\hat{\lambda}_p)\sum_{k=1}^{\infty} k\hat{\mu}_r(Z_k) = \hat{\lambda}_p c_{\tau,l}^{-1} + (1-\hat{\lambda}_p)c_{\tau,r}^{-1}.$$

Therefore, using (4.1), for $x \in \Delta$, we have

$$\begin{aligned} h_p(x) &= c_{\tau,p}(\hat{\lambda}_p\hat{h}_l(x) + (1-\hat{\lambda}_p)h_r(x)) \\ &= \hat{\lambda}_p\frac{c_{\tau,l}}{\hat{\lambda}_p + (1-\hat{\lambda}_p)c_{\tau,l}c_{\tau,r}^{-1}}\hat{h}_l(x) + (1-\hat{\lambda}_p)\frac{c_{\tau,r}}{\hat{\lambda}_p c_{\tau,r}c_{\tau,l}^{-1} + (1-\hat{\lambda}_p)}\hat{h}_r(x) \\ &= \frac{\hat{\lambda}_p}{\hat{\lambda}_p + (1-\hat{\lambda}_p)c_{\tau,l}c_{\tau,r}^{-1}}h_l(x) + \frac{(1-\hat{\lambda}_p)}{\hat{\lambda}_p c_{\tau,r}c_{\tau,l}^{-1} + (1-\hat{\lambda}_p)}\hat{h}_r(x) \\ &= \lambda_p h_l(x) + (1-\lambda_p)h_r(x). \end{aligned}$$

Using (4.1) again, for $x \in W_k$, we obtain

$$\begin{aligned} h_p(x) &= c_{\tau,p}\hat{\lambda}_p \sum_{n=k+1}^{\infty}\left(\sum_{i=2}^{3}\frac{\hat{h}_l(T_i^{-1}T_1^{-(n-k-1)}x)}{|DT^{(n-k)}(T_i^{-1}T_1^{-(n-k-1)}x)|}\right)\\ &= \hat{\lambda}_p\frac{c_{\tau,l}}{\hat{\lambda}_p+(1-\hat{\lambda}_p)c_{\tau,l}c_{\tau,r}^{-1}}\sum_{n=k+1}^{\infty}\left(\sum_{i=2}^{3}\frac{\hat{h}_l(T_i^{-1}T_1^{-(n-k-1)}x)}{|DT^{(n-k)}(T_i^{-1}T_1^{-(n-k-1)}x)|}\right)\\ &= \frac{\hat{\lambda}_p}{\hat{\lambda}_p+(1-\hat{\lambda}_p)c_{\tau,l}c_{\tau,r}^{-1}}h_l = \lambda_p h_l(x). \end{aligned}$$

□

4.2. **Main result and the strategy of our proof.** The following theorem is the main result of the paper.

Theorem 4.3. *Let h_ε be the unique invariant density of T_ε. Then*

(1)
$$\lim_{\varepsilon\to 0}||h_\varepsilon - h_p||_1 = 0.$$

(2) *Moreover,*
$$\lim_{\varepsilon\to 0}\frac{\mu_r(H_{r,\varepsilon})}{\mu_l(H_{l,\varepsilon})} = \frac{\lambda_p}{1-\lambda_p}.$$

To prove (1) of Theorem 4.3, we use the following strategy:

(1) First we estimate

$$\begin{aligned} ||h_\varepsilon - h_p||_1 \le \int_\Delta |h_\varepsilon - h_p|dx + \sum_{k=1}^{\infty}\int_{W_k\setminus(W_{k,\varepsilon}\cap W_k)}|h_\varepsilon - h_p|dx\\ + \sum_{k=1}^{\infty}\int_{W_{k,\varepsilon}\cap W_k}|h_\varepsilon - h_p|dx = (I)+(II)+(III). \end{aligned} \tag{4.2}$$

(2) In (I), we exploit the representations of h_p, h_ε on Δ, and use Remark 3.2 to conclude that the limit of (I) is zero as $\varepsilon \to 0$.

(3) In (II), we obtain an estimate like

$$\sup_{x\in W_k\setminus(W_{k,\varepsilon}\cap W_k)}|h_\varepsilon(x)| + \sup_{x\in W_k\setminus(W_{k,\varepsilon}\cap W_k)}|h_\varepsilon(x)| \lesssim k.$$

Since the left boundary point of W_k, b_k, scales like $k^{-\frac{1}{\alpha}}$, we have just recovered, with a different technique, the well known fact that the density of the intermittent map behaves like $x^{-\alpha}$ in the neighbourhood of the neutral fixed point. Consequently, this implies that

$$(II) \lesssim \sum_{k=1}^{\infty}k|W_k\setminus(W_{k,\varepsilon}\cap W_k)| \simeq \sum_{k=1}\frac{1}{k^{1/\alpha}}.$$

and the uniform convergence of the series allows us to bring the limit inside for $\varepsilon \to 0$.

(4) In (III) h_ε and h_p can be compared on $W_{k,\varepsilon}\cap W_k$ via their representations in terms of $\hat{h}_\varepsilon$ and $\hat{h}_p$ respectively. We then show that (III) is summable. This allows us to crush the limit $\varepsilon \to 0$ inside the sum to conclude that the limit of (III) equals zero. In this part, we invoke two results from the

induced system. Namely that $\lim_{\varepsilon\to 0}||\hat{h}_\varepsilon - \hat{h}_p||_1 = 0$, and the fact that $\hat{h}_p$ is Lipschitz continuous on $[a_0, b]$.

To prove (2) of Theorem 4.3, we use the representation of λ_p in Proposition 4.2 and part (1) of Theorem 4.3.

5. Proof of Theorem 4.3

Before proving Theorem 4.3, we state and prove two lemmas. We first observe that $T_\varepsilon(a_{k,\varepsilon}) = b_{k,\varepsilon}$ and $b_{k,\varepsilon} \lesssim k^{-\frac{1}{\alpha}}$, see for instance Lemma 3.2 in [16], from which $|Z_{k,\varepsilon}| \lesssim k^{-\frac{1}{\alpha}-1}$, actually $|Z_{k,\varepsilon}| \le C_\epsilon k^{-\frac{1}{\alpha}-1}$, where $C_\epsilon = 1 + O(\varepsilon)$. In the next Lemma $\tilde{C}$ will denote a constant which is independent of ε and which could change from one formula to another.

Lemma 5.1.

(1) *For* $\varepsilon \ge 0$, $\sum_{k=1}^\infty k\hat{\mu}_\varepsilon(Z_{k,\varepsilon}) \le \tilde{C}(\frac{B}{1-\beta})(\frac{\alpha}{1-\alpha})$.

(2) $\lim_{\varepsilon\to 0}|c_{\tau,\varepsilon} - c_{\tau,p}| = 0$.

Proof. (1) By Proposition 2.1 and remembering that the infinite norm (w.r.t. m) is simply bounded by the BV-norm, we have

$$\sum_{k=1}^\infty k\hat{\mu}_\varepsilon(Z_{k,\varepsilon}) \le ||\hat{h}_\varepsilon||_\infty \sum_{k=1}^\infty k|Z_{k,\varepsilon}| \le \tilde{C}(\frac{B}{1-\beta})\sum_{k=1}^\infty \frac{1}{k^{1/\alpha}} \le \tilde{C}(\frac{B}{1-\beta})(\frac{\alpha}{1-\alpha}).$$

To prove (2), we first observe that the constants $c_{\tau,\varepsilon}$ and $c_{\tau,p}$ are less or equal to 1; then

$$|c_{\tau,\varepsilon} - c_{\tau,p}| = \left|\frac{1}{\sum_{k=1}^\infty k\hat{\mu}_\varepsilon(Z_{k,\varepsilon})} - \frac{1}{\sum_{k=1}^\infty k\hat{\mu}_p(Z_k)}\right| \le \sum_{k=1}^\infty k|\hat{\mu}_\varepsilon(Z_k,\varepsilon) - \hat{\mu}_p(Z_k)|.$$

By (1) the previous series is uniformly convergent in ε. Therefore, it is enough to show that for any k, $|\hat{\mu}_\varepsilon(Z_k,\varepsilon) - \hat{\mu}_p(Z_k)|$ converges to zero as $\varepsilon \to 0$. We have

$$|\hat{\mu}_\varepsilon(Z_k,\varepsilon) - \hat{\mu}_p(Z_k)| = \frac{1}{m(\Delta)}\left|\int_{Z_{k,\varepsilon}} \hat{h}_\varepsilon dx - \int_{Z_k} \hat{h} dx\right| \le$$

$$\frac{1}{m(\Delta)}\left|\int_{Z_{k,\varepsilon}\cap Z_k} \hat{h}_\varepsilon dx + \int_{Z_{k,\varepsilon}\setminus(Z_{k,\varepsilon}\cap Z_k)} \hat{h}_\varepsilon dx - \int_{Z_k\cap Z_{k,\varepsilon}} \hat{h} dx - \int_{Z_k\setminus(Z_k\cap Z_{k,\varepsilon})} \hat{h} dx\right| \le$$

$$\frac{1}{m(\Delta)}\left[\int_{Z_{k,\varepsilon}\cap Z_k} |\hat{h}_\varepsilon - \hat{h}| dx + 2||\hat{h}_\varepsilon||_\infty m(Z_{k,\varepsilon}\Delta Z_k)\right]$$

and the first term in the square bracket goes to zero because $\lim_{\varepsilon\to 0}||\hat{h}_\varepsilon - \hat{h}||_1 = 0$. □

Lemma 5.2. *For* $\varepsilon \ge 0$, $x \in W_{k,\varepsilon}$ *and* k *large we have*

(1)
$$|DT^{(n-k)}(T_i^{-1}T_1^{-(n-k-1)}x)| \ge \left(\frac{n}{k+2}\right)^{\eta_k},$$
where $i = 2, 3$, $\eta_k = \frac{d(k+2)}{k+2+d}$ *and* $d > 1$.

(2)
$$\sum_{n=k+1}^\infty \frac{1}{|DT^{(n-k)}(T_i^{-1}T_1^{-(n-k-1)}x)|} \lesssim k.$$

Remark 5.3. Before doing the proof we need two observations:

- The same proof holds for T_ε with all the constants involved uniformly bounded in ε for ε small. Moreover it will be clear in the proof of the theorem below that we should also take x not in W_k but in one of the two similar sets adjacent to it: the proof will not change.
- It will be extremely important to have the constant d strictly larger than 1. Working with the map $T(x) = x + 4^\alpha x^{1+\alpha}$, $x \in [0, 1/4]$, such a constant will be $d = c^\alpha\ 4^\alpha(1+\alpha)$, where the constant c verifies $b_k \geq ck^{-\frac{1}{\alpha}}$. This is done in the next sublemma.

Sublemma 5.4. *Let* $b_k = T_1^{-1} b_{k-1}$*, with* $b_0 = 1/4$*. Then there exists* c *independent of* k *for which* $b_k \geq ck^{-\frac{1}{\alpha}}$*,* $k \geq 1$ *and* $d := c^\alpha\ 4^\alpha(1+\alpha) > 1$*.*

Proof. We proceed as in Lemma 3.2 in [16], but proving the lower bound. Let us choose $c = \frac{1}{4(1+\alpha)^{\frac{1}{\alpha}}} + \delta$, where δ is a small positive constant whose value will be fixed later on. Note that with this value of c, the quantity $d > 1$. We now prove the first assertion of the sublemma by induction. Suppose it is true for k; if it is not true for $k+1$ we should have

$$b_k = b_{k+1}(1 + 4^\alpha b_{k+1}^\alpha) \leq c(k+1)^{-\frac{1}{\alpha}}(1 + 4^\alpha c^\alpha (k+1)^{-1})$$

which implies that $k^{-\frac{1}{\alpha}} \leq (k+1)^{-\frac{1}{\alpha}}(1 + 4^\alpha c^\alpha (k+1)^{-1})$ or $\left(1 + \frac{1}{k}\right)^{\frac{1}{\alpha}} - 1 \leq \frac{4^\alpha c^\alpha}{k+1}$. But $\left(1 + \frac{1}{k}\right)^{\frac{1}{\alpha}} - 1 \geq \frac{1}{\alpha}\frac{1}{k+1}$, which in conclusion gives us $c^\alpha \geq \frac{1}{\alpha\ 4^\alpha}$. With the given choice $c = \frac{1}{4(1+\alpha)^{\frac{1}{\alpha}}} + \delta$, we see that for δ small enough the preceding lower bound is false and so the induction is restored provided we prove the first step of it, namely $b_1 \geq \frac{1}{4(1+\alpha)^{\frac{1}{\alpha}}} + \delta$. Now $b_1 + 4^\alpha b_1^{1+\alpha} = 1/4$; suppose b_1 will not verify the previous lower bound, then we should have

$$\frac{1}{4} \leq \frac{1}{4(1+\alpha)^{\frac{1}{\alpha}}} + \delta + 4^\alpha \left(\frac{1}{4(1+\alpha)^{\frac{1}{\alpha}}} + \delta\right)^{1+\alpha}.$$

It is easy to check that this can never be true.

Proof. (Of the Lemma 5.2) As we anticipated above, we first need (1) We have

$$\begin{aligned} |DT^{(n-k)}(T_i^{-1}T_1^{-(n-k-1)}x)| &= \Pi_{m=0}^{n-k-1}|DT(T^m T_i^{-1}T_1^{-(n-k-1)}x) \\ &\geq \Pi_{m=1}^{n-k-1} \inf_{y \in W_{k+m}} |DTy| \geq \Pi_{m=1}^{n-k-1} DT(b_{k+m+1}). \end{aligned}$$

The last estimate is true because the derivative of T is increasing on $[0, a_0)$. In particular, since $DT_1(x) = 1 + (1+\alpha)4^\alpha x^\alpha$ and $b_k \geq c\frac{1}{k^{1/\alpha}}$, where c is the constant given in the sublemma, we have

(5.1)

$$|DT^{(n-k)}(T_i^{-1}T_1^{-(n-k-1)}x)| \geq \Pi_{m=1}^{n-k-1}(1 + \frac{d}{k+m+1}) = e^{\sum_{m=1}^{n-k-1}\log(1+\frac{d}{k+m+1})}.$$

By the mean value theorem applied to the function $x \mapsto \log(1+x), x > 0$ we immediately have

(5.2)

$$|DT^{(n-k)}(T_i^{-1}T_1^{-(n-k-1)}x)| \geq e^{\frac{d}{1+\frac{d}{k+2}}\sum_{m=1}^{n-k-1}(\frac{1}{k+m+1})} \geq e^{\frac{d}{1+\frac{d}{k+2}}\log\frac{n}{k+2}} = \left(\frac{n}{k+2}\right)^{\eta_k}.$$

To prove (2) we sum over n the estimate in (5.2) and we use the fact that $d > 1$. □

Proof. (Proof of Theorem 4.3) We have

$$||h_\varepsilon - h_p||_1 \le \int_\Delta |h_\varepsilon - h_p| dx + \sum_{k=1}^{\infty} \int_{W_k \setminus (W_{k,\varepsilon} \cap W_k)} |h_\varepsilon - h_p| dx$$
$$+ \sum_{k=1}^{\infty} \int_{W_{k,\varepsilon} \cap W_k} |h_\varepsilon - h_p| dx = (I) + (II) + (III).$$

By Lemma 3.3

$$(I) = \int_\Delta |c_{\tau,\varepsilon} \hat{h}_\varepsilon - c_{\tau,p} \hat{h}_p| dx \le c_{\tau,p} \int_\Delta |\hat{h}_\varepsilon - \hat{h}_p| dx + |c_{\tau,\varepsilon} - c_{\tau,p}| \int_\Delta |\hat{h}_\varepsilon| dx.$$

Therefore, by Proposition 2.1 and Lemma 5.1, $(I) \to 0$ as $\varepsilon \to 0$. To prove that (II) converges to zero we first obtain a bound on $\sup_{x \in W_k \setminus (W_{k,\varepsilon} \cap W_k)} (|h_p(x)| + |h_\varepsilon(x)|)$. Using (4.1), Proposition 2.1 and Lemma 5.2, we have

$$\sup_{x \in W_k \setminus (W_{k,\varepsilon} \cap W_k)} |h_p(x)| \le \sup_{x \in W_k} c_{\tau,p} \sum_{n=k+2}^{\infty} \sum_{i=2}^{3} \frac{|\hat{h}_p(T_i^{-1} T_1^{-(n-k-2)} x)|}{|DT^{(n-k-1)}(T_i^{-1} T_1^{-(n-k-3)} x)|}$$
$$\lesssim (\frac{B}{1-\beta})\ k.$$

A similar bound holds for h_ε by observing that the supremum should now be taken on an adjacent cylinder of $W_{k,\varepsilon}$. Consequently since, as we already saw, $|b_k - b_{k-1}| \approx k^{-\frac{1}{\alpha}-1},\ k \ge 1$ we obtain

$$(II) \le 2(\frac{B}{1-\beta}) \cdot \text{const} \sum_{k=1}^{\infty} k |b_k - b_{k-1}| \le \text{const} \sum_{k=1}^{\infty} k^{-\frac{1}{\alpha}}$$

The uniform convergence of the series allows us to take the limit for $\varepsilon \to 0$ inside and this will cancel the second contribution since $m(W_k \setminus (W_{k,\varepsilon} \cap W_k)) \to 0$ when $\varepsilon \to 0$. For the third one we have:

$$(III) = \sum_{k=1}^{\infty} \int_{W_{k,\varepsilon} \cap W_k} |h_\varepsilon(x) - h_p(x)| dx$$
$$\le \sum_{k=1}^{\infty} \int_{W_{k,\varepsilon} \cap W_k} |\sum_{n=k+1}^{\infty} \sum_{i=2}^{3} c_{\tau,p} \frac{\hat{h}_p(T_i^{-1} T_1^{-(n-k-1)} x)}{|DT^{(n-k)}(T_i^{-1} T_1^{-(n-k-1)} x)|}$$
$$- c_{\tau,\varepsilon} \frac{\hat{h}_\varepsilon(T_{i,\varepsilon}^{-1} T_{1,\varepsilon}^{-(n-k-1)} x)}{|DT_\varepsilon^{(n-k)}(T_{i,\varepsilon}^{-1} T_{1,\varepsilon}^{-(n-k-1)} x)|}|$$
$$\le \sum_{k=1}^{\infty} |c_{\tau,p} - c_{\tau,\varepsilon}| \int_{W_{k,\varepsilon} \cap W_k} \sum_{n=k+1}^{\infty} \sum_{i=2}^{3} \frac{|\hat{h}_p(T_i^{-1} T_1^{-(n-k-1)} x)|}{|DT^{(n-k)}(T_i^{-1} T_1^{-(n-k-1)} x)|} dx$$
$$+ \sum_{k=1}^{\infty} c_{\tau,\varepsilon} \int_{W_{k,\varepsilon} \cap W_k} \sum_{n=k+1}^{\infty} \sum_{i=2}^{3} |\frac{\hat{h}_p(T_i^{-1} T_1^{-(n-k-1)} x)}{|DT^{(n-k)}(T_i^{-1} T_1^{-(n-k-1)} x)|}$$
$$- \frac{\hat{h}_\varepsilon(T_{i,\varepsilon}^{-1} T_{1,\varepsilon}^{-(n-k-1)} x)}{|DT_\varepsilon^{(n-k)}(T_{i,\varepsilon}^{-1} T_{1,\varepsilon}^{-(n-k-1)} x)|}| dx$$
$$= A_1 + A_2.$$

The quantity A_1 could be treated as the term (II) above: the integral inside the sum gives the summable contribution $k^{-\frac{1}{\alpha}}$ which will allow us to take afterwards the limit $|c_{\tau,p} - c_{\tau,\varepsilon}| \to 0$ for $\varepsilon \to 0$. The same argument shows that A_2 converges uniformly in ε, but in order to take the limit inside the series, we have first of all to split A_2 into two supplementary terms:

$$\begin{aligned}A_2 \le &\sum_{k=1}^{\infty} c_{\tau,\varepsilon} \int_{W_{k,\varepsilon}\cap W_k} \sum_{n=k+1}^{\infty} \sum_{i=2}^{3} \Big| \frac{\hat{h}_p(T_i^{-1}T_1^{-(n-k-1)}x)}{|DT^{(n-k)}(T_i^{-1}T_1^{-(n-k-1)}x)|} \\ &\qquad - \frac{\hat{h}_\varepsilon(T_{i,\varepsilon}^{-1}T_{1,\varepsilon}^{-(n-k-1)}x)}{|DT^{(n-k)}(T_i^{-1}T_1^{-(n-k-1)}x)|}\Big| dx \\ &+ \sum_{k=1}^{\infty} c_{\tau,\varepsilon} \int_{W_{k,\varepsilon}\cap W_k} \sum_{n=k+1}^{\infty} \sum_{i=2}^{3} \Big| \frac{\hat{h}_\varepsilon(T_{i,\varepsilon}^{-1}T_{1,\varepsilon}^{-(n-k-1)}x)}{|DT^{(n-k)}(T_i^{-1}T_1^{-(n-k-1)}x)|} \\ &\qquad - \frac{\hat{h}_\varepsilon(T_{i,\varepsilon}^{-1}T_{1,\varepsilon}^{-(n-k-1)}x)}{|DT_\varepsilon^{(n-k)}(T_{i,\varepsilon}^{-1}T_{1,\varepsilon}^{-(n-k-1)}x)|}\Big| dx \\ &= A_2^* + A_2^\dagger.\end{aligned}$$

To show that A_2^* converges to zero as $\varepsilon \to 0$ it will be sufficient to control the integral

$$\int_{W_{k,\varepsilon}\cap W_k} \frac{1}{|DT^{(n-k)}(T_i^{-1}T_1^{-(n-k-1)}x)|}[\hat{h}_p(T_i^{-1}T_1^{-(n-k-1)}x) - \hat{h}_p(T_{i,\varepsilon}^{-1}T_{1,\varepsilon}^{-(n-k-1)}x)) +$$

$$\hat{h}_p(T_{i,\varepsilon}^{-1}T_{1,\varepsilon}^{-(n-k-1)}x)) - \hat{h}_\varepsilon(T_{i,\varepsilon}^{-1}T_{1,\varepsilon}^{-(n-k-1)}x)]dx.$$

We now make the change of variable $y_i = T_i^{-1}T_1^{-(n-k-1)}x \in Z_n$ and set $y_i' := y_i'(y_i) = T_{i,\varepsilon}^{-1}T_{1,\varepsilon}^{-(n-k-1)}(T^{n-k}y_i)$. Then $y_i, y_i' \in Z_n \cup Z_{n\varepsilon}$ and we rewrite the previous integral as

$$\int_{Z_n} \left(|\hat{h}_p(y_i) - \hat{h}_p(y_i')| + |\hat{h}_p(y_i') - \hat{h}_\varepsilon(y_i')|\right) dy_i. \tag{5.3}$$

We first have

$$\lim_{\varepsilon\to 0} \int_{Z_n} |\hat{h}_p(y_i') - \hat{h}_\varepsilon(y_i')| dy_i \le \lim_{\varepsilon\to 0} ||\hat{h}_p - \hat{h}_\varepsilon||_1 = 0.$$

We also have $\lim_{\varepsilon\to 0}\int_{Z_n} |\hat{h}_p(y_i) - \hat{h}_p(y_i')|dy_i = 0$, since by (1) of Lemma 3.1 $\hat{h}_p$ is Lipschitz on Δ and $y_i' \to y_i$ as $\varepsilon \to 0$.

To prove that $A_2^\dagger$ converges to 0 as $\varepsilon \to 0$, it will be sufficient, after having factorized one of the inverse of the derivatives, to show that the ratio

$$\frac{DT_\varepsilon^{(n-k)}(T_{i,\varepsilon}^{-1}T_{1,\varepsilon}^{-(n-k-1)}x)}{DT^{(n-k)}(T_i^{-1}T_1^{-(n-k-1)}x)},\ x \in W_{k,\varepsilon} \cap W_k$$

goes to 1. We begin to rewrite it as

$$\Pi_{m=0}^{n-k-1} \frac{DT_\varepsilon(T^m y')}{DT(T^m y')} \frac{DT_\varepsilon(T_\varepsilon^m y)}{DT_\varepsilon(T^m y')}$$

where we put $y := T_{i,\varepsilon}^{-1} T_{1,\varepsilon}^{-(n-k-1)} x \in Z_{n,\varepsilon}$ and $y' := T_i^{-1} T_1^{-(n-k-1)} x \in Z_n$ and we also recall that $T_\varepsilon^m y \in W_{n-m,\varepsilon}$ and $T^m y' \in W_{n-m}$. The first ratio $\frac{DT_\varepsilon(T^m y')}{DT(T^m y')}$ goes to one since for any $0 \le x < b_0$: $\lim_{\varepsilon\to 0} DT_\varepsilon(x) = DT(x)$. The second ratio can now be written in the form

$$\left|\frac{DT_\varepsilon(T_\varepsilon^m y)}{DT_\varepsilon(T^m y')}\right| = \exp\left[\left|\frac{D^2 T_\varepsilon}{DT_\varepsilon}\right|_{\xi\in(T_\varepsilon^m y, T^m y')} \cdot |T_\varepsilon^m y - T^m y'|\right].$$

Recall that the first and the second derivative are finite outside the origin; so we are left with proving that $|T_\varepsilon^m y - T^m y'|$ tends to 0 when $\varepsilon \to 0$. But $|T_\varepsilon^m y - T^m y'| = |T_\varepsilon^m y - T^m y| + |T^m y - T^m y'|$ and the first term goes to zero since T_ε^m converges uniformly to T^m and the second term goes to zero by the continuity of T^m. This finishes the proof of part (1) of the theorem.
To prove (2), we first use Proposition 4.2 to obtain

$$\frac{\lambda_p}{1-\lambda_p} = \frac{\hat\lambda_p c_{\tau,r}}{(1-\hat\lambda_p)c_{\tau,l}}.$$

Using (3.4) it follows immediately that $c_{\tau,r} = 1$ and $c_{\tau,l} = \mu_l(\Delta_l)$, where Δ_l is the interval (b_0, b). Therefore,

$$\frac{\lambda_p}{1-\lambda_p} = \frac{\hat\lambda_p c_{\tau,r}}{(1-\hat\lambda_p)c_{\tau,l}} = \frac{1}{\mu_l(\Delta_l)} \lim_{\varepsilon\to 0} \frac{\hat\mu_r(\hat H_{r,\varepsilon})}{\hat\mu_l(\hat H_{l,\varepsilon})}$$

We now show that

$$\frac{1}{\mu_l(\Delta_l)} \lim_{\varepsilon\to 0} \frac{\hat\mu_r(\hat H_{r,\varepsilon})}{\hat\mu_l(\hat H_{l,\varepsilon})} = \lim_{\varepsilon\to 0} \frac{\mu_r(H_{r,\varepsilon})}{\mu_l(H_{l,\varepsilon})} \tag{5.4}$$

which leads to the formula in part (2) of the theorem. We know invoke formula (3.4) and the result which we obtained in part (1) of this theorem. We have $H_{l,\varepsilon} = I_l \cap T_\varepsilon^{-1} I_r = (I_l \cap T_\varepsilon^{-1} I_r)_l \cup (I_l \cap T_\varepsilon^{-1} I_r)_r$, where $(I_l \cap T_\varepsilon^{-1} I_r)_l = (I_l \cap T_\varepsilon^{-1} I_r) \cap (0, b_0)$ and $(I_l \cap T_\varepsilon^{-1} I_r)_r = (I_l \cap T_\varepsilon^{-1} I_r) \cap (b_0, b)$.
Now, using (3.4) we obtain

$$\mu_\varepsilon((I_l \cap T_\varepsilon^{-1} I_r)_l) = c_{\tau,\varepsilon} \sum_{n=2}^{\infty} \hat\mu_\varepsilon(T_\varepsilon^{-(n-1)}(I_l \cap T_\varepsilon^{-1} I_r)_l \cap Z_{n,\varepsilon}) =$$

$$c_{\tau,\varepsilon} \sum_{n=2}^{\infty} \hat\mu_\varepsilon(\hat T_{n,\varepsilon}^{-1} I_r \cap \Delta_l) = c_{\tau,\varepsilon}\left[\hat\mu_\varepsilon(\hat T_\varepsilon^{-1} I_r \cap \Delta_l) - \hat\mu_\varepsilon(\hat T_{1,\varepsilon}^{-1} I_r \cap \Delta_l)\right]$$

where we define $\hat T_{n,\varepsilon}^{-1} := \left(T_\varepsilon^n{}_{|Z_{n,\varepsilon}}\right)^{-1}$. On the other hand

$$\mu_\varepsilon((I_l \cap T_\varepsilon^{-1} I_r)_r) = c_{\tau,\varepsilon} \hat\mu_\varepsilon(\hat T_{1,\varepsilon}^{-1} I_r \cap \Delta_l)$$

since $(I_l \cap T_\varepsilon^{-1} I_r)_r$ is inside the domain of induction. In conclusion we have proved that

$$\mu_\varepsilon(H_{l,\varepsilon}) = \mu_\varepsilon(I_l \cap T_\varepsilon^{-1} I_r) = c_{\tau,\varepsilon} \hat\mu_\varepsilon(\hat H_{l,\varepsilon})$$

In a much easier way we immediately have

$$\mu_\varepsilon(H_{r,\varepsilon}) = \mu_\varepsilon(I_r \cap T_\varepsilon^{-1} I_l) = c_{\tau,\varepsilon} \hat\mu_\varepsilon(\hat T_\varepsilon^{-1} \Delta_l \cap I_r) = c_{\tau,\varepsilon} \hat\mu_\varepsilon(\hat H_{r,\varepsilon})$$

Therefore we have

$$\frac{\mu_\varepsilon(H_{r,\varepsilon})}{\mu_\varepsilon(H_{l,\varepsilon})} = \frac{\hat{\mu}_\varepsilon(\hat{H}_{r,\varepsilon})}{\hat{\mu}_\varepsilon(\hat{H}_{l,\varepsilon})} \tag{5.5}$$

Now we observe that for what we previously proved we have:

- $\hat{\mu}_\varepsilon(A) \to (1-\hat{\lambda}_p)\hat{\mu}_r(A)$, whenever A is a mesurable set in Δ_r.
- $\mu_r(A) \leftarrow \frac{\mu_\varepsilon(A)}{1-\lambda_p}$, whenever A is a mesurable set in I_r.
- $\hat{\mu}_\varepsilon(A) \to \hat{\lambda}_p\hat{\mu}_l(A)$, whenever A is a mesurable set in Δ_l.
- $\mu_l(A) \leftarrow \frac{\mu_\varepsilon(A)}{\lambda_p}$, whenever A is a mesurable set in I_l.

Of course the same is true if A depends on ε since, take for instance $A \subset I_r$,

$$|\hat{\mu}_\varepsilon(A) - \hat{\mu}_r(A)| \leq ||\hat{h}_\varepsilon - (1-\hat{\lambda}_p)\hat{h}_p||_1 \to 0.$$

Putting together all that and using (5.5) we get (5.4):

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\mu_r(H_{r,\varepsilon})}{\mu_l(H_{l,\varepsilon})} &= \lim_{\varepsilon\to 0}\frac{\lambda_p}{1-\lambda_p}\frac{\mu_\varepsilon(H_{r,\varepsilon})}{\mu_\varepsilon(H_{l,\varepsilon})} = \lim_{\varepsilon\to 0}\frac{\lambda_p}{1-\lambda_p}\frac{\hat{\mu}_\varepsilon(\hat{H}_{r,\varepsilon})}{\hat{\mu}_\varepsilon(\hat{H}_{l,\varepsilon})} \\ &= \lim_{\varepsilon\to 0}\frac{\hat{\lambda}_p}{(1-\hat{\lambda}_p)\mu_l(\Delta_l)}\frac{\hat{\mu}_\varepsilon(\hat{H}_{r,\varepsilon})}{\hat{\mu}_\varepsilon(\hat{H}_{l,\varepsilon})} = \lim_{\varepsilon\to 0}\frac{1}{\mu_l(\Delta_l)}\frac{\hat{\mu}_r(\hat{H}_{r,\varepsilon})}{\hat{\mu}_l(\hat{H}_{l,\varepsilon})}.\end{aligned}$$

□

6. Appendix

In the appendix we provide a method which could be used to determine the number of ergodic a.c.i.ms for maps similar to T_ε. In particular we will show that for any $\varepsilon > 0$, the map T_ε defined in (3.1) has exactly one a.c.i.m. Let $\mathcal{C}_{(T_\varepsilon)} := \{I_i\}_{i=1}^6$ be the partition on which T_ε is piecewise monotonic. We introduce a directed graph associated with the perturbed map T_ε, $\varepsilon > 0$, and we denote it by $G(T_\varepsilon)$[3].

• There is an arrow from $I_i \to I_j$ if and only if there exists a $k \geq 1$ such that $T_\varepsilon^k(I_i) \supseteq I_j$, $i, j \in \{1, \ldots, 6\}$.
• A finite sequence of *arrows* will be called a *path*.
• I_j is said to be *accessible* from I_i if there exists a *path* in $G(T_\varepsilon)$ from I_i to I_j.
• The *accessible set* from I_i, denoted by $[I_i]$, consists of all intervals I_j which are *accessible* from I_i.

Lemma 6.1. *Let μ be a T_ε ergodic a.c.i.m*[4]*. Then the support of μ contains $[I_i]$ for some $i = 1, \ldots, 6$.*

Proof. We will first show that for any interval $J \subset I$, there exists an $n \geq 1$ such that $T_\varepsilon^n(J)$ contains two partition points. Let $J \subset I_i$ for some i. Since $m(T_\varepsilon(J)) > m(J)$, there exists a $j \geq 1$ such that $T_\varepsilon^j(J)$ contains a partition point in its interior. We consider all possible cases.

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[3]A similar graph can be found in [5] which is used to get an upper bound on the number on ergodic of a.c.i.ms when the modulus of the derivative of the map is greater than 2. Since in our case $\inf_x |T_\varepsilon| = 1$, we cannot use the results found in [5].

[4]We know that there is at least one such measure since the corresponding induced map has an a.c.i.m.

(1) If $T_\varepsilon^j(J)$ contains the partition point 0, then obviously there exists a $k \geq 1$ such that $T_\varepsilon^{j+k}(J)$ contains $[0, 1/2]$.
(2) The case of the partition point 3/8 is the same as that of 0.
(3) If $T_\varepsilon^j(J)$ contains the partition point 1/4 in its interior; i.e $T_\varepsilon^j(J) \supset (p_1, p_2)$ with $1/4 \in (p_1, p_2)$. Then we observe that $T_\varepsilon^k([1/4, p_2)) \subseteq [1/4, 1]$ for all $k \geq 1$, and $\inf_{x\in[1/4,1]} |T'_\varepsilon| > 2$. Thus for some $k \geq 1$, $T_\varepsilon^k([1/4, p_2))$ must contain two partition points (otherwise the length of iterates of the image will go to ∞ since the modulus of the derivative is bigger than 2). Thus, $T_\varepsilon^{j+k}(J)$ contains two partition points.
(4) The cases of the partition points $1/2, 5/8, 13/16, 1$ are similar to that of 1/4.

Let C denote the support of μ. Since C contains an interval J, $T^n(J)$, $n \geq 1$ contains two partition points, and C is an invariant set, C must contain (mode 0) an I_i. Consequently (by invariance) C must contain (mode 0) $[I_i]$. □

Lemma 6.2. *For each $\varepsilon > 0$ T_ε has a unique ergodic a.c.i.m.*

Proof. Observe that for each $i = 1, \dots 6$,

$$[I_i] = \{I_1, I_2, I_3, I_4, I_5, I_6\}.$$

Thus by Lemma 6.1, and the fact that ergodic a.c.i.ms must have disjoint supports, T_ε has a unique a.c.i.m. □

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