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Statistical properties of intermittent maps with unbounded derivative

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Abstract
We study the ergodic and statistical properties of a class of maps of the circle and of the interval of Lorenz type which present indifferent fixed points and points with unbounded derivative. These maps have been previously investigated in the physics literature. We prove in particular that correlations decay polynomially, and that suitable limit theorems (convergence to stable laws or central limit theorem) hold for Hölder continuous observables. Moreover, we show that the return and hitting times are exponentially distributed in the limit.

Mathematics Subject Classification: 37E05, 60F05

1. Introduction

The prototype for intermittent maps of the interval is the well-known Pomeau–Manneville map T defined on the unit interval [0, 1] and which admits a neutral fixed point at 0 with local behaviour $T(x) = x + cx^{1+\alpha}$; otherwise it is uniformly expanding. The constant α belongs to (0, 1) to guarantee the existence of a finite absolutely continuous invariant probability measure and the constant c could be chosen in such a way that the map T has a Markov structure. This map enjoys polynomial decay of correlations and this property still persists even if the map is no longer Markov [33].

Another interesting class of maps of the interval are the one-dimensional uniformly expanding Lorenz-like maps (see [13, 17, 32] for their introduction and for the study of their topological properties), whose features are now the presence of points with unbounded derivatives and a lack of Markov structure: in this case one could build up towers and find various rates for the decay of correlations depending on the tail of the return time function on the base of the tower, see, for instance [9, 10]. The latter paper deals in particular with one-dimensional maps which admit critical points and, possibly, points with unbounded derivatives, but it leaves open the case where there is presence of neutral fixed points.

In this paper we are interested in maps which exhibit the last two behaviours, namely neutral fixed points and points with unbounded derivatives. Such maps have been introduced into the physics literature by Grossmann and Horner in 1985 [16]; they showed numerically a polynomial decay of correlations and they also studied other statistical properties, like the susceptibility and the $1/f$-noise. Another contribution by Pikovsky [28] showed, still with heuristic arguments, that these maps produce anomalous diffusion with square displacement growing faster than linearly. Artuso and Cristadoro [3] improved the latter result by computing the moments of the displacement on the infinite replicas of the fundamental domain and showed a 'phase transition' in the exponent of the moments growth. Recently Lorenz cusp maps arose to describe the distribution of the Casimir maximum in the Kolmogorov–Lorenz model of geofluid dynamics [27]. Despite this interesting physical phenomenology, we did not find any rigorous mathematical investigation of such maps. These maps are defined on the torus $\mathbb{T} = [-1, 1]\backslash \sim$ and depend on the parameter γ (see below); when $\gamma = 2$ the corresponding map was taken as an example of the non-summability of the first hyperbolic time by Alves and Araujo in [2]. This maps reads

$$\tilde{T}(x) = \begin{cases} 2\sqrt{x} - 1 & \text{if } x \geqslant 0 \\ 1 - 2\sqrt{|x|} & \text{otherwise} \end{cases} \tag{1}$$

and it was proved in [2] that it is topologically mixing, but no other ergodic properties were studied.

Actually, the Grossmann and Horner maps are slightly different from those investigated in [3, 28], the difference being substantially in the fact that the latter are defined on the circle instead of on the unit interval. We will study in detail the circle version of these maps in sections 2 to 5, and we will show, in section 6, how to generalize our results to the interval version: since both classes of maps are Markov, the most important information, especially in computing distortion, will come from the local behaviour around the neutral fixed points and the points with unbounded derivatives and these behaviours will be the same for both versions. There is nevertheless an interesting difference. The circle version introduced in section 1 is written in such a way that the Lebesgue measure is automatically invariant. This is not the case in general for the interval version quoted in section 6. However, the strategy that we adopt to prove statistical properties (Lai-Sang Young towers) will also give us the existence of an absolutely continuous invariant measure and we will complete it by providing information on the behaviour of the density. It is interesting to observe that in the class of maps considered by Grossmann and Horner on the interval $[-1, 1]$ (see section 6), the analogue of (1) is given by the following map:

$$\tilde{S}(x) = 1 - 2\sqrt{|x|}. \tag{2}$$

This map was investigated by Hemmer in 1984 [21]: he also computed by inspection the invariant density which is $\rho(x) = \frac{1}{2}(1 - x)$ and the Lyapunov exponent (simply equal to $1/2$), but he only argued about a slow decay of correlations. We will show in section 6 how to recover the qualitative behaviour of this density (and of all the others in the Grossmann and Horner class).

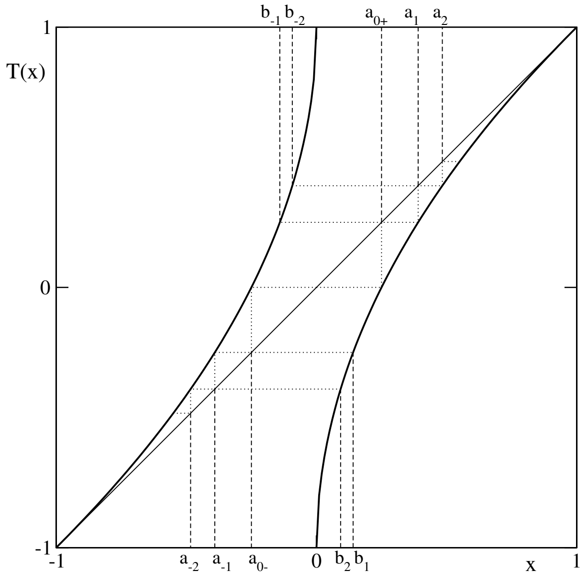

Figure 1. The map T on the circle.

In this paper we study the one-parametric family of continuous maps T (figure 1) which are C^1 on $\mathbb{T}\setminus\{0\}$, C^2 on $\mathbb{T}\setminus(\{0\}\cup\{1\})$ and are implicitly defined on the circle by the equations

$$x = \begin{cases} \frac{1}{2\gamma}(1+T(x))^{\gamma} & \text{if } 0 \leqslant x \leqslant \frac{1}{2\gamma} \\ T(x) + \frac{1}{2\gamma}(1-T(x))^{\gamma} & \text{if } \frac{1}{2\gamma} \leqslant x \leqslant 1 \end{cases} \tag{3}$$

and for negative values of x by putting $T(-x) = -T(x)$. We assume that parameter $\gamma > 1$. Note that when $\gamma = 1$ the map is continuous with constant derivative equal to 2 and is the classical doubling map. The point 1 is a fixed point with derivative equal to 1, while at 0 the derivative becomes infinite. The map leaves the Lebesgue measure ν invariant (it is straightforward to check that the Perron–Frobenius operator has 1 as a fixed point). We will prove in the next sections several statistical properties: these will follow from existing techniques, especially towers, combined with the distortion bound proved in the next section, which will tell us that the logarithm of the derivative of the first return map is (locally) Hölder on each cylinder of a countable Markov partition associated with T. Actually one could induce on a suitable interval only (called I_0^+ in the following): the proof we give is intended to provide distortion on *all* cylinders of the countable Markov partition covering *mod* 0 the whole space $[-1, 1]$, since this is necessary to get the local smoothness of the invariant density is section 6 and in order to apply the inducing technique of [6] which will give us the statistical features of recurrence studied in section 5. We now summarize the kind of statistical properties which we are going to prove and that could be useful in physical applications whenever these maps arise as the first return maps in suitable Poincaré sections; we recall that we are dealing with two class of maps, T and S which share a few properties but that also exhibit a few differences, in particular the existence of an absolutely continuous invariant measure must be proved for the maps S and the regularity of the density must be investigated. The interesting features, even for applications, are (we defer to the precise statements in the next sections for the correct choice of the parameters)

- The maps T and S enjoy polynomial decay of correlations for Hölder observables and this decay is optimal in the sense that we can exhibit a specific class of functions that vanishes in a neighbourhood of the indifferent fixed point and for which the correlations have polynomial lower bounds.
- When we consider the map S, the density of the absolutely continuous invariant measure is relevant from the physical point of view since it is related to some quantities and invariants (see, e.g., [27]). We provide a local characterization of such a density, it is Lipschitz continuous and bounded, but we believe that a precise knowledge of the map could allow one to improve the smoothness of the density.
- The process $\mathcal{S}_n/\sqrt{n}$, where $\mathcal{S}_n = \sum_{k=0}^{n-1} \phi \circ T^k$ and with ϕ an Hölder observable of zero mean (the same holds for the map S), will tend in distribution to the normal law for certain values of the parameters defining the maps; for other values of such parameters the process $\mathcal{S}_n/n^{\iota}$ will converge to a stable law and with ι depending on the parameters. Our examples enrich the list of one-dimensional maps for which the stable laws could be exactly computed.
- The process $(1/n)\mathcal{S}_n$ satisfies large deviations bounds for the maps T and S in the sense that, if μ denotes the invariant measure and given a positive ε, then the distribution
$$\mu\left(\left|\frac{1}{n}\mathcal{S}_n\right| > \varepsilon\right)$$
decays in n polynomially to zero.
- The maps T and S have exponential return and hitting time statistics with respect to the invariant measure μ and around balls whose centre is chosen μ-almost surely. Moreover, the number of visits in such balls converge to the Poissonian distribution whenever the radius of the ball goes to zero. Finally, extreme values laws hold for the distribution of partial maximum.

2. Distortion

Notations. With $a_n \approx b_n$ we mean that there exists a constant $C \geqslant 1$ such that $C^{-1}b_n \leqslant a_n \leqslant Cb_n$ for all $n \geqslant 1$; with $a_n \lesssim b_n$, or equivalently $a_n = \mathcal{O}(b_n)$ with a_n and b_n non-negative, we mean that there exists a constant $C \geqslant 1$ such that $\forall n \geqslant 1$, $a_n \leqslant Cb_n$; with $a_n \sim b_n$ we mean that $\lim_{n\to\infty} a_n/b_n = 1$. Throughout the paper ν will denote the Lebesgue measure. The letter C will denote often different constants.

There is a countable Markov partition $\{I_m\}_{m\in\mathbb{Z}^*\bigcup\{0\pm\}}$ associated with the map (3); the partition is built *mod* ν as follows: $I_m = (a_{m-1}, a_m)$ for all $m \in \mathbb{Z}^*$ and $I_0^+ = (0, a_{0+})$ and $I_0^- = (a_{0_-}, 0)$, where

$$a_{0+} = \frac{1}{2\gamma}, \quad a_{0-} = -\frac{1}{2\gamma} \qquad \text{and} \qquad a_i = T_+^{-i}a_{0+}, \quad a_{-i} = T_-^{-i}a_{-0}, \ i \geqslant 1,$$

with $T_+ = T_{|(0,1)}$, $T_- = T_{|(-1,0)}$. Then we define $\forall i \geqslant 1$:

$$b_{-i} = T_-^{-1}a_{i-1} \qquad \text{and} \qquad b_i = T_+^{-1}a_{-(i-1)}.$$

We now state without proof a few results which are direct consequences of the definition of the map.

Lemma 1.

(1) When $x \to 1^-$: $T(x) = 1 - (1-x) - \frac{1}{2\gamma}(1-x)^{\gamma} + o((1-x)^{\gamma})$

$DT(x) = 1 + \frac{1}{2}(1-x)^{\gamma-1} + o((1-x)^{\gamma-1})$

$D^2T(x) = -\frac{(\gamma-1)}{2}(1-x)^{\gamma-2} + o((1-x)^{\gamma-2})$.

(2) When $x \to 0^+$: $T(x) = -1 + (2\gamma)^{\frac{1}{\gamma}} x^{\frac{1}{\gamma}} + o(x^{\frac{1}{\gamma}})$
$DT(x) = (2\gamma)^{\frac{1}{\gamma}} \frac{1}{\gamma} x^{\frac{1}{\gamma}-1} + o(x^{\frac{1}{\gamma}-1})$
$D^2T(x) = (2\gamma)^{\frac{1}{\gamma}} \frac{1}{\gamma}(\frac{1}{\gamma} - 1)x^{\frac{1}{\gamma}-2} + o(x^{\frac{1}{\gamma}-2})$.

The derivations at point 2 are obvious since the map is explicit. The formulae for the derivatives at point 1 are obtained by computing the first and the second derivative of the local inverse of T and by using successively the local expansion of T in the neighbourhood of 1.

Lemma 2. *We have for all n,* $a_{\pm(n+1)} = a_{\pm n} \pm \frac{1}{2\gamma}(1 \mp a_{\pm n})^\gamma$ *and*

$$1 - a_n \sim \left(\frac{2\gamma}{\gamma - 1}\right)^{\frac{1}{\gamma-1}} \frac{1}{n^{\frac{1}{\gamma-1}}};$$

$$a_{-n} + 1 \sim \left(\frac{2\gamma}{\gamma - 1}\right)^{\frac{1}{\gamma-1}} \frac{1}{n^{\frac{1}{\gamma-1}}};$$

$$l_n := \text{length}[a_{n-1}, a_n] \sim \frac{1}{2\gamma}\left(\frac{2\gamma}{\gamma - 1}\right)^{\frac{\gamma}{\gamma-1}} \frac{1}{n^{\frac{\gamma}{\gamma-1}}} \qquad n > 1;$$

$$|b_{\pm n}| \sim \frac{1}{2\gamma}\left(\frac{2\gamma}{\gamma - 1}\right)^{\frac{\gamma}{\gamma-1}} \frac{1}{(n-1)^{\frac{\gamma}{\gamma-1}}}, \qquad n > 1.$$

Our next step is to induce over subsets where the first return map is mixing and has a nice topological structure. We will see that the first return map is Bernoulli on the cylinders $I_m = (a_{m-1}, a_m)$, $m \in \mathbb{Z}^*$ and $I_0^\pm$ introduced above. A distortion estimate on those cylinders is possible although quite lengthy. We proceed therefore in another way. We perform the distortion estimate on the interval $\tilde{I}_m := (a_{-m}, a_m)/\{0\}$ which turns out to be much easier, and we will show that such distortion persists over the (Bernoulli) cylinders $I_m \subset \tilde{I}_m$ (see corollary 1 below)[5]. It is important to stress that the distortion only is not enough to work on the sets of the form $\tilde{I}_m$, since, for example, on the set (a_{0-}, a_{0+}) the first return map is irreducible but not aperiodic, as it is easy to check by inspection. We then proceed to induce on the interval $\tilde{I}_m$ to get a bounded distortion estimate for the first return map. We define $Z^+_{m,1} := (b_{m+1}, a_m)$, $Z^-_{m,1} := (a_{-m}, b_{-(m+1)})$ and $Z^+_{m,p>1} := (b_{m+p}, b_{m+p-1})$, $Z^-_{m,p>1} := (b_{-(m+p-1)}, b_{-(m+p)})$. Note that $\tilde{I}_m = \cup_{p\geqslant 1} Z^\pm_{m,p}$ and that the first return map $\widehat{T}_m = \tilde{I}_m \to \tilde{I}_m$ acts on each $Z^\pm_{m,p}$ as $\widehat{T}_m = T^p$ and in particular

$$T^p(Z^+_{m,p}) = \begin{cases} (a_{-m}, a_{m-1}) & p = 1 \\ (a_{-m}, a_{-(m-1)}) & p > 1 \end{cases} \qquad T^p(Z^-_{m,p}) = \begin{cases} (a_{-(m-1)}, a_m) & p = 1 \\ (a_{m-1}, a_m) & p > 1. \end{cases}$$

We finally observe that the induced map $\widehat{T}_m$ is uniformly expanding in the sense that for each m and p there exists $\beta > 1$ such that $|D\widehat{T}_m(x)| > \beta$, $\forall x \in \tilde{I}_m$[6].

Proposition 3. *Let us induce on $\tilde{I}_m$; then there exists a constant $K > 0$ that depends on m, such that for all p and for all x, y in a cylinder of the form $Z^+_{m,p}$ or $Z^-_{m,p}$, we have*

$$\left|\frac{D\hat{T}_m(x)}{D\hat{T}_m(y)}\right| = \left|\frac{DT^p(x)}{DT^p(y)}\right| \leqslant \mathrm{e}^{K|T^p(x)-T^p(y)|} \leqslant \mathrm{e}^{2K}.$$

[5] We warmly thank the anonymous referee who suggested this strategy to us which greatly simplifies our previous distortion estimate performed on each cylinder I_m with long combinatorics.
[6] Using the chain rule we can see that $\beta \equiv \inf_{x\in Z_{m,1}} |DT(x)| > 1$.

Remark 1. The techniques of the proof allow us to get the equivalent result, especially used in section 6. Let us consider as before the interval $\tilde{I}_m$; then there exists a constant $K' > 0$ that depends on m, such that for all p and for all x in a cylinder of the form $Z^{\pm}_{m,p}$, we have (Adler's condition)

$$\frac{|D^2\hat{T}(x)|}{|D\hat{T}(x)|^2} \leqslant K'. \tag{4}$$

Proof. We work on the cylinders of the form $Z^-_{m,p}$, the other case being completely similar by symmetry. We denote with l_m the length of the interval (a_{m-1}, a_m) (when $m = 0$, $l_0 =$ length of $(0, a_{0+})$). We start by observing that

$$\left|\frac{DT^p(x)}{DT^p(y)}\right| = \exp\left[\sum_{q=0}^{p-1}\left|\frac{D^2T(\xi)}{DT(\xi)}\right||T^qx - T^qy|\right], \tag{5}$$

where ξ is a point between T^qx and T^qy.

We divide the cases $p = 1$ and $p > 1$.

- $p = 1$
  For $(x, y) \in Z^-_{m,1}$ and by using $|x - y| < |T(x) - T(y)|$, we directly get
  $$\left|\frac{DT(x)}{DT(y)}\right| \leqslant \exp\left[K_1|T(x) - T(y)|\right],$$
  where $K_1 = \sup_{(Z^-_{m,1})} D^2T = D^2T(a_m)$.
- $p > 1$
  We start with $x, y \in Z^-_{m,p}$; since $Tx, Ty \in (a_{m+p-2}, a_{m+p-1})$; $T^2x, T^2y \in (a_{m+p-3}, a_{m+p-2})$; $\ldots$; $T^{p-1}x, T^{p-1}y \in (a_m, a_{m+1})$ we finally have
  $$\begin{aligned}(5) &\leqslant \exp\left[\frac{\sup_{(Z^-_{m,p})}\big(|D^2T|\big)}{\inf_{(Z^-_{m,p})}\big(|DT|\big)}|x-y| + \sum_{q=1}^{p-1}\frac{\sup_{(a_{m+p-q-1},a_{m+p-q})}\big(|D^2T|\big)}{\inf_{(a_{m+p-q-1},a_{m+p-q})}\big(|DT|\big)}|T^qx - T^qy|\right]\\ &\leqslant \exp\left[\frac{\sup_{(Z^-_{m,p})}\big(|D^2T|\big)}{\inf_{(Z^-_{m,p})}\big(|DT|\big)}|x-y| + \sum_{q=1}^{p-1}\sup_{(a_{m+p-q-1},a_{m+p-q})}\big(|D^2T|\big)|T^qx - T^qy|\right] \end{aligned} \tag{6}$$

  To continue we need the following lemma.

Lemma 4. *For $x, y \in Z^-_{m,p}$ we have*

(i) $\sum_{q=1}^{p-1}\sup_{(a_{m+p-q-1},a_{m+p-q})}\big(|D^2T|\big)|T^qx - T^qy| \leqslant C_1|T^{p-1}Z|$

(ii) $\sup_{Z^-_{m,p}}\big(|D^2T|\big)|x-y| \leqslant C_2\frac{|T^{p-1}Z|}{l_{m+1}}$,

where we set for convenience Z the interval with endpoints x and y.

Proof.

(i) Denote $T^{p-1}x = z_x$ and $T^{p-1}y = z_y$; since the derivative is positive and decreasing on $(0, m)$ we have

$$|T^qx - T^qy| \leqslant \frac{1}{DT^{p-1-q}(a_{m+p-q})}|z_x - z_y|. \tag{7}$$

Let us now consider the term

$$DT^{p-1-q}(a_{m+p-q}) = DT(a_{m+p-q})DT(Ta_{m+p-q})\ldots DT(T^{p-2-q}a_{m+p-q}) \tag{8}$$

Since for $q \geqslant 1$ and $\xi_1 \in (a_q, a_{q+1})$

$$DT(a_q) \geqslant DT(\xi_1) = \frac{T(a_{q+1}) - T(a_q)}{a_{q+1} - a_q} = \frac{a_q - a_{q-1}}{a_{q+1} - a_q}$$

it follows that

$$\begin{aligned}(8) &\geqslant \frac{a_{m+p-q} - a_{m+p-q-1}}{a_{m+p+1-q} - a_{m+p-q}} \cdot \frac{a_{m+p-q-1} - a_{m+p-q-2}}{a_{m+p-q} - a_{m+p-q-1}} \cdots \frac{a_{m+2} - a_{m+1}}{a_{m+3} - a_{m+2}}\\ &\geqslant \frac{a_{m+2} - a_{m+1}}{a_{m+p+1-q} - a_{m+p-q}}\end{aligned}$$

and thus[7]:

$$\frac{1}{DT^{p-1-q}(a_{m+p-q})} \leqslant \frac{a_{m+p+1-q} - a_{m+p-q}}{a_{m+2} - a_{m+1}}.$$

Moreover $|z_x - z_y| \leqslant |T^{p-1}Z|$. Finally

$$(7) \leqslant \frac{a_{m+p+1-q} - a_{m+p-q}}{a_{m+2} - a_{m+1}} |T^{p-1}Z|. \tag{9}$$

Using lemmas 1 and 2 we see that there exists a constant C_0 depending only on the map T such that

$$\left(\sup_{(a_{m+q-1}, a_{m+q})} |D^2T|\right)(a_{m+q+1} - a_{m+q}) \leqslant C_0 \cdot \frac{1}{(q+m)^{\frac{\gamma-2}{\gamma-1}}(q+m)^{\frac{\gamma}{\gamma-1}}} = C_0 \cdot \frac{1}{(q+m)^2}.$$

Therefore the sum over $q = 1, 2, \ldots$ is summable and there exists a constant C_1 such that for $x, y \in Z^-_{m,p}$

$$\sum_{q=1}^{p-1}\left(\sup_{(a_{m+p-q-1}, a_{m+p-q})} |D^2T|\right)|T^q x - T^q y| \leqslant C_1 |T^{p-1}Z| \tag{10}$$

(ii) In this case we need to control the behaviour of the map close to 0. In particular, by using lemmas 1 and 2 (and the symmetry of $b_{\pm i}$) we start by noticing that

$$\left(\frac{\sup_{(b_{i+1}, b_i)} |D^2T|}{\inf_{(b_{i+1}, b_i)} |DT|}\right)|b_i - b_{i+1}| = \mathcal{O}\left(\frac{1}{i}\frac{i^{\frac{2\gamma-1}{\gamma-1}}}{i^{\frac{2\gamma-1}{\gamma-1}}}\right) = \mathcal{O}\left(\frac{1}{i}\right). \tag{11}$$

Combining (10) and (11) with (6) we get that there exists a constant D_2 so that for all $j \leqslant p-1$

$$\frac{1}{D_2} \leqslant \left|\frac{DT^j(x)}{DT^j(y)}\right| \leqslant D_2. \tag{12}$$

Let us call $t = b_{-(m+p-1)}$, $u = b_{-(m+p)}$ the end points of $Z^-_{m,p}$. For $j_1, j_2 \leqslant p-1$ there exist $\eta_1 \in (x, y)$ and $\eta_2 \in (t, u)$ such that

$$\begin{aligned}|T^{j_1}x - T^{j_1}y| &= DT^{j_1}(\eta_1)|x - y|,\\ |T^{j_2}t - T^{j_2}u| &= DT^{j_2}(\eta_2)|t - u|.\end{aligned}$$

[7] We have just proved that if ξ is any point in (a_{m+p}, a_{m+p+1}) (and the same result holds for its negative counterpart $(a_{-(m+p+1)}, a_{-(m+p)})$ as well) then $DT^p(\xi) \geqslant \frac{a_{m+2}-a_{m+1}}{a_{m+p+1}-a_{m+p}}$. In a similar way we can prove the lower bound: $DT^p(\xi) \leqslant \frac{a_{0^+}}{a_{m+p-1}-a_{m+p-2}}$, for $p \geqslant 2$.

The distortion bound (12) yields

$$\frac{\left|T^{j_1}x - T^{j_1}y\right|}{\left|T^{j_1}t - T^{j_1}u\right|} \leqslant D_2^2 \frac{\left|T^{j_2}x - T^{j_2}y\right|}{\left|T^{j_2}t - T^{j_2}u\right|}.$$

If we now choose $j_1 = 0$ and $j_2 = p - 1$ then

$$\left(\sup_{(t,u)} |D^2T|\right) |x - y| \leqslant D_2^2 \left(\sup_{(t,u)} |D^2T|\right) \frac{|t - u| \cdot |T^{p-1}x - T^{p-1}y|}{|T^{p-1}t - T^{p-1}u|}.$$

Since $|T^{p-1}t - T^{p-1}u| = l_{m+1} = a_m - a_{m+1}$ and x and y belong to Z we get

$$\left(\sup_{(t,u)} |D^2T|\right) |x - y| \leqslant D_2^2 \left(\sup_{(t,u)} |D^2T|\right) \frac{|t - u| \cdot |T^{p-1}Z|}{l_{m+1}}$$

and using distortion bound (11) once more we have that there exists a constant C_2 such that

$$\left(\sup_{(t,u)} |D^2T|\right) |x - y| \leqslant C_2 \frac{|T^{p-1}Z|}{l_{m+1}}. \qquad \square$$

By collecting lemma 4(i) and 4(ii) we see that the ratio $|DT^p(x)/DT^p(y)|$, $(x, y \in Z)$ is bounded as

$$\left|\frac{DT^p(x)}{DT^p(y)}\right| \leqslant \exp\left[C_2 \frac{|T^{p-1}Z|}{l_{m+1}} + C_1|T^{p-1}Z|\right] \leqslant \exp\left[K_2|T^{p-1}Z|\right] \tag{13}$$

with $K_2 = C_1 + C_2/l_{m+1}$.

We finish the proof of the proposition by choosing $K = \max(K_1, K_2)$. $\square$

We now return to the induction over the sets of the form $I_m = (a_{m-1}, a_m)$, $m \in \mathbb{Z}$ which, for $m = 0$ are intended to be $I_0^+ = (0, a_{0+})$ and $I_0^- = (a_{0_-}, 0)$. We then define a partition of I_m by $\mathcal{W}_m = \{W_{m,1}, W_{m,2}, \ldots, W_{m,p}, \ldots\}$, where

$$W_{m,p} = \{x \in I_m\ ,\ \tau_{I_m}(x) = p\}.$$

$\tau_{I_m}(x)$ being the first return time of x into I_m. We call $W^c_{m,p}$ any of the connected components of $W_{m,p}$ and we set $\hat{T}'_m : I_m \to I_m$ the first return map to I_m. $W_{m,p}$ is the disjoint union of its connected components $W^c_{m,p}$ and moreover $\widehat{T}(W^c_{m,p}) = T^p(W^c_{m,p}) = I_m$ and the map $\widehat{T}$ is surjective and onto[8].

[8] It easy to describe symbolically such connected components; we first give a suitable coding of each point $x \in \mathbb{T}\setminus\mathcal{S}$, where $\mathcal{S} = \cup_{i\geqslant 0} T^{-i}\{0\}$. We associate with such a point x the sequence $\underline{\omega} = (\omega_0\omega_1 \ldots \omega_n \ldots)$; $\omega_l \in \mathbb{Z}^* \cup \{0_-\} \cup \{0_+\}$ such that (from now on n will denote a positive integer) $\omega_l = n$ iff $T^l x \in (a_{n-1}, a_n)$; $\omega_l = -n$ iff $T^l x \in (a_{-n}, a_{-(n-1)})$; $\omega_l = 0_+$ iff $T^l x \in I_0^+$; $\omega_l = 0_-$ iff $T^l x \in I_0^-$.

The grammar is the following:

$$\omega_l = n > 0 \Rightarrow \omega_{l+1} = n - 1; \qquad \omega_l = -n < 0 \Rightarrow \omega_{l+1} = -(n-1),$$
$$\omega_l = 0_+ \Rightarrow \omega_{l+1} = 0_- \ or\ -n \quad (\text{any } n); \qquad \omega_l = 0_- \Rightarrow \omega_{l+1} = 0+ \ or\ n > 0 \quad (\text{any } n).$$

A cylinder $(\omega_1, \omega_2, \cdots, \omega_n)$, with $\omega_i \in \mathbb{Z}^* \bigcup\{0\pm\}$ and compatible with the grammar, will denote the open set $\bigcap_{i=1}^n T^{-i} I_{\omega_i}$. Therefore we see that every $W_{m,p}$ is the disjoint union of connected cylinders $W_{m,p}(k_1, \ldots, k_n)$ of the form

$$W_{m,p}(k_1, \ldots, k_n) = \Big(\underbrace{k_1\ k_1 - 1 \ldots 1}_{k_1}\ \underbrace{0_+\ 0_- \ldots 0_+}_{k_2}\ \underbrace{-k_3 \ldots -1}_{k_3}\ 0_-0_+ \ldots 0_-\ \underbrace{m + k_n - 1 \ldots m}_{k_n}\Big), \tag{14}$$

with $k_1 + \ldots + k_n = p + 1$ and $k_1 = m$.

Corollary 1. *Let us consider the cylinders* $I_m = (a_{m-1}, a_m), m \in \mathbb{Z}$. *Then there exists constants* K *and* K' *(possibly different from those given in proposition 3), depending on* m *such that for all* x, y *in any connected component* $W^c_{m,p}$ *we have*

$$\left|\frac{D\hat{T}'_m(x)}{D\hat{T}'_m(y)}\right| = \left|\frac{DT^p(x)}{DT^p(y)}\right| \leqslant \exp\left[K|T^p(x) - T^p(y)|\right] \leqslant e^{2K}$$

and

$$\frac{|D^2\hat{T}'_m(x)|}{|D\hat{T}'_m(x)|^2} \leqslant K'$$

Proof. We first observe that by standard arguments the first return map $\hat{T}'_m$ induced by T on I_m coincides with the first return map induced by $\hat{T}_m$ on I_m, where $\hat{T}_m$ is the first return map induced by T on $\tilde{I}_m \supset I_m$. Then we conclude by noticing that an induced map of a map satisfying the bounded distortion condition or Adler's condition (on $Z^{\pm}_{m,p}$ in our case) satisfies the bounded distortion condition or Adler's condition as well (on $W^c_{m,p}$ in our case) [5]. □

3. Decay of correlations

In this section and the next we prove several statistical properties for our map: they are basically consequences of the distortion inequality obtained in the previous section matched with established techniques.

Proposition 5. *The map* T *enjoys polynomial decay of correlations w.r.t. the invariant Lebesgue measure* ν*, for Hölder continuous functions on* $\mathbb{T}$. *More precisely, for all Hölder* $\varphi : \mathbb{T} \to \mathbb{R}$ *and all* $\psi \in L^\infty_\nu$*, we have*

$$\left|\int (\varphi \circ T^n)\, \psi \,\mathrm{d}\nu - \int \varphi \,\mathrm{d}\nu \int \psi \,\mathrm{d}\nu\right| = \mathcal{O}\left(\frac{1}{n^{\frac{1}{\gamma-1}}}\right).$$

Proof. We will use Lai-Sang Young's tower technique [33]. We build the tower over the interval I_0^+ and we define the return time function as the first return time:

$$\text{for all } x \in I_0^+, \quad \tau_{I_0^+}(x) := \min\{n \in \mathbb{N}^+;\ T^n x \in I_0^+\}.$$

The tower is thus defined by

$$\Delta = \{(x, l) \in I_0^+ \times \mathbb{N}\ ;\ l \leqslant \tau_{I_0^+}(x) - 1\}$$

and the partition of the base I_0^+ is given by the sets $W_{0_+,p}$. Recall that the dynamics on the tower is given by

$$F(x, l) = \begin{cases} (x, l+1) & \text{if } l < \tau_{I_0^+}(x) - 1, \\ (T^{\tau_{I_0^+}(x)}(x), 0) & \text{if } l = \tau_{I_0^+}(x) - 1. \end{cases}$$

According to [33], the decay of correlations is given by the asymptotics of $\nu\{x \in I_0^+\ ;\ \tau_{I_0^+}(x) > n\}$ namely

$$\nu\{x \in I_0^+\ ;\ \tau_{I_0^+}(x) > n\} = \sum_{p=n+1}^{+\infty} \nu\{x \in I_0^+\ ;\ \tau_{I_0^+}(x) = p\} = \sum_{p=n+1}^{+\infty} \nu(W_{0_+,p}).$$

Before computing this quantity explicitly, we must verify another important requirement of the theory; this will also be useful in the next section about limit theorems. Let us first introduce the separation time $s(x, y)$ between two points x and y in I_0^+. Put $\hat{T}_0'$ the first return map on I_0^+; we define $s(x, y) = \min_{n\geqslant 0}\{(\hat{T_0'}^n(x), \hat{T_0'}^n(y))$ lie in distinct $W_{0_+,p}, p \geqslant 1\}$. We ask that $\exists C > 0, \theta \in (0, 1)$ such that $\forall x, y \in W_{0_+,p}, p \geqslant 1$, we have

$$\left|\frac{D\hat{T}_0'(x)}{D\hat{T}_0'(y)}\right| \leqslant \exp[C\theta^{s(\hat{T}_0'(x),\hat{T}_0'(y))}]. \tag{15}$$

Let us prove this inequality. Suppose $s(\hat{T}_0'(x), \hat{T}_0'(y)) = n$; then since the orbits (under $\hat{T}_0'$) of the two points will be in the same cylinder of type $W_{0_+,p}, p \geqslant 1$ up to time $n - 1$ and on these cylinders $\hat{T}_0'$ is monotone and uniformly expanding, $|D\hat{T}_0'| \geqslant \beta > 1$ (see footnote 5), we have $|\hat{T}_0'(x) - \hat{T}_0'(y)| \leqslant \beta^{-(n-1)}$. Therefore by the distortion inequality (13) we get

$$\left|\frac{D\hat{T}_0'(x)}{D\hat{T}_0'(y)}\right| \leqslant \exp\left[\frac{K\beta^{-(n-1)}}{l_0}\right] \leqslant \exp[C\theta^{s(\hat{T}_0'(x),\hat{T}_0'(y))}], \tag{16}$$

where $C = \frac{K\beta}{l_0}$ and $\theta = \beta^{-1}$. This bound is often called the *local Hölder condition for* $\log|D\hat{T}_0'|$ *with exponent* θ; we will encounter it again pretty soon and when we refer to it, it will be associated with a given Markov partition of the induced space. We now come back to estimate the quantity $\nu(W_{0_+,p})$. The cylinder-set $W_{0_+,p}$ could be easily described using symbolic dynamics (see footnote 8): it will be the disjoint union of cylinders of the following form:

$$C_{p,q} = \big(0_+ \underbrace{-q \cdots -1}_{q}\ 0_-\ \underbrace{p-q-2 \cdots 1}_{p-q-2}\ 0_+\big).$$

Thus, there are $p-1$ cylinders whose first return time in I_0^+ is p. Noticing that $T^p : C_{p,q} \to I_0^+$ is surjective we know that there exists $\xi \in C_{p,q}$ such that $\nu(C_{p,q}) = \frac{\nu(I_0^+)}{|DT^p(\xi)|}$. Since as usual $DT^p(\xi) = DT(\xi)DT^q(T\xi)DT(T^{q+1}\xi)DT^{p-q-2}(T^{q+2}\xi)$ and $\xi \in (b_{-q-1}, b_{-q})$ and $T^{q+1}\xi \in (b_{p-q-2}, b_{p-q-1})$, by using the asymptotic bound on the $b_{\pm n}$ given by lemma 2 and the lower bound on the term $DT^p(x)$, $x \in (a_{p-1}, a_p)$ given in the footnote 7, we immediately get

$$\begin{aligned}
\nu(C_{p,q}) &\leqslant \frac{a_{+0}}{DT(b_{-q})DT(b_{p-q-2})}\frac{a_q - a_{q-1}}{a_1 - a_{+0}}\frac{a_{p-q-2} - a_{p-q-3}}{a_1 - a_{+0}} \\
&\leqslant \frac{a_{+0}}{DT(b_{-q})DT(b_{p-q-2})}\frac{(1 - a_{q-1})^\gamma}{(1 - \frac{1}{2\gamma})^\gamma}\frac{(1 - a_{p-q-3})^\gamma}{(1 - \frac{1}{2\gamma})^\gamma} \\
&\lesssim \frac{a_{+0}}{q(p-q-2)}\frac{1}{(q(p-q-2))^{\frac{\gamma}{\gamma-1}}} \lesssim \frac{1}{\big(q(p-q-2)\big)^{\frac{\gamma}{\gamma-1}+1}}.
\end{aligned}$$

The cases $q = 0$ (for which $C_{p,0} = (0_+\, 0_-\, (p-2) \ldots 2\ 1\ 0_+)$) and $q = p - 2$ (for which $C_{p,p-2} = (0_+\ -(p-2) \ldots -2\ -1\, 0_-\, 0_+)$ can be computed in a similar way and both are bounded by a quantity of order $\frac{1}{(p-2)^{\frac{\gamma}{\gamma-1}+1}}$.

Since there are only $p-1$ ways to place 0_- in $C_{p,q}$, we get

$$\begin{aligned}
\nu(W_{0_+,p}) &= \nu\left(\bigcup_{q=0}^{p-2} C_{p,q}\right) \\
&\lesssim \left(\sum_{q=1}^{p-3} \frac{1}{q^{\frac{\gamma}{\gamma-1}+1}(p-q-2)^{\frac{\gamma}{\gamma-1}+1}}\right) + \frac{2}{(p-2)^{\frac{\gamma}{\gamma-1}+1}} \\
&\lesssim \frac{1}{p^{\frac{\gamma}{\gamma-1}+1}}\left(\sum_{q=1}^{[\frac{p-3}{2}]} \frac{1}{q^{\frac{\gamma}{\gamma-1}+1}(1-\frac{q-2}{p})^{\frac{\gamma}{\gamma-1}+1}}\right) + \frac{2}{(p-2)^{\frac{\gamma}{\gamma-1}+1}} \lesssim \frac{1}{p^{\frac{\gamma}{\gamma-1}+1}}.
\end{aligned}$$

Finally

$$\nu\{x \in I_0^+ \ ; \ \tau_{I_0^+}(x) \geqslant n\} \lesssim \sum_{p=n}^{+\infty} \frac{1}{p^{\frac{\gamma}{\gamma-1}+1}} \lesssim \frac{1}{n^{\frac{\gamma}{\gamma-1}}}.$$

According to [33] the correlations decay satisfies $|\int(\varphi \circ T^n)\ \psi\, \mathrm{d}\nu - \int \varphi\, \mathrm{d}\nu \int \psi\, \mathrm{d}\nu| = \mathcal{O}(\sum_{k>n} \nu\{x \in I_0^+ \ ; \ \tau_{I_0^+}(x) \geqslant n\})$ and the right-hand side of this inequality behaves like $\mathcal{O}\left(\frac{1}{n^{\frac{1}{\gamma-1}}}\right)$. □

Optimal bounds. The previous result on the decay of correlations could be strengthened to produce a lower bound for the decay of correlations for integrable functions supported on the cylinders of the countable Markov partition I_m constructed in the previous section. For that we will use the renewal technique introduced by Sarig [31] and successively improved by Gouëzel [15]. In this regard we need a few assumptions that we directly formulate in our setting:

- Suppose we induce on the rectangle I_0^+; call $\mathcal{W}_0^+$ the Markov partition into the rectangles $W_{0^+,p}$ with first return p. A cylinder $[d_0, d_1, \cdots, d_{n-1}]$ with $d_i \in \mathcal{W}_0^+$ will be the set $\cap_{l=0}^{n-1} \hat{T}_0'^{-l} d_l$.
  We first need that the Jacobian of the first return map is locally Hölder continuous with exponent θ, but this is an immediate consequence of formula (16) with $\theta = \beta^{-1}$ and $C = K\beta/l_0$. Using the separation time $s(\cdot,\cdot)$, we define $D_{0^+} f = \sup |f(x) - f(y)|/\theta^{s(x,y)}$, where f is an integrable function on I_0^+ and the supremum is taken over all couples $x, y \in I_0^+$. We then put $||f||_{\mathcal{L}_{\theta,0^+}} \equiv ||f||_\infty + D_{0^+} f$. We call $\mathcal{L}_{\theta,0^+}$ the space of θ-Hölder functions on I_0^+ and we call $D_{0^+} f$ the Hölder constant of f (on I_0^+).
- We need the so-called *big image* property, which means that the Lebesgue measure of the images, under $\hat{T}_0'$, all the rectangles $W_{0^+,p} \in \mathcal{W}_0^+$ are uniformly bounded from below by a strictly positive constant. This is actually the case since all these images coincide with I_0^+.
- We finally need that $\nu(x \in I_m | \tau(x) > n) = \mathcal{O}(n^{-\chi})$, for some $\chi > 1$ (this is Gouëzel's assumption, which improves Sarig's one, asking for $\chi > 2$). But this has been proved above with $\chi = \frac{\gamma}{\gamma-1}$.

Under these assumptions, Sarig and Gouëzel proved a lower bound for the decay of correlations which we directly specialize to our map and for the interval I_0^+: the same kind of result of course holds for any other rectangle I_m.

Proposition 6. *There exists a constant C such that for all $f \in \mathcal{L}_{\theta,0^+}$ and $g \in L^\infty_\nu$ with norm $||\cdot||_\infty$ and both supported in I_0^+ we have*

$$\left|\mathrm{Corr}(f, g\circ T^n) - \left(\sum_{k=n+1}^{\infty} \nu(x\in I_0^+|\tau(x)>n)\right)\int g\,\mathrm{d}\nu \int f\,\mathrm{d}\nu\right| \leqslant C F_\gamma(n)||g||_\infty||f||_{\mathcal{L}_{\theta,0^+}},$$

where $F_\gamma(n) = \frac{1}{n^{\frac{\gamma}{\gamma-1}}}$ if $\gamma < 2$, $(\log n)/n^2$ if $\gamma = 2$ and $\frac{1}{n^{\frac{2}{\gamma-1}}}$ if $\gamma > 2$.

Moreover, if $\int f\,\mathrm{d}\nu = 0$, then $\int (g\circ T^n)\, f\,\mathrm{d}\nu = \mathcal{O}\left(\frac{1}{n^{\frac{\gamma}{\gamma-1}}}\right)$. Finally the central limit theorem holds for the observable f.

Remark 2. *The last sentence about the existence of the central limit theorem will be also obtained, using a different technique, in proposition 7, part 2, (a).*

4. Limit theorems

Let us recall the notion of stable law (see [11, 14]): a stable law is the limit of a rescaled i.i.d process. More precisely, the distribution of a random variable X is said to be stable if there exist an i.i.d stochastic process $(X_i)_{i\in\mathbb{N}}$ and some constants $A_n \in \mathbb{R}$ and $B_n > 0$ such that in distribution

$$\frac{1}{B_n}\left(\sum_{i=0}^{n-1} X_i - A_n\right) \longrightarrow X.$$

We will denote by $X(p, c, \vartheta)$ the law whose characteristic function is

$$E(\mathrm{e}^{X(p,c,\vartheta)}) = \mathrm{e}^{-c|t|^p\left(1-i\vartheta\,\mathrm{sgn}(t)\tan(\frac{p\pi}{2})\right)},$$

where $p \in (0,1)\cup(1,2]$, $c > 0$, $\vartheta \in [-1,1]$. Note that when $p \in (1,2]$ the law is of zero expectation. The case $p = 2$ corresponds to the normal law and the value of ϑ is irrelevant. The case $p = 1$ is problematic and it is not included here. We defer to [14] for a characterization of the constant c in terms of the asymptotic behaviour of the distribution of the random variable X. For one-dimensional Gibbs–Markov maps such a constant c enters the tail of the first return times, as it is showed in equation (17).

Proposition 7. *Let us denote $S_n\varphi = \sum_{k=0}^{n-1}\varphi\circ T^k$, where φ is a υ-Hölder observable, with $\int\varphi(x)\,\mathrm{d}x = 0$.*

(1) If $\gamma < 2$ then the central limit theorem holds for any $\upsilon > 0$. That is to say there exists a positive constant σ^2 such that $\frac{S_n\varphi}{\sqrt{n}}$ tends in distribution to $\mathcal{N}(0,\sigma^2)$. Moreover $\sigma^2 = 0$ iff there exists a measurable function ψ such that $\varphi = \psi\circ T - \psi$.

(2) If $\gamma > 2$ then

(a) If $\varphi(1) = 0$ and $|\varphi(x)| \leqslant \tilde{C}|x-1|_{\mathbb{T}}^{\upsilon'}$, where $|\cdot|_{\mathbb{T}}$ denotes the distance on the circle, $\tilde{C}$ is a positive constant and $\frac{1}{2}(\gamma-2) < \upsilon' < \gamma - 1$ then the central limit theorem still holds with a positive variance σ^2. Moreover $\sigma^2 = 0$ iff there exists a measurable function ψ such that $\varphi = \psi\circ T - \psi$.

(b) If $\varphi(1) \neq 0$ then $\frac{S_n\varphi}{n^{\frac{\gamma-1}{\gamma}}}$ converges in distribution to the stable law $X(p, c, \vartheta)$ with

$$p = \frac{\gamma}{\gamma-1},$$
$$c = \frac{1}{\gamma}\left(\frac{2\gamma|\varphi(1)|}{\gamma-1}\right)^{\frac{\gamma}{\gamma-1}}\Gamma\left(\frac{1}{(1-\gamma)}\right)\cos\left(\frac{\pi\gamma}{2(\gamma-1)}\right),$$
$$\vartheta = \mathrm{sgn}\,\varphi(1).$$

(3) If $\gamma = 2$ then

(a) If $\varphi(1) = 0$ then the central limit theorem holds.

(b) If $\varphi(1) \neq 0$ then there exist a constant b such that $\frac{S_n\varphi}{\sqrt{n \log n}}$ tends in distribution to $\mathcal{N}(0, b)$.

Proof.

(1) As a by-product of the tower's theory we get the existence of the central limit theorem whenever the rate of decay of correlations is summable ([33], theorem 4); this happens in our case for $\gamma < 2$. As usual we should avoid that φ is a co-boundary.

(2) (a)

We proceed as in [14] theorem 1.3 where this result was proven for the Pomeau–Manneville parabolic maps of the interval. We defer the reader to Gouezel's paper for the preparatory theory; we only prove here the necessary conditions for its application. We induce again on I_0^+ and we put $\varphi_{I_0^+}(x) := \sum_{i=0}^{\tau_{I_0^+}(x)-1} \varphi(T^i x)$. We need

- φ must be locally Hölder continuous on I_0^+ (with exponent θ).
- $\nu\{x \in I_0^+; \tau_{I_0^+}(x) = n\} = \mathcal{O}(1/n^{\eta+1})$, for some $\eta > 1$.
- $\varphi_{I_0^+} \in \mathcal{L}^2(I_0^+)$.

Recall that the induced map $\hat{T}_0'$ on I_0^+ is uniformly expanding with factor $\beta > 1$; therefore for any couple of points $x, y \in \mathbb{T}$ we have $|x - y|_{\mathbb{T}} \leqslant B\beta^{-s(x,y)}$, where B is a suitable constant and $|\cdot|_{\mathbb{T}}$ denotes the distance on the circle. Using the Hölder assumption on φ we get $|\varphi(x) - \varphi(y)| \leqslant D|x - y|_{\mathbb{T}}^{\upsilon} \leqslant E\beta^{-\upsilon s(x,y)}$, which shows that φ is locally Hölder with $\theta = \beta^{-\upsilon} < 1$.

The quantity in the second item above is exactly $\nu(W_{0+,n})$ for which we obtained in the previous section a bound of order $n^{-(\frac{\gamma}{\gamma-1}+1)}$. Hence $\eta = \gamma/(\gamma - 1)$.

To prove the third item denote $C_\varphi = \int_{I_0^+} |\varphi(x)|^2 \, dx$. As in section 3 we obtain (we simply put here $d\nu = dx$)

$$\begin{aligned}
\int_{I_0^+} |\varphi_{I_0^+}(x)|^2 \, dx &\leqslant C_\varphi + \sum_{l=2}^{+\infty} \int_{W_{0+,l}} \Big| \sum_{i=0}^{\tau_{I_0^+}-1} \varphi(T^i x) \Big|^2 dx \\
&\leqslant C_\varphi + \sum_{l=2}^{+\infty} \sum_{q=0}^{l-2} \int_{C_{l,q}} \Big| \sum_{i=0}^{l-1} \varphi(T^i x) \Big|^2 dx \\
&\lesssim C_\varphi + \tilde{C} \sum_{l=2}^{+\infty} \sum_{q=0}^{l-2} \int_{C_{l,q}} \Big| \sum_{i=0}^{l-1} |T^i x - 1|_{\mathbb{T}}^{\upsilon'} \Big|^2 dx \\
&\lesssim C_\varphi + \tilde{C} \sum_{l=2}^{+\infty} \sum_{q=0}^{l-2} \int_{C_{l,q}} \Big| \sum_{m=0}^{q} |a_{-m} + 1|^{\upsilon'} + \sum_{m=0}^{l-q-2} |a_m - 1|^{\upsilon'} \Big|^2 dx \\
&\lesssim C_\varphi + \tilde{C} \sum_{l=2}^{+\infty} \sum_{q=0}^{l-2} \nu(C_{l,q}) \Big| q^{\frac{-\upsilon'+\gamma-1}{\gamma-1}} + (l - q - 2)^{\frac{-\upsilon'+\gamma-1}{\gamma-1}} \Big|^2 \\
&\lesssim C_\varphi + \tilde{C} \sum_{l=2}^{+\infty} \sum_{q=0}^{[(l-2)/2]} \nu(C_{l,q})(l - q - 2)^{\frac{2(-\upsilon'+\gamma-1)}{\gamma-1}} \Big| 1 + \Big(\frac{q}{(l-q-2)} \Big)^{\frac{-\upsilon'+\gamma-1}{\gamma-1}} \Big|^2
\end{aligned}$$

$$\lesssim C_\varphi + \tilde{C}\sum_{l=2}^{+\infty} \frac{l^{\frac{2(-\upsilon'+\gamma-1)}{\gamma-1}}}{l^{\frac{\gamma}{\gamma-1}+1}}\left[1+\sum_{q=1}^{(l-2)/2}\frac{1}{q^{\frac{\gamma}{\gamma-1}}}\right]$$

$$\lesssim C_\varphi + \tilde{C}\sum_{l=2}^{+\infty} \frac{l^{\frac{2(-\upsilon'+\gamma-1)}{\gamma-1}}}{l^{\frac{\gamma}{\gamma-1}+1}}.$$

Finally if $\frac{2(-\upsilon'+\gamma-1)}{\gamma-1} - \frac{\gamma}{\gamma-1} - 1 < -1$ (i.e. $\upsilon' > \frac{1}{2}(\gamma-2)$) then $\varphi_{I_0^+} \in \mathcal{L}^2(I_0^+)$.

(b)

The proof proceeds as for the analogous case of theorem 1.3 in [14]; in order to use what we got at the point (a), we introduce the auxiliary function $\varphi = \varphi(1) + \tilde{\varphi}$, where $\tilde{\varphi}$ is υ-Hölder and satisfies $\tilde{\varphi}(1) = 0$. The corresponding functions on the induced space will verify $\varphi_{I_0^+} = \tilde{\varphi}_{I_0^+} + s$, where $s(x) = n\varphi(0)$ when x belongs to the cylinders $Z_{0_+,n}$ with first return n. It is argued in [14], and the same remains true in our case, that the constant c of the stable law should verify

$$\nu(s > n\varphi(1)) = \nu(\tau_{I_0^+} > n) \sim cn^g, \tag{17}$$

where g is a given exponent. Previous estimates suggest that $g = -(\frac{\gamma}{\gamma-1})$, but they are not enough to get the asymptotic equivalence prescribed to obtain the constant c. This is achieved by the following lemma.

Lemma 8. *Let us define* $R_n = (\tau_{I_0^+} > n)$*; then*

$$\nu(R_n) \sim 2b_n.$$

The proof of this lemma is postponed to the appendix; thanks to it we immediately see that $c = \frac{1}{\gamma}(\frac{2\gamma}{\gamma-1})^{\frac{\gamma}{\gamma-1}}$.

(3) This could also be obtained as in the proof of theorem 1.3 in [14]. □

Large deviations. The knowledge of the measure of the tail for the first returns on the tower (in our case built over I_0^+) will allow us to apply the results of Melbourne and Nicol [24], Melbourne [25] and Pollicott and Sharp [29] to get the large deviations property for particular classes of observables. We will apply to our map Melbourne's result [25], which states that whenever $\nu(x; \tau_{I_0^+} > n) = \mathcal{O}(n^{-(\zeta+1)})$, with $\zeta > 0$ (in our case $\zeta = \frac{1}{\gamma-1}$), then for some observables $\phi : [-1, 1] \to \mathbb{R}$ (for the regularity see below), which we take of zero mean, we have the following proposition.

Proposition 9. *If* $\zeta > 0$ *(*$\gamma > 1$*) then the map* T *verifies the following large deviations bounds:*

(I) whenever the observable ϕ *is Hölder*[9]*, then for all* $\epsilon > 0$ *there exists a constant* $C_{\phi,\epsilon}$ *such that*

$$\nu\left(\left|\frac{1}{n}\sum_{j=0}^{n-1}\phi(T^j(x))\right| > \epsilon\right) \leqslant C_{\phi,\epsilon} n^{-\frac{1}{\gamma-1}}$$

for all $n \geqslant 1$.

[9] In the paper [25] the observable ϕ is in L^∞_ν. But in this case one needs a specific assumption on the algebraic decay of correlations for L^∞_ν functions.

(II) There exists a constant c_1 and an open and dense set $\mathcal{A}$ of Hölder observables in the space of of Hölder observables, such that whenever $\phi \in \mathcal{A}$, then one can find $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ we have

$$\nu\left(\left|\frac{1}{n}\sum_{j=0}^{n-1}\phi(T^j(x))\right| > \epsilon\right) \geqslant c_1 n^{-\frac{1}{\gamma-1}}$$

for infinitely many n.

5. Recurrence

First returns. In the past ten years the statistics of first return and hitting times have been widely used as new and interesting tools to understand the recurrence behaviours in dynamical systems. Surveys of the latest results and some historical background can be found in [1, 19, 22].

Take a ball $B_r(x)$ of radius r around the point $x \in \mathbb{T}$ and consider the first return $\tau_{B_r(x)}(y)$ of the point $y \in B_r(x)$ into the ball. If we denote with ν_r the conditional measure to $B_r(x)$, we ask whether there exists the limit of the following distribution when $r \to 0$[10]:

$$F_r^e(t) = \nu_r\left(y \in B_r(x); \tau_{B_r(x)}(y)\nu(B_r(x)) > t\right).$$

The distribution $F_r^h(t)$ for the first-*hitting* time (into $B_r(x)$) is defined analogously just taking y and the probability ν on the whole space $\mathbb{T}$.

A powerful tool to investigate such distributions for non-uniformly expanding and hyperbolic systems is given by the conjunction of the following results, which reduce the computations to induced subsets.

- Suppose (T, X, μ) is an ergodic measure preserving transformation of a smooth Riemannian manifold X; take $\hat{X} \subset X$ an open set and equip it with the first return map $\hat{T}$ and with the induced (ergodic) measure $\hat{\mu}$. For $x \in \hat{X}$ we consider the ball $B_r(x)$ ($B_r(x) \subset \hat{X}$) around it and we write $\hat{\tau}_{B_r(x)}(y)$ for the first return of the point $y \in B_r(x)$ under $\hat{T}$. We now consider the distribution of the first return time for the two variables $\tau_{B_r(x)}$ and $\hat{\tau}_{B_r(x)}$ in the respective probability spaces $(B_r(x), \mu_r)$ and $(B_r(x), \hat{\mu}_r)$ (where again the subindex r means conditioning to the ball $B_r(x)$), as

  $$F_r^e(t) = \mu_r(y \in B_r(x); \tau_{B_r(x)}(y)\mu(B_r(x)) > t)$$

  and

  $$\hat{F}_r^e(t) = \hat{\mu}_r(y \in B_r(x); \hat{\tau}_{B_r(x)}(y)\hat{\mu}(B_r(x)) > t).$$

  In [6] we proved the following result: suppose that for μ-a.e. $x \in \hat{X}$ the distribution $\hat{F}_r^e(t)$ converges point wise to the continuous functions $f^e(t)$ when $r \to 0$ (remember that the previous distribution depends on x *via* the location of the ball $B_r(x)$); then we also have $F_r^e(t) \to f^e(t)$ and the convergence is uniform in t[11]. We should note that whenever we have the distribution $f^e(t)$ for the first return time we can ensure the existence of the weak-limit distribution for the first-hitting time $F_r^h(t) \to f^h(t)$, where $f^h(t) = 1 - \int_0^t f^e(s)\,\mathrm{d}s,\ t \geqslant 0$ [18].
  Note: From now on we will say that we have $f^e(t)$ as *limit distributions for balls*, if we get them in the limit $r \to 0$ and for μ-almost all the centres x of the balls $B_r(x)$.

[10] We call it distribution with abuse of language; in probabilistic terminology we should rather take 1 minus that quantity.
[11] The result proved in [6] is slightly more general since it does not require the continuity of the asymptotic distributions over all $t \geqslant 0$. We should note instead that we could relax the assumption that $\hat{X}$ is open just removing from it a set of measure zero, which happens on our induced sets I_m.

- The previous result is useful if we are able to handle the recurrence on induced subsets, see [7, 8] for a few applications. Induction for one-dimensional maps often produces piecewise monotonic maps with countably many pieces. We proved in [6], theorem 3.2, that whenever such maps are of Rychlik's type [30] (see definition 3.1 in [6] for a precise definition), then we have exponential return time statistics around balls (i.e. $f^e(t) = f^k(t) = \mathrm{e}^{-t}$). Our first return maps on the I_m are Bernoulli and expanding; in order to satisfy Rychlik's property it will be enough to show that the total variation of the potential, in our case $1/|DT(x)|$, is finite. This is again an easy consequence of the bounded distortion property[12].

We therefore have the following proposition.

Proposition 10. *The map T has exponential return and hitting time distributions with respect to the measure ν provided $\gamma > 1$.*

Number of visits. Let us come back to the general framework introduced in section 5.1 with the two probability spaces (X, T, μ) and $(\hat{X}, \hat{T}, \hat{\mu})$. We now introduce the random variables ξ_r^e and $\hat{\xi}_r^e$ which count the number of visits of the orbits of a point $y \in B_r(x)$ to the ball itself and up to a certain rescaled time. Namely

$$\xi_r^e(x, t) \equiv \sum_{j=1}^{\left[\frac{t}{\mu(B_r(x))}\right]} \chi_{B_r(x)}\left(T^j(y)\right),$$

where χ stands for the characteristic function and $x \in X$. If we take $x \in \hat{X}$ we can define in the same manner the variable $\hat{\xi}_r^e(x, t)$ by replacing the action of T with that of $\hat{T}$. We now introduce the two distributions

$$G_r^e(t, k) = \mu_r(x; \xi_r^e(x, t) = k), \qquad \hat{G}_r^e(t, k) = \hat{\mu}_r(x; \hat{\xi}_r^e(x, t) = k),$$

where again the index r for the measures means conditioning on $B_r(x)$. We proved in [6] that whenever the distribution $\hat{G}_r^e(t, k)$ converges weakly (in t) to the function $g(t, k)$ and for almost all $x \in \hat{X}$, the same happens, with the same limit, to the distribution $G_r^e(t, k)$. For systems with strong mixing properties the limit distribution is usually expected to be Poissonian [1, 19, 20, 22]: $\frac{t^k \mathrm{e}^{-t}}{k!}$.

[12] Let us consider the cylinder I_m and partition it into the cylinders $W_{m,p}$ with first return $p \geqslant 1$; then we have for the variation on I_m

$$\mathrm{Var}\frac{1}{|D\hat{T}_m'|} \leqslant \sum_{W_{m,p}} \int_{W_{m,p}} \frac{|D^2\hat{T}_m'(t)|}{|D\hat{T}_m'(t)|^2}\,\mathrm{d}\nu(t) + 2\sum_{W_{m,p}} \sup_{W_{m,p}} \frac{1}{|D\hat{T}_m'|}.$$

By the distortion bound proven in the first section we have that $K \geqslant |\log \frac{D\hat{T}_m'(x)}{D\hat{T}_m'(y)}| = |\int_x^y \frac{D^2\hat{T}_m'(t)}{D\hat{T}_m'(t)}\,\mathrm{d}\nu(t)| = \int_x^y \frac{|D^2\hat{T}_m'(t)|}{D\hat{T}_m'(t)}\,\mathrm{d}\nu(t)$ for any $x, y \in W_{m,p}$, since the first derivative is always positive and the second derivative has the same sign for all the points in the same cylinder. But this immediately implies that $\int_{W_{m,p}} \frac{|D^2\hat{T}_m'(t)|}{|D\hat{T}_m'(t)|^2}\,\mathrm{d}\nu(t) \leqslant \sup_{W_{m,p}} \frac{1}{|D\hat{T}_m'|}K$. Since $\hat{T}_m'$ maps $W_{m,p}$ diffeomorphically onto I_m there will be a point ξ for which $D\hat{T}_m'(\xi)\nu(W_{m,p}) = \nu(I_m)$. Applying the bounded distortion estimate one more time, we get $\sup_{W_{m,p}} \frac{1}{|D\hat{T}_m'|} \leqslant \mathrm{e}^K \frac{\nu(W_{m,p})}{\nu(I_m)}$. We finally obtain

$$\mathrm{Var}\frac{1}{|D\hat{T}_m'|} \leqslant \frac{\mathrm{e}^K(2+K)}{\nu(I_m)} \sum_{W_{m,p}} \nu(W_{m,p}) \leqslant \mathrm{e}^K(2+K) \leqslant \infty.$$

In [12] it was shown that Rychlik maps enjoy Poisson statistics for the limit distribution of the variables ξ_r^e and whenever the centre of the ball is taken a.e.. Hence we get the following result.

Proposition 11. *Let $\gamma > 1$. Then for ν-almost every x the number of visits to the balls $B_r(x)$ converges to the Poissonian distribution as $r \to 0$.*

Extreme values. The last quoted paper [12] contains another interesting application of the statistics of the first-hitting time that we could apply to our map T too. Let us first briefly recall the extreme value theory. Given the probability measure preserving dynamical system (X, T, μ) and the observable $\phi : X \to \mathbb{R} \cup \{\pm\infty\}$, we consider the process $Y_n = \phi \circ T^n$ for $n \in \mathbb{N}$. Then we define the partial maximum $M_n \equiv \max\{Y_0, \cdots, Y_{n-1}\}$ and we see if there are normalizing sequences $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ and $\{b_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\mu(\{x : a-n(M_n - b-n) \leqslant y\}) \to H(y)$$

for some non-degenerate distribution function H: in this case we will say that an *extreme value law* (EVL) holds for M_n. If the variables Y_n were i.i.d., the classical extreme value theory prescribes the existence of only three types of non-degenerate asymptotic distributions for the maximum M_n and under linear normalization, namely

- Type 1: $EV_1 = \mathrm{e}^{-e^{-y}}$ for $y \in \mathbb{R}$, which is called the *Gumbel* law.
- Type 2: $EV_2 = \mathrm{e}^{-y^{-\alpha}}$ for $y > 0$, $EV_2 = 0$, otherwise, where $\alpha > 0$ is a parameter, which is called *Frechet* law.
- Type 3: $EV_3 = \mathrm{e}^{-(-y)^{\alpha}}$ for $y \leqslant 0$, $EV_3 = 1$, otherwise, where $\alpha > 0$ is a parameter, which is called *Weibull* law.

In the paper [12] it has been proved, in a very general setting which applies to our situation too, that whenever the system (X, T, μ) has exponential hitting time statistics, then it satisfies an extreme value theory for the partial maximum M_n constructed on the process $\phi(x) = g(d(x, \xi))$, where d denotes the distance function, ξ is a chosen point in X and g is a strictly decreasing non-negative bijection in a neighbourhood of 0 where it attains a global maximum. In particular for some choices of g (see [12] for the details), one recovers the Gumbel, Frechet and Weibull laws.

Of course this result can be immediately applied to the mapping T under investigation in this paper.

6. Generalizations to Lorenz-like maps

As mentioned in the introduction, the original paper by Grossmann and Horner [16] dealt with different Lorentz-like *symmetric* maps S which map $[-1, 1]$ onto itself with two surjective branches defined on the half intervals $[-1, 0)$ and $(0, 1]$: we will suppose that the map is C^1 on $[-1, 1]/\{0\}$ and C^2 on $[-1, 1]/\{0\} \cup \{\pm 1\})$. Moreover we will ask for the following local behaviours (C will denote a positive constant which could take different values from one formula to the other):

$$S(x) = 1 - b|x|^{\kappa} + o(|x|^{\kappa}), \qquad x \sim 0,\ b > 0,$$
$$|DS(x)| = C|x|^{\kappa-1} + o(|x|^{\kappa-1});\ |D^2S(x)| = C|x|^{\kappa-2} + o(|x|^{\kappa-2}), \qquad x \sim 0,$$
$$S(x) = -x + a|x-1|^{\gamma} + o(|x-1|^{\gamma}), \qquad x \sim 1_-, \quad a > 0,$$
$$DS(x) = -1 + C|x-1|^{\gamma-1} + o(|x-1|^{\gamma-1});$$
$$D^2S(x) = C|x-1|^{\gamma-2} + o(|x-1|^{\gamma-2}), \qquad x \sim 1_-,$$

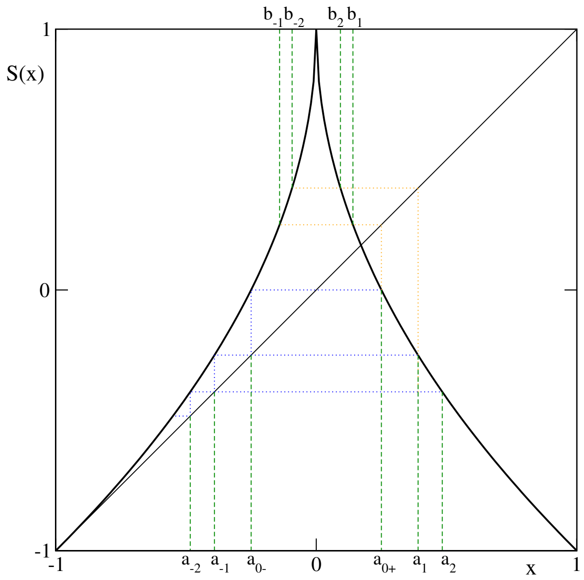

Figure 2. The Lorenz-like map S on the interval.
(This figure is in colour only in the electronic version)

$$S(x) = x + a|x+1|^{\gamma} + o(|x+1|^{\gamma}), \qquad x \sim -1_{+}, \quad a > 0,$$
$$DS(x) = 1 + C|x+1|^{\gamma-1} + o(|x+1|^{\gamma-1});$$
$$D^2S(x) = C|x+1|^{\gamma-2} + o(|x+1|^{\gamma-2}), \qquad x \sim -1_{+},$$

where $\kappa \in (0,1)$ and $\gamma > 1$ are two parameters. We also require that

(i) in all points $x \neq -1, 1$ the derivative is strictly bigger than 1.

(ii) S is strictly increasing on $[-1, 0)$, strictly decreasing on $(0, 1]$ and convex on the two intervals $(-1, 0)$, $(0, 1)$.

The map has a cusp at the origin where the left and right first derivatives diverge to $\pm\infty$ and the fixed point -1 is parabolic (figure 2). Although the map S is Markov with respect to the partition $\{[-1, 0), (0, 1]\}$ it will be more convenient to use a countable Markov partition whose endpoints are given by suitable preimages of 0 (see below).

The reflection symmetry of the map T in section 2 was related to the invariance of the Lebesgue measure. We do not really need the map S being symmetric with respect to the origin. We made this choice to get only two scaling exponents (κ and γ) in 0 and in ± 1. This implies in particular the same scalings for the preimages of 0 on $(-1, 0)$ and $(0, 1)$. If the left and right branches are not symmetric anymore, still preserving the Markov structure and the presence of indifferent points and a point with unbounded derivative, one should play with, at most, four scaling exponents giving the local behaviour of S in 0 and ± 1.

We denote by S_1 (respectively S_2) the restriction of S to $[-1, 0)$ (respectively $(0, 1]$) and define $a_{0+} = S_2^{-1}0$; $a_{0-} = S_1^{-1}0$; $a_{-p} = S_1^{-p}a_{0-}$; $a_p = S_2^{-1}S_1^{-(p-1)}a_{0-}$ for $p = 1, 2, \ldots$. It follows that $Sa_{-p} = Sa_p = a_{-(p-1)}$. In the same way as we did in the first section we define the sequence b_p, $p \geqslant 1$ as $Sb_{\pm p} = a_{p-1}$. The countable Markov partition mod ν will be $\{(a_{-p}, a_{-(p-1)}) : p \geqslant 1\} \cup \{(a_p, a_{p+1}) : p \geqslant 1\} \cup \{I_{0\pm}\}$; $I_{0+} = (0, a_{0+})$; $I_{0-} = (a_{0-}, 0)$.

From the local behaviours one gets the following scaling relations (use the symmetry to get the analogous relations for a_{-p} and b_{-p}):

$$
\begin{aligned}
1 - a_p &\sim \left(\frac{1}{a(\gamma-1)}\right)^{\frac{1}{\gamma-1}} \frac{1}{p^{\frac{1}{\gamma-1}}},\\
a_p - a_{p+1} &\sim a\left(\frac{1}{a(\gamma-1)}\right)^{\frac{\gamma}{\gamma-1}} \frac{1}{p^{\frac{\gamma}{\gamma-1}}},\\
b_p &\sim \left(\frac{1}{ab^{(\gamma-1)}(\gamma-1)}\right)^{\frac{1}{\kappa(\gamma-1)}} \frac{1}{p^{\frac{1}{\kappa(\gamma-1)}}},\\
b_{p-1} - b_p &\sim \frac{1}{\kappa(\gamma-1)}\left(\frac{1}{ab^{(\gamma-1)}(\gamma-1)}\right)^{\frac{1}{\kappa(\gamma-1)}} \frac{1}{p^{\frac{\kappa(\gamma-1)+1}{\kappa(\gamma-1)}}}.
\end{aligned}
$$

Bounded distortion, invariant measure and decay of correlations. An important difference with the map on the circle is that we are not guaranteed that the Lebesgue measure ν is invariant anymore; so we have to build an absolutely continuous invariant measure μ. Fortunately the tower's technique helps us again but we first need a useful change. As in section 2 we will induce over the disjoint union: $I_0 \equiv (a_{0-}, 0) \cup (0, a_{0-})$. The cylinders Z_p of I_0 with first return time p will have the form

$$
\begin{aligned}
Z_1 &= (a_{0-}, b_{-1}) \cup (b_1, a_{0+}),\\
Z_p &= (b_{-(p-1)}, b_{-p}) \cup (b_p, b_{p-1}) \qquad p > 1.
\end{aligned}
\tag{18}
$$

The first return map $\hat{S}$ for S on I_0 is not onto I_0 on each cylinder Z_p with prescribed first return time. In fact $\hat{S}$ maps all the cylinders (b_p, b_{p-1}) and $(b_{-(p-1)}, b_{-p})$ onto $(a_{0-}, 0)$, but it maps the cylinders (a_{0-}, b_{-1}) and (b_1, a_{0+}) onto $(0, a_{0-})$. Nevertheless $\hat{S}$ is an irreducible and aperiodic Markov map, as it is easy to check, and this is enough for the next considerations; moreover distortion can be proved exactly as in proposition 3[13].

The advantage of this induction scheme could be immediately seen in the exact scaling of the following tail (the Lebesgue measure of the points in I_0 with first return bigger than n, cf (17)), the precise form of it being essential to get stable laws later on:

$$
\nu(x \in I_0; \tau_{I_0}(x) > n) = 2\sum_{p=n+1}^{\infty}(b_{p-1} - b_p) \sim \frac{2}{\kappa(\gamma-1)}\left(\frac{1}{ab^{(\gamma-1)}(\gamma-1)}\right)^{\frac{1}{\kappa(\gamma-1)}} \frac{1}{n^{\frac{1}{\kappa(\gamma-1)}}}.
\tag{19}
$$

For the decay of correlations we invoke theorem 1 in Young's paper [33] and we get the following proposition.

Proposition 12. *Let us consider the map S depending upon the parameters γ and κ. Then for $0 < \kappa < \min(\frac{1}{\gamma-1}, 1)$, we get the existence of an absolutely continuous invariant measure μ which mixes polynomially fast on Hölder observables with rate $\mathcal{O}\big(n^{-\frac{1-\kappa(\gamma-1)}{\kappa(\gamma-1)}}\big)$.*

The map has exponential return and hitting times distributions and Poissonian statistic for the limit distribution of the number of visits in balls.

[13] If one wants a genuine first return Bernoulli map, one should induce over $(a_{0-}, 0)$: the cylinders with given first return time are simply slightly more complicated to manage. One could reduce to this situation as in corollary 1.

Density. Before continuing with the other statistical properties, we need a better knowledge of the invariant density ρ for the measure μ. We stress that our first return map $\hat{S}$ on I_0 is Gibbs–Markov according to the terminology of Aaronson and Denker in [4] (this is *a fortiori* true for the first return maps of Bernoulli type over the rectangles $(a_{-p}, a_{-(p-1)})$, (a_p, a_{p+1}), $(a_{0-}, 0)$, $(0, a_{0+})$)[14]. We now observe that as a consequence of the action of $\hat{S}$ on I_0 described above, the sigma-algebra generated by the images of the rectangles (18) is the same as that generated by the open intervals $(a_{0-}, 0)$, $(0, a_{0+})$. Hence by the Doëblin–Fortet inequality proved in proposition 1.4 in [4], the map $\hat{S}$ admits a mixing absolutely continuous invariant measure $\hat{\mu}$ whose density $\hat{\rho}$ is Lebesgue essentially bounded on I_0 and Lipschitz continuous on each of the two intervals $(a_{0-}, 0)$, $(0, a_{0+})$[15]. It is possible to get one more property of the density $\hat{\rho}$, namely it is bounded away from zero from below on the rectangles where we induced: this has been proved in [23, 26]. We now relate the densities $\hat{\rho}$ and ρ, supposing we induce over I_0, although similar results hold by inducing over the rectangles $(a_{-p}, a_{-(p-1)})$, (a_p, a_{p+1}), $(a_{0-}, 0)$, $(0, a_{0+})$. The measures μ and $\hat{\mu}$ satisfy

$$\mu(B) = C_r \sum_i \sum_{j=0}^{\tau_i - 1} \hat{\mu}(S^{-j}(B) \cap Z_i), \tag{20}$$

where B is any Borel set in $[-1, 1]$ and the first sum runs over the cylinders Z_i with prescribed first return time τ_i and whose union gives I_0. The normalizing constant $C_r = \mu(I_0)$ satisfies $1 = C_r \sum_i \tau_i \hat{\mu}(Z_i)$ and $\rho(x) = C_r \hat{\rho}(x)$, $x \in I_0$. The latter equality and the fact that $\hat{\rho}$ is essentially bounded and Lipschitz continuous over I_0 imply that the two limits

$$\lim_{x \to 0_-} \rho(x) = \rho_-; \qquad \lim_{x \to 0_+} \rho(x) = \rho_+ \tag{21}$$

exist, although they could be different. We have now to investigate the behaviour of the density $\rho(x)$ when $x \to 1_-$ (respectively $x \to -1_+$); we first note that the Lipschitz continuity being local on the rectangles (a_{-n}, a_{-n+1}) (respectively (a_n, a_{n+1})) approaching -1 (respectively 1) we cannot conclude that $\lim_{x \to 1_-} \rho(x)$ (respectively $\lim_{x \to -1_+} \rho(x)$) exist. To get at least a partial answer, we need a finer analysis. Since, as we said above, $\hat{\mu}$ is uniformly equivalent to ν on I_0, we will use the latter measure in the next computations. We first note that in order to estimate the μ-measure of a set B we need to consider only the cylinders Z_p of I_0 whose iterates will have non-empty intersection with B before they return to I_0: we use that to estimate the μ-measure of the cylinder (a_{n-1}, a_n) (for big n) near the point 1. We get that $S^{-1}(a_{n-1}, a_n) \cap Z_{n+1} = Z_{n+1}$ is the only possible non-empty intersection of the preimage $S^{-j}(a_{n-1}, a_n)$ with Z_p, for every p and for $0 \leqslant j \leqslant p - 1$. Therefore we have

$$\mu((a_{n-1}, a_n)) \approx C_r \nu(Z_{n+1}) \approx n^{-\frac{1-\kappa+\kappa\gamma}{\kappa(\gamma-1)}}.$$

The density on (a_{n-1}, a_n) will satisfy

$$\frac{1}{\nu((a_{n-1}, a_n))} \int_{(a_{n-1}, a_n)} \rho \, \mathrm{d}\nu = \frac{\mu((a_{n-1}, a_n))}{\nu((a_{n-1}, a_n))} \approx n^{-\frac{1-\kappa}{\kappa(\gamma-1)}}.$$

[14] We already checked that the partition, mod-ν, of I_0 given by the rectangles $\{Z_p\}$ is Markov and moreover the first return map $\hat{S}$ is uniformly expanding. We are left with the proof of the Adler's condition $\sup_{x \in Z_p \in I_0} \frac{|D^2 \hat{S}(x)|}{|D\hat{S}(x)|^2} < \infty$; but this can be proved along the same lines of the proof of proposition 3, as we already pointed out in remark 1.

[15] It is argued in [4] that if α is a Markov partition of the standard probability metric space $(X, \mathcal{B}, m, T)$ with distance d, then $T\alpha \subset \sigma(\alpha)$, where $\sigma(\alpha)$ denotes the sigma-algebra generated by the partition α, and therefore there exists a (possibly countable) partition β coarser than α such that $\sigma(T\alpha) = \sigma(\beta)$. Moreover if the system is Gibbs–Markov, then the space $Lip_{\infty,\beta}$ of functions $f : X \to \mathbb{R}$, $f \in L^\infty_m$ and which are Lipschitz continuous on each $Z \in \beta$ is a Banach space with the norm $||f||_{Lip_{\infty,\beta}} = ||f||_{L^\infty_m} + D_\beta f$, where $D_\beta f = \sup_{Z \in \beta} \sup_{x,y \in Z} \frac{|f(x) - f(y)|}{d(x,y)}$. The space $Lip_{\infty,\beta}$ is compactly injected into L^1_m, which gives the desired conclusions on the smoothness of the density as soon as the Doëblin–Fortet (or Lasota–Yorke) inequality is proved.

We now study the density in the neighbourhood of -1, by considering the cylinder (a_{-n}, a_{-n+1}), for large $n > 0$. The cylinders Z_p of I_0 whose iterates will have non-empty intersection with (a_{-n}, a_{-n+1}) before they return to I_0, have $p \geqslant n+2$. Therefore we get in the usual way:

$$\mu((a_{-n}, a_{-n+1})) \approx C_r \sum_{p=n+2}^{\infty} \nu(Z_p) \approx n^{-\frac{1}{\kappa(\gamma-1)}}.$$

The density on (a_{-n}, a_{-n+1}) will satisfy

$$\frac{1}{\nu((a_{-n}, a_{-n+1}))} \int_{(a_{-n},a_{-n+1})} \rho \,\mathrm{d}\nu = \frac{\mu((a_{-n}, a_{-n+1}))}{\nu((a_{-n}, a_{-n+1}))} \approx n^{-\frac{1-\kappa\gamma}{\kappa(\gamma-1)}}.$$

Now, suppose that in the last estimate the exponent of n is strictly negative, namely that $\kappa < 1/\gamma$. We want to prove that $\liminf_{x\to -1_+} \rho(x) = 0$. Suppose it is not zero, say $v > 0$; fix $0 < \epsilon < v$, then there exists $\delta > 0$ such that for all $-1 < x < -1+\delta$, we have $v - \epsilon < \rho(x)$. Take n large enough such that $(a_{-n}, a_{-n+1}) \subset (-1, -1+\delta)$. Then on such a rectangle we have that $\frac{1}{\nu((a_{-n},a_{-n+1}))} \int_{(a_{-n},a_{-n+1})} \rho \,\mathrm{d}\nu > v - \epsilon$, which is false. This argument could be applied to the various cases above to get the following result.

Proposition 13. *Let us consider the map S with $\gamma > 1$ and $0 < \kappa < \min(\frac{1}{\gamma-1}, 1)$. The density ρ of the invariant measure μ is Lipschitz continuous and bounded on the open cylinders $(a_{-p}, a_{-(p-1)})$, (a_p, a_{p+1}), $(a_{0-}, 0)$, $(0, a_{0+})$, $p \geqslant 1$. Moreover we have*

- $\liminf_{x\to 1_-} \rho(x) = 0$,
- *When $x \to -1_+$ the density verifies*
  - *(i) if $\kappa = \frac{1}{\gamma}$ then $\liminf_{x\to -1_+} \rho(x)$ and $\limsup_{x\to -1_+} \rho(x)$ are $\mathcal{O}(1)$, eventually different,*
  - *(ii) if $\frac{1}{\gamma} < \kappa$, then $\limsup_{x\to -1_+} \rho(x) = \infty$,*
  - *(iii) if $\frac{1}{\gamma} > \kappa$, then $\liminf_{x\to -1_+} \rho(x) = 0$.*
- *In the neighbourhood of 0 the limits (21) hold.*

Note that our proposition fits with the density found by Hemmer for the map (2); for this map and its circle companion (1) the correlations decay as n^{-1}.

Optimal bounds. As shown in the previous section, the result on the decay of correlations could be strengthened to produce a lower bound for the decay of correlations using the renewal technique introduced in [15, 31]. The only difference from the previous section is that now the Lebesgue measure is not any more invariant and thus we additionally need to show that the invariant density ρ is Lipschitz in the region of inducing. We established above the Lipschitz continuity of the density on the rectangles $(a_{-p}, a_{-(p-1)})$, (a_p, a_{p+1}), $(a_{0-}, 0)$, $(0, a_{0+})$, $p \geqslant 1$. We now keep, for instance, $I_{0-} = (a_{0-}, 0)$; the space of locally Hölder continuous functions with exponent θ and Hölder constant D_{0^-} (on I_{0-}), will produce the space $\mathcal{L}_{\theta,0^-,\mu}$ by adding to the Hölder constant on I_{0-} the L^∞_μ norm of the function analogously to what we did before proposition 6. Thus, we get in the same manner the following proposition.

Proposition 14. *There exists a constant C such that for all $f \in \mathcal{L}_{\theta,0^-,\mu}$ and $g \in L^\infty_\mu$ with norm $||\cdot||_\infty$, and both supported in I_{0-} we have (remember that $\gamma > 1$)*

$$\left|\mathrm{Corr}(f, g\circ S^n) - \left(\sum_{k=n+1}^{\infty} \mu(x \in I_{0-}|\tau(x) > n)\right)\int g\,\mathrm{d}\mu \int f\,\mathrm{d}\mu\right|$$
$$\leqslant C F_{\gamma,\kappa}(n) ||g||_\infty ||f||_{\mathcal{L}_{\theta,0^-,\mu}},$$

where

$$F_{\gamma,\kappa}(n) = \begin{cases} n^{-\frac{1}{\kappa(\gamma-1)}} & \text{if } 0 < \kappa < \min\left(\dfrac{1}{2(\gamma-1)}, 1\right), \\ (\log n)/n^2 & \text{if } \kappa = \dfrac{1}{2(\gamma-1)}, \\ n^{-\frac{2}{\kappa(\gamma-1)}+2} & \text{if } \dfrac{1}{2(\gamma-1)} < \kappa < \min\left(\dfrac{1}{\gamma-1}, 1\right). \end{cases}$$

Moreover, if $\int f \, \mathrm{d}\mu = 0$, *then* $\int (g \circ T^n) f \, \mathrm{d}\mu = \mathcal{O}\left(\frac{1}{n^{\frac{1}{\kappa(\gamma-1)}}}\right)$. *Finally the central limit theorem holds for the observable* f.

Remark 3. The last sentence about the existence of the central limit theorem will be also obtained in proposition 15, part 2, (a).

Limit theorems. To get the limit theorems we could induce again over I_0 since we only need that the induced map be Gibbs–Markov with a density which is eventually piecewise Lipschitz. As we stressed above, the advantage to induce over I_0 is that we easily control the Lebesgue measure of the points in I_0 with first return bigger than n, see formula (19). Passing from Lebesgue to the invariant measure μ we have to take care of the fact that the density could be discontinuous at 0. Following the corresponding arguments in section 3 we thus have the following proposition.

Proposition 15. *Let us denote* $S_n\varphi = \sum_{k=0}^{n-1} \varphi \circ S^k$, *where* φ *is an* υ*-Hölder observable, with* $\int \varphi(x) \, \mathrm{d}x = 0$.

(1) If $0 < \kappa < \min(\frac{1}{2(\gamma-1)}, 1)$, *then the central limit theorem holds for any* $\upsilon > 0$, *namely there exists a positive constant* σ^2 *such that* $\frac{S_n\varphi}{\sqrt{n}}$ *tends in distribution to* $\mathcal{N}(0, \sigma^2)$. *Moreover* $\sigma^2 = 0$ *iff there exists a measurable function* ψ *such that* $\varphi = \psi \circ S - \psi$.

(2) If $\frac{1}{2(\gamma-1)} < \kappa < \min(\frac{1}{\gamma-1}, 1)$, *then*

- *(a) If* $\varphi(-1) = 0$ *and* $|\varphi(x)| \leqslant \hat{C}|x+1|^{\upsilon''}$, *where* $\hat{C}$ *is a positive constant and* $\upsilon'' > \frac{1}{2\kappa(\gamma-1)}$ *then the central limit theorem still holds with positive variance* σ^2. *Moreover* $\sigma^2 = 0$ *iff there exists a measurable function* ψ *such that* $\varphi = \psi \circ S - \psi$.
- *(b) If* $\varphi(-1) \neq 0$ *then* $\frac{S_n\varphi}{n^{\frac{1}{p}}}$ *converges in distribution to the stable law* $X(p, c, \vartheta)$ *with*

$$p = \frac{1}{\kappa(\gamma-1)},$$

$$c = (\rho_+ + \rho_-)\left(\frac{|\varphi(-1)|}{ab^{(\gamma-1)}(\gamma-1)}\right)^{\frac{1}{\kappa(\gamma-1)}} \frac{1}{\kappa(\gamma-1)}\Gamma(1-p)\cos\left(\frac{\pi p}{2}\right),$$

$$\vartheta = \mathrm{sgn}\varphi(-1).$$

(3) If $\kappa = \frac{1}{2(\gamma-1)}$ *then*

- *(a) If* $\varphi(-1) = 0$ *then the central limit theorem holds.*
- *(b) If* $\varphi(-1) \neq 0$ *then there exist a constant* b *such that* $\frac{S_n\varphi}{\sqrt{n \log n}}$ *tends in distribution to* $\mathcal{N}(0, b)$.

Large deviations. Large deviations results can be derived following the corresponding arguments in previous sections. In particular, and by using the recent result by Melbourne [25], we can state that for Hölder observables, the large deviation property holds with polynomial decay at a rate which is given by that of the decay of correlations; for our Lorenz maps it is of order $n^{-\frac{1-\kappa(\gamma-1)}{\kappa(\gamma-1)}}$, provided that $0 < \kappa < \min(\frac{1}{\gamma-1}, 1)$.

Appendix

We prove here lemma (8).

Let us call

$$\beta_i^+ = (b_i, b_{i-1}); \qquad \beta_i^- = (b_{-(i-1)}, b_{-i});$$

these sets are such that $T^i(\beta_i^{\pm}) = I_0^{\mp}$.

Put $H(x)$ the first-hitting map from I_0^+ in I_0^-: $H(x) = T^i(x)$ if $x \in \beta_i^+$ and define $A_{p,q} := (T_+^{-1}T_-^{-(p-1)}(\beta_q^-)) \cap \beta_p^+$, that is the subset of β_p^+ that will go exactly in β_q^- under the action of H and will return in I^+ in $p+q$ iterations. Note that $\cup_{q=1}^{\infty} A_{p,q} = \beta_p^+$.

The set that will return after n is $C_n = \sum_{p=1}^{n-1} A_{p,n-p}$ and

$$R_n = \sum_{m=n+1}^{\infty} C_m = \sum_{m=n+1}^{\infty} \sum_{p=1}^{m-1} A_{p,m-p}.$$

Let us decompose the set R_n in

$$\begin{aligned} R_n &= \sum_{p=n}^{\infty}\sum_{q=1}^{\infty} A_{p,q} + \sum_{p=1}^{n}\sum_{q=n}^{\infty} A_{p,q} + \sum_{p+q>n;\, p,q<n} A_{p,q} \\ &= \sum_{p=n}^{\infty}\sum_{q=1}^{\infty} A_{p,q} + \sum_{p=1}^{\infty}\sum_{q=n}^{\infty} A_{p,q} - \sum_{p=n+1}^{\infty}\sum_{q=n}^{\infty} A_{p,q} + \sum_{p+q>n;\, p,q<n} A_{p,q}. \end{aligned} \tag{22}$$

Note that the measure of the sets of the first two sums in (22) is

$$\sum_{p=n}^{\infty}\sum_{q=1}^{\infty} \nu(A_{p,q}) = \sum_{p=n}^{\infty} \nu(\beta_p^+) = \nu(0, b_n) \tag{23}$$

and

$$\begin{aligned} \nu(b_{-n}, 0) = \sum_{q=n}^{\infty} \nu(\beta_q^-) &= \sum_{q=n}^{\infty} \nu(T_+^{-1}\beta_q^-) + \nu(T_-^{-1}\beta_q^-) \\ &= \sum_{q=n}^{\infty} \nu(A_{1,q}) + \nu(T_-^{-1}\beta_q^-) \\ &= \sum_{q=n}^{\infty} \nu(A_{1,q}) + \nu(T_+^{-1}T_-^{-1}\beta_q^-) + \nu(T_L^{-2}\beta_q^-) \\ &= \sum_{q=n}^{\infty} \nu(A_{1,q}) + \nu(A_{2,q}) + \nu(T_-^{-2}\beta_q^-) \\ &\dots \\ &= \sum_{q=n}^{\infty}\sum_{p=1}^{\infty} \nu(A_{p,q}). \end{aligned} \tag{24}$$

We already showed in the proof of proposition (5) that the measure of each $A_{p,q}$ scales as

$$\nu(A_{p,q}) \lesssim (pq)^{-\xi} \qquad \text{with } \xi = \gamma/(\gamma-1)+1$$

and thus the remaining two terms in (22) scale faster than $n^{-2(\xi-1)}$, that is faster than $b_n \sim n^{-(\xi-1)}$ and therefore they can be neglected.

Using (23) and (24) we have finally

$$\nu(R_n) \sim 2b_n.$$

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