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Computing the Pressure for Axiom-A Attractors by Time Series and Large Deviations for the Lyapunov Exponent

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For the Axiom-A attractors a relation is given between the topological pressure and the spectrum of the generalized Lyapunov exponents. As a consequence, a simple formula is found to compute the topological entropy of the attractor by means of a time series. The results are used to compute the large deviations for positive Lyapunov exponents.

KEY WORDS: Hyperbolic attractors; topological pressure; thermodynamic formalism; topological entropy; Lyapunov exponents; large deviations.

1. INTRODUCTION

The characterization of the ergodic and fractal properties of the invariant sets under the iteration of maps relies on two fundamental objects: the measurement of the Lyapunov exponents and of (any sort) of fractal dimension. There are, however, other quantities, such as the metric entropy, the topological entropy, and the Hausdorff dimension, which often give more precise informations about the degree of chaoticity of the system and the shape of its asymptotic limit sets. Recently, several authors[(1-4)] proposed a new technique, based on the computation of the so-called generalized Lyapunov exponent (see below), which allows one to estimate the above quantities by time averaging on the orbit of each point. The crucial fact is the identification of the generalized Lyapunov exponent with the topological pressure, which, at least for hyperbolic invariant sets, contains all the information about the Lyapunov exponents, the entropies, and the Hausdorff dimension. The first object of this paper is to prove

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rigorously this identification for Axiom-A attractors. The second step is the identification of the generalized Lyapunov exponent with another quantity studied in probability theory: the free energy. This leads to the investigation of the large deviations for positive Lyapunov exponent through the exact computation of the relative deviation function.

Let T be a C^2 diffeomorphism of a compact connected Riemannian manifold M into itself and J an Axiom-A attractor for T. Axiom-A means that J is uniformly hyperbolic and it is also the closure of the fixed points of $T^n_{|J}$, $n \geqslant 0$.[5] If E^u_x is the expanding subspace at the point $x \in J$, we put $\phi(x) = -\log \Lambda(x)$, where $\Lambda(x) = \mathrm{Jac}(DT \mid E^u_x)$ is the Jacobian of the linear map $DT\colon E^u_x \to E^u_{Tx}$, using inner products given by a Riemannian metric adapted to J. By μ we denote a T-invariant and ergodic Borel probability measure with support J. The generalized Lyapunov exponent of order β with respect to the measure μ will be defined for any real β by the formula[6]

$$\begin{aligned} L_\mu(\beta) &= \limsup_{n \to +\infty} \frac{1}{n} \log \int_J \exp \sum_{l=0}^{n-1} -\beta\phi(T^l x)\, d\mu(x) \\ &= \limsup_{n \to +\infty} \frac{1}{n} \log \int_J \mathrm{Jac}(DT^n \mid E^u_x)^\beta\, d\mu(x) \end{aligned} \tag{1.1}$$

The other basic object of consideration is the topological pressure $P(\beta)$ of the function $-\beta \log \Lambda(x)$. Since the latter is Hölder continuous,[5] there exists only one T-invariant and ergodic probability measure μ_β on J such that $P(\beta)$ is given by[5,8]

$$P(\beta) = h(\mu_\beta) + \beta \int_J \phi(x)\, d\mu_\beta(x) \tag{1.2}$$

where $h(\mu_\beta)$ is the μ_β-Kolmogorov metric entropy. We call μ_β the Gibbs measure[7] corresponding to the function $\beta\phi(x)$; the function $P(\beta)$ is convex, nonincreasing and real analytic for $\beta \in \mathbb{R}$[8]: we strongly use these properties in Section 2. We recall that the integral in (1.2) equals the sum of the positive μ_β-Lyapunov exponents of $T_{|J}$. The measure corresponding to $\beta = 1$ is particularly important. First of all one has

$$h(\mu_1) = \int_J \log \Lambda(x)\, d\mu_1(x)$$

with $P(1) = 0$.[5,9] Then, for any continuous function $g\colon M \to \mathbb{R}$ and for almost all the points x in the basin of attraction of J, with respect to the Riemannian measure on M, one has[9]

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{l=0}^{n-1} g(T^l x) = \int_J g(x)\, d\mu_1(x) \tag{1.3}$$

Finally, μ_1 has absolutely continuous conditional measures on the unstable manifolds: μ_1 is often called the Sinai–Bowen–Ruelle measure and denoted μ_{SBR}. In this paper we prove the following theorem conjectured in ref. 2.

Theorem 1. If we put the μ_{SBR} measure on the attractor, then the limit (1.1) exists, the convergence is uniform in β on any compact subset of $\mathbb{R}$, and

$$L_{\mu_{SBR}}(1-\beta) = P(\beta) \tag{1.4}$$

Remark 1. If we put $\beta = 0$, we get an interesting formula for computing the topological entropy h_{TOP} of $T_{|J}$ [recall that $P(0) = h_{TOP}$ [5,8]]:

$$h_{TOP} = \lim_{n \to +\infty} \frac{1}{n} \log \int_J \mathrm{Jac}(DT^n \mid E_x^u)\, d\mu_{SBR}(x) \tag{1.5}$$

Using the ergodic average (1.3), it is possible to compute numerically the limits (1.1) and (1.5) with respect to the μ_{SBR} measure. In a series of papers,[2-4] we used this procedure to compute the topological entropy, the pressure, and some fractal indices connected to it for polynomial maps of the plane, such as the Hénon and the Lozi ones. Although these maps are not Axiom-A, we found results in good agreement with other, different kinds of computation. For example, for the Hénon map we obtained $h_{TOP} = 0.445$ and for the Lozi map, which is quasihyperbolic,[10] $h_{TOP} = 0.488$. This suggests that the relation (1.5) could be more general and we think that it is true for C^2 diffeomorphisms of a smooth, compact, Riemannian manifold in the sense that if μ is an ergodic measure absolutely continuous with respect to the unstable foliation, then the limit on the right-hand side of (1.5) exists and gives the topological entropy of the support of μ. Is the converse also true?

Remark 2. For one-dimensional expanding maps, a formula like (1.4) was rigorously derived in refs. 2 and 11 computing the generalized Lyapunov exponents with respect to all the Gibbs measures (see also ref. 12 for similar rigorous results). During the completion of this work, P. Walters communicated a brief sketch of an (unpublished) proof of (1.5) using the theory of the Ruelle–Perron–Frobenius operator on the subshift of finite type. Finally, a bound of the type (1.5) involving the Riemannian measure on M was proved in all generality by Przytycki.[13] Theorem 1 will be proved in Section 3.

2. LARGE DEVIATIONS FOR THE LYAPUNOV EXPONENTS

The existence of the limit (1.1) allow us to study the large-deviation property for positive Lyapunov exponent. To be more precise and more transparent, let us consider an Axiom-A attractor J in a two-dimensional manifold with a one-dimensional unstable subspace E_x^u at $x \in J$. The set J equipped with the normalized Borel measure μ_{SBR} gives a probability space (J, μ_{SBR}) and we define on it the discrete random process

$$W_n(x) = \log \|D_x T^n \mid E_x^u\|, \qquad n \in \mathbb{Z}^+ \tag{2.1}$$

where $\|D_x T^n \mid E_x^u\|$ is the Riemannian norm of the tangent map of T^n restricted to E_x^u. The limit of $W_n(x)/n$ for $n \to +\infty$ gives for μ_{SBR}-almost all $x \in J$ the positive Lyapunov exponent of the Sinai–Bowen–Ruelle measure: $\lambda^+(\mu_{\mathrm{SBR}}) = \lambda^+(\mu_1)$. The large-deviation theory (see ref. 14 for all the concepts which we are using) studies, in our case, the fluctuations of the "finite-time Lyapunov exponent" $(1/n)\, W_n(x)$ from the true value $\lambda^+(\mu_{\mathrm{SBR}})$ [if the dimension of the manifold M is greater than two, we must replace $\|D_x T^n \mid E_x^u\|$ with $\mathrm{Jac}(DT^n \mid E_x^u)$, getting the large deviations of the volumes on the unstable subspaces or, which is the same, of the sum of the positive Lyapunov exponents]. The finite-time deviation is measured by the distribution of the process $(1/n)\, W_n(x)$ on $\mathbb{R}$; this means that for any Borel subset B of $\mathbb{R}$ we must consider the probability

$$Q_n(B) = \mu_{\mathrm{SBR}} \left\{ x \in J \Big/ \frac{1}{n} W_n(x) \in B \right\} \tag{2.2}$$

Then we introduce the functions

$$c_n(\beta) = \frac{1}{n} \log \int_J \|D_x T^n \mid E_x^u\|^\beta \, d\mu_{\mathrm{SBR}}(x) \tag{2.3}$$

The hyperbolic properties of $T_{|J}$ guarantee that each function $c_n(\beta)$ is finite and differentiable for $\beta \in \mathbb{R}$ and Theorem 1 proves the limit of $c_n(\beta)$ toward $c(\beta) = P(1-\beta)$: $c(\beta)$ is also called the free energy. The central step of the theory is now to introduce the Legendre–Fenchel transform of the free energy, or deviation function $I(x)$:

$$I(x) = \sup_{\beta \in \mathbb{R}} \{\beta x - c(\beta)\}, \qquad x \in \mathbb{R}$$

and to prove, under the regularity conditions for $c(\beta)$ satisfied in our case, the large-deviation bounds:

$$\limsup_{n \to +\infty} \frac{1}{n} \log Q_n(K) \leqslant -\inf_{x \in K} I(x) \qquad \text{when} \quad K \subset \mathbb{R} \text{ is closed} \tag{2.4a}$$

and

$$\liminf_{n \to +\infty} \frac{1}{n} \log Q_n(G) \geqslant -\inf_{x \in G} I(x) \qquad \text{when} \quad G \subset \mathbb{R} \text{ is open} \tag{2.4b}$$

For the Axiom-A attractors endowed with the μ_{SBR}-measure, we are able to compute the deviation function $I(x)$. First of all we need some facts about the Lyapunov exponents of the Gibbs measures on J. Since the map is expanding along the unstable directions and $\|D_x T | E_x^u\|$ is continuous on the compact set J, there will be two positive constants F_1 and F_2 (in the metric adapted to J) such that $1 < F_1 \leqslant \|D_x T | E_x^u\| \leqslant F_2$, $\forall x \in J$. This implies that the positive Lyaunov exponent $\lambda^+(\mu)$ of any ergodic measure μ on J will be bounded as $\log F_1 \leqslant \lambda^+(\mu) \leqslant \log F_2$. Now, if $\lambda^+(\mu_\beta)$ denotes the positive Lyapunov exponent corresponding to the Gibbs measure μ_β for the function $-\beta \log \|D_x T | E_x^u\|$, we have from thermodynamic formalism[8]

$$\left.\frac{dP(\delta)}{d\delta}\right|_{\delta=\beta} = -\lambda^+(\mu_\beta) \tag{2.5}$$

Moreover, by Ex. 5.5c and Corollary 7.12 in ref. 8, if $d^2P(\delta)/d\delta^2|_{\delta=\beta} = 0$ for some β, then $\lambda^+(\mu_\beta) = -dP(\delta)/d\delta|_{\delta=\beta} = \lambda^+ = \text{const}$ for all $\beta \in \mathbb{R}$. We avoid this trivial case, putting $d^2P(\beta)/d\beta^2 > 0$. In fact, in that case we have $P(\beta) = -\lambda^+\beta + h_{\mathrm{TOP}}$ and $I(x) = +\infty$, $\forall x \in \mathbb{R} \backslash \{\lambda^+\}$, and, since $P(1) = 0$, $I(x) = (\lambda^+ - h_{\mathrm{TOP}}) = 0$ when $x = \lambda^+$. An example of this type is the piecewise linear mapping of the interval generating the ternary Cantor set[11] for which $P(\beta) = -\beta \log 3 + \log 2$ (but in this case the invariant set is not an attractor). Then it follows that $d\lambda^+(\mu_\beta)/d\beta < 0$, so that $\lambda^+(\mu_\beta)$ is a nonincreasing, strictly monotone, real analytic function of β. This and the bounds on the Lyapunov exponent imply the existence of the two limits

$$\lim_{\beta \to +\infty} \lambda^+(\mu_\beta) = \Lambda_\infty^+ \geqslant \log F_1 \tag{2.6a}$$

$$\lim_{\beta \to -\infty} \lambda^+(\mu_\beta) = \Lambda_\infty^- \leqslant \log F_2 \tag{2.6b}$$

Moreover, since $h(\mu_\beta) = P(\beta) + \beta\lambda^+(\mu_\beta)$, the limits $\lim_{\beta \to \pm\infty} h(\mu_\beta)$ exist and we put

$$\lim_{\beta \to +\infty} h(\mu_\beta) = H_\infty^+ < \Lambda_\infty^+ \tag{2.6c}$$

and

$$\lim_{\beta \to -\infty} h(\mu_\beta) = H_\infty^- < \Lambda_\infty^- \tag{2.6d}$$

The strict inequalities on the right-hand sides of (2.6c) and (2.6d) are due to the fact that putting $g(\beta) = \lambda^+(\mu_\beta) - h(\mu_\beta)$, we have

$$\frac{dg(\beta)}{d\beta} = (\beta - 1)\frac{d^2P(\beta)}{d\beta^2}$$

so that, with $d^2P(\beta)/d\beta^2$ strictly positive by the above assumptions, $g(\beta)$ is always different from zero unless $\beta = 1$ (it is not evident that $h(\mu_\beta) \to 0$ for $\beta \to \pm\infty$).

We call Λ_L the open interval $(\Lambda_\infty^+, \Lambda_\infty^-)$.

Theorem 2. For $x > \Lambda_\infty^-$ and $x < \Lambda_\infty^+$ the deviation function $I(x) = +\infty$.

For $x \in \Lambda_L$ the deviation function is given by

$$I(x) = \lambda^+(\mu_{1-\beta}) - h(\mu_{1-\beta}) \tag{2.7}$$

where β is the unique real number satisfying $\lambda^+(\mu_{1-\beta}) = x$.

For $x = \Lambda_\infty^\pm$ we have $I(\Lambda_\infty^\pm) = \Lambda_\infty^\pm - H_\infty^\pm$.

Finally, the function $I(x)$ is differentiable for $x \in \Lambda_L$ with a minimum in $x = \lambda^+(\mu_{\mathrm{SBR}})$.

Remark 3. An incomplete version of Theorem 2 was already obtained in ref. 3 in a nonrigorous way and without using the large-deviation theory. In that paper we computed the deviation function for the positive Lyapunov exponent with respect to any Gibbs measure and not only for the Sinai–Bowen–Ruelle one (see also Section 4). The results quoted in ref. 3 extend also to conformal mixing repellers and they were independently discovered in refs. 12 and 15 (see ref. 16 for a review of some applications of large-deviation theory to dynamical systems). The statements of Theorem 2 allow us to graph the deviation function $I(x)$ of the positive Lyapunov exponent computed with respect to the μ_{SBR}-measure: see Fig. 1. This theoretical curve is very similar to the graph

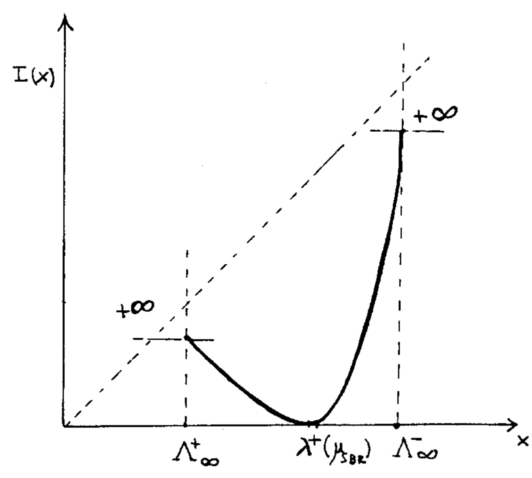

Fig. 1. Deviation function for positive Lyapunov exponent for Axiom-A attractors.

of the deviation function for the Lozi attractor obtained numerically in ref. 3, Fig. 1b, just computing the Lyapunov exponent and the entropy of many Gibbs measures and then using (2.7). Theorem 2 will be proved in Section 3.

3. PROOF OF THE THEOREMS

3.1. Proof of Theorem 1

Let us consider a Markov partition $\mathscr{R}^{(0)} = \{R_1^{(0)},..., R_m^{(0)}\}$ of J and construct the dynamical Markov partition $\mathscr{R}^{(n)} = \bigvee_{i=0}^{n-1} T^{-i}\mathscr{R}^{(0)}$: the elements of $\mathscr{R}^{(n)}$ are of the form $C_0 \cap \cdots \cap C_{n-1}$ for $C_i \in T^{-i}\mathscr{R}^{(0)}$ and $\mu(C_0 \cap \cdots \cap C_{n-1}) \neq 0$, where μ is any Gibbs measure on J. Then let A be the transition matrix associated with $\mathscr{R}^{(0)}$, that is, the matrix defined by

$$A_{ij} = \begin{cases} 1 & \text{if } \operatorname{int} R_i^{(0)} \cap T^{-1} \operatorname{int} R_j^{(0)} \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

where int R is the interior of R as a subset of J. We put

$$\Sigma_A = \{\mathbf{x} = \{x_i\}_{i=-\infty}^{+\infty} \in \{1,..., m\}^{\mathbb{Z}};\ A_{x_i x_{i+1}} = 1,\ \forall i \in \mathbb{Z}\}$$

and endow it with the compact open topology. It is well-known result of symbolic dynamics applied to Axiom-A diffeomorphisms[5] that there exists a map $\pi: \Sigma_A \to J$ associating to any $\mathbf{x} \in \Sigma_A$ the point $J \ni \pi(\mathbf{x}) = \bigcap_{i \in \mathbb{Z}} T^{-i} R_{x_i}^{(0)}$ and satisfying $\pi \circ \sigma = T \circ \pi$, where σ is ths shift on Σ_A: $\sigma\{x_i\} = \{x_{i+1}\}$. Moreover, π is onto J and is one-to-one over the residual set $\tilde{J} = J \setminus \bigcup_{i \in \mathbb{Z}} T^i(\partial^s \mathscr{R}^{(0)} \cup \partial^u \mathscr{R}^{(0)})$, where $\partial^s \mathscr{R}^{(0)}$ (resp. $\partial^u \mathscr{R}^{(0)}$) denotes the global stable (resp. unstable) boundary of $\mathscr{R}^{(0)}$.

We note that $\tilde{J}$ is T invariant and any Gibbs measure of $\tilde{J}$ is equal to 1. It is easy to see that a necessary and sufficient condition for the set $R_\alpha^{(n)}$ to be an element of $\mathscr{R}^{(n)}$ is that it is of the form

$$R_\alpha^{(n)} = \bigcap_{i=0}^{n-1} T^{-i} R_{x_i^\alpha}^{(0)}$$

where $x_0^\alpha,..., x_{n-1}^\alpha$ is a word with $A_{x_i^\alpha x_{i+1}^\alpha} = 1$, $i = 0,..., n-2$; then $\alpha = 1,..., \#(\mathscr{R}^{(n)})$. Now we put μ_{SBR}^* the Gibbs measure induced on Σ_A, that is, the measure defined by $\mu_{\text{SBR}}(E) = \mu_{\text{SBR}}^*(\pi^{-1}E)$, where E is a Borel subset of J and with $P^*(\beta)$ the pressure of the function $\beta\phi^*(\mathbf{x})$ [where $\phi^*(\mathbf{x}) = \phi(\pi\mathbf{x})$] computed with respect to the map $T \circ \pi$ on the shift space Σ_A.

We can, without loss of generality and by Smale's decomposition theorem (see Section 4 of ref. 5), consider the action of T on J topologically mixing: under this condition we have $P^*(\beta)=P(\beta)$.

Now, since any point $\mathbf{x}\in\Sigma_A$ with $x_i=x_i^\alpha$, $\forall i\in[0,n-1]$, belongs to $R_\alpha^{(n)}$, we have, splitting the integral

$$I_n(\beta)=\int_J \exp\sum_{l=0}^{n-1} -(1-\beta)\,\phi(T^l x)\,d\mu_{\mathrm{SBR}}(x)$$

over the rectangles $R_\alpha^{(n)}$,

$$\begin{aligned}I_n(\beta)&=\sum_\alpha\int_{\Sigma_A^\alpha}\exp\sum_{l=0}^{n-1} -(1-\beta)\,\phi(T^l\pi\mathbf{x})\,d\mu^*_{\mathrm{SBR}}(\mathbf{x})\\&=\sum_\alpha\int_{\Sigma_A^\alpha}\exp\sum_{l=0}^{n-1} -(1-\beta)\,\phi^*(\sigma^l\mathbf{x})\,d\mu^*_{\mathrm{SBR}}(\mathbf{x})\end{aligned}$$

where $\Sigma_A^\alpha=\pi^{-1}(R_\alpha^{(n)}\cap\tilde{J})$. Since

$$\Sigma_A^\alpha\subset\bar{\Sigma}_A^\alpha=\{\mathbf{x}\in\Sigma_A;\,x_i=x_i^\alpha,\,\forall i\in[0,n-1]\}$$

and $\bar{\Sigma}_A^\alpha$ is compact, we can bound the continuous function in the integral from above and below by the function itself computed at two points, say $\mathbf{x}_M^\alpha$ and $\mathbf{x}_m^\alpha$ belonging to $\bar{\Sigma}_A^\alpha$. Then we use Theorem (1.4) in ref. 5 stating that there exist two strictly positive constants C_1 and C_2 such that

$$\begin{aligned}C_2\exp&\left[-P^*(1)n+\sum_{l=0}^{n-1}\phi^*(\sigma^l\mathbf{x}^\alpha)\right]\\&\leqslant\mu^*_{\mathrm{SBR}}(\Sigma_A^\alpha)=\mu^*_{\mathrm{SBR}}(\bar{\Sigma}_A^\alpha)\\&\leqslant C_1\exp\left[-P^*(1)n+\sum_{l=0}^{n-1}\phi^*(\sigma^l\mathbf{x}^\alpha)\right]\end{aligned}\tag{3.1}$$

where $\mathbf{x}^\alpha\in\bar{\Sigma}_A^\alpha$ and for each α and $n\geqslant 0$. We choose the point $\mathbf{x}^\alpha$ in such a way as to maximize the function $\sum_{l=0}^{n-1}\beta\phi^*(\sigma^l\mathbf{x})$ on $\bar{\Sigma}_A^\alpha$. By Lemma (1.15) in ref. 5, there exists a positive finite constant C_3 independent of α and n such that, for each pair of points $\mathbf{x}$ and $\mathbf{y}$ belonging to $\bar{\Sigma}_A^\alpha$, we have

$$\left|\sum_{l=0}^{n-1}\phi^*(\sigma^l\mathbf{x})-\sum_{l=0}^{n-1}\phi^*(\sigma^l\mathbf{y})\right|\leqslant C_3\tag{3.2}$$

This allows us to replace the points $\mathbf{x}_M^\alpha$ and $\mathbf{x}_m^\alpha$ with $\mathbf{x}^\alpha$ up to two positive factors of the form $\exp[\mp(1-\beta)\,C_3]$: we suppose here without any restriction that $1-\beta<0$; for $\beta=1$, Theorem 1 is trivial, with $L_{\mu_{\mathrm{SBR}}}(0)=0$ and $P(1)=0$. Using this last fact, we finally get

$$e^{(1-\beta)C_3}C_2\sum_{\alpha}\exp\sum_{l=0}^{n-1}\beta\phi^*(\sigma^l\mathbf{x}^\alpha)$$
$$\leqslant I_n(\beta)\leqslant e^{-(1-\beta)C_3}C_1\sum_{\alpha}\exp\sum_{l=0}^{n-1}\beta\phi^*(\sigma^l\mathbf{x}^\alpha)$$

As we have already noted, the sum over α is equivalent to the sum over all the possible combinations of numbers $x_0,\dots,x_{n-1}$ such that $A_{x_ix_{i+1}}=1$; by definition, the thermodynamic limit of $\sum_\alpha\exp\sum_{l=0}^{n-1}\beta\phi^*(\sigma^l\mathbf{x}^\alpha)$ just gives the pressure $P^*(\beta)$ (ref. 5, p. 30). This proves that

$$\lim_{n\to+\infty}\frac{1}{n}\log I_n(\beta)=P(\beta) \tag{3.3}$$

pointwise for $\beta\in\mathbb{R}$. Since the functions $(1/n)\log I_n(\beta)$ are convex (as is easy to verify applying Hölder's inequality), it follows that the convergence (3.3) is uniform on each compact subset of $\mathbb{R}$.

3.2. Proof of Theorem 2

From now on we put, with abuse of notation, $\dot{f}(\beta)=df(\delta)/d\alpha\,|_{\delta=\beta}$. We write $D(\beta)=\beta x-P(1-\beta)$, $x\in\mathbb{R}$; we have $\dot{D}(\beta)=x-\lambda^+(\mu_{1-\beta})$ and $\ddot{D}(\beta)=\dot{\lambda}^+(\mu_{1-\beta})<0$; here we used (2.5). If $x>\Lambda_\infty^-$, we have $\dot{D}(\beta)>0$, so that

$$I(x)=\lim_{\beta\to+\infty}D(\beta)=\lim_{\beta\to+\infty}\beta[x-\lambda^+(\mu_{1-\beta})]=+\infty$$

In a similar way, when $x<\Lambda_\infty^+$, $\dot{D}(\beta)<0$, and then $I(x)=\lim_{\beta\to-\infty}D(\beta)=+\infty$. When $x\in\Lambda_L$, since $D(\beta)$ is concave, we compute the supremum taking the derivative and equting it to zero. Using (2.5) again, we get that the supremum in (2.4) will be attained by the unique $\bar{\beta}$ satisfying $\lambda^+(\mu_{1-\bar{\beta}})=x$. This allows us to express the deviation function as

$$\begin{aligned}I(x)&=\bar{\beta}\lambda^+(\mu_{1-\bar{\beta}})-P(1-\bar{\beta})\\&=\bar{\beta}\lambda^+(\mu_{1-\bar{\beta}})-h(\mu_{1-\bar{\beta}})+(1-\bar{\beta})\,\lambda^+(\mu_{1-\bar{\beta}})\\&=\lambda^+(\mu_{1-\bar{\beta}})-h(\mu_{1-\bar{\beta}})\end{aligned}$$

where we decomposed $P(1-\bar{\beta})$ using (1.2). Since the deviation function is lower semicontinuous,[14] by (2.6) we get $I(\Lambda_\infty^\pm)=\Lambda_\infty^\pm-H_\infty^\pm$. Now, since the function $\beta \mapsto x(\beta) = \lambda^+(\mu_{1-\beta})$ is strictly monotone and differentiable for each $\beta \in \mathbb{R}$ with its inverse $\beta(x)$, we get that $I(x)$ is continuous and differentiable for $x \in \Lambda_L$, too, and, moreover,

$$\dot{I}(x) = \dot{\beta}(x)x + \beta(x) - \dot{P}(1-\beta)\,\dot{\beta}(x) = \dot{\beta}(x)x + \beta(x) - \dot{\beta}(x)x = \beta(x)$$

This implies that when $\beta > 0$, which is equivalent to $x > \lambda^+(\mu_1)$, we get $\dot{I}(x) > 0$ and for $\beta < 0$, which is equivalent to $x < \lambda^+(\mu_1)$, we get $\dot{I}(x) < 0$. The deviation function has its minimum, as expected, for x equal to the Lyapunov exponent of the μ_{SBR}-measure: $I(\lambda^+(\mu_{\mathrm{SBR}})) = 0$ (see Fig. 1). It is now easy to compute the infimum of $I(x)$ on open or closed subsets B of the open intervals $(\lambda^+(\mu_{\mathrm{SBR}}), \Lambda_\infty^-)$ and $(\Lambda_\infty^+, \lambda^+(\mu_{\mathrm{SBR}}))$. In this case the sets B are sets of I-continuity, that is, $\inf_{x \in \mathrm{cl}\, B} I(x) = \inf_{x \in \mathrm{int}\, B} I(x)$, and the limits (2.4) properly exist, giving[14]

$$\lim_{n \to +\infty} \frac{1}{n} \log Q_n(B) = -\inf_{x \in B} (x)$$

4. EXTENSION AND CONCLUDING REMARKS

Theorem 1 can be easily extended, comparing the pressure with the generalized Lyapunov exponents computed with respect to any Gibbs measure μ_δ, $\delta \in \mathbb{R}$. The proof is similar to that of Theorem 1: it is only necessary to replace the term $P^*(1)$ with $P^*(\delta)$ and $\phi^*(\sigma^l \mathbf{x}^\alpha)$ with $\delta\phi^*(\sigma^l \mathbf{x}^\alpha)$ in inequality (3.1). This gives the following formula for each real β and δ:

$$L_{\mu_\delta}(\beta) = P(\delta - \beta) - P(\delta) \tag{4.1}$$

The deviation function $I_\delta(x)$ computed on the probability space (J, μ_δ) for $x \in \Lambda_L$ now reads

$$I_\delta(x) = \delta\lambda^+(\mu_{\delta-\beta}) - h(\mu_{\delta-\beta}) + P(\delta) \tag{4.2}$$

where β is the unique real root of $x = \lambda^+(\mu_{\delta-\beta})$.

However, the only observable L_{μ_δ} will be that corresponding to $\delta = 1$: in fact, in this case the measure can be approximated by an ergodic average in an accessible neighborhood of the attractor. We point out that the attractive nature of our invariant set J is expressed by the only condition $P(1) = 0$; for other kinds of Axiom-A basic sets (for example, hyperbolic horseshoes or repellers) the expressions (4.1) and (4.2) are unchanged provided that $P(\delta = 1) \neq 0$. In this case the quantity $P(1)$ is interpreted as the "escape rate" from the basic set.[17] The topological meaning of the condition $P(1) = 0$ is that the unstable manifolds foliate the attractor and the μ_{SBR}-measure conditioned on them is absolutely continuous with respect to the induced Riemannian volume. It is possible to make a proof of Theorem 1 avoiding symbolic dynamics, but using only the preceding topological prescriptions. The proof is particularly easy for Anosov diffeomorphisms using the techniques of Theorem (14.1) in ref. 18.

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