TOPOLOGICAL SYNCHRONISATION OR A SIMPLE ATTRACTOR?

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Abstract. A few recent papers introduced the concept of topological synchronisation. We refer in particular to [13], where the theory was illustrated by means of a skew product system, coupling two logistic maps. In this case, we show that the topological synchronisation could be easily explained as the birth of an attractor for increasing values of the coupling strength and the mutual convergence of two marginal empirical measures. Numerical computations based on a careful analysis of the Lyapunov exponents suggest that the attractor supports an absolutely continuous physical measure (acpm). We finally show that for some unimodal maps such acpm exhibit a multifractal structure.

1. Introduction

The recent paper [13], which also garnered some press attention [14, 15], introduced the concept of topological synchronisation which occurs when, in a dynamical system, it is possible to identify two or more attractors which become very similar when the system evolves. This situation is for instance met in coupled lattice map, where each site of the lattice brings its own attractor. It is written in [13] that *during the gradual process of topological adjustment in phase space, the multifractal structures of each strange attractor of the two coupled oscillators continuously converge, taking a similar form, until complete topological synchronization ensues.* As an example of this process of synchronisations, the authors in [13] studied a skew system whose base is a logistic (master) map T of the interval $[-1, 1]$ and the other map (the slave), is another logistic map on the same interval which is coupled with the master in a convex way in order to be confined to the interval $[-1, 1]$. As an indicator of the *closeness* of the attractors of the master and slave maps when the coupling strength, say k, increases, the authors in [13] used the spectrum D_q of generalized dimensions. They showed in particular the interesting phenomenon, which they called the *zipper effect*, where the dimensions begin to synchronise at negative q, with low values of k before becoming similar, for positive values of q, when k arrives at the threshold of complete synchronisation of the attractors. They interpreted this fact by saying that *the road to complete synchronization starts at low coupling with topological synchronization of the sparse areas in the attractor and continues with topological synchronizations of much more dense areas in the attractor until complete topological synchronization is reached for high enough coupling.*

The object of our note is to show that, in the case of the skew system where the master and the slave map are both in the logistic family, if we denote with $\{x_n\}_{n\geq 0}$ the trajectory

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of the master system and with $\{y_n\}_{n\geq 0}$ that of the slave system, the topological synchronisation is easily interpreted as the presence of an invariant set in the neighborhood of the diagonal of the square $[-1,1]^2$ to which $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ converge when the coupling strength tends to 1, in the sense of (6). Moreover, we show that the empirical measure computed along the trajectories of the slave system approaches, in the limit as the number of the iterations tends to infinity, the physical measure of the master map. We compute numerically the Lyapunov exponent of the master map T and we show that it is positive for the parameter values considered in [13], which implies that the attractor in the master space is a finite union of intervals. We therefore discuss the real occurrence of a multifractal spectrum for the empirical measure of unimodal maps. We prove the existence of a non-trivial multifractal spectrum for the Benedicks-Carleson type maps investigated in [3] and where the invariant density has at most countably many poles. We show that the generalized dimensions are constant and equal to 1 for $q < 2$, and so in particular for negative q and this explains easily the presence of the *zipper effect.* We also give a toy-model example of an invariant density generating a multifractal spectrum on a Cantor set of poles. Finally in the last section we return on the relationship between the empirical measure for the slave dynamics and the invariant measure of the master map, say μ. We will give a Markov chain interpretation of the dynamics in the slave direction, perturbing its evolution by means of an additive noise whose distribution is given by μ. We will show that this random dynamical system admits at least one stationary measure which converges weakly to μ when the coupling strength tends to 1.

2. The attractor

The skew system studied in [13] is defined on the square $[-1,1]^2$ and has the form for $0 < k < 1$

$$\begin{cases} x_{n+1} = T_1(x_n) \\ y_{n+1} = (1-k)T_2(y_n) + kT_1(x_n), \end{cases} \tag{1}$$

where T_1 and T_2 are two maps of the interval $[-1,1]$ into itself. Set

$$\Delta_n := |x_n - y_n|;$$

it is immediate to see that for any $n \geq 1$:

$$\Delta_n \leq 2(1-k) \sup_{i=1,2} \sup_{x\in[-1,1]} |T_i(x)|, \tag{2}$$

and therefore the sequences x_n, y_n approach each other when $k \to 1$. We now specialize to the example investigated in [13], namely $[-1,1] \ni x \longmapsto T_i(x) := c_i(1-2x^2) \in [-1,1]$, and show how to improve the previous bound. The skew system now reads:

$$\begin{cases} x_{n+1} = c_1(1-2x_n^2) \\ y_{n+1} = (1-k)c_2(1-2y_n^2) + c_1k(1-2x_n^2). \end{cases} \tag{3}$$

We have

$$\Delta_{n+1} = |(1-k)c_1(1-2x_n^2) - (1-k)c_2(1-2y_n^2)| \leq \\ (1-k)|c_1 - 2c_1x_n^2 - c_2 + 2c_2y_n^2|.$$

We add and subtract the term $2c_1y_n^2$ and we easily obtain

$$\Delta_{n+1} \leq (1-k)|(c_1-c_2)(1-2y_n^2)+2c_1(y_n^2-x_n^2)|.$$

We now put $\Delta_c := |c_1-c_2|$. Since x_n, y_n are in the interval $[-1,1]$, we have

$$\Delta_{n+1} \leq (1-k)\Delta_c|1-2y_n^2| + 4c_1(1-k)\Delta_n \leq (1-k)\Delta_c + 4c_1(1-k)\Delta_n.$$

We now iterate it and we finally get

$$\Delta_{n+1} \leq (1-k)\Delta_c \sum_{l=0}^{n} (1-k)^l(4c_1)^l + (1-k)^{n+1}(4c_1)^{n+1}.$$

We then require

$$k > 1 - \frac{1}{4c_1}, \tag{4}$$

and we define the quantity

$$W_\infty(k) := \Delta_c(1-k)\frac{1}{1-(1-k)4c_1}, \text{ such that } \lim_{k\to 1} W_\infty(k) = 0. \tag{5}$$

By sending $n \to \infty$ we finally get

$$\limsup_{n\to\infty} \Delta_n \leq W_\infty(k). \tag{6}$$

We now use the following values taken in [13]:

$$c_1 = 0.89, \;\; c_2 = 0.8373351.$$

First of all we note that with these values (4) gives $k > 0.72$, which is consistent with what was used in [13]. As in the latter we now take $k = 0.9$, which is the value where, according to [13], the system reaches complete topological synchronization. By substituting into $W_\infty(k)$ we get

$$W_\infty(0.9) = 0.0082,$$

which implies that the projections x_n and y_n are really very close. The bound (2) instead gives, still for $k = 0.9$,

$$\sup_{n\geq 1} |\Delta_n| \leq (1-k)[c_1+c_2] = 0.17273351.$$

3. The measures

In order to justify the closeness of the asymptotic behaviors of the master and slave dynamics, the paper [13] uses the spectrum of the generalized dimensions. These dimensions are defined in terms of a probability measure, see, e.g., [11] and [16, 8] for a rigorous treatment. Roughly speaking, if μ denotes a probability measure, and $B(x,r)$ a ball of center x and radius r on the phase space M, the generalized dimensions D_q are defined by the following scaling of the correlation integral

$$\int_M \mu(B(x,r))^{q-1} d\mu \sim r^{D_q(q-1)}, \;\; r \to 0. \tag{7}$$

The importance of the generalized dimension is that in several cases, see [16] and the references therein for rigorous results, if we denote by

$$d_\mu(x) := \lim_{r\to 0} \frac{1}{\log r} \log \mu(B(x,r)),$$

the local dimension of the measure μ at the point x, provided the limit exists, then

$$D_q(q-1) = \inf_\alpha \{q\alpha - f(\alpha)\}, \tag{8}$$

where $f(\alpha)$ denotes the Hausdorff dimension of the set of points x in the support of μ for which $d_\mu(x) = \alpha$[1].
The master map, which to ease the notation in the present section we denote by T instead of T_1, has several invariant probability measures; we pick one, namely the *physical measure* μ which is given by the weak limit of the probability measures

$$\mu_n := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{T^i x}, \tag{9}$$

where x is chosen Lebesgue almost everywhere on the unit interval (see for instance [10] Chapter V.1). In the following we will forget about the initial condition x, provided it is taken Lebesgue almost everywhere, and simply write $x_i = T^i(x)$. The *slave* sequence $\{y_n\}_{n\geq 0}$ could be seen as a non-autonomous, or *sequential*, dynamical system and it is not clear what probability measure we should associate to it. We argue that in [13] the authors used the sequence of probability measures

$$\nu_n := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{y_i}, \tag{10}$$

where y_i is the point associated to x_i in (3). We call μ_n and ν_n the *empirical measures*.
There are now two questions: (i) does the sequence ν_n converge weakly? and, in the affirmative case, (ii) is that weak limit point equal to μ? This is in fact what the numerical simulations on the generalized dimensions seem to indicate in [13].[2]

To study weak convergence, we have to integrate the probability measures against continuous functions defined on the interval $[-1, 1]$. Let f be one of this function; since it is

[1]It is worth noticing that in the next section we will compute the D_q spectrum in a few cases by using the characterization (8) and *not* the definition (7) in terms of the correlation integral.

[2]We point out, however, that it is in general not enough to have a weak convergence of the measures to ensure the convergence of the D_q spectrum. Suppose for instance that the master system has an absolutely continuous invariant measure $d\mu(x) = h(x)dx$ and that, for k close enough to 1, so does the measure of the slave system $d\mu_k(x) = h_k(x)dx$. If $h(x) \sim_{x_0} \text{const}|x - x_0|^\alpha$, with $-1 < \alpha < 0$ as, for instance, it is the case for some quadratic map along the orbit of the critical point, then the local dimension of μ at x_0 is $\alpha + 1 < 1$ and it is easily seen (see the detailed computations in the section 4.2) that the D_q spectrum is not constant. Moreover, if we further assume that, for all $k < 1$, h_k is a piecewise constant function converging in L^1 to h, it is easy to see that the D_q spectrum for μ_k is constant equal to 1 for all $k < 1$, so that there is no convergence to the spectrum of the master map.

also uniformly continuous, given $\varepsilon > 0$, call δ_ε the quantity such that $|f(x) - f(y)| < \frac{\varepsilon}{2}$, when $|x - y| < \delta_\varepsilon$. Let $k_\varepsilon \in (0, 1)$ such that

$$2W_\infty(k_\varepsilon) < \delta_\varepsilon. \tag{11}$$

For values of k such that $k_\varepsilon < k < 1$, we define $n_{k,\varepsilon}$ as

$$(1 - k)^{n_{k,\varepsilon}+1}(4c_1)^{n_{k,\varepsilon}+1} \leq W_\infty(k),$$

such that for all $n > n_{k,\varepsilon}$, $\Delta_n \leq \delta_\varepsilon$. By weak-compactness there will be a subsequence n_l for which $\{\nu_{n_l}\}_{l\geq 1}$ will converge weakly to a probability measure μ^*. Then for any continuous function f on the unit interval and for n_l sufficiently large, say $n_l > n^*$, we have that $|\frac{1}{n_l}\sum_{i=0}^{n_l-1} f(y_i) - \mu^*(f)| \leq \varepsilon/2$. Then

$$|\mu^*(f) - \mu(f)| \leq \left|\frac{1}{n_l}\sum_{i=0}^{n_l-1} f(y_i) - \mu^*(f)\right| + \left|\frac{1}{n_l}\sum_{i=0}^{n_l-1} f(y_i) - \mu(f)\right|$$

We now estimate the second expression on the r.h.s. of the previous inequality:

$$\frac{1}{n_l}\sum_{i=0}^{n_l-1} f(y_i) - \mu(f) = \frac{1}{n_l}\sum_{i=0}^{n_l-1} f(y_i) - \mu(f) + \frac{1}{n_l}\sum_{i=0}^{n_l-1} f(x_i) - \frac{1}{n_l}\sum_{i=0}^{n_l-1} f(x_i).$$

Now, for $n_l \geq n_{k,\varepsilon} + 2$, consider the difference

$$\frac{1}{n_l}\sum_{i=0}^{n_l-1}[f(y_i) - f(x_i)] = \frac{1}{n_l}\sum_{i=0}^{n_{k,\varepsilon}}[f(y_i) - f(x_i)] + \frac{1}{n_l}\sum_{i=n_{k,\varepsilon}+1}^{n_l-1}[f(y_i) - f(x_i)]. \tag{12}$$

By exploiting the uniform continuity of f on the unit interval we have

$$\left|\frac{1}{n_l}\sum_{i=n_{k,\varepsilon}+1}^{n_l-1}[f(y_i) - f(x_i)]\right| \leq \frac{n_l - n_{k,\varepsilon} - 2}{n_l}\varepsilon/2.$$

The other sum in the right hand side of equation (12) gives

$$\left|\frac{1}{n_l}\sum_{i=0}^{n_{k,\varepsilon}}[f(y_i) - f(x_i)]\right| \leq 2\max|f|\frac{n_{k+\varepsilon} + 1}{n_l}.$$

By sending $l \to \infty$, we finally get that $|\mu^*(f) - \mu(f)| \leq \varepsilon$, and this result is independent of the subsequence we choose. We thus have

Proposition 3.1. *(i) For any $\varepsilon > 0$, let $k_\varepsilon \in (0, 1)$ be given as in (11); then for all k such that $k_\varepsilon < k < 1$, we have*

$$\left|\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(y_i) - \mu(f)\right| \leq \varepsilon.$$

(ii) As a consequence we get

$$\inf_{0<k<1}\left|\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(y_i) - \mu(f)\right| = 0.$$

This is the best result we could get without further information on the system and it justifies the numerical evidence that the empirical measures constructed along the x and y axis become very close to each other when $k \to 1$.

4. The nature of the master's physical invariant measure

We said above that μ, the invariant measure for the master map T, is a physical measure; the paper [13] claims that such a measure has a multifractal structure for the prescribed values of c_1, where *the master has a dense strange attractor*, [ibid]. Before exploring and commenting such a possibility, we should remind a few important properties of the quadratic maps: first that they usually depend on a parameter, in our case c since the map in [13] is of the form

$$[-1, 1] \ni x \longmapsto T(x) = c(1 - 2x^2) \in [-1, 1], \tag{13}$$

with $0 < c \leq 1$. We refer in particular to the nice review paper by Thunberg [19], which contains a clear and exhaustive list of all the relevant results on unimodal maps and a rich bibliography. First of all, we define the attractor Ω_c of the map T as the unique set of accumulation points of the orbit of the point x, whenever this point is chosen Lebesgue almost everywhere. Then it is well known, see [7] or Theorem 6 in [19], that for our kind of logistic maps, the attractor could be of three types:
(1) an attracting periodic orbit; (2) a Cantor set of measure zero; (3) a finite union of intervals with a dense orbit.
Still in the quadratic case, we could classify the preceding three different types of attractors in terms of the set of parameters c. Following section 2.2 in [19] we have:

(1) $\mathcal{P} := \{c \in \mathbb{R} : \ \Omega_c \ \text{ is a periodic cycle}\}$ is open and dense in parameter space and consists of countably infinitely many nontrivial intervals.

(2) $\mathcal{C} := \{c \in \mathbb{R} : \Omega_c \ \text{ is a Cantor set}\}$ is a completely disconnected set of Lebesgue measure zero.

(3) $\mathcal{I} := \{c \in \mathbb{R} : \Omega_c \ \text{ is a union of intervals}\}$ is a completely disconnected set of positive Lebesgue measure.

The physical measures, constructed according to the prescription (9) exist and are parametrized by c in the following way, see section 2.3 in [19]:

(1) If $c \in \mathcal{P}$, the physical measure consists of normalized point masses on the periodic cycle Ω_c.

(2) If $c \in \mathcal{C}$, the support of the physical measure equals the Cantor attractor Ω_c, and it is singular with respect to Lebesgue measure.

(3) (a) There is a full-measure subset $\mathcal{S} \subset \mathcal{I}$ such that for all $c \in \mathcal{I}$, the physical measure is absolutely continuous with respect to Lebesgue measure and its support equals the interval attractor Ω_c.
(b) There are uncountably many parameters in $\mathcal{I} \setminus \mathcal{C}$ for which the physical measure may fail to exist.

We now have a very efficient numerical test to determine the nature of a physical measure. It is based on the following two rigorous results:

(i) the first says that if T has a non-flat critical point, as in our case, and it admits an absolutely continuous invariant probability measure μ, then it is the weak-limit of the sequence μ_n given in (9) and therefore it is a physical measure (see Chapter V.1 in [10]).

(ii) The second result is taken from the paper [12]. Let us define the number

$$\lambda_T := \limsup_{n\to\infty} \frac{1}{n} \log |(T^n)'(x)|. \tag{14}$$

This quantity exists for x chosen *Leb*-almost everywhere and it is strictly positive if and only if T has an absolutely continuous invariant measure.
From the joint use of (i) and (ii) it follows immediately the following important fact which we could summarize as

Theorem 4.1 ([10]+[12]). *Suppose that the map T has a non-flat critical point. Then, if the sequence*

$$\lambda_T^n := \frac{1}{n} \log |(T^n)'(x)| = \frac{1}{n} \sum_{i=0}^{n-1} \log |T'(x_i)| \tag{15}$$

has a positive limit for Leb-a.e. x, then the sequence of empirical measures μ_n in (9) converges weakly to an absolutely continuous invariant probability measure and therefore the attractor Ω_c will be a finite union of intervals and not a Cantor set.

In fig. 2, we represent the bifurcation diagram of T, and its Lyapunov exponent for different values of parameter c. This quantity is non-positive whenever the attractor is a periodic cycle or a Cantor set. We computed in particular the limit of λ_T^n for $c = c_1 = 0.89$, still called λ_T, and we got a positive value of ≈ 0.35, confirming the fact that μ *is not supported on a Cantor set.* We performed the same computation with $c = c_2$ and, denoting from now on by $\tilde{\mu}$ the associated physical measure, there are strong numerical evidences that it is again absolutely continuous, with a strictly positive Lyapunov exponent.

5. Multifractal spectrum for absolutely continuous measures

5.1. **Multifractal spectrum for unimodal maps.** Let us summarize: by choosing the parameter c with positive (Lebesgue) probability, we could get a periodic cycle or union of intervals. On the other hand Dirac measures with finitely many masses on the periodic cycles cannot have a multifractal spectrum, since $D_q = 0$ for all q in that case. Finally the Lyapunov exponent for $c = c_1$ is positive showing that the attractor cannot be a Cantor set and consequently the physical measure is absolutely continuous. The question is, therefore, if such a measure μ could exhibit a multifractal spectrum. Let us consider unimodal maps of Benedicks-Carleson type, which are known to preserve an absolutely continuous invariant measure μ [20]. Let us denote $z_k = f^k(z_0)$, where z_0 is the critical point. Under additional assumptions on the dynamics of the critical point[3] (see [3] for details), their density has the form:

[3]The map f is of class C^4 and it must be:

- a *Collet-Eckmann* S-unimodal map verifying $|(f^k)'(f(c))| > \lambda_c^k$, with $\lambda_c > 1$, $\forall k > H_0$, where H_0 is a constant larger than 1.
- a *Benedicks-Carleson* map: $\exists 0 < \gamma < \frac{\log \lambda_c}{14}$ such that $|f^k(c) - c| > e^{-\gamma k}, \forall k > H_0$.

$$h(x) = \psi_0(x) + \sum_{k\geq 1} \frac{\varphi_k(x)\chi_k(x)}{\sqrt{|x-z_k|}},$$

with $\psi_0 \in C^1$, $\varphi_k \in C^1$ is such that $||\varphi_k||_\infty \leq e^{-ak}$ for some $a > 0$ and $\chi_k = 1_{[1,z_k]}$ if f^k has a local maximum at z_0, while $\chi_k = 1_{[z_k,1]}$ if f^k has a local minimum at z_0. Notice that in this case the Lyapunov exponent (14) is surely positive by the Keller Theorem [12] quoted in section 4.

Proposition 5.1. *Suppose that f satisfies the hypothesis of Proposition 2.7 in* [3]*, see also footnote 3. Then, the generalized dimensions spectrum of μ is given by:*

$$D_q = \begin{cases} 1 \ \textit{if}\ q < 2 \\ \frac{q}{2(q-1)} \ \textit{otherwise.} \end{cases} \tag{16}$$

Proof. In the following proof the constants $a_j, j = 1, 2..$ will be independent of x and r. Let $x \in \mathrm{supp}(\mu)$, the support of μ. The measure of a ball centered at x of radius r is given by

$$\mu(B(x,r)) := \int_{x-r}^{x+r} \psi_0(y)dy + \sum_{k\geq 1} \int_{x-r}^{x+r} \frac{\varphi_k(y)\chi_k(y)}{\sqrt{|y-z_k|}} dy.$$

Let us take $m_n = \frac{n}{2a}$ and $\delta_n = m_n^{-\log n}$. Let

$$\Gamma_n = \{x : \exists k < m_n \text{ such that } |x - z_k| < \delta_n\}.$$

Given $r > 0$, we take n the smaller integer such that $r < e^{-n}$. Since the functions φ_k are bounded, the integrals in the sum are bounded above by $a_1||\varphi_k||_\infty\sqrt{r}$ when $|x - z_k| < \delta_n$ and by $a_2||\varphi_k||_\infty r/\sqrt{\delta_n}$ otherwise , where $a_1, a_2 > 0$. We get:

$$\mu(B(x,r)) \leq 2r||\psi_0||_\infty + a_1\sqrt{r} \sum_{k:|x-z_k|<\delta_n} ||\varphi_k||_\infty + a_2\frac{r}{\sqrt{\delta_n}} \sum_{k:|x-z_k|>\delta_n} ||\varphi_k||_\infty \tag{17}$$

For $x \notin \Gamma_n$, the first sum starts at least at m_n is therefore at most $a_3e^{-am_n} \leq a_3\sqrt{r}$. The second geometric sum is bounded par a_3. Thus there exists $a_4 > 0$ such that,

$$\mu(B(x,r)) \leq a_4(r + \frac{r}{\sqrt{\delta_n}}).$$

If $x \notin \Gamma_n$ for n large enough, then $d_\mu(x) = 1$, since $-\log\delta_n$ is of order $(\log\log(1/r))^2 < \log(1/r)$, so the second term does not affect the dimension. Therefore $d_\mu(x) = 1$ in the set

$$G = \bigcup_p \bigcap_{n>p} \Gamma_n^c.$$

Let $\Gamma = G^c = \bigcap_p \bigcup_{n>p} \Gamma_n$, the set of x such that there exists an infinity of n such that $x \in \Gamma_n$. Γ is covered by the union of balls

$$\bigcup_n \bigcup_{k<m_n} B(z_k, \delta_n).$$

Now, for all $\varepsilon > 0$, we have

$$\sum_n \sum_{k<m_n} \delta_n^\varepsilon = \sum_n (\frac{n}{2a})^{1-\varepsilon \log n} < \infty.$$

So the Hausdorff measure $H^\varepsilon(\Gamma)$ is finite, which show that $\dim_H(\Gamma) \leq \varepsilon$.

It is easily seen from (17) that for all $x \in [-1, 1]$,

$$\mu(B(x,r)) \leq a_5\sqrt{r}. \tag{18}$$

This shows that the infimum of the local dimensions is larger or equal to 1/2. On the other hand, since for all k, φ_k is C^1 and h is integrable, the singularities are of type $|x - z_k|^{-1/2}$ either to the left or to the right of z_k. Therefore, if the density admits a singularity at z_k,

$$\mu(B(z_k,r)) \geq a_6\sqrt{r}. \tag{19}$$

Combining the last two estimates, we get that $d_\mu(z_k) = 1/2$. We can now compute the generalized dimensions. $D_q(q-1)$ is defined as the Legendre transform of the function $f(\alpha) := d_H\{x \in \mathrm{supp}(\mu); d_\mu(x) = \alpha\}$, where d_H denotes the Hausdorff dimension. Since $h(x) > 0$ for all $x \in \mathrm{supp}(\mu)$ (see Theorem 2 in [20]), then the local dimensions are bounded above by 1. In our case, we have $f(1) = 1$ and, for all $\alpha < 1$, either $f(\alpha) = 0$, or $f(\alpha)$ is not defined, so that

$$D_q = (q-1)^{-1} \inf_\alpha \{q\alpha, q-1\}.$$

Since $\min_{x\in \mathrm{supp}(\mu)} d_\mu(x) = 1/2$, we obtain our result. □

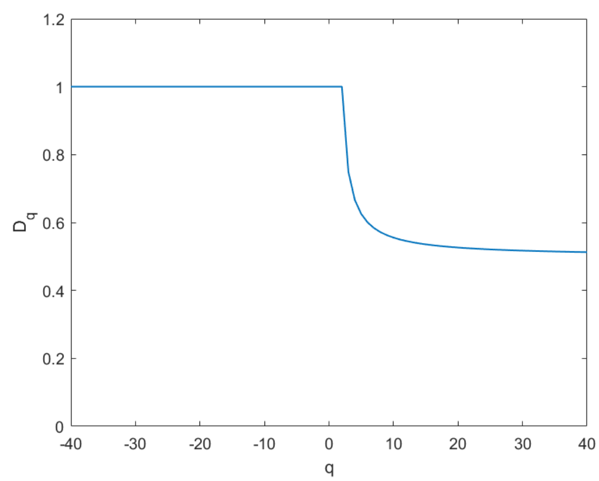

FIGURE 1. D_q spectrum associated with the family of unimodal maps considered in Proposition 5.1

For $c = 1$ for instance, the associated density

$$h(x) = \frac{1}{\pi\sqrt{1-x^2}}$$

has two poles at -1 and 1 and its D_q spectrum is that of the previous proposition.
For our class of quadratic maps (13), depending upon the parameter c, the assumptions stated in the footnote 3 are satisfied for a positive measure set of values of the parameter c, [6, 4]. It is therefore plausible, although not certain, that the previous proposition applies to the physical measure of our master map. As the latter has a density bounded away from 0 on its support [20], we surely expect its generalized dimensions spectrum to be constant for negative values of q and not differentiable (see Fig. 1), although its numerical approximation shows a smooth behavior (see Fig. 5.1 in [2], which investigated the fully quadratic map with only one divergent singularity for the density). It is enough for the measure of the slave system to have a density bounded away from 0 as k approaches 1, to yield $D_q = 1$ for negative q. If this happens in the synchronization process, before the appearance of singularities of the form $|x_0 - r|^{-1/2}$ in the slave's density, this would explain the observed *zipper effect* described in [13] for this particular example.

5.2. **Densities with a singularity spectrum defined on an interval.** In this section, we construct a density having singularities distributed on a Cantor set, that has a non trivial singularity spectrum. This example does not relate directly to the density of unimodal maps, but is intended to show that non trivial multifractal features can arise from absolutely continuous invariant measures.

Theorem 5.2. *There exists an absolutely continuous invariant measure whose singularity spectrum is defined on a non-trivial interval.*

Proof. Let us construct such an example. Let T be a C^2 expanding map of the unit circle I with three full branches, coded by 0, 1 and 2, based on three intervals I_0, I_1, I_2 making a Markov partition for T. Each point $x \in I$ is encoded by a sequence $w(x) \in \{0,1,2\}^{\mathbb{N}}$. We note $u = -\log|T'|$ and K the Cantor set constituted of the set of points whose codes do not contain 1. We denote by $p = P_T(u|K)$ the topological pressure of u on K. Let μ_u be the Gibbs measure of u on K. We recall the Gibbs property[4]

$$\mu_u(Z_n^{w(x)}) \asymp \exp(-np + S_n u(x)),$$

where Z_n^w is the cylinder of length n containing the code w and $S_n u(x) = \sum_{k=0}^{n-1} u(T^k x)$. Note that $p < P_T(u|I) = 0$. We fix $\alpha \in (0, -p)$ and define a density with respect to the Lebesgue measure, for $z \notin K$, as

$$h(z) = \exp[-k(p + \alpha)],$$

where k is the smallest integer such that $w(z)_k = 1$. The measure μ with density h with respect to Lebesgue is a finite measure: Indeed, by definition and the Gibbs property of μ_u we have

$$\int_I h(z)dz = \sum_{k=0}^{\infty} \sum_{S \in \{0,2\}^k} \mathrm{Leb}(S1) \exp(-k(p+\alpha)) \asymp \sum_{k=0}^{\infty} \exp(-k\alpha) \sum_{S \in \{0,2\}^k} \mu_u(S) < \infty,$$

where $S1$ is the concatenation of the cylinders S and [1].

[4] With the symbol $a \asymp b$ we mean that a is bounded from below and above as $C_1 a \le b \le C_2 a$, with C_1, C_2 two positive constants.

We have the following properties

Lemma 5.3. *For $x \in K$ and n integer:*

(1) $\mu(Z_n^{w(x)}) \asymp \mu_u(Z_n^{w(x)}) \exp(-\alpha n)$.
(2) $\operatorname{diam} Z_n^{w(x)} \asymp \exp(np)\mu_u(Z_n^{w(x)})$.
(3) $\mu(Z_n^{w(x)}) \asymp \mu(B(x,r))$, *when* $n = n_r(x)$ *is the smallest integer such that* $\operatorname{diam} Z_n^{w(x)} < r$.

Proof. Let $x \in K$, n an integer and note $Z = Z_n^{w(x)}$. Since K has zero Lebesgue measure we have

$$\mu(Z) = \sum_{k\geq n+1} \sum_{|S|=k-n-1} \mu(ZS1) = \sum_{k\geq n+1} \sum_{|S|=k-n-1} C\exp[-k(p+\alpha)]\mathrm{Leb}(ZS1),$$

where S runs through all the cylinders of size $k-n-1$ with no 1, and $ZS1$ is the concatenation of the three cylinders Z, S and $[1]$. By distortion and the Gibbs property we have, for any $y \in ZS$,

$$e^{-kp}\mathrm{Leb}(ZS1) \asymp e^{-kp}\mathrm{Leb}(ZS) \asymp \exp(-kp + S_k u(y)) \asymp \mu_u(ZS).$$

Summing up over the cylinders S, which form a partition of the support of μ_u, gives $\mu_u(Z)$ hence

$$\mu(Z) \asymp \mu_u(Z) \sum_{k\geq n+1} \exp(-k\alpha) \asymp \mu_u(Z)\exp(-\alpha n).$$

This proves the first point. For the second one, we observe that by distortion and the Gibbs property

$$\mathrm{diam}(Z) = \mathrm{Leb}(Z) \asymp \exp(S_n u(x)) = \exp(np)\exp(-np + S_n u(x)) \asymp \exp(np)\mu_u(Z).$$

For the last point, it is first obvious that $Z \subset B(x,r)$ thus $\mu(Z) \leq \mu(B(x,r))$. To bound from above the measure of the ball, we consider the left L and right R n-cylinders adjacent to Z. If $Z = 0^n$ or 2^n we set $L = \emptyset$ or $R = \emptyset$. Otherwise, the cylinder Z is either equal to $Z'20^\ell$ or $Z''02^\ell$ for some Z', Z'', $\ell \geq 1$. In the first case, we set $L = Z'12^\ell$ and $R = Z'20^{\ell-1}1$; $L = Z''02^{\ell-1}1$ and $R = Z''10^\ell$ in the second case. Their diameter is at least cr by distortion (recall that the map is C^2 on the circle), where c is a constant depending only on T. Therefore $B(x,cr) \subset L \cup Z \cup R$. In the first case, we have

$$\mu(L) = \mu(Z'12^\ell) = C\exp(-(n-\ell)(p+\alpha))\mathrm{Leb}(L) \leq C\exp(-n(p+\alpha))\mathrm{Leb}(L) \asymp \mu(Z).$$

In addition,

$$\mu(R) = \mu(Z'20^{\ell-1}1) = C\exp(-\alpha n)\mathrm{Leb}(R) \asymp \mu(Z).$$

Therefore $\mu(L \cup Z \cup R) \asymp \mu(Z)$. The second case may be treated similarly. Finally the factor c can be removed since the difference $n_{cr}(x) - n_r(x)$ is uniformly bounded therefore by (1) $\mu(Z_{n_r(x)}^{w(x)}) \asymp \mu(Z_{n_{cr}(x)}^{w(x)})$. $\square$

This lemma implies that the local dimension for the measure μ of points in K satisfies

$$d_\mu(x) = \lim_{n\to\infty} \frac{-(\alpha+p)n + S_n u(x)}{S_n u(x)} = 1 - (p+\alpha)\left(\lim_{n\to\infty} \frac{1}{n} S_n u(x)\right)^{-1}.$$

Therefore the dimension spectrum of the measure μ is determined by the Lyapunov multifractal spectrum of the map T on K. The latter is non-trivial for many systems (e.g. generically, see [17]): the set L_λ of points $x \in K$ having for local Lyapunov exponent

$$\lambda(x) := \lim_{n\to\infty} \frac{1}{n} \log(T^n)'(x) = \lambda$$

has a Hausdorff dimension $g(\lambda)$ which is non trivial in an interval of values of λ, and for these points the local dimension of the measure μ is exactly

$$1 + \frac{p+\alpha}{\lambda} < 1.$$

We obtain

$$f(1 + \frac{p+\alpha}{\lambda}) = g(\lambda),$$

for an interval of values of λ.

□

One could wonder if such a measure μ constructed in the proof is invariant for some non-trivial map of the interval. Indeed the map $S = F^{-1} \circ A \circ F$, with $F(x) = \mu([0,x])$ (invertible since μ has full support) and $A(x) = 2x \mod 1$ preserves μ. Thus, there are dynamical systems whose associated invariant measure is absolutely continuous and exhibits non-trivial multifractal behavior.

6. On the nature of the slave measure

Our proposition 3.1 suggests that the sequence of empirical measures ν_n for the slave non-autonomous evolution converges weakly to μ. One may wonder if other measures could be associated with the slave dynamics. We present here two other possible alternatives, which raise a few questions which we believe would be of some interest.

6.1. **A skew system.**

Let us consider our coupled system in the form (3) and define now the function

$$\psi_x(y) = (1-k)T_2(y) + kT_1(x).$$

Then, setting $I := [-1, 1]$, the coupled system (3) can be written as a skew system on the product space I^2, that is

$$\Theta(x,y) = (T_1(x), \psi_x(y)) .$$

Let μ denote the invariant measure for T_1 with density h. Then it is well known, see for instance [18], that under very general assumptions which are verified in our case[5], we can construct an invariant measure ρ for Θ such that

$$\bar{\pi}_* \rho = \mu,$$

where $\bar{\pi} : I^2 \to I$ projects on the first coordinate, and we have the existence and uniqueness of a family of conditional measures ρ_x defined through

$$\int_{I^2} f(x,y) d\rho = \int_I \int_I f(x,y) d\rho_x(y) d\mu(x)$$

[5] The continuity of T_1 and of T_2 and therefore of ψ is sufficient.

for any ρ-summable f. If we replace T_1 and T_2 with our unimodal maps, the conditional measures will depend on k, therefore we write $\rho_x^{(k)}$. We can now formulate our first question:

Question I: Does $\rho_x^{(k)}$ converges weakly, when $k \to 1$, to μ for $x - \mu$ almost everywhere?

6.2. **A random analogue.** Consider the iteration $y_{n+1} = (1-k)T_2(y_n)$, where $T_2(x) = c_2(1-2x^2)$; at each time step n we now add the number $k\omega_{n+1}$, where $\{\omega_n\}_{n\geq 1}$ is a sequence of i.i.d. random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in I distributed according to the invariant measure μ for the map T_1 whose density is h.[6] Given $k \in [0,1]$, we then get a random dynamical system perturbed with additive noise, namely

$$y_{n+1} = (1-k)T_2(y_n) + k\omega_{n+1} \ , \ n \geq 0 \ . \tag{20}$$

More precisely, if $\Omega := I^{\mathbb{N}}, \mathcal{F}$ is the σ-algebra generated by the cylinder sets

$$\{\omega \in \Omega : (\omega_1, .., \omega_n) \in B\}, \ n \in \mathbb{N}, \ B \in \mathcal{B}(I^n) \ , \tag{21}$$

where $\mathcal{B}(I^n)$ is the Borel σ-field on I^n, and $\mathbb{P} := \mu^{\otimes\mathbb{N}}$, for any $k \in [0,1]$, we consider the skew system

$$I \times \Omega \ni (y, \omega) \longmapsto \Psi_k(y, \omega) := (F^{(k)}_{\pi\omega}(y), S\omega) \in I \times \Omega \ ,$$

where $\pi : \Omega \to I$ is such that $\pi\omega = \omega_1, S : \Omega \to \Omega$ is the left shift, so that $\forall n \geq 0, \omega_n = \pi S^n \omega$, and

$$I \ni y \longmapsto F^{(k)}_{\pi\omega}(y) := (1-k)T_2(y) + k\omega_1 \in I \ .$$

Let us denote by $F_j^{(k)} := F^{(k)}_{\pi S^j \omega}$. According to the theory of random transformations, see Kifer [9], or Arnold [1], for any $k \in [0,1]$, we can construct a homogeneous Markov chain $Y_n^{(k)}$ in the following way. Take an initial point y_0 which we could also consider as a random variable $Y_0^{(k)}$ independent of the $F_j^{(k)}$, and define the stochastic process $\{Y_n^{(k)}\}_{n\geq 0}$ such that, for any $n \geq 0$,

$$Y^{(k)}_{n+1} = F^{(k)}_{n+1}(Y^{(k)}_n) \ . \tag{22}$$

Then, for $y \in [-1,1], z \in \mathbb{R}$:

$$\mathbb{P}(Y^{(k)}_{n+1} \leq z | Y^{(k)}_n = y) = \mathbb{P}(F^{(k)}_{n+1}(y) \leq z) = \mathbb{P}(F^{(k)}_1(y) \leq z) \tag{23}$$

since all the maps $F_j^{(k)}$ have the same distribution. Then we have

$$\begin{aligned} P_y^{(k)}(z) &:= \int_{-1}^{1} \mathbf{1}_{(-\infty,z]}((1-k)T_2(y) + k\omega)h(\omega)d\omega \\ &= \int_{-1}^{1} \mathbf{1}_{(-\infty, \frac{z-(1-k)T_2(y)}{k}]}(\omega)h(\omega)d\omega \\ &= \int_{-1}^{\frac{z-(1-k)T_2(y)}{k}\wedge 1} h(\omega)d\omega. \end{aligned} \tag{24}$$

[6] In other words we are supposing that the $\{\omega_n\}_{n\geq 1}$ are mutually independent. This is of course not true when ω_n is distributed as $T_1^n(x)$, with x chosen Leb-a.e., but it becomes asymptotically true since T mixes exponentially fast with respect to μ.

Hence the transition probability kernel of $\{Y_n^{(k)}\}_{n\geq 0}$ is

$$p_k(y,z) := \frac{d}{dz}P_y^{(k)}(z) = \frac{1}{k}h\left(\frac{z-(1-k)T_2(y)}{k}\right)\mathbf{1}_{[-1,1]}\left(\frac{z-(1-k)T_2(y)}{k}\right). \tag{25}$$

Any stationary measure for the chain will have a density with respect to the Lebesgue measure which is a fixed point of the Markov operator L_k defined as $(L_k g)(z) = \int_I p_k(y,z)g(y)dy$, where, denoting by m the Lebesgue measure, $g \in L^1(m)$. The proof of the following proposition has been provided to us by Y. Nakano and uses results from his last paper [5].

Proposition 6.1. *For any $k \in (0,1]$, the following integral equation:*

$$(L_k g)(z) = \int_I \frac{1}{k}h\left(\frac{z-(1-k)T_2(y)}{k}\right)\mathbf{1}_I\left(\frac{z-(1-k)T_2(y)}{k}\right)g(y)dy = g(z). \tag{26}$$

admits finitely many solutions $g_1,..,g_r$ such that $\forall i=1,..,r$ the probability measures with density g_i are ergodic. Moreover,

$$m\left(I\setminus\bigcup_{i=1}^r B_\omega(g_i)\right) = 0\ ,\ \mathbb{P}-a.s.\ , \tag{27}$$

where,

$$B_\omega(g_i) := \left\{y \in I : \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\delta_{F^{(\cdot)}_{\pi S^j\omega}\circ\cdots\circ F^{(\cdot)}_{\pi S^0\omega}(y)} = g_i m\right\}\ ,\ i=1,..,r\ . \tag{28}$$

Proof. The proof follows from Theorem A in [5] once we have shown that 1 is an eigenvalue of the operator L_k.

Note first that 1 is in the spectrum of L_k, since $\mathbf{1}_I$ is the eigenfunction with eigenvalue 1 of the adjoint operator

$$L^\infty(m) \ni \varphi \mapsto L_k^* := \int_I \varphi((1-k)T_2(y)+k\omega)h(\omega)d\omega \in L^\infty(m) \tag{29}$$

where $L^\infty(m) \cong (L^1(m))^*$.

On the other hand, by the definition of L_k,

$$\|L_k g\|_{L^1} \leq \int_I dz L_k|g|(z) = \int_I\int_I |g(y)|dyh(\omega)d\omega = \|g\|_{L^1}\ , \tag{30}$$

so that the spectral radius of L_k is 1.

Notice that $F_{\pi\omega}^{(k)} : I \to I$ is continuous and $p_k(x,\cdot)$, defined by $p_k(x,A) := \mu(\{\omega \in I : (1-k)T_2(x)+k\omega \in A\})$, is absolutely continuous with respect to m for all x. In fact, since

$$\{\omega \in I : (1-k)T_2(x)+k\omega \in A\} = \left\{\frac{y-(1-k)T_2(x)}{k} : y \in A\right\}\cap I\ , \tag{31}$$

it follows from the absolute continuity of μ with respect to m that

$$m(A)=0 \implies m(\{\frac{y-(1-k)T_2(x)}{k} : y\in A\}\cap I) = 0 \implies p_k(x,A)=0. \tag{32}$$

(cf. Remark 1.4 of [5]). Therefore, the condition in Remark 1.2 of the just mentioned paper is satisfied so that the hypothesis in Theorem A and Theorem B in [5] are satisfied too. Thus, L_k is eventually compact and therefore also quasi-compact. Consequently, the essential spectral radius of L_k is strictly smaller than 1, so that 1 is an eigenvalue. □

Question II Is there a unique solution to (26)?

This would be the case if the transition probability kernel was strictly positive, but the support of the density h is a subset of the *dynamical core* $[T_1^2(0), T_1(0)]$.

For any $k \in [0,1]$, if $g^{(k)}$ is a solutions of (26), let us denote by $\bar{\nu}^{(k)}$ the associated probability distribution on $(I, \mathcal{B}(I))$, where we have made explicit the dependence on the parameter k. We could now consider the stochastic stability property.

Proposition 6.2. *In the limit of $k \to 1, \bar{\nu}^{(k)}$ converges weakly to μ. Moreover, if, in the limit of $k \to 0, \bar{\nu}^{(k)}$ has a weak limit, this is a probability distribution $\bar{\nu}$ on $(I, \mathcal{B}(I))$ invariant under the evolution defined by T_2.*

Proof. Given ϱ a probability distribution on $(I, \mathcal{B}(I))$, let us denote by $\mathbb{R} \ni t \longmapsto \varphi_\varrho(t) := \int \varrho(dx) e^{itx} \in \mathbb{C}$ the associated characteristic function. In particular, we write φ_g for the characteristic function of a probability distribution with density g. Hence, from (26), since the support of $h, S(h)$, is contained in I, we get

(33)
$$\int dze^{itz} \int \frac{1}{k} h\left(\frac{z-(1-k)T_2(y)}{k}\right) \mathbf{1}_{S(h)}\left(\frac{z-(1-k)T_2(y)}{k}\right) g^{(k)}(y)dy = \int dzg^{(k)}(z)e^{itz}$$

Setting $u = \frac{z-(1-k)T_2(y)}{k}$ we obtain

(34)
$$\begin{aligned}
\varphi_{g^{(k)}}(t) &= \int_I dyg^{(k)}(y) \int_I duh(u) e^{it[ku+(1-k)T_2(y)]} = \int_I dyg^{(k)}(y) \int_I duh(u) e^{it(1-k)T_2(y)} e^{itku} \\
&= \int_I dyg^{(k)}(y) e^{it(1-k)T_2(y)} \int_I duh(u) e^{itku} = \int_I dv \left(\mathcal{L}_{T_2} g^{(k)}\right)(v) e^{it(1-k)v} \int_I duh(u) e^{itku} \\
&= \varphi_{\mathcal{L}_{T_2} g^{(k)}}((1-k)t) \varphi_h(kt) \ ,
\end{aligned}$$

where $\mathcal{L}_{T_2}$ is the Perron-Frobenius operator associated with the map T_2, which, in probabilistic terms, can be read in the following way: the r.v. ξ_k with density $g^{(k)}$ is equal in distribution to the convex combination $k\zeta + (1-k)\eta_k$ of two independent r.v.'s ζ and η_k such that η_k has probability density $f_k := \mathcal{L}_{T_2} g^{(k)}$ and $\zeta \overset{d}{=} \omega_1$. Therefore (26) can be rewritten in the form

$$\left(L_k g^{(k)}\right)(z) = \int duh(u) \left(\mathcal{L}_{T_2} g^{(k)}\right)\left(\frac{z-ku}{1-k}\right) \frac{1}{1-k} = g^{(k)}(z) \ . \tag{35}$$

Since η_k is bounded, then $(1-k)\eta_k \underset{k\uparrow 1}{\longrightarrow} 0$ almost surely. Hence, $\varphi_{f_k}((1-k)t)$ is equal to the characteristic function $\mathbb{E}\left[e^{it(1-k)\eta_k}\right]$ of $(1-k)\eta_k$ which, in the limit as $k \uparrow 1$, for any $t \in \mathbb{R}$, converges to 1 by the Lévy continuity theorem. Then, by weak-compactness, given a sequence $\{\bar{\nu}^{(k_n)}\}_{n\geq 1} \subset \{\bar{\nu}^{(k)}, k \in [0,1]\}$, where $\{k_n\}_{n\geq 1} \uparrow 1$, let $\{\bar{\nu}^{(k_{n_l})}\}_{l\geq 1}$, where $\{k_{n_l}\}_{l\geq 1} \subset \{k_n\}_{n\geq 1}$, be a weakly convergent subsequence. Taking the limit along $\{k_{n_l}\}_{l\geq 1}$ on both sides of the (34), if the weak limit of $\{\bar{\nu}^{(k_{n_l})}\}_{l\geq 1}$ is $\bar{\nu}$, the l.h.s. converges pointwise to $\varphi_{\bar{\nu}}$ while the r.h.s. converges pointwise to φ_h. Thus $\varphi_{\bar{\nu}} = \varphi_h$ and, since the characteristic functions uniquely determine the associated probability distributions, $\bar{\nu} = \mu$. From this follows that every weakly convergent subsequence of $\{\bar{\nu}^{(k_n)}\}_{n\geq 1}$ converges to μ and therefore the whole sequence has μ as weak limit. Since this is true for any $\{\bar{\nu}^{(k_n)}\}_{n\geq 1} \subset \{\bar{\nu}^{(k)}, k \in [0,1]\}$, we get that the weak limit as $k \uparrow 1$ of $\bar{\nu}^{(k)}$ is μ. On the other hand, in the limit as k tends to 0, denoting once again by $\bar{\nu}$ the weak limit of $\bar{\nu}^{(k)}$, since $\mathcal{L}_{T_2} g^{(k)} = (T_2)_* \bar{\nu}^{(k)}$, where $(T_2)_* \bar{\nu}^{(k)}$ is the pushforward of $\bar{\nu}^{(k)}$ under T_2, by (34) we obtain $\varphi_{\bar{\nu}} = \varphi_{(T_2)_* \bar{\nu}}$. □

Remark 6.3. We stress that if $\bar{\nu}$, the weak limit of $\bar{\nu}^{(k)}$ as k tends to 0, has density $\bar{h}$ we get $\bar{h} = \frac{d(T_2)_* \bar{\nu}}{dm}$ i.e. $\bar{h}$ satisfies $\mathcal{L}_{T_2} \bar{h} = \bar{h}$. This is, in particular, the case for our choice of T_2 which admits a unique invariant density. As a matter of fact, in this case, we do not need to assume the existence of the weak limit of $\bar{\nu}^{(k)}$ to $\bar{h}m$ as $k \to 0$ since the same argument that lead us to prove the existence of the weak limit of $\bar{\nu}^{(k)}$ as $k \to 1$ applies also in the limit of $k \to 0$.

Let us now consider the empirical measure (10) which we now write as $\nu_n^{(k)}$ to emphasize the dependence on k and set $\bar{\nu}_n^{(k)} := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{F^{(\cdot)}_{\pi S^j \omega} \circ \cdots \circ F^{(\cdot)}_{\pi S^0 \omega}(y_0)}$. Remember that the measure $\nu_n^{(k)}$ depends on two (independent) initial conditions x_0 and y_0, while $\bar{\nu}_n^{(k)}$ depends on y_0 and the realization $\omega \in \Omega$. Here is our last question:

Question III Assume that for any $k \in [0,1]$ the solution of (26), (35) are unique. Put $\Delta_n^{(k)}$ the total variation distance between $\nu_n^{(k)}$ and $\bar{\nu}_n^{(k)}$; do we have

$$\limsup_{k\to 1} \limsup_{n\to\infty} \Delta_n^{(k)} = 0$$

for Lebesgue almost all choices of x_0, y_0 and $\mathbb{P}$ almost all ω? If Questions II and III have positive answers, we expect that numerically the sequence $\nu_n^{(k)}$ would converge to an absolutely continuous measure, which is also suggested by our Proposition 3.1. This is confirmed by Fig. 4 (A), which shows the evolution with k of the support of the limiting measure of the $\nu_n^{(k)}$. Fig. 5 shows the empirical measures $\nu_n^{(k)}$ for $n = 10^9$ at $k = 0$ and $k = 0.5$ and $k = 1$ (at which $\nu_n = \mu$). The different histograms are compatible with $\nu_n^{(k)}$ being absolutely continuous with densities having a large number of singularities on a large set of points, which may fit with the simple formal models presented in section 5, and therefore with the findings of a non-trivial D_q spectrum, as found in [13].

To gauge the convergence of the measure $\nu_n^{(k)}$ to μ, we plot in Fig. 3 (A) the evolution of the empirical Lyapunov exponent (we set $\tilde{\lambda}_{T_1}$ the limiting value):

$$\tilde{\lambda}_{T_1}^n := \frac{1}{n} \sum_{i=0}^{n-1} \log |T_1'(y_i)|, \; y_{i+1} = (1-k)c_2(1-2y_{i-1}^2) + c_1 k(1-2x_{i-1}^2), i \geq 0,$$

with respect to the master parameter c_1, and where the initial values x_0, y_0 are chosen independently on I and Lebesgue almost everywhere.

For the values of c_1 and c_2 prescribed in [13], the dependence of $\tilde{\lambda}_{T_1}$ vs k is made explicit in Fig. 3 (B). We see that when $k \to 1$ the empirical Lyapunov exponent $\tilde{\lambda}_{T_1}$ converges to λ_{T_1}.

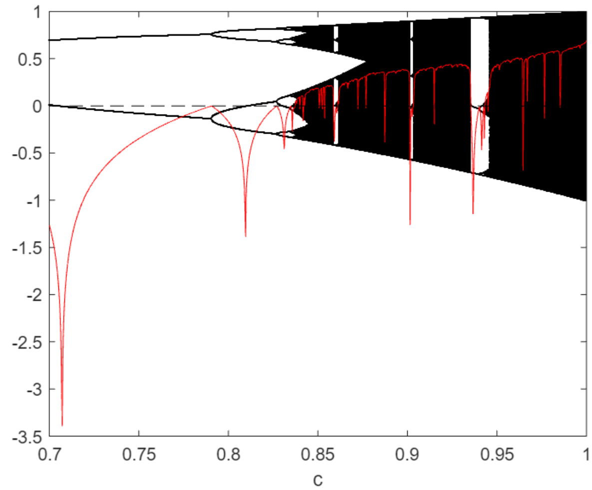

FIGURE 2. Bifurcation diagram for the map $T(x) = c(1-2x^2)$ and its associated λ_T computed over different values of the parameter c.

This supports a positive answer to Question III, although in principle it could not be applied to $\log|T_1'|$, which is not even bounded on $[-1,1]$.

7. Conclusions

The paper [13] used the spectrum of the generalized dimensions to follow the process of synchronization in master/slave systems. We showed that for the parameter values of the quadratic map considered in the aforementioned paper, the master map has an absolutely continuous invariant measure and the attractor is not a Cantor set. We did not find in the literature any result on the multifractal spectrum of such a measure. We instead gave examples of densities allowing a multifractal structure. In those cases, the function $q \to D_q$ is continuous but not smooth, which is not what was observed in [13], unless smoothness was a consequence of numerical approximations. Moreover, our examples suggest that the dimensions are constant for negative q, since the invariant densities are bounded away from zero, which supports the presence of the so called *zipper effect* highlighted in [13].
We presented a detailed study of the Lyapunov exponent of the master map T_1, of the slave map T_2 and of the dynamics of the coupled system in the slave direction in the limit $k \to 1$ believing that it is a much more reliable technique, besides being more theoretically founded, to describe the synchronization process.

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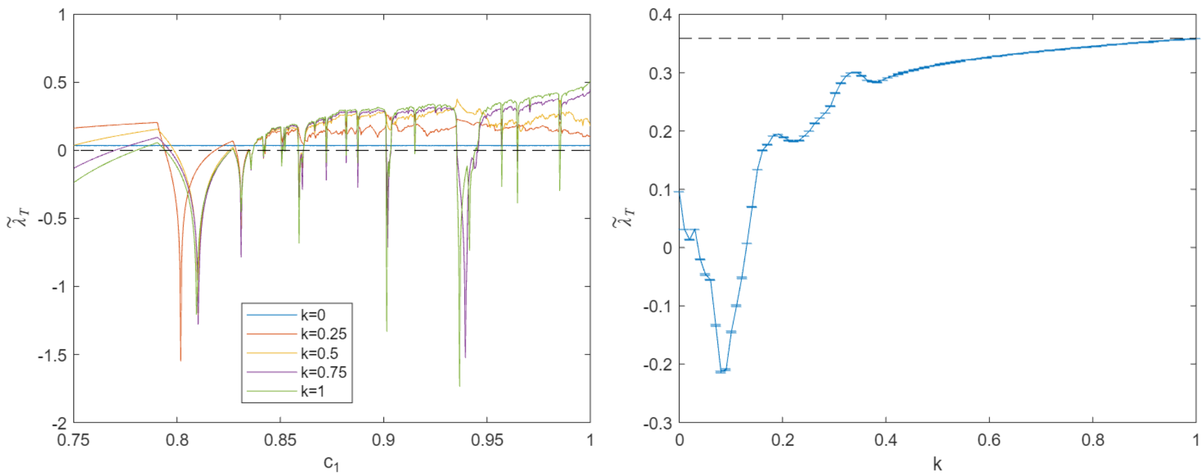

(A) (B)

FIGURE 3. Evolution of $\tilde{\lambda}_{T_1}$ with c_1 for different values of k (left) and with k for the fixed value of $c_1 = 0.89$ (right). For both figures, we set $c_2 = 0.8373351$.

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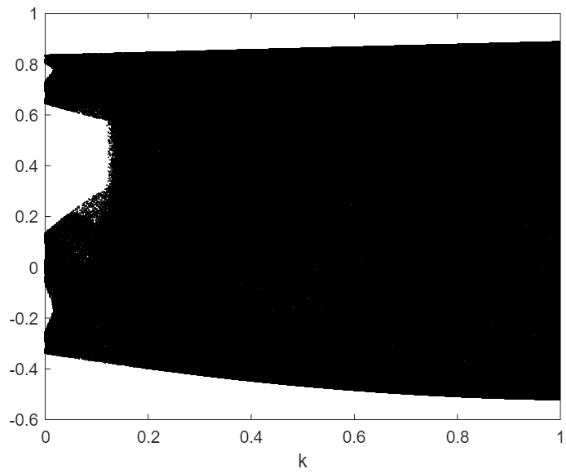
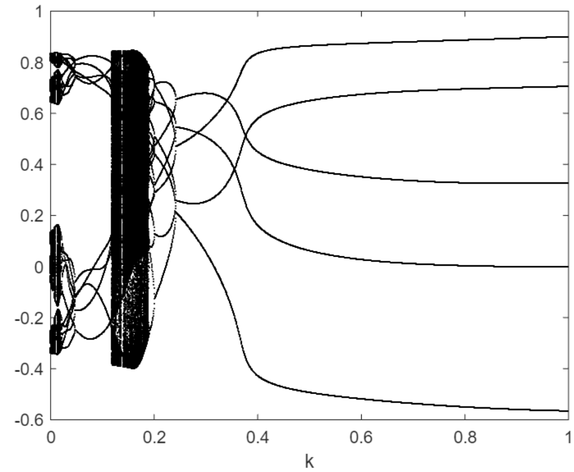
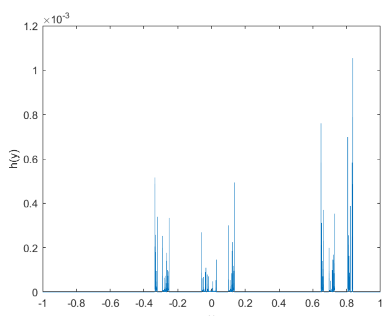


(A) $c_1 = 0.89$; $c_2 = 0.8373351$

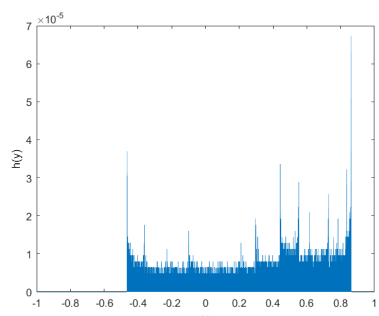


(B) $c_1 = 0.901502$; $c_2 = 0.8373351$

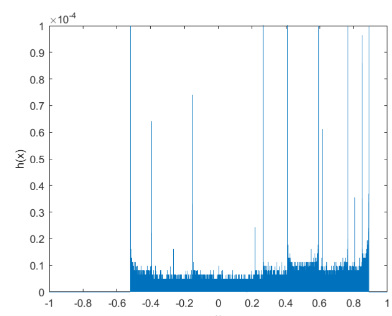
FIGURE 4. Bifurcation diagrams of the dynamics of y_n with the parameter k, for two different values of c_1. On the left, the master measure is supported on an interval, and on the right on a set of 5 points.



(A) $k = 0$



(B) $k = 0.5$



(C) $k = 1$

FIGURE 5. Numerical estimation of the empirical densities $g^{(k)}$ for different values of k.

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