IL NUOVO CIMENTO VOL. 111 B, N. 4 Aprile 1996

Scaling exponents for turbulent scalar fields: analytic results

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(ricevuto l'8 Giugno 1995; approvato il 19 Settembre 1995)

Summary. — We extend the analytic results recently found by Constantin and Fefferman for the scaling exponents of the velocity structure functions in fluid turbulence to the scaling exponents for passive scalar fields such as the temperature.

PACS 30.40.Gc – Fluid dynamics: general mathematical aspects.
PACS 47.25.Cg – Isotropic turbulence; homogeneous turbulence.

1. – Introduction

In the recent paper [1], Constantin and Fefferman obtained some rigorous bounds for the generalized velocity structure functions by a direct investigation of the Navier-Stokes equations. The importance of such a statistical analysis for a real understanding of turbulence was pointed out by Kolmogorov in his famous K-41 paper [2]. The contribution of Kolmogorov was twofold: first, he got an ordinary differential equation for the third-order moment of the velocity difference by assuming isotropy (local and global) and homogeneity. Then, he generalized the theory to higher moments by dimensional arguments, obtaining that the scaling exponents depend linearly on the order of the moments. The stated experimental deviations from this linear law [3] gave rise to various theories and models, some of which are quoted in the (unexhaustive) list [7].

The purpose of [1] was to analyse the moments of the velocity difference by taking care of the following facts:

i) A good averaging procedure for the fields, solution of the Navier-Stokes equations, is introduced. This step is easily achieved by weighting the volume

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measure with a smooth cut-off function of compact support, which makes the spatial integration limited to a ball of finite diameter. This allows to use rigorously a series of tricks (such as integrating by parts) which have already been invoked in the standard statistical calculus of turbulence [4].

ii) There is no assumption of statistical nature, in particular, no assumption of isotropy and homogeneity. This yields, in general, bounds rather than equalities for the moments unlike the previously recalled Kolmogorov's result.

iii) The analysis is carried out in the so-called inertial range. In [1] this range is defined by introducing a scaling law: First an inertial range for each moment is defined and the bottom of this range is related to the Reynolds number by an exponential law. The exponent of this law is independent of the Reynolds number and is characteristic of the moment. As pointed out by the authors, there is actually no way «to prove that scaling occurs at all» ([1], p. 46).

iv) One of the key ideas of [1] is to introduce a local Reynolds number corresponding to each structure function and assume it bounded in the limit of a large (global) Reynolds number. This point is discussed in sect. 1 of [1].

In the present paper we will carry out the same analysis for an advected passive scalar in a turbulent flow. In particular, we will consider the conservation equation for the temperature with external sources. The structure functions will, therefore, be the moments of temperature increments over a small distance r. We shall give rigorous bounds for the second moment and some estimates on higher moments, following closely the framework developed in [1]. Our development will be simpler since we do not have to estimate the pressure term, which presented the most difficult technical problem in [1]. The main results of the analysis are the following:

a) We show that the dissipation rate of the temperature remains bounded if the thermal diffusivity becomes small, which is analogous of what was proved in [1] concerning the dissipation rate of energy for vanishing viscosity.

b) The hypothesis stated at point iii) above will be reformulated in terms of the Peclet number. In analogy with point iv), we will also introduce a local Peclet number based, again, on the value of the velocity structure function at the scale r. In doing so, the underlying assumption we use is that the temperature, as a passive scalar, does not perturb the velocity field. Differently from of [1] we do not consider the boundedness of the local Peclet number as a separate hypothesis. In accordance with the usual considerations yielding the Kolmogorov length scale, we rather use a chosen value of the local Peclet number (say one) to define the bottom of the inertial range.

As a consequence, we obtain bounds of the scaling exponents of the temperature structure functions depending on the scaling exponents of the velocity structure functions. We show that these relations are in agreement with some experimental facts related to the deviations of the scaling exponents for temperature and velocity from the linear law given by the Kolmogorov-Obukhov-Corssin (KOC) theory.

c) We show that, under a certain number of assumptions similar to those invoked in [1], we are able to recover the KOC theory on the linear scaling for the moments of the temperature differences. We, moreover, point out a certain number of inequalities which, if they become strict, could give anomalous scaling, *i.e.* a deviation from the linear KOC law. These inequalities are rigorously derived from the differential equation for the convection and diffusion of the temperature field without any additional statistical (or multifractal) assumptions and should, in principle, be verifiable experimentally.

2. – Second-order structure function

The partial differential equation governing the temperature $\theta(\boldsymbol{x}, t)$, where $\boldsymbol{x} \in R^3$ and $t \geq 0$, is [4]

$$\frac{\partial \theta(\boldsymbol{x}, t)}{\partial t} + \boldsymbol{u}(\boldsymbol{x}, t) \cdot \nabla \theta(\boldsymbol{x}, t) = \kappa_0 \nabla^2 \theta(\boldsymbol{x}, t) + F_\theta(\boldsymbol{x}, t), \tag{2.1}$$

where κ_0 is the thermal diffusivity and $\boldsymbol{u}(\boldsymbol{x}, t)$ is the divergence-free solution of the Navier-Stokes equations:

$$(\partial_t + \boldsymbol{u} \cdot \nabla) \boldsymbol{u}(\boldsymbol{x}, t) + \nabla p = \nu \nabla^2 \boldsymbol{u}(\boldsymbol{x}, t) + \boldsymbol{F}_u(\boldsymbol{x}, t), \tag{2.2}$$

ν being the viscosity.

As in [1] we will consider two ensembles $\mathscr{E}_\theta$ and $\mathscr{E}_u$ of $C^\infty(R^3 \times R_+)$ solutions of (2.1) and (2.2), respectively, belonging to $L^\infty(R^3 \times R_+)$, *i.e.* admitting finite bounds Θ and U such that

$$\sup_{\boldsymbol{u} \in \mathscr{E}_u,\, \boldsymbol{x} \in R^3,\, t \in R_+} |\boldsymbol{u}(\boldsymbol{x}, t)| \leq U \tag{2.3}$$

and

$$\sup_{\theta \in \mathscr{E}_u,\, \boldsymbol{x} \in R^3,\, t \in R_+} |\theta(\boldsymbol{x}, t)| \leq \Theta . \tag{2.4}$$

F_θ and $\boldsymbol{F}_u$ are the external heat source and the external forces, respectively, that we shall also consider bounded:

$$\sup_{\boldsymbol{x} \in R^3,\, t \in R_+} |F_\theta(\boldsymbol{x}, t)| \leq F, \tag{2.5}$$

$$\sup_{\boldsymbol{x} \in R^3,\, t \in R_+} |\boldsymbol{F}_u(\boldsymbol{x}, t)| \leq \Gamma. \tag{2.6}$$

Equation (2.3) implies that the kinetic energy density $|\boldsymbol{u}|^2$ is bounded, too. Finally, we shall assume that also the dissipation rate of the temperature $|\nabla \theta|^2$ is bounded for each $t \geq 0$. This last condition will be used in sect. **4**.

The averaging operation $M_\varrho[f]$ will be defined for a smooth scalar field $f(\boldsymbol{x}, t)$, $(\boldsymbol{x} \in R^3, t \geq 0)$ as

$$M_\varrho[f] = \sup_{\boldsymbol{x} \in R^3} \frac{1}{|B_\varrho(\boldsymbol{x}_0)|} \mathrm{Av}\left\langle \limsup_{T \to \infty} \frac{1}{T} \int_0^T \mathrm{d}t \int_{B_\varrho(\boldsymbol{x}_0)} f(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x} \right\rangle, \tag{2.7}$$

where Av means ensemble average and $B_\varrho(\boldsymbol{x}_0)$ denotes the ball of radius ϱ and centre $\boldsymbol{x}_0$, of volume $|B_\varrho(\boldsymbol{x}_0)| = 4/3\,\pi\varrho^3$. $M_\varrho[.]$ has the linear properties $M_\varrho[f+g] \leq M_\varrho[f] + M_\varrho[g]$ and $M_\varrho[cf] = cM_\varrho[f]$, $c > 0$ and satisfies the translation invariance $M_\varrho[\tau_{\boldsymbol{y}}(f)] = M_\varrho[f]$, where $\tau_{\boldsymbol{y}}(f)(\boldsymbol{x}, t) = f(\boldsymbol{x} + \boldsymbol{y}, t)$.

We shall be interested in the generalized structure functions of the temperature increments defined as

$$S_m(\varrho, \boldsymbol{r}) = M_\varrho[\,|\theta(\boldsymbol{x}+\boldsymbol{r}, t) - \theta(\boldsymbol{x}, t)|^m\,]. \tag{2.8}$$

We prefer not to extract the m-th root of M_ϱ which yields a slightly different notation if compared with [1]. Our notation corresponds, however, to that used commonly in physical literature.

In this section we shall consider the case $m = 2$ in detail. Our first result corresponds to eq. (49) of [1]:

Lemma 1.

$$S_m(\varrho, \boldsymbol{r}) \leqslant |r|^m M_\varrho[\,|\nabla\theta(\boldsymbol{x}, t)|^m\,], \qquad m > 0, \tag{2.9}$$

Proof. The mean-value theorem applied to the temperature difference yields

$$\theta(\boldsymbol{x}+\boldsymbol{r}, t) - \theta(\boldsymbol{x}, t) = \sum_{i=1}^{3} r_i \int_0^1 \frac{\partial\theta}{\partial z_i}\bigg|_{z_i = x_i + \xi r_i} \mathrm{d}\xi\,. \tag{2.10}$$

The substitution of eq. (2.10) into (2.7) and (2.8), exchange of the order of integration and the use of the Cauchy inequality and the translational invariance of the mean value on the r.h.s. of eq. (2.10) give immediately the lemma. □

We now estimate the second factor on the r.h.s. of (2.9) for $m = 2$; a similar proof already appeared in [8], we give it here since proposition 4 below will be proved in the same manner.

Proposition 1.

$$M_\varrho[\,|\nabla\theta(\boldsymbol{x}, t)|^2\,] \leqslant C\cdot\Theta^2\left[\frac{1}{\varrho^2} + \frac{U}{\kappa_0\varrho}\right] + \frac{\Theta F_\theta}{\kappa_0}, \tag{2.11}$$

where C is a constant of geometric nature.

Proof. We multiply eq. (2.1) by $\theta(\boldsymbol{x}, t)$ getting

$$\frac{1}{2}\left[\frac{\partial\theta^2(\boldsymbol{x}, t)}{\partial t} + \boldsymbol{u}(\boldsymbol{x}, t)\cdot\nabla\theta^2(\boldsymbol{x}, t) - \kappa_0\nabla^2\theta^2(\boldsymbol{x}, t)\right] = \tag{2.12}$$

$$= -\kappa_0\,|\nabla\theta(\boldsymbol{x}, t)|^2 + F_\theta(\boldsymbol{x}, t)\,\theta(\boldsymbol{x}, t)\,.$$

From this it follows that

$$\frac{1}{|B_\varrho(\boldsymbol{x}_0)|}\int_{B_\varrho(\boldsymbol{x}_0)} |\nabla\theta(\boldsymbol{x}, t)|^2\,\mathrm{d}\boldsymbol{x} \leqslant \frac{1}{|B_\varrho(\boldsymbol{x}_0)|}\int_{R^3} |\nabla\theta(\boldsymbol{x}, t)|^2\chi_{\boldsymbol{x}_0, \varrho}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \leqslant \tag{2.13}$$

$$\leqslant |\mathcal{J}_1| + |\mathcal{J}_2| + |\mathcal{J}_3| + |\mathcal{J}_4|\,,$$

where

$$\chi_{x_0,\varrho}(x) = \chi\left(\frac{\|x - x_0\|}{2\varrho}\right),$$

χ being a C^∞ cut-off function equal to 1 on the interval $[0, 1]$ and 0 on $[2, \infty]$. We introduce

$$\mathcal{J}_1 = -\frac{1}{2\kappa_0 |B_\varrho|}\int_{R^3} \frac{\partial\theta^2}{\partial t}(x, t)\chi_{x_0,\varrho}(x)\,\mathrm{d}x\,, \tag{2.14a}$$

$$\mathcal{J}_2 = -\frac{1}{2\kappa_0 |B_\varrho|}\int_{R^3} u(x, t)\cdot\nabla\theta^2(x, t)\chi_{x_0,\varrho}(x)\,\mathrm{d}x\,, \tag{2.14b}$$

$$\mathcal{J}_3 = \frac{1}{2|B_\varrho|}\int_{R^3} \nabla^2\theta^2(x, t)\chi_{x_0,\varrho}(x)\,\mathrm{d}x\,, \tag{2.14c}$$

$$\mathcal{J}_4 = \frac{1}{\kappa_0 |B_\varrho|}\int_{R^3} F_\theta(x, t)\,\theta(x, t)\chi_{x_0,\varrho}(x)\,\mathrm{d}x\,; \tag{2.14d}$$

(2.7) yields directly

$$\mathcal{J}_1 = 0\,. \tag{2.15}$$

The Gauss theorem applied to the r.h.s. of eq. (2.14*b*) and the use of the continuity equation $\nabla u = 0$ yield the estimate

$$|\mathcal{J}_2| \leqslant C_2' \frac{\Theta^2 U}{\kappa_0\varrho}\,, \tag{2.16}$$

where C_2' is a constant involving the integral of $|\mathrm{d}\chi(r)/\mathrm{d}r|$ on the bounded set $[0, 2]$. In a similar way, by using twice the Gauss theorem, we find

$$|\mathcal{J}_3| \leqslant C_3' \frac{\Theta^2}{\varrho^2}\,. \tag{2.17}$$

The inequality

$$|\mathcal{J}_4| \leqslant \frac{F\Theta}{\kappa_0} \tag{2.18}$$

is straightforward.

Taking the time and ensemble averages and the supremum over x_0 we finally get

$$M_\varrho[\,|\nabla\theta(x, t)|^2\,] \leqslant C\Theta^2\left[\frac{1}{\varrho^2} + \frac{U}{\kappa_0\varrho}\right] + \frac{\Theta F}{\kappa_0}\,, \tag{2.19}$$

where $C = \max\{C_2', C_3'\}$. □

Remark 1. Note that the term on the l.h.s. of eq. (2.11) mutiplied by κ_0 is nothing but the quantity $\overline{N}$ in Monin-Yaglom [4] and represents the «dissipation rate of temperature inhomogeneities». The bound (2.19) states that for small κ_0 there exists a κ_0-independent bound of $\overline{N}$ which is similar to what happens to the dissipation of energy for small viscosity ν [1].

Coming back to $S_2(\varrho, \boldsymbol{r})$, we thus get

$$S_2(\varrho, \boldsymbol{r}) \leq Cr^2 \Theta^2 \left[\frac{1}{\varrho^2} + \frac{U}{\kappa_0 \varrho} \right] + \frac{\Theta F r^2}{\kappa_0} . \tag{2.20}$$

Now, we introduce the Peclet number Pe defined as

$$\mathrm{Pe} = \frac{U_\theta L_\theta}{\kappa_0} ,$$

where L_θ is the length over which there is an appreciable change in the mean temperature and U_θ is a typical change in the mean velocity over the distance L_θ. In accordance with the typical physical situation, we shall identify U_θ and L_θ with U and L, characteristic velocity and distance, defining the Reynolds number

$$\mathrm{Re} = \frac{UL}{\nu} . \tag{2.21}$$

As a result,

$$\mathrm{Pe} = \frac{UL}{\kappa_0} . \tag{2.22}$$

In analogy with [1], let us denote

$$S_m(\varrho, \boldsymbol{r}, \boldsymbol{u}) = M_\varrho [|\boldsymbol{u}(\boldsymbol{x} + \boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})|^m] \tag{2.23}$$

the m-th order velocity structure function. We now define the «local Peclet number» at the length scale r based on the m-th order velocity structure function in the following way:

Definition 1.

$$\mathrm{Pe}_{\mathrm{loc}}^{(m)}(r) = \frac{r}{\kappa_0} \sup_{|\boldsymbol{r}| = r} [S_m(\varrho, \boldsymbol{r}, \boldsymbol{u})]^{1/m} . \tag{2.24}$$

The local Peclet number will be used to define the «bottom length scale» of the inertial range much in the same way as the Kolmogorov length does. We, however, differentiate the order of the considered structure functions, having, in principle, different local Peclet numbers, and thus different limits of the inertial range, for different structure function orders.

Definition 2. r_m, the *bottom length scale of the inertial range* for m-th–order passive scalar structure function is defined by the relation

$$\mathrm{Pe}_{\mathrm{loc}}^{(m)}(r_m) = 1 . \tag{2.25}$$

Definition 3. The *sub-inertial range* for the m-th–order passive scalar structure function is defined as

$$r_m \leq |\boldsymbol{r}| \leq 2r_m < L . \tag{2.26}$$

Let us constrain the size of the ball over which we take the spatial average as

$$1 \leqslant \frac{\varrho}{L} \leqslant 2\,. \tag{2.27}$$

In this section, we shall limit our consideration to the second-order moments. The higher-order moments will be taken up in the next section. If we reword our definitions on the basis of the velocity structure functions and Reynolds, instead of Peclet, numbers we obtain the bottom length scale of inertial range denoted $r_m^{(v)}$ defined by

$$\frac{r_m^{(v)}}{\nu} \sup_{|r|=r} \left[S_m(\varrho, \boldsymbol{r}, \boldsymbol{u})\right]^{1/m} = 1\,. \tag{2.28}$$

For this length scale and the corresponding sub-inertial range we can transpose the results of [1]. It has been shown there that, assuming the Reynolds number to be related to the bottom length scale $r_2^{(v)}$ of the inertial range via

$$\mathrm{Re} = \left(\frac{r_2^{(v)}}{L}\right)^{-b_2^{(v)}}, \tag{2.29}$$

where $b_2^{(v)}$ is a positive constant independent of Re and $r_2^{(v)}/L$, any lower bound written in the form

$$\frac{S_2(\varrho, \boldsymbol{r}, \boldsymbol{u})}{U^2} \geqslant \overline{C}\left(\frac{|\boldsymbol{r}|}{L}\right)^{\xi_2^{(v)}}, \tag{2.30}$$

supposed to hold in the inertial sub-range $r_2^{(v)} \leqslant |\boldsymbol{r}| \leqslant 2r_2^{(v)}$ satisfies the inequality

$$\xi_2^{(v)} \geqslant 2 - b_2^{(v)}\,.$$

Our present task is to show an analogical result for the passive scalar structure function $S_2(\varrho, \boldsymbol{r})$. We shall first state the following:

Assumption 2.1.

$$0 < \lim_{\mathrm{Re}\to\infty} \frac{\mathrm{Pe}}{\mathrm{Re}} < \infty\,, \tag{2.31}$$

which is equivalent to the physical statement that the Prandtl number remains bounded and non-zero for high Reynolds numbers. Comparing (2.28) to (2.24), (2.25), this assumption means that there exist constants K_1, K_2, K_1', K_2', independent of the Reynolds number and m, such that

$$K_1 r_m^{(v)} \leqslant r_m \leqslant K_2 r_m^{(v)}\,, \qquad K_1' r_m \leqslant r_m^{(v)} \leqslant K_2' r_m\,. \tag{2.32}$$

Next we assume

Assumption 2.2. The Peclet number defined by eq. (2.22) can be expressed as

$$\mathrm{Pe} = k_2\left(\frac{r_2}{L}\right)^{-b_2}, \tag{2.33}$$

where k_2 and b_2 are positive constants independent of Pe and r_2/L.

This assumption states the physical fact that the bottom length scale of the inertial zone decreases proportionally to some negative power of the Peclet number.

The results of this paper concern the bounds of scaling exponents for the passive scalar structure functions assuming that there exist scaling laws:

Assumption 2.3. There exist positive constants c_2 and ξ_2, independent of Pe, such that

$$\forall \boldsymbol{r}, \quad r_2 \leqslant |\boldsymbol{r}| \leqslant 2r_2: \frac{S_2(\varrho, \boldsymbol{r})}{\Theta^2} \geqslant c_2\left(\frac{|\boldsymbol{r}|}{L}\right)^{\xi_2}. \tag{2.34}$$

The proposition 1 can now be shown to yield

Proposition 2. Under assumptions 2.1, 2.2 and 2.3,

i)

$$\xi_2 \geqslant 2 - b_2, \tag{2.35}$$

ii)

$$\xi_2^{(v)} \geqslant 2(b_2 - 1) \tag{2.36}$$

and, combining (2.35) with (2.36),

iii)

$$\xi_2 \geqslant 1 - \frac{\xi_2^{(v)}}{2}. \tag{2.37}$$

Proof. Assumption 2.3 and the inequalities (2.26), (2.27) and (2.20) yield

$$c_2\left(\frac{r_2}{L}\right)^{\xi_2} \leqslant c_2\left(\frac{|\boldsymbol{r}|}{L}\right)^{\xi_2} \leqslant \frac{S_2(\varrho, \boldsymbol{r})}{\Theta^2} \leqslant r^2\left[C\left(\frac{1}{L^2} + \frac{U}{\kappa_0 L}\right) + \frac{F}{\Theta\kappa_0}\right].$$

Introducing the Peclet number (2.22) and using, once more, the inequality (2.26), the last expression can be estimated by

$$4\left(\frac{r_2}{L}\right)^2\left[C(1 + \mathrm{Pe}) + \mathrm{Pe}\,\frac{FL}{\Theta U}\right] \leqslant C^*\,\mathrm{Pe}\left(\frac{r_2}{L}\right)^2,$$

where

$$C^* = 8C + \frac{FL}{\Theta U}. \tag{2.38}$$

As a result, by assumption 2.2:

$$c_2\left(\frac{r_2}{L}\right)^{\xi_2} \leqslant K_2 C^*\left(\frac{r_2}{L}\right)^{2 - b_2}.$$

The constants being independent of Pe (1), the last inequality necessarily implies (2.25) for $r_2/L \to 0$.

The definition of $r_2^{(v)}$ (2.28) and the estimate (2.30) for the second-order velocity structure function can be rewritten as

$$1 = \frac{r_2^{(v)}}{\nu} \sup_{|r|=r} [S_2(\varrho, \boldsymbol{r}, \boldsymbol{u})]^{1/2} \geqslant \frac{\overline{C}UL}{\nu}\left(\frac{r_2^{(v)}}{L}\right)^{1+\xi_2^{(v)}/2} =$$

$$= \overline{C}\,\frac{\mathrm{Re}}{\mathrm{Pe}}\left(\frac{r_2^{(v)}}{r_2}\right)^{1+\xi_2^{(v)}/2} \mathrm{Pe}\left(\frac{r_2}{L}\right)^{1+\xi_2^{(v)}/2} \geqslant \bar{c}\,\mathrm{Pe}\left(\frac{r_2}{L}\right)^{1+\xi_2^{(v)}/2},$$

where $\bar{c}$ is a Pe-independent constant resulting from assumption 2.1 (inequalities (2.31), (2.32)). Using again assumption 2.2 (eq. (2.33)), we arrive at the inequality

$$1 \geqslant \bar{c}k_2\left(\frac{r_2}{L}\right)^{1-b_2+\xi_2^{(v)}/2}$$

supposed to hold for all r_2/L. For $r_2/L \to 0$ we thus obtain (2.36). □

Remark 2. The estimate on $\xi_2^{(v)}$ obtained in [1] was $\xi_2^{(v)} \geqslant 2/3$, $2/3$ being the value of the scaling exponent predicted by the Kolmogorov theory under the assumption that the local Reynolds number at the length scale $r_2^{(v)}$ remains bounded as $\mathrm{Re} \to \infty$. Equation (2.37) implies that, whenever $\xi_2^{(v)}$ is larger than $2/3$, ξ_2 is allowed to have smaller values than $2/3$; this reciprocal behaviour has, indeed, been detected experimentally and predicted by the shell model [5].

3. – Higher-order structure functions

In order to estimate the higher-order structure functions we can follow the analysis carried out in sect. 5 of [1]. The starting point is again lemma 1, this time with an arbitrary $m > 0$. In the same way as in [1] we assume that

Assumption 3.1.

$$\frac{M_\varrho[\,|\nabla\theta|^m]}{(M_\varrho[\,|\nabla\theta|^2])^{m/2}} \leqslant C_m \mathrm{Pe}^{m\beta_m}, \tag{3.1}$$

where C_m and β_m are Pe-independent constants.

(1) The analogous result of ref. [1] (Theorem 1.3) involves a Reynolds-number–dependent «constant» originating in the estimate of the pressure term. The estimate of the pressure term necessitates to postulate an additional assumption on its asymptotic ($\mathrm{Re} \to \infty$) behaviour. As can be seen, the present situation is much simpler.

By (2.19), (2.33) and (2.26) we have (see the first part of the proof of proposition 2.2):

$$\frac{S_m(\varrho, \boldsymbol{r})}{\Theta^m} \leqslant \widehat{C}\left(\frac{|\boldsymbol{r}|}{L}\right)^m \mathrm{Pe}^{m(\beta_m + 1/2)}, \tag{3.2}$$

where $\widehat{C}$ is a positive Pe-independent constant.

We now generalize the assumptions of the previous section in the following way:

Assumption 3.2. The Peclet number can be related to r_m, for arbitrary m, via

$$\mathrm{Pe} = k_m \left(\frac{r_m}{L}\right)^{-b_m}, \tag{3.2}$$

where k_m and b_m are positive constants independent of Pe and r_m/L.

Assumption 3.3. There exist positive constants c_m and ξ_m, independent of Pe, such that

$$\forall \boldsymbol{r}, \quad r_m \leqslant |\boldsymbol{r}| \leqslant 2r_m: \ \frac{S_m(\varrho, \boldsymbol{r})}{\Theta^m} \geqslant c_m \left(\frac{|\boldsymbol{r}|}{L}\right)^{\xi_m}. \tag{3.3}$$

Using these assumptions and repeating the considerations of the proof of proposition 2, we arrive at the following

Proposition 3. Under assumptions 3.1, 3.2 and 3.3,

i)

$$\xi_m \geqslant m\left[1 - b_m\left(\beta_m + \frac{1}{2}\right)\right], \tag{3.4}$$

ii)

$$\xi_m^{(v)} \geqslant m(b_m - 1) \tag{3.5}$$

and

iii)

$$\xi_m \geqslant m\left[1 - \left(\frac{\xi_m^{(v)}}{m} + 1\right)\left(\beta_m + \frac{1}{2}\right)\right]. \tag{3.6}$$

If the ratio defined in (3.1) is bounded independently of Pe (*i.e.* $\beta_m = 0$) and the same happens for the corresponding ratio of velocity gradients (see eq. (52) in [1]), and also the local Reynolds numbers at the bottom of the scaling ranges are bounded, it was argued in [1] that $\xi_m^{(v)} = m/3$ for all $m \geqslant 3$ and (3.6) thus yields

$$\xi_m \geqslant \frac{m}{3}, \qquad \forall m \geqslant 2. \tag{3.7}$$

Remark 2. We call the set of requirements leading to (3.7) K-L hypothesis, since they gave, for the velocity field, the Kolmogorov linear scaling just combining, through the Hölder inequality, the bound $\xi_m^{(v)} \geqslant m/3$ proved in sect. 6 of [1] with the old Kolmogorov statistical result $\xi_3^{(v)} = 1$ [4]. At this point we have two important observations to do:

i) If the K-L hypotheses are not satisfied, one could expect a non-Kolmogorov picture of turbulence, *i.e.* the anomalous scaling, as pointed out in sect. 1 of [1] for the velocity field.

ii) Even under the K-L hypotheses our bound (3.7) does not imply a linear scaling since we have not a statistical result for the temperature field for $m = 3$. However, we shall show in sect. 4 that a new statistical analysis based on a theory recently developed by us [6] allows to show that

$$\bar{\xi}_4 \leqslant \frac{8}{3} - \bar{\xi}_4^{(v)} \tag{3.8}$$

which, combined with (3.7) and for $\xi_4^{(v)} = 4/3$, gives a Kolmogorov linear scaling for $m \geqslant 4$.

Remark 3. It is interesting to note that if we define, in analogy with (3.2),

$$\mathrm{Re} = \left(\frac{r_m^{(v)}}{L}\right)^{-b_m^{(v)}}, \tag{3.9}$$

the exponents $b_m^{(v)}$ related to the velocity (cf. [1] p. 55) satisfy

$$1 + \xi_m^{(v)} - b_m^{(v)} \geqslant 0\,. \tag{3.10}$$

Our inequality (3.5) gives another bound involving $\xi_m^{(v)}$, this being expressed in terms of the exponent b_m for the temperature field.

4. – Upper bound estimate of the fourth scaling exponent

In this section we prove the bound (3.8) anticipated in remark 2 of the preceding section.

The following analysis will be performed assuming the forcing term F to be zero. This will allow us to use the statistical theory developed in [6] and will give immediately a crucial statistical relation, the validity of which was only assumed on the basis of physical arguments in [1].

The starting point is the equation [6]

$$\frac{\partial}{\partial t}(\Delta\theta)^2 + \boldsymbol{u}\cdot\nabla(\Delta\theta)^2 - \kappa_0\nabla^2(\Delta\theta)^2 + \kappa_0\,|\nabla(\Delta\theta)|^2 = -2\,\Delta\theta\,\Delta\boldsymbol{u}\cdot\nabla\theta(\boldsymbol{x}+\boldsymbol{r},\,t), \tag{4.1}$$

where $\Delta\theta = \theta(\boldsymbol{x}+\boldsymbol{r},\,t) - \theta(\boldsymbol{x},\,t)$ and $\Delta\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},\,t) - \boldsymbol{u}(\boldsymbol{x},\,t)$.

Averaging according to (2.7) and using, repeatedly, the Hölder inequality, translational invariance and considerations analogous to those leading to proposition 1 we get:

Proposition 4.

$$(4.2)\qquad M_\varrho[\,|\nabla(\Delta\theta)|^2\,] \leqslant$$

$$\leqslant C\left[\frac{\Theta^2 U}{\kappa_0 \varrho} + \frac{\Theta^2}{\varrho^2} + \frac{1}{\kappa_0} M_\varrho^{1/2}[\,|\nabla\theta|^2\,]\cdot M_\varrho^{1/4}[(\Delta\theta)^4\,] + \cdot M_\varrho^{1/4}[\,|\Delta \boldsymbol{u}|^4\,]\right],$$

where C is a positive constant of geometric nature in the same way as that of the inequality (2.11).

We now bound $M_\varrho^{1/2}[\,|\nabla\theta|^2\,]$ using (2.19):

$$M_\varrho^{1/2}[\,|\nabla\theta|^2\,] \leqslant \text{const}\,\mathrm{Pe}^{1/2}\,\frac{\Theta}{L}$$

and we assume that

Assumption 4.1.

$$(4.3)\qquad M_\varrho^{1/4}[(\Delta\theta)^4\,] \leqslant \text{const}\,\Theta\left(\frac{|\boldsymbol{r}|}{L}\right)^{\bar{\xi}_4/4},$$

$$(4.4)\qquad M_\varrho^{1/4}[\,|\Delta\boldsymbol{u}|^4\,] \leqslant \text{const}\,U\left(\frac{|\boldsymbol{r}|}{L}\right)^{\bar{\xi}_4^{(v)}/4},$$

where $\bar{\xi}_4$ and $\bar{\xi}_4^{(v)}$ are scaling exponents, in general different from (3.3) through (3.6) with $m = 4$, and «const» stands for $|\boldsymbol{r}|/L$ independent constants. We shall consider the r.h.s. of (4.3) and (4.4) with $|\boldsymbol{r}|$ equal to the Kolmogorov scales (bottoms of the inertial zones) defined above. As far as the l.h.s. of (4.2) is concerned, we found in [6] (eq. (2.9) of [6]) that

$$(4.5)\qquad \langle\,|\nabla(\Delta\theta)|^2\,\rangle = -\frac{1}{\kappa_0}\langle \Delta\theta\,\Delta\boldsymbol{u}\cdot\nabla\theta(\boldsymbol{x}+\boldsymbol{r})\rangle .$$

Though the mean of ref. [6] was defined slightly differently, the manipulations used in [6] can be transposed to the average (2.7) in a straightforward manner (2). Local isotropy and homogeneity can then be used to project eq. (4.5) along the direction of $\boldsymbol{r}$

(2) Note that (4.5) is deduced with boundary conditions, which amounts to neglect the first two terms in the square bracket on the r.h.s. of (4.2). This, of course, does not affect the final result of this section.

and to obtain (see [6], the l.h.s. of eq. (2.12) therein (3)):

$$\langle \Delta\theta\, \Delta \boldsymbol{u} \cdot \nabla\theta(\boldsymbol{x}+\boldsymbol{r})\rangle = \frac{1}{2}\left(\frac{2}{r} + \frac{\partial}{\partial r}\right)\langle (\Delta\theta)^2\, \Delta u_L\rangle, \tag{4.6}$$

where $r = |\boldsymbol{r}|$ and Δu_L is the projection of $\Delta \boldsymbol{u}$ along the direction of $\boldsymbol{r}$. The well-known Yaglom formula ([4], Vol. II, p. 400) valid in the inertial range

$$\langle (\Delta\theta)^2\, \Delta u_L\rangle = -\frac{4}{3}\overline{N}, \tag{4.7}$$

where

$$\overline{N} = \kappa_0 \langle |\nabla\theta|^2\rangle$$

can be applied to the r.h.s. of eq. (4.6). As a result,

$$\overline{N} \leqslant C\left[\frac{\Theta^2 U}{\varrho} + \frac{\kappa_0 \Theta^2}{\varrho^2} + \mathrm{Pe}^{1/2}\,\frac{\Theta^2 U}{L}\left(\frac{r}{L}\right)^{(\bar{\xi}_4 + \bar{\xi}_4^{(v)})/4}\right]. \tag{4.8}$$

We now take r at the bottom of the inertial ranges of the fourth-order moments of the temperature and velocity increments r_4 and $r_4^{(v)}$, respectively, knowing that both values are related via (2.32). Moreover, we suppose that the exponents b_m and $b_m^{(v)}$ defined in (3.2) and (3.9), respectively, are the same and equal to $4/3$, as argued in [1] (p. 57). We thus set

$$\mathrm{Pe} = \mathrm{Re} = \left(\frac{r_4}{L}\right)^{-4/3}. \tag{4.9}$$

We then rewrite (4.8) as (remember that $1 \leqslant \varrho/L \leqslant 2$):

$$\overline{N} \leqslant C\cdot \frac{\Theta^2 \kappa_0}{L^2}\mathrm{Pe}\left[1 + \frac{1}{\mathrm{Pe}} + \mathrm{Pe}^{1/2 - (3/16)(\bar{\xi}_4 + \bar{\xi}_4^{(v)})}\right], \tag{4.10}$$

Next we assume that the dissipation rate $\overline{N}$ bounds (uniformly for $\mathrm{Pe} \to \infty$) the quantity

$$\overline{N} \geqslant \mathrm{const}\,\frac{\Theta^2 U}{L} \tag{4.11}$$

(3) For even moments, eq. (2.12) of [6] reads (see also [9]):

$$\frac{1}{\overline{N}2n(2n-1)}\left(\frac{2}{r} + \frac{\partial}{\partial r}\right)\langle [\Delta\theta(\boldsymbol{r},\boldsymbol{x})]^{2n}\, \Delta u_L(\boldsymbol{r},\boldsymbol{x})\rangle =$$
$$= -2\langle [\Delta\theta(\boldsymbol{r},\boldsymbol{x})]^{2n-2}\, |\nabla\theta(\boldsymbol{x})|^2\rangle + \frac{2\kappa_0}{\overline{N}2n(2n-1)}\left(\frac{2}{r} + \frac{\partial}{\partial r}\right)\frac{\partial}{\partial r}\langle [\Delta\theta(\boldsymbol{r},\boldsymbol{x})]^{2n}\rangle.$$

which is equivalent to the bound (63) of [1]. Formula (4.10) can thus be rewritten as

$$\text{const}\,\frac{\Theta^2 U}{L} \leqslant \overline{N} \leqslant C\frac{\Theta^2 U}{L}\left[1 + \frac{1}{\mathrm{Pe}} + \mathrm{Pe}^{1/2-(3/16)(\bar{\xi}_4+\bar{\xi}_4^{(v)})}\right]. \tag{4.12}$$

In the limit of large Pe (4.12) implies

$$\bar{\xi}_4 \leqslant \frac{8}{3} - \bar{\xi}_4^{(v)}\,. \tag{4.13}$$

which, in the K-L hypothesis for the velocity field ($\xi_4^{(v)} = 4/3$), allows us to recover a linear scaling for the temperature field for $m \geqslant 4$, as explained in remark 2 of sect. 3.

5. – Conclusion

In a recent paper Kraichnan [10] presented a model for scaling exponents of a passive scalar using the assumption of a rapidly varying velocity field and the theory developed in ref. [11] yielding a sub-linear behaviour for the scaling exponents. It is natural to expect that a large variety of models can be obtained depending on assumptions on the velocity field behaviour. The phenomenon of internal intermittency of the velocity field being, so far, still poorly understood and even the most promising models [12, 13] being only weakly related to the Navier-Stokes equations, it is important to seek bounds, imposed by the equations of motions, such models have necessarily to satisfy. This approach has been chosen by Constantin and Fefferman [1] to obtain bounds for the scaling exponents of the velocity field and is applied, in the present paper, to the advected scalar field. Naturally, a relation between the scaling exponents of the velocity structure functions and those for the passive scalar structure functions results. The general lower bound pertinent for anomalous scaling is given in (3.6). It shows that, in addition to assumptions on the velocity field scaling exponents, an additional parameter originating in the high-Peclet-number estimate of the powers of the scalar field gradients is needed.

It has been shown, as an example, that the classical linear Kolmogorov scaling of both the passive scalar field and the velocity field is compatible with the theory presented above provided the additional assumption of a Peclet-number–independent bound of the mean values of the powers of the scalar field gradients is adopted.

Given the fact that the velocity field determines the solution of the underlying transport equation (2.1), it is not surprising that the velocity and advected passive scalar turbulent characteristics are inter-linked. Our analysis shows that, to be complete, models of anomalous scaling for a passive scalar field should predict both the velocity and the scalar-field scaling exponents.

REFERENCES

[1] CONSTANTIN P. and FEFFERMAN C., *Nonlinearity*, **7** (1994) 41.
[2] KOLMOGOROV A. N., *Dokl. Akad. Nauk SSSR*, **30** (1941) 209; see also: FRISCH U., «*Turbulent and Stochastic Processes: Kolmogorov's Ideas 50 Years Later on*» (Royal Society, London) 1992, p. 89.
[3] ANSELMET F., GAGNE Y., HOPFINGER E. J. and ANTONIA A. R., *J. Fluid. Mech.*, **140** (1984) 63.

[4] MONIN A. S. and YAGLOM A. M., *Statistical Fluid Mechanics* (MIT Press, Cambridge) 1971.
[5] JENSEN M. H., PALADIN G. and VULPIANI A., *Phys. Rev. A*, **45** (1992) 7214; WILLIAMS R. M. and PAULSON C. A., *J. Fluid Mech.*, **83** (1977) 547; HILL R. J., *J. Fluid Mech.*, **88** (1978) 541; KERR R. M., *J. Fluid Mech.*, **211** (1990) 309.
[6] VAIENTI S., OULD-ROUIS M., ANSELMET F. and LE GAL P., *Physica D*, **73** (1994) 99.
[7] CONSTANTIN P., FOIAS C., MANLEY O. and TEMAM R., *J. Fluid Mech.*, **150** (1985) 427; CONSTANTIN P. and PROCACCIA I., *Phys. Rev. Lett.*, **70** (1993) 3416; FRISCH U., SULEM P. L. and NELKIN M., *J. Fluid Mech.*, **87** (1978) 719; HENTSCHEL H. G. E. and PROCACCIA I., *Phys. Rev. A*, **29** (1984) 1461; KRAICHNAN R. H., *J. Fluid Mech.*, **62** (1974) 305; MANDELBROT B., *J. Fluid Mech.*, **62** (1974) 331; MENEVEAU C. and SREENIVASAN K. R., *Phys. Rev. Lett.*, **59** (1987) 1424; PALADIN G. and VULPIANI A., *Phys. Rep.*, **156** (1987) 147; PARISI G. and FRISCH U., in *Turbulence Predictions in Geophysical Fluid Dynamics* (North-Holland, Amsterdam) 1985, p. 84; SCHERTZER D. and LOVEJOY S., *J. Geophys. Res.*, **92** (1987) 9693; SREENIVASAN K. R., *Ann. Rev. Fluid Mech.*, **23** (1991) 539.
[8] CONSTANTIN P. and PROCACCIA I., *Phys. Rev. E*, **47** (1993) 3307.
[9] OULD-ROUIS M., ANSELMET F., LE GAL P. and VAIENTI S., *Physica D*, **85** (1995) 405.
[10] KRAICHNAN R. H., *Phys. Rev. Lett.*, **72** (1994) 1016.
[11] KRAICHNAN R. H., *Phys. Fluids*, **11** (1968) 945.
[12] SHE Z. S. and LÉVÊQUE E., *Phys. Rev. Lett.*, **72** (1994) 336.
[13] DUBRULLE B., *Phys. Rev. Lett.*, **73** (1994) 959.