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Localization Properties of the Chalker–Coddington Model

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We dedicate this work to the memory of our friend and colleague Pierre Duclos

Abstract. The Chalker–Coddington quantum network percolation model is numerically pertinent to the understanding of the delocalization transition of the quantum Hall effect. We study the model restricted to a cylinder of perimeter $2M$. We prove first that the Lyapunov exponents are simple and in particular that the localization length is finite; secondly, that this implies spectral localization. Thirdly, we prove a Thouless formula and compute the mean Lyapunov exponent, which is independent of M.

1. Introduction

We start with a mathematical then a physical description of the model. Fix the parameters

$$r, t \in [0,1], \quad \text{such that, } r^2 + t^2 = 1,$$

denote by $\mathbb{T}$ the complex numbers of modulus 1 and for $q = (q_1, q_2, q_3) \in \mathbb{T}^3$ by $S(q)$ the general unitary $U(2)$ matrix depending on these three phases

$$S(q) := \begin{pmatrix} q_1 q_2 & 0 \\ 0 & q_1 \overline{q_2} \end{pmatrix} \begin{pmatrix} t & -r \\ r & t \end{pmatrix} \begin{pmatrix} q_3 & 0 \\ 0 & \overline{q_3} \end{pmatrix}.$$

Let $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}}\right)$ be the probability space: $\widehat{\Omega} := \left(\mathbb{T}^6\right)^{(2\mathbb{Z})^2}$, $\widehat{\mathbb{P}} := \otimes_{(2\mathbb{Z})^2} d^6 l$ where dl is the normalized Lebesgue measure on $\mathbb{T}$, and $\widehat{\mathcal{F}}$ the σ-algebra generated by the cylinder sets. With

$$p \in \widehat{\Omega}, \quad p(2j, 2k) =: (\underbrace{p_1, p_2, p_3}_{p_e(2j,2k)}, \underbrace{p_4, p_5, p_6}_{p_o(2j+1,2k+1)}) \qquad (j, k \in \mathbb{Z})$$

and the basis vectors $e_\mu(\rho) := \delta_{\mu,\rho}$ $(\mu, \rho \in \mathbb{Z}^2)$, the family of unitary operators

$$\widehat{U}(p) : l^2(\mathbb{Z}^2) \to l^2(\mathbb{Z}^2)$$

is defined by its matrix elements $\widehat{U}_{\mu;\nu} = \langle e_\mu, \widehat{U} e_\nu \rangle$:

$$\widehat{U}_{\mu;\nu} := 0 \text{ except for the blocks}$$
$$\begin{pmatrix} \widehat{U}(p)_{(2j+1,2k);(2j,2k)} & \widehat{U}(p)_{(2j+1,2k);(2j+1,2k+1)} \\ \widehat{U}(p)_{(2j,2k+1);(2j,2k)} & \widehat{U}(p)_{(2j,2k+1);(2j+1,2k+1)} \end{pmatrix} := S(p_e(2j, 2k)) \qquad (1)$$
$$\begin{pmatrix} \widehat{U}_{(2j+2,2k+2);(2j+2,2k+1)} & \widehat{U}_{(2j+2,2k+2);(2j+1,2k+2)} \\ \widehat{U}_{(2j+1,2k+1);(2j+2,2k+1)} & \widehat{U}_{(2j+1,2k+1);(2j+1,2k+2)} \end{pmatrix} := S(p_o(2j+1, 2k+1)).$$

Note that $\widehat{U}$ is an ergodic family of random unitary operators; indeed, $\widehat{U}^*\widehat{U} = \mathbb{I} = \widehat{U}\widehat{U}^*$ because of the unitarity of the blocks; further denote by $\widehat{\Theta}$ the action of $\mathbb{Z}^2$ on functions f on $\mathbb{Z}^2$:

$$(\widehat{\Theta}_{(l,m)} f)(\mu) := f(\mu + (2l, 2m)) \quad (\mu \in \mathbb{Z}^2, (l, m) \in \mathbb{Z}^2),$$

and, by abuse of notation, the corresponding shift on $\widehat{\Omega}$. Then $\widehat{\Theta}$ is measure preserving and ergodic on $\widehat{\Omega}$ and

$$\widehat{U}(\widehat{\Theta} p) = \widehat{\Theta} \widehat{U}(p) \widehat{\Theta}^{-1}.$$

This model was introduced in the physics literature by Chalker and Coddington [10], see [21] for a review, in order to study essential features of the quantum Hall transition in a quantitative way. $\widehat{U}$ describes the dynamics of a 2D electron in a strong perpendicular magnetic field and a smooth bounded random electric potential which is supposed to have some array of hyperbolic fixed points forming the nodes of a graph.

In this picture, the electron moves on the directed edges of the graph whose nodes are "even": $\{(1/2, 1/2) + (2j, 2k), j, k \in \mathbb{Z}\}$ or "odd": $\{(1/2, 1/2) + (2j+1, 2k+1), j, k \in \mathbb{Z}\}$ with edges connecting the even (odd) nodes to their nearest odd (even) neighbors. $\widehat{U}$ describes the evolution at time one of the electron. The edges are labeled by their midpoints. They are directed in such a way that $\widehat{U}$ models the tunneling near the hyperbolic fixed points of the potential, see Fig. 1. The tunneling is described by the scattering matrices S associated with the even, respectively odd, nodes. The i.i.d. random phases associated with each node take into account the deviation of the random electric potential from periodicity.

Following the literature on tunneling near a hamiltonian saddle point [11,14], the parameter t is $\frac{1}{\sqrt{1+e^{\varepsilon}}}$ where ε is the distance of the electron's energy to the nearest Landau Level. An application of a finite size scaling

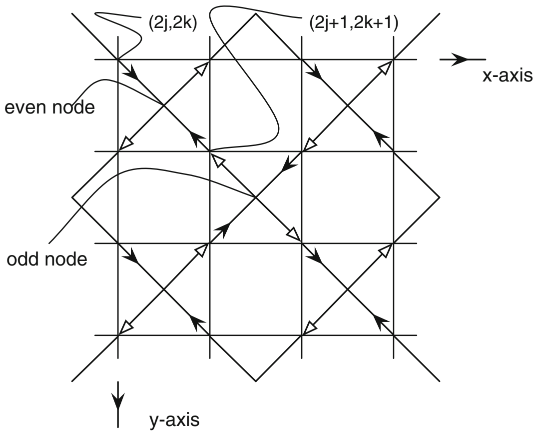

Figure 1. The network model with its incoming (*solid arrows*) and outgoing links

method to their numerical observations led Chalker and Coddington [10], see also [21], to conjecture that the localization length diverges as $t/r \to 1$ as

$$\left(\frac{1}{\ln|\frac{t}{r}|}\right)^{\alpha},$$

where the critical exponent α exceeds substantially the exponent expected when a classical percolation model is applied to the problem [25]; the values advocated for α are 2.5 ± 0.5 for the quantum and $4/3$ for the classical case.

Because of its importance for the understanding of the integer quantum Hall effect the one electron magnetic random model in two dimensions was and continues to be heavily studied in the mathematical literature. Mathematical results concerning the full Schrödinger Hamiltonian can be traced from the following contributions and their references: [27] for percolation, [16] for the existence of the localization–delocalization transition [2,8,15] for the general theory of the quantum Hall effect. For results concerning a 2D electron in a magnetic field and periodic potential, which corresponds to the absence of phases here, see [18,26]. For recent work on Lyapunov exponents on Hamiltonian strip models see [6,7,23].

Our results concern the restriction of the model to a strip of width $2M$ and periodic boundary conditions; they are presented as follows. In Sect. 2, we analyze the extreme cases, $r = 0$ and $r = 1$. Then, for the case where all phases are chosen to be 1, we give a description of the spectrum. Questions related to transfer matrix formalism are handled in Sects. 3, 4, 5. In Sect. 6, we prove simplicity of the Lyapunov spectrum and finiteness of the localization length. In Sect. 7, we prove a Thouless formula and show that the density of states is flat, which implies our results on the mean Lyapunov exponent. In Sect. 8, we prove complete spectral localization.

2. Some Properties of the Model

2.1. Extreme Cases

Note that in case of complete "reflection" or "transmission" the system localizes completely:

Proposition 2.1. *Let $rt = 0$. Then, for any $p \in \widehat{\Omega}$, the spectrum of $\widehat{U}(p)$ is pure point.*

Proof. Assume $r = 0$, $p \in \widehat{\Omega}$ and define the family of subspaces $(\mathcal{H}_{j,k})_{(j,k)\in\mathbb{Z}^2}$ as:

$$\mathcal{H}_{j,k} = \mathrm{Ran}(e_{2j,2k}, e_{2j+1,2k}, e_{2j+1,2k-1}, e_{2j,2k-1}).$$

These subspaces are invariant under $\widehat{U}(p)$ and

$$\oplus_{(j,k)\in\mathbb{Z}^2} \mathcal{H}_{j,k} = l^2(\mathbb{Z}^2), \tag{2}$$

which means the operator $\widehat{U}(p)$ is pure point. The case $t = 0$ is treated similarly. □

On the other hand, one has complete propagation if all the phases are equal to one; define $\widehat{\Omega} \ni p = (\ldots, 1, 1, 1, \ldots)$ by $p(2j, 2k) := (1, 1, 1, 1, 1, 1)$ then we have:

Proposition 2.2. *Let $rt \neq 0$. Then, the spectrum of $\widehat{U}(\ldots, 1, 1, 1, \ldots)$ is purely absolutely continuous.*

Proof. We make use of a decomposition similar to (2) and define the unitary V from $l^2(\mathbb{Z}^2)$ to $l^2(\mathbb{Z}^2) \otimes \mathbb{C}^4$ by $Ve_{2j,2k} := e_{j,k} \otimes e_1, Ve_{2j+1,2k+1} := e_{j,k} \otimes e_2, Ve_{2j,2k+1} := e_{j,k} \otimes e_3, Ve_{2j+1,2k} := e_{j,k} \otimes e_4$. Let P be the projection $P := \mathbb{I} \otimes (|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|)$. From the definition of $\widehat{U}$ in (1) one reads that $V\widehat{U}^2V^{-1}$ commutes with P and that $PV\widehat{U}^2V^{-1}P$ is equivalent to

$$\begin{pmatrix} rt(T_{0,1} - T_{1,0}) & r^2T_{1,0} + t^2T_{0,1} \\ t^2T_{0,-1} + r^2T_{-1,0} & rt(T_{-1,0} - T_{0,-1}) \end{pmatrix}$$

with the translations on $l^2(\mathbb{Z}^2)$ defined by

$$T_{n,m}\psi(j,k) := \psi(j+n, k+m) \quad (n, m \in \mathbb{Z}).$$

The Fourier transform $\mathcal{F} : l^2(\mathbb{Z}^2) \to L^2(\mathbb{T}^2)$ transforms the translations to multiplication operators: $\mathcal{F}T_{n,m}\mathcal{F}^{-1} = \exp(-i(nx + my))$, thus the restriction to the range of P of $\mathcal{F}PV\widehat{U}^2V^{-1}P\mathcal{F}^{-1}$ is equivalent to a matrix-valued multiplication operator

$$\begin{pmatrix} rt(e^{-iy} - e^{-ix}) & r^2e^{-ix} + t^2e^{-iy} \\ t^2e^{iy} + r^2e^{ix} & rt(e^{ix} - e^{iy}) \end{pmatrix}. \tag{3}$$

The trace of this matrix is not constant, its determinant is -1 hence the spectral bands are not flat, thus the spectrum of the restriction of $\widehat{U}^2$ is purely absolutely continuous. By an analogous argument this also holds for the restriction to $P^{\perp}$. □

Remark that a more general periodic distribution of phases leads to matrix-valued translation operators with periodic coefficients thus to non-trivial Hofstadter like problems.

3. Restriction to a Cylinder, Transfer Matrices

Let $M \in \mathbb{N}$. Use the notation $\mathbb{Z}_{2M} := \mathbb{Z}/(2M\mathbb{Z})$ for the discrete circle of perimeter $2M$. Consider the restriction of the model to the cylinder $\mathbb{Z} \times \mathbb{Z}_{2M}$:

$$U(p) : l^2(\mathbb{Z} \times \mathbb{Z}_{2M}) \to l^2(\mathbb{Z} \times \mathbb{Z}_{2M})$$

defined by its matrix elements with respect to the canonical basis

$$U(p)_{\mu,\nu} := \widehat{U}_{(\mu_1,\mu_2 \ mod \ 2M);(\nu_1,\nu_2 \ mod \ 2M)}. \tag{4}$$

Remark that $U(p)$ has the same spectral properties for the extreme cases as $\widehat{U}(p)$, the model on the full lattice:

Proposition 3.1. *Let $rt = 0$. Then, for any $p \in \Omega$, the spectrum of $U(p)$ is pure point.*

Proof. Similar to the proof of Proposition 2.1. □

Proposition 3.2. *Let $rt \neq 0$. Then the spectrum of $U(\ldots, 1, 1, 1, \ldots)$ is purely absolutely continuous.*

Proof. In the proof of Proposition 2.2 note that V now acts from $l^2(\mathbb{Z} \times \mathbb{Z}_{2M})$ to $l^2(\mathbb{Z} \times \mathbb{Z}_M) \otimes \mathbb{C}^4$ and replace $\mathcal{F}$ by the Fourier transform from $l^2(\mathbb{Z} \times \mathbb{Z}_M)$ to $L^2(\mathbb{T} \times \mathbb{Z}_M)$ defined by

$$\mathcal{F}\psi(x, \kappa) = \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}_M} \psi_{j,k} e^{ixj} e^{i\frac{2\pi}{M}\kappa k}$$

which diagonalizes the translations. Then, setting $y = \frac{2\pi}{M}\kappa$ ($\kappa \in \mathbb{Z}_M$), the matrix-valued multiplication operator obtained in (3) is understood as a family of matrix-valued operators indexed over $\mathbb{Z}_M$. The spectral bands are not flat by the same argument. □

From now on we restrict the discussion to the case

$$rt \neq 0.$$

In the following z denotes a complex number; also, unless otherwise stated, all indices in the second variable are to be understood *mod* $2M$, e.g.:

$$\psi_{2j,2k+1} = \psi_{2j,2k+1 \ mod \ 2M} = \psi_{2j,2k+1[2M]}.$$

A standard approach to the spectral problem of U is the transfer matrix method. Though this is well known, we wish to recall the construction explicitly for the model at hand.

Proposition 3.3. *For $z \neq 0$, $q = (q_1, q_2, q_3) \in \mathbb{T}^3$ define*

$$T_{eo}(z,q) := \begin{pmatrix} q_1 q_2 & 0 \\ 0 & q_3 \end{pmatrix} \frac{1}{t} \begin{pmatrix} z^{-1} & -r \\ -r & z \end{pmatrix} \begin{pmatrix} q_3 & 0 \\ 0 & \overline{q_1} q_2 \end{pmatrix},$$

$$T_{oe}(z,q) := \begin{pmatrix} \overline{q_3} & 0 \\ 0 & q_1 q_2 \end{pmatrix} \frac{1}{r} \begin{pmatrix} z & -t \\ t & -z^{-1} \end{pmatrix} \begin{pmatrix} \overline{q_1} q_2 & 0 \\ 0 & \overline{q_3} \end{pmatrix}.$$

Then

1. *For $\psi : \mathbb{Z} \times \mathbb{Z}_{2M}$ it holds:*

$$\sum_{\nu \in \mathbb{Z} \times \mathbb{Z}_{2M}} U_{\mu\nu} \psi_\nu = z \psi_\mu \quad \forall \mu \in \mathbb{Z} \times \mathbb{Z}_{2M}$$
$$\Longleftrightarrow$$
$$\begin{pmatrix} \psi_{2j+1,2k} \\ \psi_{2j+1,2k+1} \end{pmatrix} = T_{eo}\left(z, p_e(2j,2k)\right) \begin{pmatrix} \psi_{2j,2k} \\ \psi_{2j,2k+1} \end{pmatrix}$$

and

$$\begin{pmatrix} \psi_{2j+2,2k+1} \\ \psi_{2j+2,2k+2} \end{pmatrix} = T_{oe}\left(z, p_o(2j+1,2k+1)\right) \begin{pmatrix} \psi_{2j+1,2k+1} \\ \psi_{2j+1,2k+2} \end{pmatrix}.$$

2. *For $z \in \mathbb{T}$, it holds that $T_{oe}, T_{eo} \in U(1,1)$, the Lorentz group defined as a subset of the complex 2×2 matrices by*

$$U(1,1) := \left\{ B \in \mathbb{M}_{2,2}(\mathbb{C}); B^* J B = J, \quad J := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

Proof. By definition of U, we have for the "even" nodes:

$$\begin{pmatrix} (U\psi)_{2j+1,2k} \\ (U\psi)_{2j,2k+1} \end{pmatrix} = S\left(p_e(2j,2k)\right) \begin{pmatrix} \psi_{2j,2k} \\ \psi_{2j+1,2k+1} \end{pmatrix} = z \begin{pmatrix} \psi_{2j+1,2k} \\ \psi_{2j,2k+1} \end{pmatrix},$$

and, for the "odd" nodes:

$$S\left(p_o(2j+1,2k+1)\right) \begin{pmatrix} \psi_{2j+2,2k+1} \\ \psi_{2j+1,2k+2} \end{pmatrix} = z \begin{pmatrix} \psi_{2j+2,2k+2} \\ \psi_{2j+1,2k+1} \end{pmatrix}.$$

For a matrix

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \quad \text{with } S_{22} S_{21} \neq 0$$

it holds:

$$\begin{pmatrix} a \\ b \end{pmatrix} = S \begin{pmatrix} x \\ y \end{pmatrix} \Longleftrightarrow \begin{pmatrix} a \\ y \end{pmatrix} = \widehat{S} \begin{pmatrix} x \\ b \end{pmatrix} \Longleftrightarrow \begin{pmatrix} x \\ a \end{pmatrix} = \check{S} \begin{pmatrix} b \\ y \end{pmatrix}$$

with

$$\widehat{S} = \frac{1}{S_{22}} \begin{pmatrix} \det S & S_{12} \\ -S_{21} & 1 \end{pmatrix}, \quad \check{S} = \frac{1}{S_{21}} \begin{pmatrix} 1 & -S_{22} \\ S_{11} & -\det S \end{pmatrix}.$$

Now

$$\widehat{\begin{pmatrix} t & -r \\ r & t \end{pmatrix}} = \frac{1}{t} \begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix}; \quad \check{\begin{pmatrix} t & -r \\ r & t \end{pmatrix}} = \frac{1}{r} \begin{pmatrix} 1 & -t \\ t & -1 \end{pmatrix}$$

so

$$z\begin{pmatrix} a \\ b \end{pmatrix} = q_1 \begin{pmatrix} q_2 & 0 \\ 0 & \overline{q_2} \end{pmatrix} \begin{pmatrix} t & -r \\ r & t \end{pmatrix} \begin{pmatrix} q_3 & 0 \\ 0 & \overline{q_3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\iff \begin{pmatrix} a \\ y \end{pmatrix} = \begin{pmatrix} q_1 q_2 & 0 \\ 0 & q_3 \end{pmatrix} \frac{1}{t} \begin{pmatrix} z^{-1} & -r \\ -r & z \end{pmatrix} \begin{pmatrix} q_3 & 0 \\ 0 & \overline{q_1 q_2} \end{pmatrix} \begin{pmatrix} x \\ b \end{pmatrix}$$
$$\iff \begin{pmatrix} x \\ a \end{pmatrix} = \begin{pmatrix} \overline{q_3} & 0 \\ 0 & q_1 q_2 \end{pmatrix} \frac{1}{r} \begin{pmatrix} z & -t \\ t & -z^{-1} \end{pmatrix} \begin{pmatrix} \overline{q_1 q_2} & 0 \\ 0 & \overline{q_3} \end{pmatrix} \begin{pmatrix} b \\ y \end{pmatrix}$$

from which the first claim follows.

Denote by $\mathbb{I}$ the identity matrix in $\mathbb{C}^2$. S is a unitary matrix if and only if the pullback of the quadratic form in $\mathbb{C}^4$ associated with $Q = \begin{pmatrix} \mathbb{I} & \\ & -\mathbb{I} \end{pmatrix}$ (blanks stand for 0 entries) to the graph of S: $\{(u, Su) \in \mathbb{C}^4, u \in \mathbb{C}^2\}$ is zero. The mapping from (x, y, a, b) to (x, b, a, y) transforms Q to $\begin{pmatrix} J & \\ & -J \end{pmatrix}$. The pullback of the corresponding form to the graph of T_{eo} being zero, it follows that T_{eo} and, by the analogous argument, T_{oe}, belong to the Lorentz group. □

For later use we fix the following notation

Definition 3.4. Denote by $\mathbb{J}$ the $2M \times 2M$ block diagonal matrix consisting of M non-zero diagonal blocks equal to J and by

$$U_M(1,1) := \{B \in \mathbb{M}_{2M,2M}(\mathbb{C}); B^*\mathbb{J}B = \mathbb{J}\}.$$

the unitary group of the hermitian form defined by $\mathbb{J}$.

Note that $U_M(1,1)$ is isomorphic to the classical unitary group $U(M,M)$ of the hermitian form $|z_1|^2 + \cdots + |z_M|^2 - |z_{M+1}|^2 \cdots - |z_{2M}|^2$.

4. Relevant Phases

Because of the uniform distribution, it is possible to reduce the number of relevant phases in the model to two phases per node. Before proceeding we do this reduction. We shall repeatedly make use of

Lemma 4.1. *Let $\varphi_1, \ldots, \varphi_n$ be independent and uniformly distributed random variables on $\mathbb{R}/\mathbb{Z}$ and let $A \in \mathbb{M}_{m,n}(\mathbb{Z})$. Then, $\theta_1, \ldots, \theta_m$ defined by $\vec{\theta} = A\vec{\varphi}$ are independent and uniformly distributed if and only if Rank A is maximal.*

Proof. For $\vec{k} \in \mathbb{Z}^m$ it holds

$$\mathbb{E}(e^{i\langle \vec{k}, \vec{\theta}\rangle}) = \mathbb{E}(e^{i\langle \vec{k}, A\vec{\varphi}\rangle}) = \mathbb{E}(e^{i\langle A^t\vec{k}, \vec{\varphi}\rangle}) = \delta_{A^t\vec{k},0}.$$

Thus, the $\vec{\theta}$ are independent and uniformly distributed if and only if $\mathbb{E}(e^{i\langle \vec{k}, \vec{\theta}\rangle}) = \delta_{\vec{k},0}$ if and only if $\mathrm{Ker} A^t = \{0\}$, equivalently, if and only if Rank A is maximal. □

Proposition 4.2. *There exists* $g : \widehat{\Omega} \to \mathbb{T}^{\mathbb{Z}^2}$ *such that for* $p \in \widehat{\Omega}$ *the evolution* $\widehat{U}(p)$ *defined by* (1) *is unitarily equivalent to*

$$D(g(p))\mathbb{S} \quad on\ l^2(\mathbb{Z}^2)$$

where $D(q)$ *is diagonal,* $D(q)_{(j,k);(j,k)} = q_{j,k}$*, and* $\mathbb{S} = \widehat{U}(\ldots,1,1,1,\ldots)$*. Moreover, the image measure of* $\otimes_{(2\mathbb{Z})^2} d^6 l$ *by* g *is* $\otimes_{\mathbb{Z}^2} dl$*.*

Proof. By (1), $\widehat{U}(p)$ is of the form $\widehat{U}(p) = D^{(1)}(p)\mathbb{S}D^{(2)}(p)$ where $D^{(j)}(p)$ are diagonal, and defined by their diagonal elements:

$$\begin{aligned} D^{(1)}(p)_{2j+1,2k} &= p_1 p_2(2j,2k),\\ D^{(1)}(p)_{2j+2,2k+2} &= p_4 p_5(2j+1,2k+1),\\ D^{(2)}(p)_{2j,2k} &= p_3(2j,2k),\\ D^{(2)}(p)_{2j+2,2k+1} &= p_6(2j+1,2k+1),\\ D^{(1)}(p)_{2j,2k+1} &= p_1\bar{p}_2(2j,2k),\\ D^{(1)}(p)_{2j+1,2k+1} &= p_4\bar{p}_5(2j+1,2k+1),\\ D^{(2)}(p)_{2j+1,2k+1} &= \bar{p}_3(2j,2k),\\ D^{(2)}(p)_{2j+1,2k+2} &= \bar{p}_6(2j+1,2k+1). \end{aligned}$$

Hence, $\widehat{U}(p)$ is unitarily equivalent to $D^{(2)}(p)D^{(1)}(p)\mathbb{S}$ which has the asserted shape. Define $q = g(p)$ by

$$\begin{aligned} q(2j+1,2k) &:= \overline{p}_6(2j+1,2k-1)p_1p_2(2j,2k),\\ q(2j,2k+1) &:= p_6(2j-1,2k+1)p_1\bar{p}_2(2j,2k),\\ q(2j+2,2k+2) &:= p_3(2j+2,2k+2)p_4p_5(2j+1,2k+1),\\ q(2j+1,2k+1) &:= \overline{p}_3(2j,2k)p_4\bar{p}_5(2j+1,2k+1). \end{aligned}$$

Now, an application of Lemma 4.1 shows the q's are i.i.d. and uniformly distributed. □

Remark 4.3. Note that the unitary transformation just constructed is diagonal and thus does not affect the localization properties of the model.

In the following, we abuse notations and call for $q \in \mathbb{T}^{\mathbb{Z}^2}$ the matrix operator $D(q)\mathbb{S}$ again $\widehat{U}(q)$; same abuse for the restriction to the cylinder.

5. Characteristic Exponents

We now define and analyze the transfer matrices and in particular the localization length. Consider

$$U(p) = D(p)\mathbb{S} \quad \text{on } l^2(\mathbb{Z}\times\mathbb{Z}_{2M}) \tag{5}$$

with identically distributed uniformly distributed phases in $\mathbb{T}^{\mathbb{Z}\times\mathbb{Z}_{2M}}$, $\otimes_{\mathbb{Z}\times\mathbb{Z}_{2M}} dl$ and the cylinder set algebra.

We use the unitary equivalence

$$\begin{aligned} l^2(\mathbb{Z}\times\mathbb{Z}_{2M}) \cong l^2(\mathbb{Z}\times\{0,\dots,2M-1\}) \to l^2(\mathbb{Z}; l^2(\mathbb{Z}_{2M})) \cong l^2(\mathbb{Z},\mathbb{C}^{2M}) \\ \psi \mapsto \Psi \\ \Psi_j := (\psi_{j,0},\dots,\psi_{j,2M-1}). \end{aligned} \tag{6}$$

Note that with the reduced phases the building blocks of the transfer matrices read with phases p, q

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{t} \begin{pmatrix} z^{-1} & -r \\ -r & z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \frac{1}{r} \begin{pmatrix} z & -t \\ t & -z^{-1} \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}.$$

As we shall explain below, the previous analysis leads us to deal with the following random dynamical system:

Consider the probability space defined by $\Omega = (\mathbb{T}^{4M})^{2\mathbb{Z}}, \mathbb{P} = \otimes_{\mathbb{Z}} d^{4M}l$, and $\mathcal{F}$, the cylinder set algebra. The shift

$$\Theta : \Omega \to \Omega, \quad \Theta p(2m) := p(2(m+1)) \quad (m \in \mathbb{Z})$$

is measure preserving and ergodic. For $p \in \Omega$ define the following elements of $(\mathbb{T}^{2M})^{2\mathbb{Z}}$

$$\begin{aligned} p_r &= (1, p_1, 1, p_3, \dots, 1, p_{2M-1}) \\ p_l &= (p_0, 1, p_2, 1, \dots, p_{2M-2}, 1) \\ p_m &= (p_{2M}, p_{2M+1}, \dots, p_{4M-1}). \end{aligned}$$

Denote for $q \in \mathbb{T}^{2M}$ the unitary diagonal matrix

$$D(q) := \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_{2M} \end{pmatrix},$$

(where 0 valued matrix entries are represented by blanks) and for $z \neq 0$ the $2M \times 2M$ matrices

$$M_1(z) := \frac{1}{t} \begin{pmatrix} z^{-1} & -r & & & \\ -r & z & & & \\ & & \ddots & & \\ & & & z^{-1} & -r \\ & & & -r & z \end{pmatrix},$$

$$M_2(z) := \frac{1}{r} \begin{pmatrix} -z^{-1} & & & & & & t \\ & z & -t & & & & \\ & t & -z^{-1} & & & & \\ & & & \ddots & & & \\ & & & & z & -t & \\ & & & & t & -z^{-1} & \\ -t & & & & & & z \end{pmatrix}.$$

Define for a fixed $z \neq 0$

$$\begin{aligned} &A_z : \Omega \to U_M(1,1) \\ &A_z(p) := D\left(p_l\right) M_2(z) D\left(p_m\right) M_1(z) D\left(p_r\right). \end{aligned} \tag{7}$$

Then A generates the cocycle Φ over the ergodic dynamical system

$$(\Omega, \mathcal{F}, \mathbb{P}, (\Theta^n)_{n\in\mathbb{Z}})$$

defined by $\Phi_z : \mathbb{Z} \times \Omega \to U_M(1,1)$

$$\Phi_z(n,p) := \begin{cases} A(\Theta^{n-1}p)\ldots A(p) & n > 0 \\ \mathbb{I} & n = 0 \\ A^{-1}(\Theta^n p)\ldots A^{-1}(\Theta^{-1}p) & n < 0 \end{cases}.$$

Oseledets theorem holds for Φ, see [1], Theorem 3.4.11 and Remark 3.4.10 (ii):

Definition 5.1. Let $z \neq 0$. There exists an invariant subset of full measure of $p \in \Omega$ such that the limits

$$\lim_{n\to\infty} \left(\Phi_z^*(n,p)\Phi_z(n,p)\right)^{1/2n} = \lim_{n\to-\infty} \left(\Phi_z^*(n,p)\Phi_z(n,p)\right)^{1/2|n|} =: \Psi_z(p)$$

exist. Denote by $\gamma_k(p,z)$, $k \in \{1,\ldots,2M\}$, the eigenvalues of $\Psi_z(p)$ arranged in decreasing order. Due to ergodicity there exists $\gamma_k(z) \geq 0$ such that $\gamma_k(p,z) = \gamma_k(z)$ on an invariant subset of full measure. The characteristic exponents are defined by $\lambda_k(z) := \log \gamma_k(z)$.

Due to the Lorentz symmetry of the transfer matrices for $z \in \mathbb{T}$ we have

Proposition 5.2. 1. *Let $B \in U_M(1,1)$. Then for the singular values $SV(B)$ it holds*

$$\gamma \in SV(B) \Longleftrightarrow \frac{1}{\gamma} \in SV(B).$$

2. *For $\lambda_j := \log \gamma_j, \gamma_j \in SV(B)$ arranged in decreasing order it holds*:

$$\lambda_{j+M} = -\lambda_{M-j+1} \quad \forall j \in \{0,\ldots,M\}.$$

Proof. We have $B^*\mathbb{J}B{=}\mathbb{J}$. In particular, $\det B \neq 0$, so $\gamma \neq 0$ and $\mathbb{J}^{-1}B^* = B^{-1}\mathbb{J}^{-1}$ as well as $B\mathbb{J} = \mathbb{J}B^{*-1}$. Now,

$$\det\left(B^*B - z^2\right) = 0 \Longleftrightarrow \det\left(\mathbb{J}^{-1}B^*B\mathbb{J} - z^2\right) = 0 \Longleftrightarrow$$

$$\det\left((B^*B)^{-1} - z^2\right) = 0 \Longleftrightarrow z^{4M}\det(B^*B)^{-1}\det\left(\frac{1}{z^2} - B^*B\right) = 0.$$

From which the two claims follow. □

Thus, we restrict our discussions to the first M non-negative Lyapunov exponents

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M \geq 0$$

which we shall call for simplicity "the" Lyapunov exponents in the sequel.

We show that due to the translation invariance of the uniform distribution, the exponents are independent of z:

Lemma 5.3. *For any* $w \in \mathbb{T}$,

$$A_{wz}(p) = A_z(w \odot p),$$

where $w \odot p$ *is defined by* $w \odot p_{2j} := w^{-1}p_{2j}$*, and* $w \odot p_{2j+1} := wp_{2j+1}$.

Proof. Write $D(p_m)$ in (7) as $D(p_l')D(p_r')$, where

$$p_r' := (1, p_{2M+1}, 1, p_{2M+3}, \ldots, 1, p_{4M-1})$$
$$p_l' := (p_{2M}, 1, p_{2M+2}, 1, \ldots, p_{4M-2}, 1).$$

Thus, $A_z(p)$ is the product of the block diagonal matrices $D(p_l)M_2(z)D(p_l')$ and $D(p_r')M_1(z)D(p_r)$ whose blocks are

$$a_z^l(p,q) := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \frac{1}{r} \begin{pmatrix} z & -t \\ t & -z^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} = \frac{1}{r} \begin{pmatrix} z & -qt \\ pt & -pqz^{-1} \end{pmatrix}$$
$$a_z^r(p,q) := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \frac{1}{t} \begin{pmatrix} z^{-1} & -r \\ -r & z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} = \frac{1}{r} \begin{pmatrix} z^{-1} & -qr \\ -pr & pqz \end{pmatrix}.$$

For any $w \in \mathbb{T}$, these matrices satisfy

$$a_{wz}^l(p,q) = wa_z^l(w^{-1}p, w^{-1}q), \quad a_{wz}^r(p,q) = w^{-1}a_z^l(wp, wq),$$

from which the result follows. □

Therefore, for any fixed $w \in \mathbb{T}$, the matrices $A_z(w \odot p)$ have the same distribution as $A_{wz}(p)$. As a consequence,

Corollary 5.4. *All characteristic exponents* $\lambda_k(z) = \lambda_k$ *are independent of* $z \in \mathbb{T}$.

Proof. $\lambda_k = \mathbb{E}\left(\log(\gamma_k(z,p))\right) = \mathbb{E}\left(\log(\gamma_k(1, \frac{1}{z} \odot p))\right) = \mathbb{E}\left(\log(\gamma_k(1,p))\right)$. □

Definition 5.5. The localization length $\xi_M \in [0, \infty]$ is defined as

$$\xi_M := \frac{1}{\lambda_M}.$$

Remark 5.6. In the physics literature, see [21], ξ_M in assumed to be finite for all parameters; a change of the asymptotic behavior as $M \to \infty$ is conjectured when the parameters of the model approach the critical point $t = r$. This conjecture is supported by a numerical finite size scaling method and is supposed to reflect the divergence of the localization length of the full system at the critical point. Thus a first step to support these heuristics is to prove finiteness of ξ_M and to establish precise information of its behavior as a function of M.

The announced equivalence to the propagation problem is the content of the following

Proposition 5.7. *Let* $U(p)$ *be the ergodic family of unitary operators defined in* (5) *over the probability space* $\Gamma := \mathbb{T}^{\mathbb{Z}\times\mathbb{Z}_{2M}}, \otimes_{\mathbb{Z}\times\mathbb{Z}_{2M}} dl$ *and the cylinder set algebra. Let* $f : \Gamma \to \Omega$ *be defined for* $j \in \mathbb{Z}$ *by*

$$f(p)(2j) := (p_{2j,0}, p_{2j+2,1}, p_{2j,2}, p_{2j+2,3} \cdots p_{2j,2M-2}, p_{2j+2,2M-1}$$
$$p_{2j+1,0}, p_{2j+1,1} \cdots p_{2j+1,2M-1}).$$

The image measure by f is the measure on Ω and it holds

$$U(p)\psi = z\psi \Longleftrightarrow \Psi_{2N} = \Phi_z(N, f(p))\Psi_0$$

for Ψ defined in (6).

Proof. The construction of f follows from Proposition (3.3). The image measure follows from Lemma (4.1). □

6. Finiteness of the Localization Length

Using the methods exposed in [5], see also [17], we prove that all Lyapunov exponents are distinct and in particular that the localization length for the cylinder in finite.

Theorem 6.1. *For $rt \neq 0, z \in \mathbb{T}$ it holds*

$$\lambda_1 > \lambda_2 > \cdots > \lambda_M > 0.$$

Proof. We follow the strategy exposed in [5] and prove the theorem in several steps making use of lemmata to be proven below. Denote by

$$G := \text{the smallest subgroup of } U_M(1,1) \quad \text{generated by } \{A(p), p \in \Omega\}.$$

By Lemma 6.2

$$G = U_M(1,1).$$

In particular it is then known:

$$G \, \text{is connected.}$$

Furthermore, see also [23], G is isomorphic to the complex symplectic group. Indeed : denote by $\daleth$ the $2M \times 2M$ block diagonal matrix consisting of M non-zero blocks $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; we write: $\daleth = \oplus_1^M \sigma$ for short; denote by

$$Sp(M, \mathbb{C}) := \{B \in \mathbb{M}_{2M,2M}(\mathbb{C}); B^* \daleth B = \daleth\}$$

the complex symplectic group. From

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}^* J \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = i\sigma$$

it follows defining $C := \bigoplus_1^M \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ that

$$G = U_M(1,1) = C Sp(M, \mathbb{C}) C^*.$$

In order to freely use results in [5] we shall do our argument for real matrices. To this end we separate real and imaginary parts and consider

$$\tau : \mathbb{M}_{2M,2M}(\mathbb{C}) \to \mathbb{M}_{4M,4M}(\mathbb{R})$$
$$x = a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

It holds: $\tau(x+y) = \tau(x)+\tau(y); \tau(xy) = \tau(x)\tau(y); \tau(x^*) = \tau(x)^t; ker\{\tau\} = \{0\}; \det \tau(x) = |\det x|^2$, thus

$$\tau\left(C^*GC\right) \subset Sp(2M, \mathbb{R})$$

with the real symplectic group

$$Sp(2M, \mathbb{R}) := \{B \in \mathbb{M}_{4M,4M}(\mathbb{R}); B^t \daleth B = \daleth\}$$

for $\daleth = \oplus_1^{2M} \sigma$.

$\det \tau(x) = |\det x|^2$ implies that $\tau(x)$ shares its eigenvalues with x with the degeneracies doubled. So the Lyapunov exponents γ defined by the $\tau - C$ transformed products of transfer matrices are

$$\gamma_1 = \gamma_2 = \lambda_1 \geq \cdots \geq \gamma_{2p-1} = \gamma_{2p} = \lambda_p \geq \cdots \geq \gamma_{2M-1} = \gamma_{2M} = \lambda_M.$$

As $\tau\left(C^*GC\right)$ is connected one can infer from [5] Theorem 3.4 and Exercice 2.9 for $p \in \{1, \ldots, M\}$:

$$\left.\begin{array}{ll} \tau\left(C^*GC\right) & L_{2p}\,\text{irreducible} \\ \text{and} & \\ \tau\left(C^*GC\right) & 2p\,\text{contracting} \end{array}\right\} \Longrightarrow \gamma_{2p} = \lambda_p > \lambda_{p+1} = \gamma_{2p+1}$$

in particular for $p = M$: $\lambda_M > 0$. Now by Lemmas 6.3 and 6.4, the group $\tau\left(C^*GC\right)$ is $2p$ irreducible and $2p$ contracting for all $p \in \{1, \ldots, M\}$ so all Lyapunov exponents are distinct and $\lambda_M > 0$. □

The following lemmata complete the proof of Theorem 6.1, we use the notations introduced in the above proof.

Lemma 6.2.

$$G = U_M(1,1).$$

Proof. By definition $G \subset U_M(1,1)$ is a closed subgroup of $Gl(2M, \mathbb{C})$ thus G is a Lie group. By connectedness of $U_M(1,1)$ it is sufficient to show that the Lie algebras $\mathfrak{g}$ and $\mathfrak{u}_M(1,1)$ coincide. Now

$$\mathfrak{u}_M(1,1) = \{A \in \mathbb{M}_{2M,2M}(\mathbb{C}); A_{jk} = -\overline{A}_{kj}(-1)^{k+j}\}$$

whose dimension as a real vector space equals $4M^2$.

Denote by $D_j(t) = \mathrm{diag}(1, 1, \ldots, 1, e^{it}, 1, \ldots, 1)$ the unitary matrix where the phase sits at the j'th slot, for $j = 1, 2, \ldots, 2M$ and use the M_j as defined in Sect. 5. For $z \in \mathbb{T}$ the matrices

$$i|j\rangle\langle j|, \quad iM_2(z)|j\rangle\langle j|M_2(z)^{-1}, \quad iM_1(z)^{-1}|j\rangle\langle j|M_1(z) \tag{8}$$

belong to $\mathfrak{g}$, for $j = 1, 2, \ldots, 2M$ as they are the generators of the curves $D_j(t)$, $M_2(z)D_j(t)M_2^{-1}(z)$, $M_1^{-1}(z)D_j(t)M_1(z)$, which lie in G as

$$\begin{aligned} D_j(t) &= D_j(t)M_2(z)M_1(z)(M_2(z)M_1(z))^{-1}, \\ M_2(z)D_j(t)M_2^{-1}(z) &= M_2(z)D_j(t)M_1(z)(M_2(z)M_1(z))^{-1}, \\ M_1^{-1}(z)D_j(t)M_1(z) &= (M_2(z)M_1(z))^{-1}M_2(z)D_j(t)M_1(z). \end{aligned}$$

The generators in (8) have the same block structure as the M_j. We compute the relevant blocks. For $iM_2(z)|j\rangle\langle j|M_2(z)^{-1}$ we get

$$\frac{i}{r^2}\begin{pmatrix} z & -t \\ t & -\bar{z}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix} \bar{z} & -t \\ t & -z\end{pmatrix} = \frac{i}{r^2}\begin{pmatrix} 1 & -tz \\ t\bar{z} & -t^2\end{pmatrix}$$
$$\frac{i}{r^2}\begin{pmatrix} z & -t \\ t & -\bar{z}\end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} \bar{z} & -t \\ t & -z\end{pmatrix} = \frac{i}{r^2}\begin{pmatrix} -t^2 & tz \\ -t\bar{z} & 1\end{pmatrix}.$$

Similarly, for $iM_1(z)^{-1}|j\rangle\langle j|M_1(z)$, the blocks take the form

$$\frac{i}{t^2}\begin{pmatrix} z & r \\ r & \bar{z}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix} \bar{z} & -r \\ -r & z\end{pmatrix} = \frac{i}{t^2}\begin{pmatrix} 1 & -rz \\ r\bar{z} & -r^2\end{pmatrix}$$
$$\frac{i}{t^2}\begin{pmatrix} z & r \\ r & \bar{z}\end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} \bar{z} & -r \\ -r & z\end{pmatrix} = \frac{i}{t^2}\begin{pmatrix} -r^2 & rz \\ -r\bar{z} & 1\end{pmatrix}.$$

Now using these matrices for $z \notin \mathbb{R}$ and the diagonal matrix $i|j\rangle\langle j|$, $j = 1, 2$ one gets by taking suitable real linear combinations of the matrices above that, in both cases, the relevant blocks are generated by

$$\left\{ i\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}, i\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}, i\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\right\}.$$

For real z, use the curves $D_j(t)$, $D_e M_2 D_j(t) M_2^{-1} D_e^{-1}$, $D_o^{-1} M_1^{-1} D_j(t) \cdot M_1 D_o$, with $D_e = \oplus \begin{pmatrix} 1 & 0 \\ 0 & w\end{pmatrix}$ and $D_o = \oplus \begin{pmatrix} w & 0 \\ 0 & 1\end{pmatrix}$ for $w \in \mathbb{T}$, which amounts to perform the change $z \mapsto w^{-1}z$.

Taking into account the shift in the blocks and the period $2M$ of the indices in the matrices, we get that the restrictions of $\mathfrak{g}$ and $\mathfrak{u}_M(1,1)$ to their tridiagonal elements, mod $2M$ coincide.

To go off the diagonals we use commutators, i.e. we exploit that $X, Y \in \mathfrak{g}$ implies $[X, Y] \in \mathfrak{g}$.

Let $A_k = |k+1\rangle\langle k| + |k\rangle\langle k+1| \in \mathfrak{g}$, for $k \in \mathbb{Z}_{2M}$. Considering $[A_j, A_k]$ for all values of j, k, we generate a basis of all anti self-adjoint matrices that have non-zero real matrix elements at distance two away from the diagonal (and in the corners, by periodicity). By commuting A_k with $\tilde{A}_j = i(|j+1\rangle\langle j| - |j\rangle\langle j+1|) \in \mathfrak{g}$, we get a basis of self-adjoint matrices with non-zero purely imaginary elements on the same upper and lower diagonals (plus corners) only. These matrices correspond to the restriction of all matrices in $\mathfrak{u}_M(1,1)$ to these diagonals.

We generalize the argument as follows: Assume we already generated a basis of all matrices $A \in \mathfrak{u}_M(1,1)$ such that $A_{jk} = 0$ if $|j-k| > m$, m fixed. Again, periodicity is implicit here.

Let $\mathfrak{u}_M(1,1) \ni B_j^{\pm}(m) = |j+m\rangle\langle j| \pm |j\rangle\langle j+m|$. We compute

$$[A_{j+m}, B_j^{\pm}(m)] = B_j^{\mp}(m+1).$$

This way we generate all matrices $A \in \mathfrak{u}_M(1,1)$ such $A_{jk} = 0$ if $|j-k| > m+1$.

Hence by induction, we see that $\mathfrak{g} = \mathfrak{u}_M(1,1)$, so that $G = U_M(1,1)$. □

Lemma 6.3. *$\tau(C^*GC) = \tau(Sp(M,\mathbb{C}))$ is L_{2p} irreducible for $p \in \{1,\dots,M\}$.*

Proof. Denote e_i, $i \in \{1,\dots,4M\}$ the canonical basis vectors of $\mathbb{R}^{4M}$. By definition (see [5] with adaptation to our symplectic form)

$$L_q := span\left\{v \in \Lambda^q \mathbb{R}^{4M}; v = Me_1 \wedge Me_3 \wedge \cdots \wedge Me_{2q-1}, M \in Sp(2M,\mathbb{R})\right\}$$

for $q \le 2M$. Remark that the set of directions in L_q corresponds to the set of isotropic subspaces of $\mathbb{R}^{4M}$.

$\tau(C^*GC)$ is L_q-irreducible if there is no proper linear subspace $V \subset L_q$ invariant under $\Lambda^q\tau(C^*GC)$.

Consider M real numbers

$$a_1 > a_2 > \cdots a_M > 1.$$

The $4M \times 4M$ diagonal matrix

$$A = \operatorname{diag}\left(a_1, \frac{1}{a_1}, a_2, \frac{1}{a_2}, \dots, a_M, \frac{1}{a_M}, a_1, \frac{1}{a_1}, a_2, \frac{1}{a_2}, \dots, a_M, \frac{1}{a_M}\right)$$

belongs to $\tau(C^*GC)$ and $e_1 \wedge e_3 \wedge \cdots \wedge e_{2q-1}$ is an eigenvector of $(\Lambda^q A)^n$ for all n with simple dominant eigenvalue > 1. Thus for an invariant subspace V of L_q either $e_1 \wedge e_3 \wedge \cdots \wedge e_{2q-1} \in V$ which implies $\Lambda^q M(e_1 \wedge e_3 \wedge \cdots \wedge e_{2q-1}) \in V, \forall M$, thus $V = L_q$, or $e_1 \wedge e_3 \wedge \cdots \wedge e_{2q-1} \in V^\perp$, which implies for all $w \in V$

$$0 = \left\langle \Lambda^q M^t w, e_1 \wedge e_3 \wedge \cdots \wedge e_{2q-1}\right\rangle = \left\langle w, \Lambda^q M e_1 \wedge e_3 \wedge \cdots \wedge e_{2q-1}\right\rangle$$

thus $V^\perp = L_q \iff V = \{0\}$. Thus we conclude the claimed irreducibility for $q = 2p$. □

Lemma 6.4. *$\tau(C^*GC) = \tau(Sp(M,\mathbb{C}))$ is $2p$ contracting for $p \in \{1,\dots,2M-1\}$.*

Proof. For any $a \in \mathbb{R}\backslash 0$ there exist $x, y \in \mathbb{R}$ with $x^2 - y^2 = 1$ such that $\begin{pmatrix} x & y \\ y & x \end{pmatrix}$, which belongs to $U(1,1)$, has eigenvalues $a, 1/a$. Taking such matrices as blocks one sees that there exists an element of $U_M(1,1)$ whose singular values are distinct: $a_1 > a_2 > \cdots > a_M > 1 > 1/a_M \cdots$ and thus an element of $\tau(C^*GC)$ with $2M$ distinct singular values

$$b_1 = b_2 = a_1 > \cdots > b_{2p-1} = b_{2p} = a_p > \cdots > b_{2M-1} = b_{2M} = a_M > 0.$$

Thus $b_{2p+1}^n / b_{2p}^n \to_{n\to\infty} 0$ and it follows from Proposition 2.1, p. 81 of [5] that $\tau(C^*GC)$ is $2p$ contracting. □

Remark 6.5. To summarize, we have proved that if the transfer matrices generate the complex symplectic group $Sp(M,\mathbb{C})$ then the results of [5] apply, i.e.: the Lyapunov spectrum is simple. The results in [5] are stated for real groups only. While it is remarked in their introduction that these results should hold in the complex case, this seems not to be obvious to specialists in the field.

7. Thouless Formula and the Mean Lyapunov Exponent

In this section we shall prove the announced identity in a series of lemmata.

Theorem 7.1. *Let $M \in \mathbb{N}$. For the first M Lyapunov exponents associated with U defined in Definition (5.1) with $z \in \mathbb{T}$ it holds*:

$$\frac{1}{M}\sum_{i=1}^{M}\lambda_i(z) = \frac{1}{2}\log\frac{1}{rt} \geq \frac{1}{2}\log 2$$

Proof. Let $z \in \mathbb{T}$. Denoting by $P_{2L}(z)$ the propagator

$$P_{2L}(z) : \Omega \to U_M(1,1)$$
$$P_{2L}(z)(p) := \Phi_z(L,p)\,(\Phi_z(-L,p))^{-1}$$

we have for $m \in \{1,\ldots,M\}$

$$\sum_{i}^{m}\lambda_i = \lim_{L\to\infty}\frac{1}{4L}\log \| \wedge^m P_{2L}(z)(p)\| \quad p \text{ a.e.} \tag{9}$$

where $\wedge^m$ denotes the mth exterior product (cf. [1], ch. 3).

We analyze the above limit in Proposition 7.2 below and show:

$$\frac{1}{M}\sum_{i}^{M}\lambda_i = 2\int_{\mathbb{T}}\log|z-x|\mathrm{d}l(x) + \frac{1}{2}\log\frac{1}{rt}.$$

The assertion follows by an explicit calculation proving that

$$\int_{\mathbb{T}}\log|z-x|\mathrm{d}l(x) = 0.$$

Proposition 7.2. (Thouless formula) *Let $M \in \mathbb{N}, z \in \mathbb{C}\backslash 0$ then*

$$\frac{1}{M}\sum_{i}^{M}\lambda_i(z) = 2\int_{\mathbb{T}}\log|z-x|\,dl(x) + \frac{1}{2}\log\frac{1}{rt} - \log|z|$$

Proof. Will be done in Appendix 1. □

Remark 7.3. We prove in particular that the density of states is the Lebesgue measure, see Lemma 9.3 below.

7.1. Bounds on the Localization Length

We now use the Thouless formula and an M independent bound on the largest Lyapunov exponent to derive a bound on the localization length. We remark that this bound is very crude and that more involved techniques should be established to get more detailed information; cf. [23] and references therein.

First observe that a lower bound on the mean Lyapunov exponent together with a tight upper bound on the largest, implies a lower bound on all.

Lemma 7.4. *Let* $\kappa > 0, \delta > 0$ *such that* $\forall M \in \mathbb{N}, z \in \mathbb{T}$

$$\frac{1}{M}\sum_{j=1}^{M}\lambda_j \geq \kappa, \quad \textit{and} \quad \lambda_1 \leq \kappa + \delta,$$

then, for all $j = 0, 1, \ldots, M-1$,

$$\lambda_{j+1} \geq \kappa - \frac{j\delta}{M-j}. \tag{10}$$

Proof. First note that $\lambda_1 \geq \frac{1}{M}\sum_{j=1}^{M}\lambda_j$. Thus $\lambda_1 \geq \kappa$, which corresponds to (10) for $j = 0$. Similarly, using also the upper bound on λ_1, we have for any $1 \leq j \leq M-1$,

$$M\kappa \leq \left(\sum_{k=1}^{j} + \sum_{k=j+1}^{M}\right)\lambda_k \leq j(\kappa+\delta) + \sum_{k=j+1}^{M}\lambda_k$$

so that

$$\lambda_{j+1} \geq \frac{1}{M-j}\sum_{k=j+1}^{M}\lambda_k \geq \kappa - \frac{j\delta}{M-j}.$$

□

Remark. In view of localization properties, the estimate is useful only if

$$\kappa > (M-1)\delta. \tag{11}$$

We now estimate the cocycle to derive an upper bound on the largest Lyapunov exponent, which is uniform in the quasienergy and width of the strip M.

Proposition 7.5. *Let* $M \in \mathbb{N}$

1. *For the generator of the cocycle defined in* (7)*, it holds*
$$\|A(p)\| \leq \frac{1}{rt}(1+r)(1+t);$$
2. *It follows:* $2\lambda_1 \leq \log\left(\frac{1}{rt}\right) + \log\left((1+r)(1+t)\right)$.
3. *There exists a* $c > 0$ *such that for* $M \in \mathbb{N}$ *it holds*:
$$dist(r, \{0,1\}) < e^{-cM} \Longrightarrow$$
$$\xi_M = \frac{1}{\lambda_M} \leq \frac{2}{\log\left(\frac{1}{rt}\right) - (M-1)\log\left((1+r)(1+t)\right)}.$$

Proof. The estimate on A follows from its definition. The estimate on λ_1 is obtained using the equality (9). Finally, from the estimate (10) it follows

$$\xi_M = \frac{1}{\lambda_M} \leq \frac{2}{\log\left(\frac{1}{rt}\right) - (M-1)\log\left((1+r)(1+t)\right)}.$$

The bound is symmetric around $t = r = \frac{1}{\sqrt{2}}$ and finite for r sufficiently away from the critical point $\frac{1}{\sqrt{2}}$ because of the singularity of $\log 1/rt$. □

8. Spectral Localization

We follow the strategy which was successfully employed for the case of one dimensional Schrödinger operators: polynomial boundedness of generalized eigenfunctions, positivity of the Lyapunov exponent and spectral averaging. We lean on the work of [4,19]. Our result is:

Theorem 8.1. *Let $M \in \mathbb{N}$, $rt \neq 0$. Then, the Chalker Coddington model on the cylinder exhibits spectral localization throughout the spectrum, almost surely. More precisely,*

1. *the almost sure : spectrum Σ, continuous spectrum Σ_c and pure point spectrum Σ_{pp} of $U(p)$ satisfy*
$$\Sigma = \Sigma_{pp} = \mathbb{T} \quad \textit{and} \quad \Sigma_c = \emptyset;$$
2. *the eigenfunctions decay exponentially, almost surely.*

Proof. We prove the theorem in Appendix 2. □

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9. Appendix 1

We follow the strategy of [13] and first prove the lower bound

$$\frac{1}{M}\sum_{i}^{M}\lambda_i(z) \geq 2\int_{\mathbb{T}}\log|z-x|\mathrm{d}l(x) + \frac{1}{2}\log\frac{1}{rt} - \log|z| \tag{12}$$

for $0 \neq z \in \mathbb{C}\backslash\mathbb{T}$ which follows from Lemma 9.3 Eq. 15 below in the limit $L \to \infty$.

Lemma 9.1. *Denote by U^D the unitary operator defined by restriction of U to $l^2\left(\{-2L,\ldots,2L\}, l^2(\mathbb{Z}_{2M})\right)$ with reflecting boundary conditions: the scattering picture for the links which are incoming to walls at $-(2L+1)$ and $2L+1$ reads*

$$U^D e_{-2L,2k+1} = e_{-2L,2k+2}, \quad U^D e_{2L,2k} = e_{2L,2k+1}.$$

For $z \in \mathbb{C}$ let

$$F_z := \left\{\psi \in l^2(\mathbb{Z}_{2M}); \psi_{2k+1} = z\psi_{2k+2}, k \in \mathbb{Z}_M\right\},$$
$$G_z := \left\{\psi \in l^2(\mathbb{Z}_{2M}); z\psi_{2k+1} = \psi_{2k}, k \in \mathbb{Z}_M\right\}$$

and denote by Q_F the orthogonal projection to a subspace F. It holds:

$$\begin{aligned} &z \textit{ is an eigenvalue of } U^D \\ &\iff \Psi_{2L} = P_{2L}(z)\Psi_{-2L} \textit{ and } \Psi_{-2L} \in F_z \textit{ and } \Psi_{2L} \in G_z \\ &\iff Ker\left(Q_{G_z^\perp} P_{2L}(z) Q_{F_z}\right) \neq \{0\} \end{aligned}$$

Proof. It holds

$$U^D\psi = z\psi \Longrightarrow$$
$$\psi_{-2L,2k+1} = z\psi_{-2L,2k+2} \quad \text{and} \quad z\psi_{2L,2k+1} = \psi_{2L,2k}$$

so

$$\Psi_{-2L} \in F_z \quad \text{and} \quad \Psi_{2L} \in G_z.$$

The identity

$$\Psi_{2L} = P_{2L}(z)\Psi_{-2L}$$

holds by construction of the transfer matrices so

$$U^D\psi = z\psi \Longleftrightarrow Q_{G_z^\perp} P_{2L}(z) Q_{F_z}\Psi = 0.$$

□

Lemma 9.2. *Denote the "even" subspace of $l^2(\mathbb{Z}_{2M})$ by*

$$E := span\{e_{2k}; k \in \mathbb{Z}_M\}.$$

For $z \neq 0$ there exist invertible operators V_z, W_z on $l^2(\mathbb{Z}_{2M})$ such that $W_z(E) = F_z$ and $V_z(E) = G_z^\perp$ such that

1. $$z \text{ is an eigenvalue of } U^D \Longleftrightarrow$$
$$\det\left(Q_E V_z^{-1} P_{2L}(z) W_z Q_E\right) = 0$$

   where we understand the determinant to apply to the restriction to E.

2. *For $z \neq 0$; $\{z_1, \ldots, z_{(4L+1)2M}\}$ the eigenvalues of U^D it holds:*

$$|z|^{(4L+1)M} |\det\left(Q_E V_z^{-1} P_{2L} W_z Q_E\right)| = \frac{1}{(rt)^{2LM}} \Pi_{i=1}^{(4L+1)2M} |z - z_i|. \quad (13)$$

Proof. Fix $0 \neq z \in \mathbb{C}$.

In the following N_j, D_j denote generic, z independent matrices whose precise values may change from line to line. The D_j are diagonal.

The transfer matrix A_z defined in (7) is of the form

$$A_z = \frac{1}{rt}\left(z^2 D_1 Q_O + z^{-2} D_2 Q_E + z N_1 + N_2 + z^{-1} N_3\right)$$

where O denotes the "odd" subspace defined by $O + E = l^2\left(\mathbb{Z}_{2M}\right)$. Thus

$$(rt)^{2L} P_{2L} = z^{4L} D_1 Q_O + z^{-4L} D_2 Q_E + \sum_{j=-4L+1}^{4L-1} z^j N_j.$$

Note that

$$F_z = \text{span}\left\{\frac{1}{\sqrt{2}}\left(z e_{2k+1} + e_{2k+2}\right); k \in \mathbb{Z}_M\right\}$$
$$G_z = \text{span}\left\{\frac{1}{\sqrt{2}}\left(e_{2k} + z^{-1} e_{2k+1}\right); k \in \mathbb{Z}_M\right\}.$$

On $l^2(\mathbb{Z}_{2M})$ define the operators

$$W_z := \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}_M} |ze_{2k+1} + e_{2k+2}\rangle\langle e_{2k+2}| + |-e_{2k+1} + z^{-1}e_{2k+2}\rangle\langle e_{2k+1}|$$

$$V_z := \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}_M} |-e_{2k+1} + ze_{2k}\rangle\langle e_{2k}| + |e_{2k} + z^{-1}e_{2k+1}\rangle\langle e_{2k+1}|.$$

Then $W_zQ_E = Q_{F_z}W_z$ and $V_zQ_E = Q_{G_z^{\perp}}V_z$. Moreover, one checks that

$$W_z^2 = \mathbb{I}, \quad V_z^{-1} = KV_zK \tag{14}$$

with $K := \sum_{k\in\mathbb{Z}_M} |e_{2k+1}\rangle\langle e_{2k}| + |e_{2k}\rangle\langle e_{2k+1}|$. It follows:

$$z \text{ is eigenvalue of } U^D \Longleftrightarrow ker\left(Q_E V_z^{-1} P_{2L} W_z Q_E\right) \neq \{0\}.$$

Now

$$\begin{aligned} &Q_E V_z^{-1} P_{2L} W_z Q_E \\ &= \frac{1}{2}\sum_{k,m} |e_{2k}\rangle \left\langle -e_{2k+1} + \frac{1}{z}e_{2k}, P_{2L}\left(ze_{2m+1} + e_{2m+2}\right)\right\rangle \langle e_{2m+2}| \\ &= \frac{1}{(rt)^{2L}}\left(z^{4L+1}D_1Q_O + z^{-4L-1}D_2Q_E + \sum_{|j|<4L+1} z^j N_j\right). \end{aligned}$$

Multiplication by z^{4L+1} implies that for some $a_j \in \mathbb{C}$

$$z^{(4L+1)M} \det\left(Q_E V_z^{-1} P_{2L} W_z Q_E\right) = \frac{1}{(rt)^{2LM}} \sum_{0}^{(8L+2)M} z^j a_j.$$

D_1 is unitary thus, in particular, $|a_{(8L+2)M}| = 1$. $z^{(4L+1)M}$ det ... being a polynomial of degree $(8L+2)M$ whose leading coefficient has modulus $(rt)^{-2LM}$ and which is zero on the $(4L+1)2M$ eigenvalues of U^D the formula for the determinant follows. □

We now prove convergence of the finite volume ($L < \infty$) density of states μ_L^M as $L \to \infty$ to a non-random measure: the density of state. Then we show that this measure equals the Lebesgue measure.

Lemma 9.3. *Denote $\mu_L^{(M)}$ the measure defined by*

$$\frac{1}{(4L+1)2M} trf(U^D) =: \int_{\mathbb{T}} f(x)d\mu_L^M(x) \quad (f \in C(\mathbb{T})).$$

Then

1.

$$\mu_L^M \to_{L\to\infty}^{vaguely} dl$$

the Lebesgue measure on $\mathbb{T}$.

For $M \in \mathbb{N}$ there exists $c_M > 0$ such that for all $L \in \mathbb{N}, 0 \neq z \in \mathbb{C}\backslash\mathbb{T}$

$$\begin{aligned}\frac{1}{M}\frac{1}{4L}\log \| \wedge^M P_{2L}\| \\ \geq \frac{1}{2}\log\frac{1}{rt} + \left(2+\frac{1}{2L}\right)\int_{\mathbb{T}} \log|z-x|d\mu_L^M(x) \\ -\left(1+\frac{1}{4L}\right)\log|z| - \frac{c_M}{L}. \end{aligned} \tag{15}$$

Proof. 1. We first prove the existence of a non-random limit measure. The first step consists in showing that p a.e,

$$\lim_{L\to\infty}\int_{\mathbb{T}} f(U^D(p))\mathrm{d}\mu_L^M = \frac{1}{4}\left\{\mathbb{E}\left(\langle e_{0,0}, f(U)e_{0,0}\rangle\right) + \mathbb{E}\left(\langle e_{1,1}, f(U)e_{1,1}\rangle\right)\right. \\ \left. + \mathbb{E}\left(\langle e_{1,0}, f(U)e_{1,0}\rangle\right) + \mathbb{E}\left(\langle e_{0,1}, f(U)e_{0,1}\rangle\right)\right\} =: \int f\mathrm{d}\mu_\infty^M,$$

for all $f \in C(\mathbb{T})$. This follows from a classical argument based on ergodicity, separability of $C(\mathbb{T})$ and from the fact that $U - U^D$ has norm and rank uniformly bounded in L, see e.g. [20] for the details for the unitary case.

In order to identify μ_∞^M recall that the normalized Lebesgue measure dl on $\mathbb{T}$ is uniquely characterized by :

$$\int_{\mathbb{T}} p_n \mathrm{d}l = \delta_{n,0} \quad n \in \mathbb{Z}$$

where $p_n(x) := x^n$. Consider the space of loops of euclidean length n starting at $(0,0)$:

$$\Gamma_{(0,0)} = \{\gamma : \{0,\ldots,n\} \to \{-2L,\ldots,2L\} \times \mathbb{Z}_{2M}, \gamma(0) = \gamma(n) = (0,0)\}.$$

Then because of the structure of U

$$\langle e_{0,0}, p_n(U)e_{0,0}\rangle = \sum_{\gamma\in\Gamma_{(0,0)}} \langle e_{0,0}, Ue_{\gamma(1)}\rangle \ldots \langle e_{\gamma(n-1)}, Ue_{0,0}\rangle.$$

Now $\langle e_{\gamma(j)}, U(p)e_{\gamma(j+1)}\rangle = l(p)t^\alpha r^\beta$ for some $\alpha, \beta \geq 0$ and l a uniformly distributed random variable. Thus, $\mathbb{E}\left(\langle e_{0,0}, p_n(U)e_{0,0}\rangle\right) = \delta_{n,0}$. Applying the same argument to $\langle e_{1,1}, f(U)e_{1,1}\rangle, \langle e_{0,1} \ldots$, we conclude:

$$d\mu_\infty^M = dl$$

2. By formula (13):

$$\begin{aligned}
&\frac{1}{4LM}\log\left|\det\left(Q_E V_z^{-1} P_{2L} W_z Q_E\right)\right| \\
&\quad = \frac{1}{2}\log\frac{1}{rt} + \left(2+\frac{1}{2L}\right)\int\limits_{\mathbb{T}} \log|z-x|\mathrm{d}\mu_L^M(x) - \left(1+\frac{1}{4L}\right)\log|z| \\
&\quad \le \frac{1}{4LM}\log\|\wedge^M\left(Q_E V_z^{-1} P_{2L} W_z Q_E\right)\| \\
&\quad \le \frac{1}{M}\frac{1}{4L}\log\|\wedge^M P_{2L}\| + \frac{1}{L}\underbrace{\frac{1}{4M}\left(\log\|\wedge^M Q_E V_z^{-1}\| + \log\|\wedge^M W_z Q_E\|\right)}_{=:c_M}
\end{aligned}$$

where we used the identity

$$\det Q_E A Q_E = \left\langle e_0\wedge\cdots\wedge e_{2M-2}, \wedge^M A e_0\wedge\cdots\wedge e_{2M-2}\right\rangle.$$

From this, the claim follows. □

We turn now to the proof of the opposite inequality:

$$\frac{1}{M}\sum_i^M \lambda_i(z) \le 2\int\limits_{\mathbb{T}}\log|z-x|\mathrm{d}l(x) + \frac{1}{2}\log\frac{1}{rt} - \log|z| \tag{16}$$

for $0\neq z\in\mathbb{C}\backslash\mathbb{T}$:

Proposition 9.4. *Suppose that for any choice of sets of vectors* $\left\{d_0^-, d_2^-,\ldots,d_{2M-2}^-\right\}$ *and* $\left\{d_0^+, d_2^+,\ldots,d_{2M-2}^+\right\}$ *in the "odd" subspace* O

$$\begin{aligned}
&\limsup_{L\to\infty}\frac{1}{M(4L+1)} \\
&\qquad\times\log\left|\left\langle(e_0+d_0^-)\wedge\cdots\wedge(e_{2M-2}+d_{2M-2}^-),\right.\right. \\
&\qquad\left.\left.\wedge^M(V_z^{-1}P_{2L}(z)W_z)(e_0+d_0^+)\wedge\cdots\wedge(e_{2M-2}+d_{2M-2}^+)\right\rangle\right| \\
&\quad\le\frac{1}{2}\ln\frac{1}{rt}+2\int\limits_{\mathbb{T}}\log|x-z|dl(x)-\log|z|
\end{aligned}\tag{17}$$

then, for all $0\neq z\in\mathbb{C}\backslash\mathbb{T}$,

$$\frac{1}{M}\sum_i^M\lambda_i(z)\le\frac{1}{2}\log\frac{1}{rt}+2\int\limits_{\mathbb{T}}\log|x-z|dl(x)-\log|z|$$

Proof. The vectors of the form $\{(e_0+d_0)\wedge\cdots\wedge(e_{2M-2}+d_{2M-2}); d_0,\ldots,d_{2M-2}\in O\}$ span $\wedge^M\mathbb{C}^{2M}$. On the other hand, given any spanning sets S_1 and S_2 in $\wedge^M\mathbb{C}^{2M}$, the mapping $\|\cdot\|_S$ defined by

$$\|A\|_S\equiv\sup_{\phi\in S_1,\psi\in S_2}|\langle\phi,A\psi\rangle|$$

defines a norm over the algebra of operators in $\wedge^M \mathbb{C}^{2M}$. It follows that there exists $c > 0$, which depends on S_1 and S_2, such that for any matrix A, $\|A\|_S \geq c\|A\|$, hence that

$$\frac{1}{M}\sum_{i}^{M} \lambda_i(z) \leq \frac{1}{2}\log\frac{1}{rt} + 2\int_{\mathbb{T}} \log|\zeta - z| \mathrm{d}l(\zeta) - \log|z|.$$

□

We now prove that the inequality (17) is satisfied. This will be achieved in two steps. In order to keep track of the L dependence denote by U_L^D the former U^D. Now reinterpret the left hand side of Eq. 17 as the characteristic polynomial of a deformation of U_L^D denoted V_L^D; more precisely: we aim at Eq. 18 below. The problem is then reduced to the proof of the weak convergence of the associated sequence of counting measures $(\nu_{L,z}^M)$ towards μ^M.

9.1. Deformation of U_L^D

Let L in $\mathbb{N}$ and define the matrix V_{L+1}^D on $l^2(\{-2L-2,\ldots,2L+2\}, l^2(\mathbb{Z}_{2M}))$ by: $\forall \psi \in l^2(\{-2L-2,\ldots,2L+2\}, l^2(\mathbb{Z}_{2M}))$,

$$\begin{aligned}
(V_{L+1}^D\psi)_{2L+2,2k} &= \sum_{l=0}^{M-1} B_{2k,2l}^{+}\psi_{2L,2l}\\
(V_{L+1}^D\psi)_{2L+1,k} &= \psi_{2L+1,k}\\
(V_{L+1}^D\psi)_{2L,2k+1} &= \sum_{l=0}^{M-1} C_{2k+1,2l+1}^{+}\psi_{2L+2,2l+1},\\
(V_{L+1}^D\psi)_{-2L,2k} &= \sum_{l=0}^{M-1} B_{2k,2l}^{-}\psi_{-2L-2,2l}\\
(V_{L+1}^D\psi)_{-2L-1,k} &= \psi_{-2L-1,k}\\
(V_{L+1}^D\psi)_{-2L-2,2k+1} &= \sum_{l=0}^{M-1} C_{2k+1,2l+1}^{-}\psi_{-2L,2l+1},
\end{aligned}$$

with the same reflecting boundary conditions as U_{L+1}^D and $(V_{L+1}^D\psi)_{\mu,\nu} = (U_{L+1}^D\psi)_{\mu,\nu}$ for any values of (μ,ν) which were not described previously. The matrix V_{L+1}^D is a deformation of the matrix U_{L+1}^D, but its structure remains close to the structure of U_{L+1}^D. Note that $\mathrm{span}\{e_{2L+1,k}, e_{-2L-1,j}; j,k \in \mathbb{Z}_{2M}\}$ belongs to $\ker\left(V_{L+1}^D - \mathbb{I}\right)$. For ψ an eigenvector of V_{L+1}^D associated with the eigenvalue z,

$$V_{L+1}^D\psi = z\psi.$$

This implies that either $\psi \in \mathrm{span}\{e_{2L+1,k}, e_{-2L-1,j}; j,k \in \mathbb{Z}_{2M}\}$ and $z = 1$, or $\psi_{-2L-2} \in F_z$ and $\psi_{2L+2} \in G_z$ and

$$\begin{aligned}\psi_{2L} &= P_{2L}(z)\psi_{-2L}\\ \psi_{2L+2} &= A^+(z)\psi_{2L} = z^{-1}Q_E B^+ Q_E\psi_{2L} + zQ_O {C^+}^{-1} Q_O\psi_{2L}\\ \psi_{-2L} &= A^-(z)\psi_{-2L-2} = z^{-1}Q_E B^- Q_E\psi_{-2L-2} + zQ_O {C^-}^{-1} Q_O\psi_{-2L-2}.\end{aligned}$$

The transfer matrices $A^+(z)$ and $A^-(z)$ are deformations of the matrices A_z. This construction is useful to establish the following lemma. In the following, z will be fixed as a parameter.

Lemma 9.5. *Let $(d^+_{2k})_{k\in\{0,\dots,M-1\}}$ and $(d^-_{2k})_{k\in\{0,\dots,M-1\}}$ two families of vectors belonging to the "odd" subspace O. These families are the columns of two corresponding matrices denoted D^+ and D^- respectively. Assume that for $z\neq 0$, $\max(\|D^-\|,\|D^+\|)\leq \max\left(\frac{1}{|z|},|z|\right)$ and consider the matrix V_L^D parametrized by z*

$$\begin{aligned}B_z^+ &= z + TD^+\\ C_z^+ &= z(1-z^{-1}D^+T)^{-1}\\ B_z^- &= z + z^2 D^{-*}\\ C_z^- &= (z^{-1}-D^{-*})^{-1}\end{aligned}$$

with

$$T = \begin{pmatrix} 0 & & & & 1\\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & & & \\ & & & 1 & 0\end{pmatrix}.$$

Then,

$$\begin{aligned}&\langle (e_0+d_0^-)\wedge\cdots\wedge(e_{2M-2}+d^-_{2M-2}),\\ &\qquad \wedge^M(V_z^{-1}P_{2L}(z)W_z)(e_0+d_0^+)\wedge\cdots\wedge(e_{2M-2}+d^+_{2M-2})\rangle\\ &\quad = \langle e_0\wedge\cdots\wedge e_{2M-2}, \wedge^M(V_z^{-1}A^-P_{2L}(z)A^+W_z)e_0\wedge\cdots\wedge e_{2M-2}\rangle. \qquad (18)\end{aligned}$$

Proof. By (14), $W_z^2 = I$, $V_z^{-1} = KV_zK$, $z\neq 0$. It follows for all $k\in\{0,\dots,M-1\}$:

$$\begin{aligned}A^+W_z e_{2k} &= W_z(e_{2k}+d^+_{2k})\\ A^{-*}V_z^{-1*}e_{2k} &= V_z^{-1*}(e_{2k}+d^-_{2k}).\end{aligned}$$

□

Given z, D^+ and D^- and the associated matrix V^D_{L+1}, we consider the corresponding eigenvalue problem:

$$V^D_{L+1}\psi = z'\psi$$

The complex number z' is an eigenvalue of V^D_{L+1} iff

$$(z'-1)^{4M}\langle e_0\wedge\cdots\wedge e_{2M-2}, \wedge^M(V_{z'}^{-1}A_z^-(z')P_{2L}(z')A_z^+(z')W_{z'})\\ e_0\wedge\cdots\wedge e_{2M-2}\rangle = 0$$

where $A_z^{\pm}(z') = z'^{-1}Q_E B_z^{\pm}Q_E + z'Q_O {C_z^{\pm}}^{-1}Q_O$.

Once multiplied by $z'^{(4L+5)M}$, the left-hand side is a polynomial of degree $2M(4L+5)$ in z'. Following the Thouless argument, we get for the logarithm of the modulus divided by $4ML$:

$$\frac{1}{2}\log\frac{1}{rt} - \log|z'| + \left(2+\frac{1}{2L}\right)\int_{\mathcal{B}(0,R_z)} \log|x-z'|\mathrm{d}\nu^M_{L,z}(x) + O\left(\frac{1}{L}\right),$$

where the family of measures $\nu^M_{L,z}$ are supported on some closed ball $\mathcal{B}(0,R_z)$, due to the fact that $\sup_L \|U^D_L - V^D_L\| < \infty$. Note that if $z'=z$, $A^+_z(z) = A^+$ and $A^-_z(z) = A^-$.

9.2. End of Proof of Inequality (17)

We split the proof in the two following lemmas, whose proof is an adaptation of the argument given in [13].

Lemma 9.6. *If $(\nu_L)_{L\in\mathbb{N}}$ and μ are measures supported on $\mathcal{B}(0,R)$ for some $R>0$, and if (ν_L) converge weakly to μ, then for any $z\in\mathbb{C}$*

$$\int_{\mathbb{T}} \log|\zeta - z|d\nu_L(\zeta) \le \int_{\mathbb{T}} \log|\zeta - z|d\mu(\zeta).$$

Proof. Given $z\in\mathbb{C}$, let f_ϵ be defined by:

$$\begin{cases} f_\epsilon(\zeta) = \log|\zeta - z| & \text{if } |\zeta - z| \ge \epsilon \\ f_\epsilon(\zeta) = \log|\epsilon| & \text{if } |\zeta - z| \le \epsilon \end{cases}$$

Since the support is compact,

$$\lim_{L\to\infty}\int_{\mathbb{T}} f_\epsilon(\zeta)\mathrm{d}\nu_L(\zeta) = \int_{\mathbb{T}} f_\epsilon(\zeta)\mathrm{d}\mu(\zeta).$$

On the other hand, for any ζ in $\mathbb{T}$,

$$\log|\zeta - z| \le f_\epsilon(\zeta)$$

so that:

$$\limsup_{L\to\infty}\int_{\mathbb{T}} \log|\zeta - z|\mathrm{d}\nu_L(\zeta) \le \limsup_{L\to\infty}\int_{\mathbb{T}} f_\epsilon(\zeta)\mathrm{d}\nu_L(\zeta)\\ = \int_{\mathbb{T}} f_\epsilon(\zeta)\mathrm{d}\mu(\zeta).$$

The result follows by monotone convergence theorem when ϵ goes to zero. □

Remark. Let us note that the $*$-algebra of trigonometric polynomials $\mathcal{F}$ defined by:

$$\mathcal{F} = \{f \in C(\mathcal{B}(0,R)); f(r,\theta) = \sum_{k_1+|k_2|\le N} a_{k_1,k_2} r^{k_1} e^{ik_2\theta}, \quad N \in \mathbb{N}_0,\, k_1 \in \mathbb{N}_0, k_2 \in \mathbb{Z}\}$$

separates points and contains the constants. Its closure under the supremum norm is $C(\mathcal{B}(0,R))$. The weak convergence of the measures is equivalent to have for all f in $\mathcal{F}$,

$$\lim_{L\to\infty} \int_{\mathcal{B}(0,R)} f(\zeta)\mathrm{d}\nu_L(\zeta) = \int_{\mathcal{B}(0,R)} f(\zeta)\mathrm{d}\mu(\zeta).$$

Lemma 9.7. *As a Borel measure on $\mathbb{C}$, the sequence of measures $(\nu^M_{L,z})$ parametrized by z, M converges almost surely weakly to dl as L tends to infinity.*

Proof. Let $(r_{j,z}e^{i\xi_{j,z}})_{j=1}^{2M(4L+1)}$ and $(e^{i\lambda_j})_{j=1}^{2M(4L+1)}$ be the eigenvalues of the problems with reflecting boundary conditions for V and U_D, which correspond respectively to the modified and unmodified "potentials". We have that:

$$\nu^M_{L,z} = \frac{1}{2M(4L+1)} \sum_j \delta_{r_{j,z}e^{i\xi_{j,z}}}$$
$$\mu^M_L = \frac{1}{2M(4L+1)} \sum_j \delta_{e^{i\lambda_j}}.$$

These measures are supported on some $\mathcal{B}(0,R_z)$. We will drop the z subscript in the sequel. Since we already know that (μ^M_L) converges almost surely weakly to dl, we only need to show that for any non-negative integer k_1 and any integer k_2,

$$\lim_{L\to\infty} \frac{1}{2M(4L+1)} \sum_{j=1}^{2M(4L+1)} e^{ik_2\lambda_j} - r_j^{k_1} e^{ik_2\xi_j} = 0.$$

Actually, it is enough to prove it for non-negative integers k_1, k_2. Let us fix such a couple (k_1,k_2) and decompose the term on the left-hand side as follows:

$$\frac{1}{2M(4L+1)} \sum_{j=1}^{2M(4L+1)} e^{ik_2\lambda_j} - r_j^{k_1} e^{ik_2\xi_j} = T_1(L) + T_2(L)$$

$$\text{where} \quad T_1(L) = \frac{1}{2M(4L+1)} \sum_{j=1}^{2M(4L+1)} (r_j^{k_2} - r_j^{k_1})e^{ik_2\xi_j}$$
$$T_2(L) = \frac{1}{2M(4L+1)} \mathrm{Tr}(U_D^{k_2} - V^{k_2}).$$

If $k = \min(k_1, k_2)$ and $l = \max(k_1, k_2)$, we have

$$|T_1(L)| \leq \frac{1}{2M(4L+1)} \sum_{j=1}^{2M(4L+1)} r^k |r_j^{l-k} - 1|$$
$$\leq \frac{R_z^k}{2M(4L+1)} \sum_{j=1}^{2M(4L+1)} |r_j^{l-k} - 1|.$$

Following [13], we first prove that:

$$\lim_{L\to\infty} \frac{1}{2M(4L+1)} \sum_{j=1}^{2M(4L+1)} |r_j - 1| = 0.$$

We know that there exists two orthonormal bases $(\phi_j)_{j=1}^{2M(4L+1)}$ and $(\phi'_j)_{j=1}^{2M(4L+1)}$ such that:

$$V_L^D = \sum_{j=1}^{2M(4L+1)} \mu_j(V_L^D)|\phi'_j\rangle\langle\phi_j|,$$

where $(\mu_j(V_L^D))$ are the singular values of the operator V_L^D. Actually, $\mu_j(V_L^D) = r_j$ and we assume them to be ordered: $\mu_{j+1}(V_L^D) \geq \mu_j(V_L^D) \geq 0$. Note that:

$$\{\mu_j^2(V_L^D); j \in \{1, \ldots, 2M(4L+1)\}\} = \sigma({V_L^D}^* V_L^D)\backslash\{0\}.$$

Since for each $j \in \{1, \ldots, 2M(4L+1)\}$, $\mu_j(U_L^D) = 1$ we deduce from the remark following Theorem 1.20 in [24] that:

$$\sum_{j=1}^{2M(4L+1)} |r_j - 1| = \sum_{j=1}^{2M(4L+1)} |\mu_j(V_L^D) - \mu_j(U_L^D)| \leq \sum_{j=1}^{2M(4L+1)} \mu_j(V_L^D - U_L^D).$$

Since $V_L^D - U_L^D$ has rank and norm uniformly bounded in L, we obtain that:

$$\lim_{L\to\infty} T_1(L) = 0.$$

The term $T_2(L)$ will be treated in a similar way. The operator $U_L^D - V_L^D$ has rank and norm uniformly bounded in L. This implies that for all integer k_2, ${U_L^D}^{k_2} - {V_L^D}^{k_2}$ has also rank and norm uniformly bounded in L. So,

$$\lim_{L\to\infty} T_2(L) = 0,$$

which concludes the proof. □

The above lemmata together with Eq. 18 establish the inequality (17) which implies (16). We finish with the proof of the Thouless formula on $\mathbb{T}$:

Lemma 9.8. *For all* $z \in \mathbb{T}$,

$$\frac{1}{M}\sum_{i}^{M} \lambda_i(z) = \frac{1}{2}\log\frac{1}{rt} + 2\int_{\mathbb{T}} \log|x - z| dl(x)$$

Proof. We note with [12] that $\lim_{L\to\infty} \frac{1}{4L} \log \| \wedge^M (P_{2L})(z)\|$ is subharmonic in $\mathbb{C}\backslash\{0\}$ and $\int_{\mathbb{T}} \log |z-x| \mathrm{d}l(x)$ subharmonic on $\mathbb{C}\backslash\mathbb{T}$. The two exceptional sets are of measure zero in $\mathbb{C}$, these quantities must agree everywhere. □

Remark 9.9. We note that the above proof does not depend on the specific form of the density of states.

10. Appendix 2

Now we prove Theorem 8.1 in several steps.

By Theorem 6.1 the localization length is finite for all values of the parameters. Note that the spectrum is characterized by the existence of generalized eigenfunctions:

Suppose that the support of $E_p(\cdot)$, the spectral resolution of $U(p)$, is the whole circle $\mathbb{T}$.

Proposition 10.1. *For $M \in \mathbb{N}$, $p \in \Omega$ the spectrum of $U(p)$ is the closure of the set*

$$S_p = \{z \in \mathbb{T}; U(p)\phi = z\phi \textit{ has a non-trivial polynomially bounded solution}\}$$

and $E_p(\mathbb{T}\backslash S_p) = 0$.

Proof. The stated behaviour at infinity of the generalized eigenvectors and the spectrum of $U(p)$ are related by Sh'nol's Theorem. This well known deterministic fact for self-adjoint operators was proven in [4] to hold in the unitary setup for band matrices on $l^2(\mathbb{Z})$. It is straightforward to check that the result holds for band matrices on $l^2(\mathbb{Z}, \mathbb{C}^{2M})$, with M finite. □

Secondly we prove the existence of a finite cyclic subspace:

Lemma 10.2. *Let $M \in \mathbb{N}, rt \neq 0$. Denote $I_0 := \{0\} \times \mathbb{Z}_{2M}$. The vectors $\{e_\mu; \mu \in I_0\}$ span a cyclic subspace of $l^2(\mathbb{Z} \times \mathbb{Z}_{2M})$.*

Proof. The only non-vanishing elements in U are the blocks given in Eq. 1. Denoting generically the elements of S by

$$S =: \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

and observing that $U^{-1}_{\mu,\nu} = \overline{U_{\nu,\mu}}$ we have

$$\begin{pmatrix} U_{(2j+1,2k);(2j,2k)} & U_{(2j+1,2k);(2j+1,2k+1)} \\ U_{(2j,2k+1);(2j,2k)} & U_{(2j,2k+1);(2j+1,2k+1)} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$
$$= \begin{pmatrix} U_{(2j+2,2k+2);(2j+2,2k+1)} & U_{(2j+2,2k+2);(2j+1,2k+2)} \\ U_{(2j+1,2k+1);(2j+2,2k+1)} & U_{(2j+1,2k+1);(2j+1,2k+2)} \end{pmatrix}$$

and

$$\begin{pmatrix} U^{-1}_{(2j,2k);(2j+1,2k)} & U^{-1}_{(2j,2k);(2j,2k+1)} \\ U^{-1}_{(2j+1,2k+1);(2j+1,2k)} & U^{-1}_{(2j+1,2k+1);(2j,2k+1)} \end{pmatrix} = \begin{pmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta} \end{pmatrix}$$
$$= \begin{pmatrix} U^{-1}_{2j+2,2k+1;2j+2,2k+2} & U^{-1}_{2j+2,2k+1;2j+1,2k+1} \\ U^{-1}_{2j+1,2k+2;2j+2,2k+2} & U^{-1}_{2j+1,2k+2;2j+1,2k+1} \end{pmatrix}.$$

Computing $Ue_{(0,2k)} = \alpha e_{(1,2k)} + \gamma e_{(0,2k+1)}$ and the corresponding expressions for $U^{-1}e_{(1,2k)}, Ue_{(0,2k+1)}, U^{-1}e_{(-1,2k+1)}$ we infer:

$$\begin{aligned} e_{(1,2k)} &= \frac{1}{\alpha}\left(Ue_{(0,2k)} - \gamma e_{(0,2k+1)}\right) \\ e_{(1,2k+1)} &= \frac{\beta}{\alpha}e_{(0,2k)} - \frac{\gamma}{\overline{\beta}}U^{-1}e_{(0,2k+1)} \\ e_{(-1,2k+1)} &= \frac{1}{\gamma}\left(Ue_{(0,2k+1)} - \alpha e_{(0,2k+2)}\right) \\ e_{(-1,2k+2)} &= \frac{\delta}{\gamma}e_{(0,2k+1)} - \frac{\alpha}{\overline{\delta}}U^{-1}e_{(0,2k+2)}. \end{aligned}$$

Thus vectors with indices in $\{\pm 1\} \times \mathbb{Z}_{2M}$ belong to the subspace generated by $U^{\pm 1}(I_0)$. The lemma follows by induction. □

Let $I = \{0,1\} \times \mathbb{Z}_{2M}$, $\overline{\Omega} = \mathbb{T}^{\mathbb{Z}^{4M}\setminus I}$, $\overline{\mathbb{P}} = \otimes_{k\in\mathbb{Z}^{4M}\setminus I}\, dl$, $\overline{p} = \{\overline{p}_j\}_{j\in\mathbb{Z}^{4M}\setminus I} \in \overline{\Omega}$, and $\Theta_I = \{\theta_j\}_{j\in I}$. We shall use the notation $\Omega \ni p = (\overline{p}, \Theta_I)$.

Denote

$$\lambda_M(p,z) := \lim_{L\to\infty}\left(\frac{1}{4L}\log(\| \wedge^M P_{2L}(z)(p)\|) - \frac{1}{4L}\log(\| \wedge^{M-1} P_{2L}(z)(p)\|)\right),$$

if the limit exists.

By construction, 5.1, it holds for almost every p

$$\lambda_M = \lambda_M(p)$$

By definition $\lambda_M(p,z)$ is independent of the finitely many Θ_I, if $p = (\overline{p}, \Theta_I)$. By Theorem 6.1 there exists $\overline{\Omega}(z) \subset \overline{\Omega}$ with $\overline{\mathbb{P}}(\overline{\Omega}(z)) = 1$ such that for any $z \in \mathbb{T}\setminus\mathbb{R}$

$$\lambda_M((\overline{p}, \Theta_I), z) = \lambda_M > 0,$$

for all $\theta_j \in \Theta_I$ and all $\overline{p} \in \overline{\Omega}(z)$. We can apply Fubini to the measure $\overline{\mathbb{P}} \times dl$ to get the existence of $\overline{\Omega}_0 \in \overline{\Omega}$ with $\overline{\mathbb{P}}(\overline{\Omega}_0) = 1$ such that for every $\overline{p} \in \overline{\Omega}_0$ there is $B_{\overline{p}} \in \mathbb{T}$ with $l(B_{\overline{p}}) = 0$ and

$$\lambda_M((\overline{p}, \Theta_I), z) > 0 \quad \text{for all } \theta_j \in \Theta_I, \text{ and all } z \in {B_{\overline{p}}}^C. \tag{19}$$

Then we show that for $\overline{p} \in \overline{\Omega_0}$, ${B_{\overline{p}}}^C$ is a support of the spectral resolution of $U((\overline{p}, \Theta_I))$ for almost every $\theta_j \in \Theta_I$ w.r.t. $d^{|I|}l$ on $\mathbb{T}^{|I|}$.

For any fixed $j \in I$, we introduce the spectral measures μ_p^j associated with $U(p) = \int_{\mathbb{T}} x \mathrm{d}E_p(x)$ defined for all Borel sets $\Delta \in \mathbb{T}$ by

$$\mu_p^j(\Delta) = \langle e_j | E_p(\Delta) | e_j \rangle.$$

Since $U(p) = D(p)\mathbb{S}$, where $D(p)$ is diagonal, the variation of a random phase at one site is described by a rank one perturbation. More precisely, dropping the variable p temporarily, we define $\widetilde{D}$ by taking $\theta_j = 1$ in the definition of D:

$$\widetilde{D} = D + |e_j\rangle\langle e_j|(1 - \theta_j) = \mathrm{e}^{\log(\overline{\theta}_j)|e_j\rangle\langle e_j|}D,$$

so that, with the obvious notations,

$$\widetilde{U} = \widetilde{D}\mathbb{S} = \mathrm{e}^{\log(\overline{\theta}_j)|e_j\rangle\langle e_j|}U.$$

The unitary version of the spectral averaging formula, see [9] and [3], reads in our case: for any $f \in L^1(\mathbb{T})$,

$$\int_{\mathbb{T}} \mathrm{d}l(\theta_j) \int_{\mathbb{T}} f(x)\mathrm{d}\mu^j_{(\overline{p},\Theta_I)}(x) = \int_{\mathbb{T}} f(x)\mathrm{d}l(x).$$

Applied to $f = \chi_{B_{\overline{p}}}$, the characteristic function of $B_{\overline{p}}$, this yields

$$0 = l(B_{\overline{p}}) = \int_{\mathbb{T}} \mu^j_{(\overline{p},\Theta_I)}(B_{\overline{p}}), \mathrm{d}l(\theta_j). \tag{20}$$

Consequently,

$$\mu^j_{(\overline{p},\Theta_I)}(B_{\overline{p}}) = 0, \quad \text{for every } \theta_k \in \Theta_I,\ k \neq j \text{ and Lebesgue-a.e. } \theta_j.$$

Therefore, for all $\overline{p} \in \overline{\Omega}_0$, there exists $J_{\overline{p}} \subset \mathbb{T}^{|I|}$ s.t. $l({J_{\overline{p}}}^C) = 0$ and

$$\Theta_I \subset J_{\overline{p}} \Rightarrow \mu^j_{(\overline{p},\Theta_I)}(B_{\overline{p}}) = 0, \quad \forall j \in I. \tag{21}$$

Now fix $\overline{p} \in \overline{\Omega}_0$ and $\Theta_I \subset J_{\overline{p}}$ and consider $p = (\overline{p}, \Theta_I)$. By Lemma 10.2 and (21) we deduce that $E_p(B_{\overline{p}}) = 0$. If S_p is the set from Sh'nol's Theorem 10.1, then the set $S_p \cap {B_{\overline{p}}}^C$ is a support for $E_p(\cdot)$.

Now take $z \in S_p \cap {B_{\overline{p}}}^C$. By Theorem 10.1, $U(p)\psi = z\psi$ has a non-trivial polynomially bounded solution ψ. On the other hand, by (19), $\lambda_M(p,z) > 0$. Thus, by Osceledec's Theorem, every solution which is polynomially bounded necessarily has to decay exponentially both at $+\infty$ and $-\infty$, and therefore it is an eigenfunction of $U(p)$. In other words, every $z \in S_p \cap {B_{\overline{p}}}^C$ is an eigenvalue of $U(p)$, hence $S_p \cap {B_{\overline{p}}}^C$ is countable. Therefore $E_p(\cdot)$ has countable support thus $U(p)$ has pure point spectrum. With

$$\Omega_0 := \{(\overline{p}, \{\theta_j\}_{j\in I}) \text{ s.t. } \overline{p} \in \overline{\Omega}_0,\ \{\theta_j\}_{j\in I} \subset \Theta_I \subset J_{\overline{p}}\},$$

we have

$$p \in \Omega_0 \quad \Rightarrow \quad \sigma_c(U(p)) = \emptyset. \tag{22}$$

Also, from $l({J_{\overline{p}}}^C) = 0$ we have

$$(\otimes_{j\in I}\, dl)(J_{\overline{p}}) = (\otimes_{j\in I}\, dl)(\mathbb{T}^{|I|}) = 1. \tag{23}$$

As $\overline{\mathbb{P}}(\overline{\Omega}_0) = 1$, we conclude from (22) and (23) that

$$\mathbb{P}(\sigma_c(U(p)) = \emptyset) \geq \mathbb{P}(\Omega_0) = \int_{\overline{\Omega}_0} d\overline{\mathbb{P}}(\overline{p})(\otimes_{j\in I}\mathrm{d}l)(J_{\overline{p}}) = 1,$$

which proves that $U(p)$ has almost surely pure point spectrum. The fact that the support of the density of state coincides with the almost sure spectrum, see [20], shows that $\Sigma_{pp} = \mathbb{T}$.

We finally show that almost surely all eigenfunctions decay exponentially. Note that we actually have shown above that the event "all eigenvectors of

$U(p)$ decay at the rate of the smallest Lyapunov exponent" has probability one, since this is true for all $p \in \Omega_0$. Measurability of this event was proven for the case of ergodic one-dimensional Schrödinger operators by Kotani and Simon in Theorem A.1 of [22]. The proof of this fact provided in [22] carries over to the CC model as well. It is enough to note that, due to Lemma 10.2, we may use $\rho_p = \sum_{j\in I} \mu_p^j$ as spectral measures in their argument. □

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