Who's Talking First? Consensus or Lack Thereof in Coevolving Opinion Formation Models

Cecilia Nardini,[1,2] Balázs Kozma,[1] and Alain Barrat[1,3]

[1]*LPT, CNRS, UMR 8627, and Univ Paris-Sud, Orsay, F-91405 (France)*

[2]*Universitá di Padova, dipartimento di Fisica "G. Galilei" (Italy)*

[3]*Complex Networks Lagrange Laboratory, ISI Foundation, Turin, Italy*

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We investigate different opinion formation models on adaptive network topologies. Depending on the dynamical process, rewiring can either (i) lead to the elimination of interactions between agents in different states, and accelerate the convergence to a consensus state or break the network in noninteracting groups or (ii), counterintuitively, favor the existence of diverse interacting groups for exponentially long times. The mean-field analysis allows us to elucidate the mechanisms at play. Strikingly, allowing the interacting agents to bear more than one opinion at the same time drastically changes the model's behavior and leads to fast consensus.

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In recent years, agent based models have been used more and more in the area of social sciences. Through a rather simple modeling approach for the individual processes of social influence, these models focus on the emergence of social behavior at the global population level. Statistical physics models and tools provide therefore a natural framework for such studies, and have been widely applied, leading to the appearance of the field called sociophysics (see [1] for a recent review on the application of statistical physics models to social dynamics).

The growing field of complex networks [2–4] has allowed us to obtain a better knowledge of social networks [5,6], and, in particular, to show that their typical topology is not regular. Many studies have since considered the evolution of models of interacting agents embedded on more realistic networks, and studied the influence of complex topologies on their dynamical behavior [7]. An additional feature of networks, that may have a strong impact on the model's behavior, lies in their dynamical nature on various time scales. The evolution of the topology and the dynamical processes can then drive each other with complex feedback effects. Studies of this coevolution are more recent and still limited [8–16].

In this Letter, we provide new insights into such feedback effects by an investigation of Voter-like models (VM), in which agents update their opinions by imitating their neighbors, and can also break and establish connections with other agents. More precisely, we show how apparently slight modifications in the evolution rule, which have minor consequences if the topology of interactions is kept fixed, can change drastically the model's behavior as soon as the topology can evolve on the same time scale as the agents' opinions. On the other hand, the simple fact of allowing agents to have several opinions at the same time, in the spirit of the Naming Game [17] or of the AB model [18], leads to more robust behavior.

The Voter model [19] considers a population of N individuals, each carrying an opinion $s = \pm 1$; only two opposite opinions are allowed here (for example a political choice between two parties) [20]. Starting from a random configuration of opinions, the dynamical evolution of the *direct* VM (d VM) is the following: at each elementary step, an agent (i) is randomly selected, chooses one of its neighbors (j) at random and adopts its opinion; i.e., s_i is set equal to s_j (one time step consists of N such updates). In the *reverse* case (r VM), the first agent i instead convinces its neighbor j (s_j is set equal to s_i). The distinction between d and r VM is necessary since the two interacting nodes do not play the same role. Moreover, the degrees of the first and the second chosen nodes have different distributions, and the second is a large-degree node with larger probability [2]. The asymmetry in the opinion update between the two interacting nodes can then couple to the asymmetry between a randomly chosen node and its randomly chosen neighbor, leading to different dynamical properties. No important difference is expected on homogeneous networks but, on heterogeneous networks, the probability for a hub to update its state will vary strongly from one rule to the other. The basic imitation process of the VM mimics the homogenization of opinions but, since interactions are binary and random, do not guarantee the convergence to a uniform state. Since a consensus in which all individuals share the same opinion is an absorbing state of the dynamics, any finite population reaches a consensus, but the time needed $t_c(N)$ depends on its size N and on the topology of interactions, and diverges as $N \to \infty$. On static networks, $t_c(N)$ grows as a power-law of N, with an exponent depending on the degree distribution, and on the updating rule [21–23]. On homogeneous networks, in particular, $t_c(N) \propto N$ for both d and r VM.

In this Letter, we consider the scenario in which agents can rewire their "unsatisfied" connections. More precisely, the initial configuration is given by a random homogeneous network of interacting agents, with average number of neighbors $\langle k \rangle$ and random opinions. At each time step, an agent i and one of its neighbors j are chosen. With probability Φ, an attempt to rewire the link is made, if $s_i \neq s_j$. A new agent k is then chosen at random and the link (i, j) is rewired to (i, k) [24]. With probability $1 - \Phi$, an opinion update takes place instead. The rewiring, which conserves the total number of links, is made at random: the new link is established without prior knowledge of the new neighbor's opinion [25].

If the frequency of rewirings is small ($\Phi \to 0$), the system still reaches a global consensus and the network remains connected. For fast rewiring rates on the other hand, ($\Phi \to 1$) the network breaks into (typically two for the VM) separate connected components, each one with a local consensus. These two regimes are separated by a nonequilibrium phase transition at a critical value Φ_c of the rewiring probability. Similar transitions have already been reported in coevolving models of opinion formation [12,15,16] and we will not focus on this aspect here.

A more surprising aspect of the dynamics is revealed by the behavior of the convergence time $t_c(N)$, which grows linearly with N on a static network for both the d and the r VM [22,23]. Strikingly, the network's adaptivity has completely opposed effects in these models (Fig. 1). Consensus is strongly favored in the d VM, for which $t_c(N)$ becomes $\propto \ln N$ [26]; in contrast, for the r VM $t_c(N)$ grows exponentially with the system size. The system therefore remains for exponentially long times in a state in which two groups of different opinions coexist and remain connected to each other. It is noteworthy that the system therefore is not frozen, with agents continuously updating their links and opinions.

In order to understand the different behavior of the d and r VM on adaptive networks, we note that the state of the system is characterized by three independent quantities: (i) the density n_+ of agents with opinion $+1$, or equivalently the magnetization $m = n_+ - n_-$ ($n_- = 1 - n_+$ is the density of agents with opinion -1); (ii) the number of links joining agents in the $+$ opinion, Nl_{++}; and (iii) the number of links joining agents of opposite opinions, i.e., of *active* links $Nl_{-+} = Nl_{+-}$ (since the total number of links is preserved, $\langle k\rangle/2 = l_{++} + l_{+-} + l_{--}$). At the mean-field (MF) level, we can derive the evolution equation of these quantities. Let us first consider the magnetization: it changes of $-2/N$ when an agent changes its states from $+$ to $-$, and of $+2/N$ in the opposite case. For the d VM, the probability of the first event is proportional to the density n_+ of agents in the $+$ state, times the probability that it chooses to interact with a neighbor that has $-$ opinion, i.e. k_{+-}/k_+ where k_+ is the average degree of a $+$ node, and $k_{+-} = l_{+-}/n_+$ is the average number of $-$ neighbors of a $+$ node. The probability of the second event ($- \to +$) is obtained in the same way, and, finally,

$$\left\langle \frac{dm}{dt} \right\rangle_{d\mathrm{VM}} = -\frac{2(1-\Phi)}{N} l_{+-} \left(\frac{1}{k_+} - \frac{1}{k_-} \right). \qquad (1)$$

In the case of the r VM, the probabilities of the two processes are simply interchanged: $\langle dm/dt\rangle_{r\mathrm{VM}} = -\langle dm/dt\rangle_{d\mathrm{VM}}$. On an adaptive network, it is essential to distinguish k_+ from k_-: as shown in Fig. 2, one has indeed $k_+ > \langle k\rangle > k_-$ if $n_+ > n_-$. In other words, the nodes of the majority opinion have more neighbors. This is a simple consequence of the rewiring dynamics: if $m > 0$, any rewiring event $(i, j) \to (i, k)$ has a higher chance to randomly pick a $+$ node as a new neighbor due to their larger number. Therefore, nodes of the larger group gain new links with larger probability. Equation (1) then immediately shows that for $m > 0$, $\langle dm/dt\rangle_{d\mathrm{VM}} > 0$ and $\langle dm/dt\rangle_{r\mathrm{VM}} < 0$. In summary, the coevolution of opinions and topology generates a positive feedback for the d VM driving the system to a consensus state, $m_{\mathrm{stable}} = \pm 1$, and a negative feedback for the r VM resulting in $m_{\mathrm{stable}} = 0$. This readily explains the strong differences between these

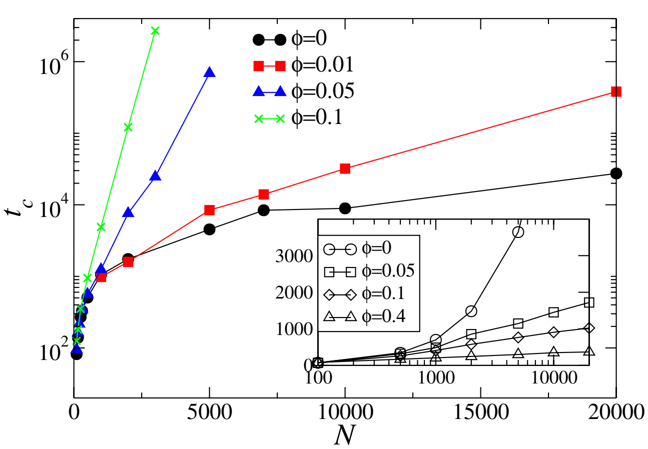

FIG. 1 (color online). Convergence time for the r VM vs N, for various rewiring probabilities. In all figures, the simulations are averaged over 100 realizations. Inset: d VM.

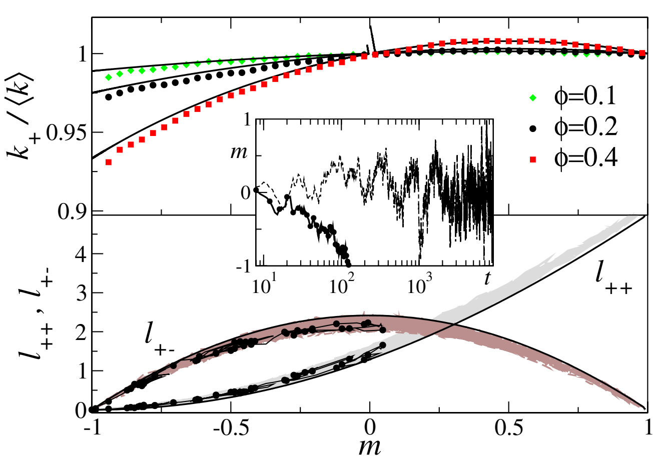

FIG. 2 (color online). Top: $k_+/\langle k\rangle$ vs m for the VM. Symbols: averages obtained from numerical simulations with $N = 1000$, $\langle k\rangle = 10$. Continuous lines: numerical solution of the MF equations for the evolution of $\mathbf{x} = (m, l_{+-}, l_{++})$, starting from initial conditions with m close to 0. Bottom: l_{++} and l_{+-} vs m. The continuous black lines correspond to the numerical solution of the MF equations. The black symbols and the gray and brown lines correspond to single runs of the d and r VM, respectively, ($N = 500$, $\langle k\rangle = 10$, $\Phi = 0.4$). The inset shows the evolution of m for the same runs (symbols for d VM and dashed line for r VM).

models. For the d VM the adaptivity leads to an accelerated consensus, while it hinders the convergence for the r VM and keeps the system in a dynamically evolving state with zero average magnetization.

It is moreover possible to write the evolution equations for the various types of links. It is easy to understand that, according to the model's definition, the vector $\mathbf{x} = (m, l_{+-}, l_{++})$ can evolve in 4 ways at each elementary update: $\mathbf{x} \to \mathbf{x} + \mathbf{v}^a$, $a = 1, \cdots, 4$, with respective probabilities w^a. Let us start with the d VM. The displacement vectors and the associated probabilities read then: $N\mathbf{v}^1 = (2, k_{--} - k_{-+}, k_{-+})$, $w_1 = (1 - \Phi)n_- k_{-+}/k_-$; $N\mathbf{v}^2 = (-2, k_{++} - k_{+-}, -k_{++})$, $w_2 = (1 - \Phi)n_+ k_{+-}/k_+$; $N\mathbf{v}^3 = (0, -1, 0)$, $w_3 = \Phi n_-^2 k_{-+}/k_-$; $N\mathbf{v}^4 = (0, -1, +1)$, $w_4 = \Phi n_+^2 k_{+-}/k_+$. $\mathbf{v}^1$ and $\mathbf{v}^2$ correspond to opinion changes, for which the change in magnetization ($\pm 2/N$) is associated with changes in the densities of links. For example, when a $-$ node is transformed to $+$, its $--$ links become $+-$ and its $+-$ links become $++$ ones (hence l_{+-} varies of $(k_{--} - k_{+-})/N$). The corresponding probabilities w_1 and w_2 are obtained as for Eq. (1). $\mathbf{v}^3$ and $\mathbf{v}^4$ correspond to rewiring events: when a $+-$ link is rewired it can be either transformed to $--$ ($\mathbf{v}^3$) or to $++$ ($\mathbf{v}^4$). For the r VM, the displacement vectors are exactly the same as for the d VM, but the transition probabilities w_1 and w_2 are interchanged. w_3 and w_4 remain the same since the rewiring rules are the same for both models. Figure 2 shows the result of the numerical integration of the evolution equations $d\mathbf{x}/dt = \sum_a \mathbf{v}^a w^a$, compared with numerical simulations of the models. It is clear that these equations correctly account for the difference between k_+ and k_- and for the system's evolution in the phase space. Of course, the real systems are moreover submitted to fluctuations that are not taken into account in the MF description. In particular, looking at single runs (Fig. 2) shows clearly the difference between the d and r VM. For the d VM, the density of active links decreases rapidly to 0 and the system is driven to one of the consensus states. For the r VM on the contrary, the system performs a random walk in a sort of potential well around $m = 0$ with a nonzero density of active links, which ends only because of a finite-size fluctuation which leads it into one of the absorbing boundaries at $m = \pm 1$.

Let us now consider that agents cannot pass directly from one opinion to another, but can keep both opinions in their "memory", being then in an intermediate state that we call 0. This is the case in the Naming Game (NG) model, in which agents try to agree on the name to assign to a given object [17], or also of the AB model [18]. If only two names are available (that we can call $+$ and $-$ for simplicity), the dynamical rules of the *direct* NG (d NG) are the following: at each time step, an agent i and one of its neighbors, j, are chosen at random to be, respectively, the hearer (H) and Speaker (S). S proposes a name to H. If S has both names in memory, it chooses one at random. Let us suppose for instance that S proposes $+$. If H does not know the name uttered (i.e., it is in state $-$), it absorbs this possibility by changing to the intermediate state, 0. If H instead has the name in memory (i.e., it is in state $+$ or 0), the interaction is successful and both H and S agree on this particular name and set in state $+$ after the interaction. In the reverse case (r NG), the first randomly selected agent is S and its neighbor is H, and the update rules remain the same [27]. When agents interact on a static topology, these dynamical rules lead to a global consensus. On homogeneous networks, we obtain $t_c(N) \sim \ln N$ (while $t_c(N) \sim N$ for the VM). The difference between the two models is due to the fact that in the VM, consensus is reached by a finite-size fluctuation of the average magnetization while in the NG, consensus is reached due to the surface-tension introduced by the 0 states, which tends to minimize the interface between the agents of different opinions and hence drive the system to a homogeneous consensus state [18,28]. For adaptive networks, Fig. 3 clearly shows that the convergence time remains logarithmic for both the direct and reverse version, even if the r NG is slower. The MF analysis allows to understand this strong difference with the VM. We can indeed write the evolution equation for the magnetization $n_+ - n_-$, by introducing the average degree of 0 nodes k_0 and the density of $+0$ and -0 links, as

$$\left\langle \frac{dm}{dt} \right\rangle_{\mathrm{dNG}} = \frac{1}{2} \left\langle \frac{dm}{dt} \right\rangle_{\mathrm{dVM}} + \frac{1 - \Phi}{N} \left(\frac{l_{+0}}{k_0} - \frac{l_{-0}}{k_0} \right) \quad (2)$$

$$\left\langle \frac{dm}{dt} \right\rangle_{\mathrm{rNG}} = \frac{1}{2} \left\langle \frac{dm}{dt} \right\rangle_{\mathrm{rVM}} + \frac{1 - \Phi}{N} \left(\frac{l_{+0}}{k_+} - \frac{l_{-0}}{k_-} \right). \quad (3)$$

The first terms on the right-hand side represent the change in the magnetization mediated by the l_{+-} links. The factor $1/2$ stems from the fact that $+$ and $-$ nodes are not transformed instantly to their opposite counterpart but to the intermediate state 0. The remaining terms correspond to the transformation of the 0 nodes to $\pm$ ones. For example, in the d NG, the second term on the right-hand side is generated by the process when a 0 node is converted to $+$ by first picking a 0 node, with probability n_0, then one of its

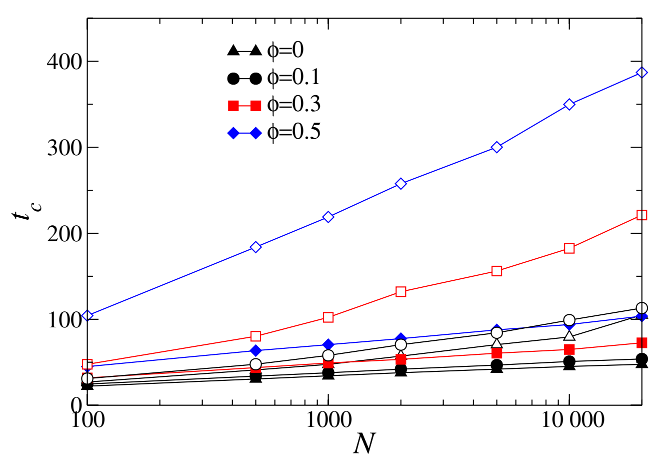

FIG. 3 (color online). Convergence time for the direct (filled symbols) and reverse (open symbols) NG.

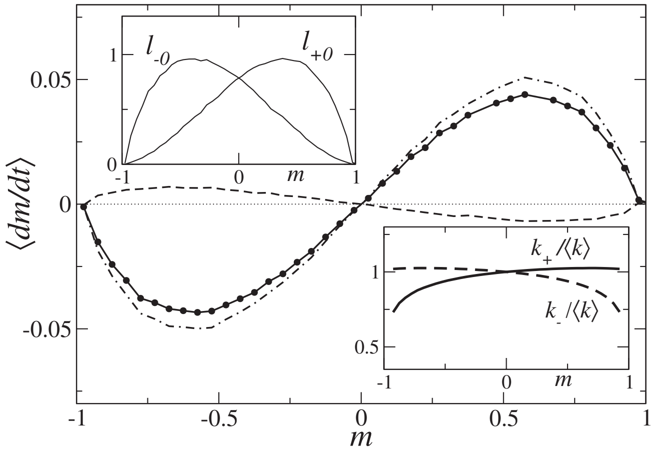

FIG. 4. $\langle dm/dt \rangle$ vs m (symbols) for the r NG. According to Eq. (3), changes in the magnetization come from r VM-like interactions (dashed line) and those mediated by the 0-links, l_{+0} and l_{-0} (dash-dotted line). The upper inset gives l_{+0} and l_{-0}, the other $k_+/\langle k \rangle$, $k_-/\langle k \rangle$. $\Phi = 0.2$, $\langle k \rangle = 10$, $N = 10^4$.

+ neighbors, with probability k_{0+}/k_0. Even though the first terms in Eqs. (2) and (3) change sign for the d and the r variants of the NG just as for the VM, this effect is suppressed by the terms associated with the transitions ($0 \rightarrow \pm$) which will always generate a positive feedback to the change of magnetization. As shown in Fig. 4 indeed, $l_{+0} - l_{-0}$ is of the sign of m, which is expected since then $n_+ > n_-$. This effect overcomes the difference between k_+ and k_-, and $\langle dm/dt \rangle$ remains of m's sign even in the r NG, leading to logarithmic convergence times. Thus, the possibility for agents to remain in an intermediate state before updating their opinion strongly enhances the trend towards consensus.

In summary, we have shown how modifications of the interaction rules, which could seem minor at first sight, can in fact have drastic effects on the behavior of opinion formation models in the case of dynamically evolving networks, due to the coupling of the asymmetry between the interacting agents to the asymmetry in their degrees. Such coupling is known to change the scaling of the convergence time in heterogeneous static networks [22,23], but is not crucial in homogeneous networks. In strong contrast, and even if the adaptive network remains globally homogeneous, with no divergence of the moments of the degree distribution, the fact that the majority has a slightly larger average degree suffices to change from a very fast convergence in logarithmic time for the d VM to a dynamical state surviving for exponentially long times for the r VM. Interestingly, if the agents cannot change opinion so easily, and have to go through an intermediate state, such as in the NG or AB models, convergence to consensus is enhanced also for adaptive networks, and irrespective of the order of interactions (d NG vs r NG). The connections with nodes in the intermediate state determine then the dominant evolution of the magnetization, leading to a more robust behavior.

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