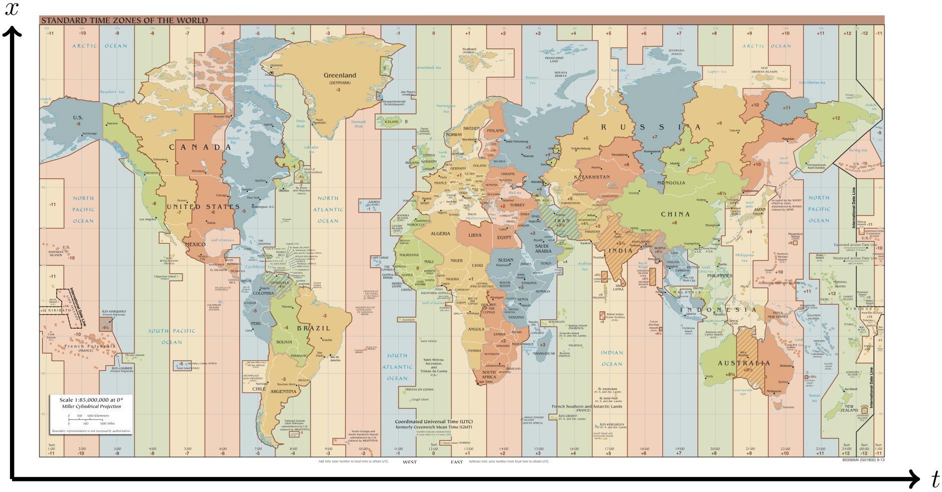
Adapted coordinates for light signals in cosmology

Fabien Nugier
法比恩

LEUNG CENTER FOR COSMOLOGY AND PARTICLE ASTROPHYSICS
National Taiwan University

Large Scale Structure & Galaxy Flows
XII$^{\text{th}}$ Rencontres du Vietnam, ICISE
8 July 2016 – Quy Nhon, Vietnam

Adapted coordinates ?



A bit of history

1938 : Temple's "optical co-ordinates"

> The use of optical co-ordinates considerably simplifies general optical theory in relativistic form. Fermat's Principle can be rigorously established and the treatment of the various astronomical determinations of distance
>
> can be put in a simple and concise form. Some previous discussions of these topics have been open to question in view of their unjustified use of Riemannian normal co-ordinates on the null cone of the base point. The general results obtained are applied to the case of greatest practical interest—the isotropic expanding universe.

1958 : Joseph's "optical co-ordinates"

> The result established here is exact and invariant. It has been found convenient to use the physically significant system of coordinates known as optical coordinates (9) but the results are independent of this choice. The
>
> Since $l^a l^a = 1$ by (2.4), the equations (2.3) define a transformation of coordinates from the system x^μ to "optical coordinates" (7) $\lambda, \tau, \alpha, \beta$ (α, β being any two independent functions of l^a) in a domain D of space-time where the forward null geodesics through points on C do not intersect one another. Let the

It should be realized that the optical coordinates λ, τ, α, β are observable quantities with direct physical significance in the sense that an operational procedure can be specified for measuring them. They are constructed in an invariant manner depending only on the choice of world line C, and not on any pre-existing coordinate system in space-time. τ, α, β can be realized by the single observer

a geodesic in Minkowskian space-time. With the world line C of the star as base line, the space-time metric can be written in optical coordinates as

$$ds^2 = d\tau^2 + 2\, d\tau\, d\lambda - \lambda^2(d\alpha^2 + \sin^2\alpha\, d\beta^2). \tag{4.1}$$

1968 : Saunders "observer's polar coordinates"

In the cosmic co-ordinate system (1), unfortunately, the constants of equation (3) will in general vary from geodesic to geodesic within the pencil, and this makes the calculation of the partial derivatives $\partial k^\alpha / \partial x^\alpha$ difficult. It is therefore convenient to introduce a second co-ordinate system. Let P be an observer moving along some

normal triad. This co-ordinate system will be referred to as 'observer's polar co-ordinates'; except for a change in parametrization they are the same as the 'optical co-ordinates' of Joseph (1958). Note that in observer's polar co-ordinates Q_2 lies on a null geodesic Γ with $\bar{x}^\alpha$ = const.

null geodesics. The cosmic co-ordinates are Eulerian co-ordinates for this fluid, while the observer's polar co-ordinates are the Lagrangian co-ordinates. It is well

1984 : Maartens detailed "observational coordinates"

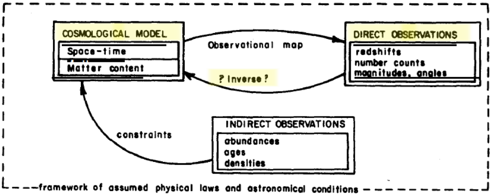
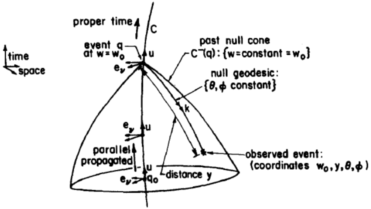


Fig. 2. The aim of observational cosmology: to determine the space–time and its matter content, from the astronomical observations (using indirect observations to confirm the model thus arrived at). This is the inverse of the usual procedure, where cosmological models are used to predict observational relations, which are then compared with actual observations.

In order to carry out a systematic investigation, it is convenient to introduce observational coordinates (cf. Temple [26], Kristian and Sachs [1]). We base these coordinates on the world line C (representing the history of our galaxy). We assume this is a geodesic at the time when the observations are made (when we may assume $p \simeq 0$). The coordinates $\{x^i\} = \{w, y, \theta, \phi\}$ are defined in the following



See also Bester, Larena, Bishop '13 & '15 for numerical applications.

The Geodesic Light-Cone (GLC) coordinates :

$$ds^2_{\rm GLC} = \Upsilon^2 dw^2 - 2\Upsilon dw d\tau + \gamma_{ab}(d\theta^a - U^a dw)(d\theta^b - U^b dw)$$

(6 arbitrary functions : Υ, U^a, γ_{ab})

Properties :

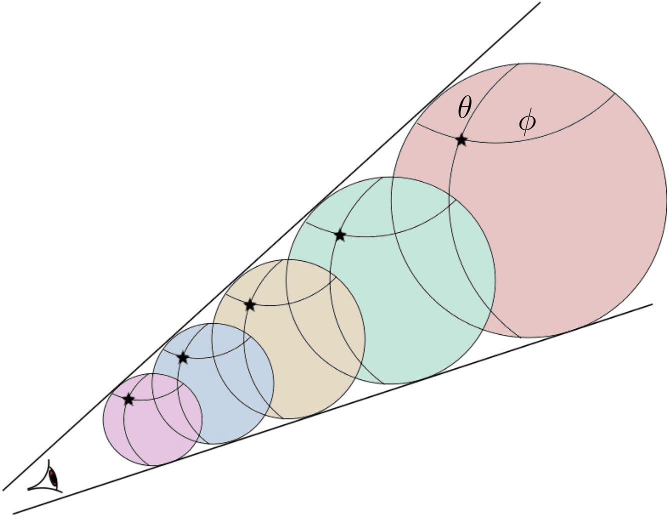
- w is a null coordinate : $\partial_\mu w \, \partial^\mu w = 0$,
- $\partial_\mu \tau$ defines a geodesic flow : $(\partial^\nu \tau) \nabla_\nu (\partial_\mu \tau) \equiv 0$ (from $g^{\tau\tau}_{GLC} = -1$) ,
- photons travel at $(w, \theta^a) = \overrightarrow{cst}$ and their path is orthogonal to $\Sigma(w, z)$.

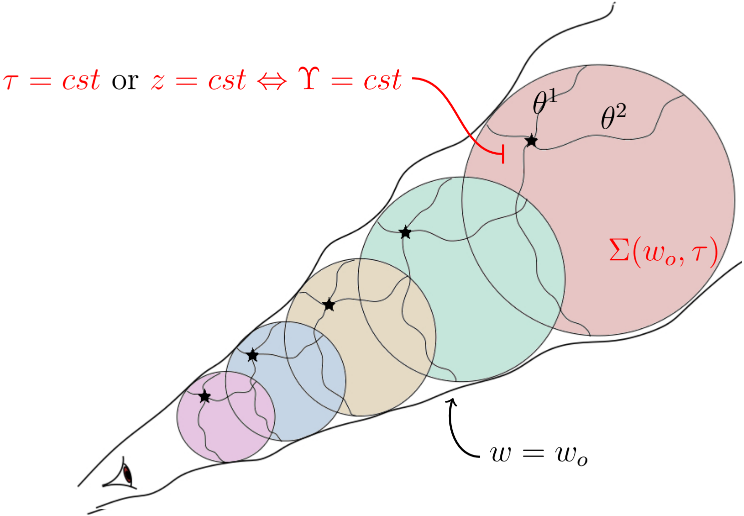
Interpretation :

Υ is like an **inhomogeneous scale factor** (lapse function), U^a is a **shift-vector** and γ_{ab} the **metric inside the 2-sphere** $\Sigma(\tau, w)$.

FLRW : $w = \eta + r$, $\tau = t$, $(\theta^1, \theta^2) = (\theta, \phi)$,

$\Upsilon = a(t)$, $U^a = 0$, $\gamma_{ab} = a^2 r^2 \text{diag}(1, \sin^2\theta)$.


θ
ϕ


$\tau = cst$ or $z = cst \Leftrightarrow \Upsilon = cst$
θ^1
θ^2
$\Sigma(w_o, \tau)$
$w = w_o$

Direct simplifications

$$ds^2_{GLC} = \Upsilon^2 dw^2 - 2\Upsilon dw d\tau + \gamma_{ab}(d\theta^a - U^a dw)(d\theta^b - U^b dw)$$

$\Rightarrow$ **Redshift perturbation** :

$$(1+z_s) = \frac{(k^\mu u_\mu)_s}{(k^\mu u_\mu)_o} = \frac{(\partial^\mu w \partial_\mu \tau)_s}{(\partial^\mu w \partial_\mu \tau)_o} = \frac{\Upsilon(w_o, \tau_o, \theta^a)}{\Upsilon(w_o, \tau_s, \theta^a)} \equiv \frac{\Upsilon_o}{\Upsilon_s}$$

where $u_\mu = -\partial_\mu \tau$ is the peculiar velocity of the **comoving** observer/source and $k_\mu = \partial_\mu w$ is the photon momentum. Cf. 1202.1247.

$\Rightarrow$ **(exact) Angular distance** (with homogeneous observer neighborhood) :

$$d_A = \gamma^{1/4} \left(\sin\theta^1\right)^{-1/2} \quad \text{with} \quad \gamma \equiv \det(\gamma_{ab}) = |\det(g_{\mathrm{GLC}})|/\Upsilon^2$$

which, combined with the redshift, gives the **distance-redshift relation**.

Hubble diagram

Magnitude :

$$m = -2.5\ \log_{10}\left(\frac{\Phi}{\Phi_{\text{ref}}}\right)$$

Flux :

$$\boxed{\Phi = \frac{L}{4\pi d_L^2}}$$

Absolute Mag. :

$$M = -2.5\ \log_{10}\left(\frac{\Phi(10\text{pc})}{\Phi_{\text{ref(pc)}}}\right)$$

Distance Modulus :

$$\boxed{\mu = m - M = 5\log_{10}(d_L) + cst}$$

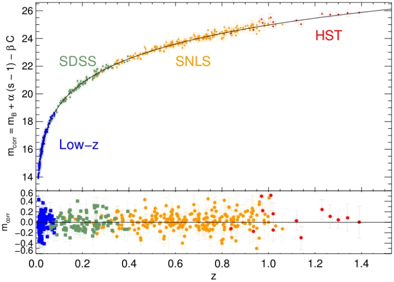
Two assumptions in SMC :

- GR valid on all scales,
- Isotropy + Homogeneity,

$\Rightarrow$ FLRW model.

Luminosity Distance (for $K = 0$) :

$$d_L^{FLRW}(z) = \frac{1+z}{H_0}\int_0^z \frac{\mathrm{d}z'}{\left[\Omega_{\Lambda 0} + \Omega_{m0}(1+z')^3\right]^{1/2}}$$



Distance-redshift relation at $\mathcal{O}(2)$

The GLC metric allows to compute the $d_L(z)$ relation to $\mathcal{O}(2)$ in NG :

$$ds^2_{NG} = a^2(\eta)\left(-(1+2\Phi)\mathrm{d}\eta^2 + (1-2\Psi)(\mathrm{d}r^2 + \gamma^{(0)}_{ab}\mathrm{d}\theta^a\mathrm{d}\theta^b)\right)$$

with $\gamma^{(0)}_{ab} = r^2 \mathrm{diag}\left(1, \sin^2\theta\right)$, and $\Phi = \psi + \frac{1}{2}\phi^{(2)}$, $\Psi = \psi + \frac{1}{2}\psi^{(2)}$ (Bardeen).

$\psi^{(2)}, \phi^{(2)} \propto \nabla^{-2}(\partial_i\psi\partial^i\psi)$, $\partial_i\psi\partial^i\psi$ (cf. Bartolo, Matarrese, Riotto, 2005)

Distance-redshift relation at $\mathcal{O}(2)$

The GLC metric allows to compute the $d_L(z)$ relation to $\mathcal{O}(2)$ in NG :

$$ds^2_{NG} = a^2(\eta)\left(-(1+2\Phi)\mathrm{d}\eta^2 + (1-2\Psi)(\mathrm{d}r^2 + \gamma^{(0)}_{ab}\mathrm{d}\theta^a\mathrm{d}\theta^b)\right)$$

with $\gamma^{(0)}_{ab} = r^2 \mathrm{diag}\left(1, \sin^2\theta\right)$, and $\Phi = \psi + \frac{1}{2}\phi^{(2)}$, $\Psi = \psi + \frac{1}{2}\psi^{(2)}$ (Bardeen).

$$\psi^{(2)}, \phi^{(2)} \propto \nabla^{-2}(\partial_i\psi\partial^i\psi)\ ,\ \partial_i\psi\partial^i\psi$$ (cf. Bartolo, Matarrese, Riotto, 2005)

FULL transformation GLC $\leftrightarrow$ NG at second order in PT :

$$(\tau, w, \widetilde{\theta}^1, \widetilde{\theta}^2) = f(\eta, r, \theta, \phi)$$
$$\Downarrow$$
$$(\Upsilon, U^a, \gamma^{ab}) = f(\psi, \psi^{(2)}, \phi^{(2)})$$
$$\Downarrow$$
$$d_L = (1+z)^2\,\gamma^{1/4}\left(\sin\tilde{\theta}^1\right)^{-1/2}$$ up to $\mathcal{O}(2)$:

$$\boxed{d_L(z_s, \theta^a) = d_L^{FLRW}(z_s)\left(1 + \delta^{(1)}_S(z_s, \theta^a) + \delta^{(2)}_S(z_s, \theta^a)\right)}$$

At $\mathcal{O}(1)$:

$$\delta_S^{(1)}(z_s,\theta^a) \sim \text{SW} + \text{ISW} + \text{Doppler} - \left(\psi_s^{(1)} + \int_{\eta_+}^{\eta_-} \mathrm{d}x\ \psi\right) - \text{Lensing}^{(1)}$$

$$\text{Lensing}^{(1)} = \frac{1}{2}\nabla_a\theta^{a(1)} = \int_{\eta_s^{(0)}}^{\eta_o} \frac{d\eta}{\Delta\eta}\ \frac{\eta-\eta_s^{(0)}}{\eta_o-\eta}\Delta_2\psi(\eta,\eta_o-\eta,\bar{\theta}^a)$$

$$\text{Doppler} = \left(1-\frac{1}{\mathcal{H}_s\Delta\eta}\right)(\mathbf{v}_o-\mathbf{v}_s)\cdot\hat{n} \quad , \quad \mathbf{v} \equiv \int_{\eta_{\text{in}}}^{\eta} \mathrm{d}\eta'\frac{a(\eta')}{a(\eta)}\nabla\psi(\eta',r,\theta^a)$$

At $\mathcal{O}(2)$, full calculation :

- **Dominant terms** : (Doppler)^2, (Lensing)^2 !!!
- Combinations of $\mathcal{O}(1)$-terms : ψ_s^2, ([I]SW)^2, [I]SW $\times$ Doppler, $(\psi_s, \int_{\eta_+}^{\eta_-} dx\ \psi)\times$ (Lensing, [I]SW, Doppler) ...
- Genuine $\mathcal{O}(2)$-terms : $\psi_s^{(2)}$, Lensing$^{(2)} = \frac{1}{2}\nabla_a\theta^{a(2)}$, $Q_s^{(2)}$...
- A LOT of other contributions : New integrated effects, Angle deformations, Redshift perturbations($\subset$ transverse peculiar velocity), Lens-Lens coupling, corrections to Born approximation, ... *See 1209.4326, also Umeh 1402.1933*.

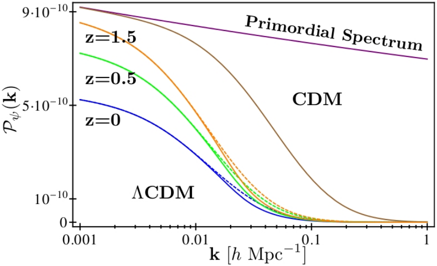
Stochastic average of inhomogeneous realizations

Inhomogeneities :

$$\psi(\eta,\vec{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\, e^{i\vec{k}\cdot\vec{x}}\psi_k(\eta)E(\vec{k})$$

with E a unit R.V. which is **homogeneous**$\left(E^*(\vec{k}) = E(-\vec{k})\right)$ and **gaussian**$\left(\overline{E(\vec{k})} = 0,\ \overline{E(\vec{k}_1)E(\vec{k}_2)} = \delta(\vec{k}_1+\vec{k}_2)\right)$.

Spectrum : $|\psi_k(\eta)|^2 = 2\pi^2\mathcal{P}_\psi(k)/k^3$



Light-cone average is combined with a stochastic average. In CDM :

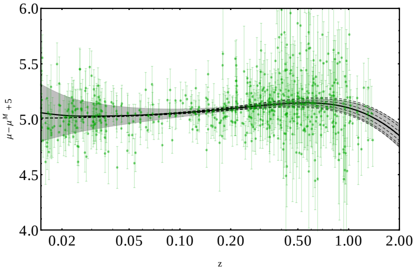
$$\overline{\langle d_L\rangle} = \int_0^\infty \frac{\mathrm{d}k}{k}\mathcal{P}_\psi(k)C(k\Delta\eta)$$

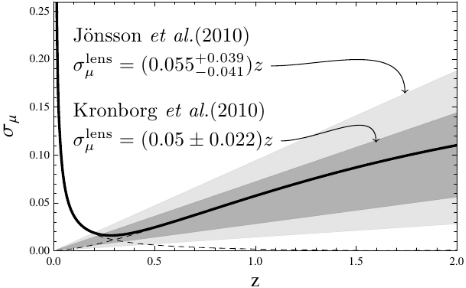
We do the same $\forall$ terms in $\overline{\left\langle \delta_S^{(1)}\right\rangle}$ and $\overline{\left\langle \delta_S^{(2)}\right\rangle}$ in ΛCDM... with approximations.

Kaiser & Peacock 2015 for precise discussion on 'directional' / 'source' averaging.

The averaged modulus $\overline{\langle\mu\rangle}$ depends on $\overline{\langle(\Phi_1/\Phi_0)^2\rangle}$ while the standard deviation $\sigma_\mu = \sqrt{\overline{\langle\mu^2\rangle} - \overline{\langle\mu\rangle}^2} = 2.5(\log_{10} e)\sqrt{\overline{\langle(\Phi_1/\Phi_0)^2\rangle}}$ with

$$\overline{\langle(\Phi_1/\Phi_0)^2\rangle} \sim \overline{\langle(\text{Doppler})^2\rangle} + \overline{\left\langle\left(\text{Lensing}^{(1)}\right)^2\right\rangle}$$





With the Union 2 dataset :

- small z : Velocities explain quite well the scatter.
- large z : Lensing is too weak to explain data's scatter ($\sim\%\,\Omega_{\Lambda 0}$).

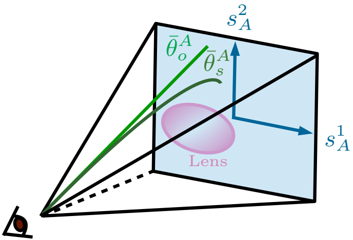
- The total effect is well approximated by Doppler ($z \leq 0.2$) + Lensing ($z > 0.3$),
- Lensing prediction is in great agreement with experiments so far !

Cf. Ben-Dayan 2014, 1401.7973 for effect on H_o.

Lensing quantities in GLC coordinates

Amplification matrix $((\ldots)^{\bullet} \equiv \partial_\tau(\ldots))$:

$$\mathcal{A}^A_B(\lambda_s,\lambda_o) = \frac{s^A_a(\lambda_s)\,\left[2u_\tau(\dot{\gamma}_{ab})^{-1}\right]_o s^B_b(\lambda_o)}{\bar{d}_A(\lambda_s)} = \begin{pmatrix} 1-\kappa-\hat{\gamma}_1 & -\hat{\gamma}_2+\hat{\omega} \\ -\hat{\gamma}_2-\hat{\omega} & 1-\kappa+\hat{\gamma}_1 \end{pmatrix}$$



The **angular distance** and **lensing quantities** become :

$$d_A \propto (\gamma\gamma_o)^{1/4} \quad , \quad \hat{\mu} = (\det\mathcal{A})^{-1} = \left(\frac{\bar{d}_A}{d_A}\right)^2 \quad ,$$

involving $\bar{d}_A = a(\tau)r$ with $r = w - \int a^{-1}(\tau)\mathrm{d}\tau$ measured from the observer,

$$\left\{ \begin{array}{c} (1-\kappa)^2+\hat{\omega}^2 \\ \hat{\gamma}_1^2+\hat{\gamma}_2^2 \end{array} \right\} = \left(\frac{u_{\tau_o}}{\bar{d}_A}\right)^2 \left\{ \left[\frac{\gamma\,\dot{\gamma}_{ab}\gamma^{bc}\dot{\gamma}_{cd}}{\left(\det^{ab}\dot{\gamma}_{ab}\right)^2}\right]_o \gamma\,\gamma^{ad} \pm 2\frac{\sqrt{\gamma\,\gamma_o}}{\left(\det^{ab}\dot{\gamma}_{ab}\right)_o} \right\}$$

Examples of applications :

Study of the **inhomogeneous** Lemaître-Tolman-Bondi spacetime (LTB) with off-center observer (*Fanizza, Nugier 2014*, 1408.1604) :

- non-perturbative d_A, κ, $|\hat{\gamma}|$, $\hat{\mu}$ in terms of LTB coordinates (t, r, θ, ϕ),
- numerical resolution of $t(z)$, $r(z)$ with ansatz for $H_o(r)$ $\Rightarrow$ plots of lensing quantities wrt observation angle from the bubble center.

Examples of applications :

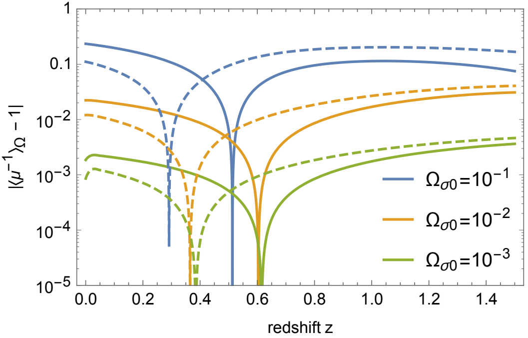
Study of the **inhomogeneous** Lemaître-Tolman-Bondi spacetime (LTB) with off-center observer (*Fanizza, Nugier 2014*, 1408.1604) :

- non-perturbative d_A, κ, $|\hat{\gamma}|$, $\hat{\mu}$ in terms of LTB coordinates (t, r, θ, ϕ),
- numerical resolution of $t(z)$, $r(z)$ with ansatz for $H_o(r) \Rightarrow$ plots of lensing quantities wrt observation angle from the bubble center.

Study of an **anisotropic** spacetime, Bianchi I (*Fleury, Nugier, Fanizza 2016* arXiv :1602.0446) :

- precise residual gauge fixing conditions for GLC coordinates,
- d_A and GLC functions written in terms of Bianchi I coordinates,
- numerical calculations indicate a **violation of the flux conservation** ! $\langle \hat{\mu}^{-1} \rangle_\Omega \equiv \frac{1}{4\pi} \int \hat{\mu}^{-1}(\tau_{\rm fixed}, w_o, \theta^a)\, \mathrm{d}^2\Omega \neq 1$ with $\hat{\mu} = \left(\bar{d}_A / d_A\right)^2 = \Phi / \bar{\Phi}$

$$ds^2_{BI} = -dt^2 + a^2(t)e^{2\beta_i(t)}\delta_{ij}dx^i dx^i$$



Axisymmetric Bianchi I
$\beta_i = (\beta, \beta, -2\beta)$
filled by dust and $\Lambda = 0$.

Associating FLRW with Bianchi I : either identifying H_0 (solid), or identifying ρ_0 (dashed).

Curves are first ≥ 0 and then ≤ 0.

Here we find that $\langle\hat{\mu}^{-1}\rangle_\Omega - 1 \sim \Omega_{\sigma 0} \equiv \frac{(a/a_o)^4}{6\mathcal{H}_o^2}\sum_i(\partial_\eta\beta_i)^2$

$\Rightarrow \mathcal{O}(2)$-**violation** of $\langle\hat{\mu}^{-1}\rangle_\Omega = 1$ from **large-scale anisotropy of Bianchi I !**

Similar to inhom. Swiss-Cheese (Lavinto, Räsänen 2013), but not like perturbative approaches for which $\langle\hat{\mu}^{-1}\rangle_\Omega - 1 = \mathcal{O}(4)$ (Ben-Dayan 2013, Bonvin 2015). Low $z \sim$ **Malmquist-like bias** (Kaiser & Hudson 2015).

Double Light-Cone (DLC) coordinates

GLC for which **we replace τ by a future null coordinate w_u !**

$$ds^2_{GLC} = \Upsilon^2 dw^2 - 2\Upsilon dw d\tau + \gamma_{ab}(d\theta^a - U^a dw)(d\theta^b - U^b dw)$$

$$\Downarrow$$

$$\boxed{ds^2_{\rm DLC} = -\widetilde{\Upsilon}^2 dw_u dw_v + \gamma_{ab}(d\theta^a - \widetilde{U}^a dw_v)(d\theta^b - \widetilde{U}^b dw_v)}$$

$$\widetilde{U}^a \equiv \gamma^{ab}\widetilde{U}_b = U^a + \Upsilon\gamma^{ab}\frac{\partial\tau}{\partial\theta^b} \quad ,$$

$$\frac{\partial\tau}{\partial w_u} = \frac{\widetilde{\Upsilon}^2}{2\Upsilon} \quad , \quad \frac{\partial\tau}{\partial w_v} = \frac{\Upsilon}{2} - \widetilde{U}^a\frac{\partial\tau}{\partial\theta^a} + \frac{\Upsilon}{2}\gamma^{ab}\frac{\partial\tau}{\partial\theta^a}\frac{\partial\tau}{\partial\theta^b} \quad .$$

$$g^{\rm GLC}_{\mu\nu} = \begin{pmatrix} 0 & -\Upsilon & \vec{0} \\ -\Upsilon & \Upsilon^2 + U^2 & -U_b \\ \vec{0}^{\,T} & -U_a^T & \gamma_{ab} \end{pmatrix} \quad , \quad g^{\mu\nu}_{\rm GLC} = \begin{pmatrix} -1 & -\Upsilon^{-1} & -U^b/\Upsilon \\ -\Upsilon^{-1} & 0 & \vec{0} \\ -(U^a)^T/\Upsilon & \vec{0}^{\,T} & \gamma^{ab} \end{pmatrix} \quad ,$$

$$g^{\rm DLC}_{\mu\nu} = \begin{pmatrix} 0 & -\widetilde{\Upsilon}^2/2 & \vec{0} \\ -\widetilde{\Upsilon}^2/2 & \widetilde{U}^2 & -\widetilde{U}_b \\ \vec{0}^{\,T} & -\widetilde{U}_a^T & \gamma_{ab} \end{pmatrix} \quad , \quad g^{\mu\nu}_{\rm DLC} = \begin{pmatrix} 0 & -2/\widetilde{\Upsilon}^2 & -2\widetilde{U}^b/\widetilde{\Upsilon}^2 \\ -2/\widetilde{\Upsilon}^2 & 0 & \vec{0} \\ -2(\widetilde{U}^a)^T/\widetilde{\Upsilon}^2 & \vec{0}^{\,T} & \gamma^{ab} \end{pmatrix} \quad .$$

See *Nugier 2016*, arXiv :1606.08296.

Properties of DLC coordinates

- Share the **advantages of GLC** (e.g. z_s still given by Υ_o/Υ_s)
- Can describe **static black holes** (also Kerr ?)
- Can describe **trajectories** around static BHs
- Can derive the **time delay between UR particles** (like done in GLC in *Fanizza, Gasperini, Marozzi, Veneziano 2016*, arXiv :1512.0848)
- w_u **is well defined from GLC** coordinates considering perturbations around an FLRW spacetime.

They are **equivalent to the double-null coordinates** of *Brady, Droz, Israel and Morsink 1995* after a gauge fixing !

Conclusions

Applications of GLC coordinates :

- **luminosity distance** at $\mathcal{O}(2)$, inhomogeneities effect on **Hubble diagram**, **weak lensing**, **averaging** on our past light cone,
- Specific models : **LTB** and **Bianchi I**.
- Link to **Double-Null coordinates !** ($\tau \to w_u$)
- **Number counts** and bispectrum at $\mathcal{O}(2)$ (Di Dio *et al.* 1407.0376, 1510.04202) **! ! !**
- Time-of-flight of **ultra-relativistic particles**.

⇒ Good coordinates to handle light signals !

Adapted systems of coordinates are very useful ! ☺

But not often employed in calculations ☹...

Thank You!

Collaborators :

- G. Veneziano, G. Marozzi, M. Gasperini, P. Fleury, G. Fanizza, I. Ben-Dayan,
- Other researches : B. Metcalf, P. Chen, Hsu-Wen Chiang, E. Romano.