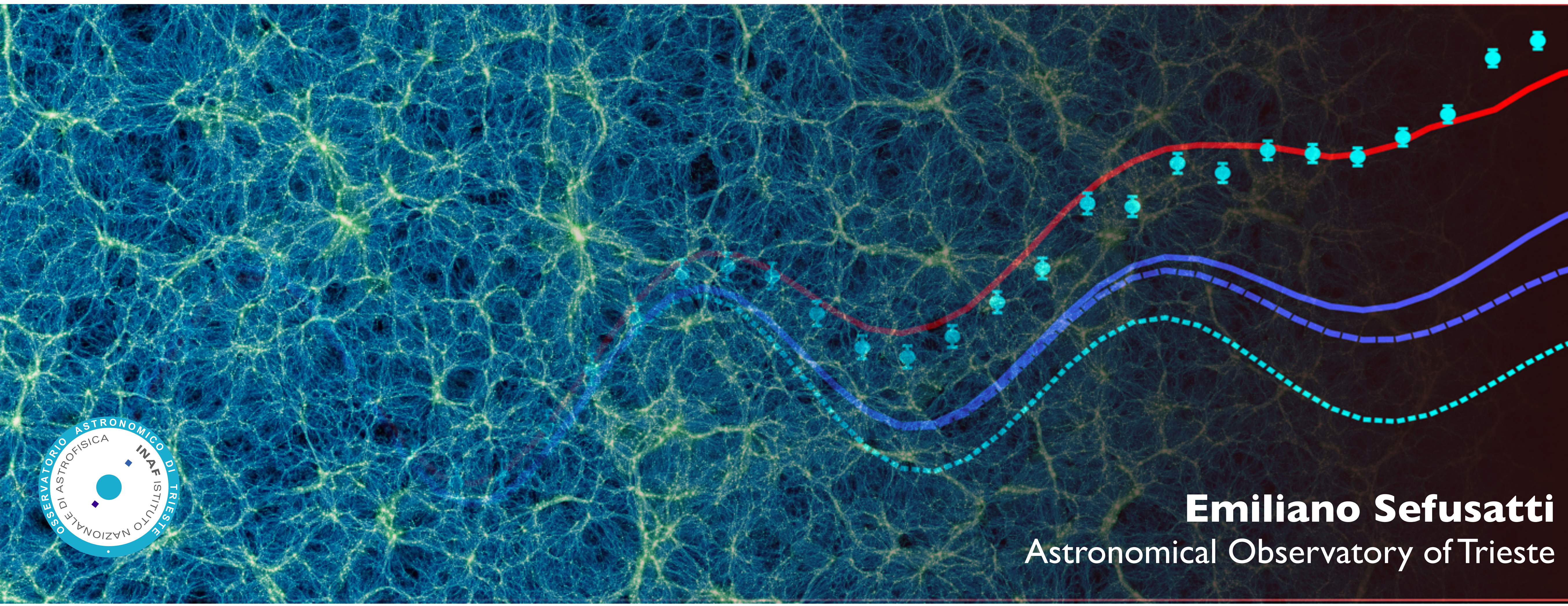
Galaxy Clustering:
Observables & Theoretical Modelling
Part 1



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Future Cosmology
Ecole thématique du CNRS
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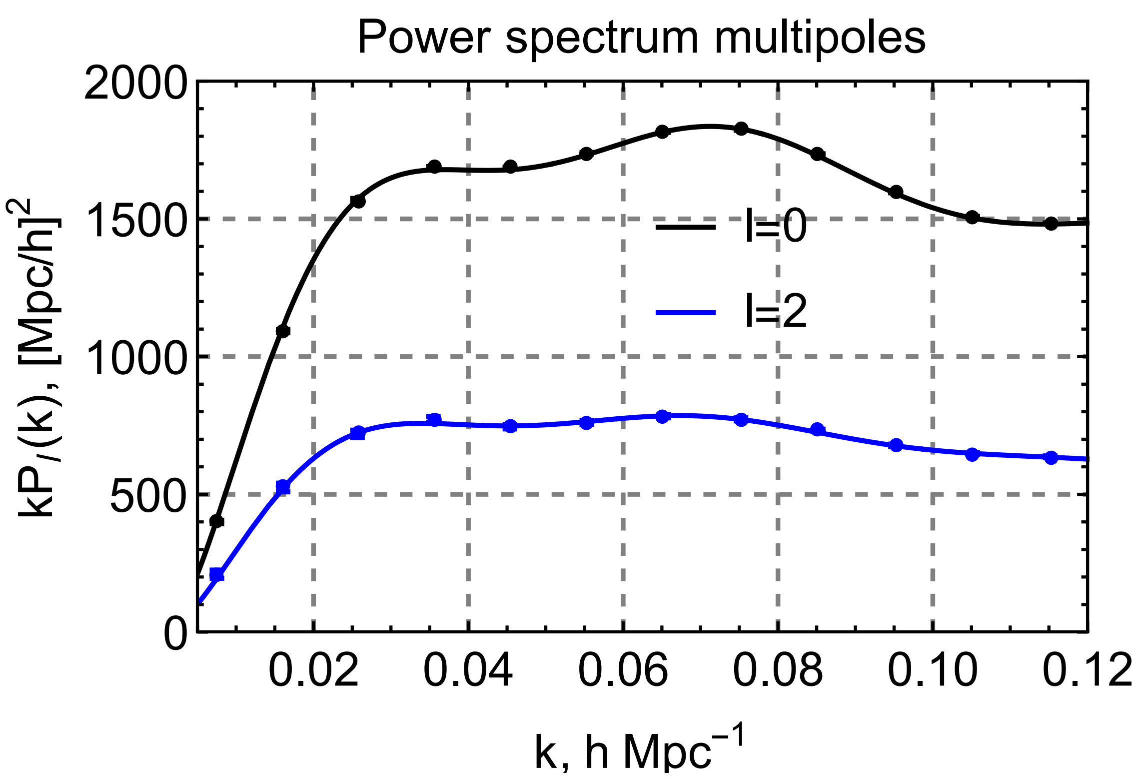
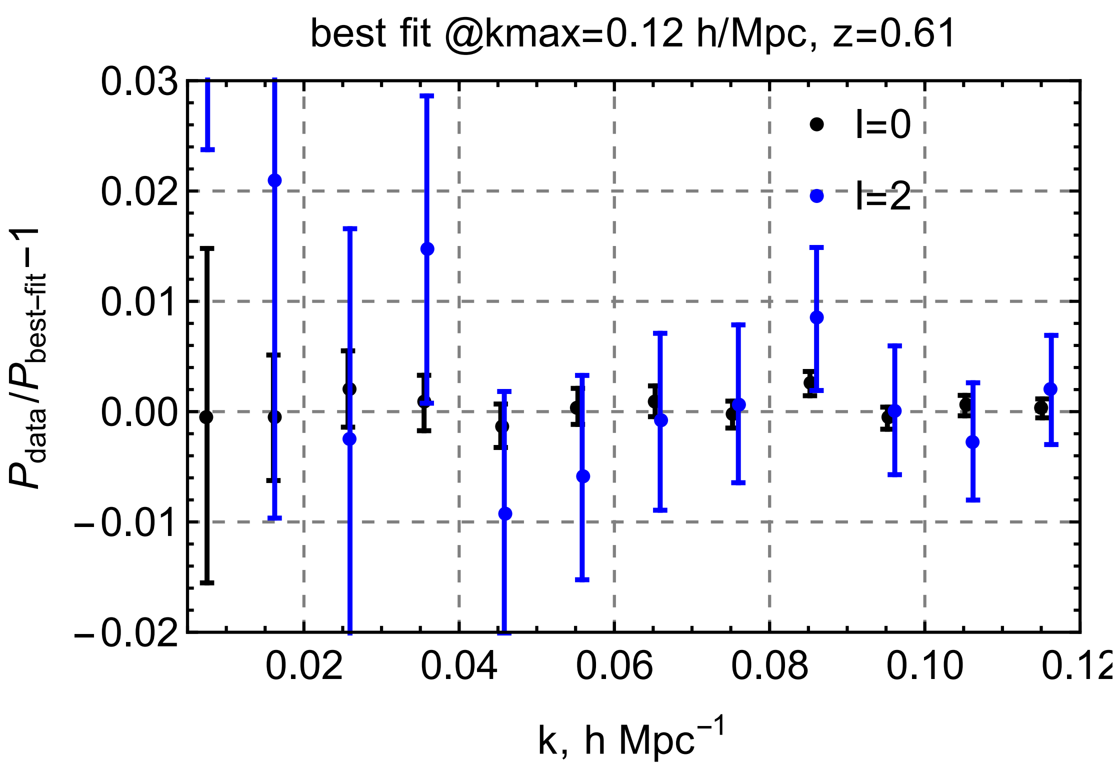
The goal

Describe and motivate the state-of-the-art **modelling** in **perturbation theory** of the redshift-space galaxy **power spectrum**,
a main observables in **galaxy spectroscopic surveys**

PT Challenge: test on simulations over a volume of $600\ \mathrm{Gpc}^3/h^3$





Nishimichi *et al* (2006)

Lecture 1

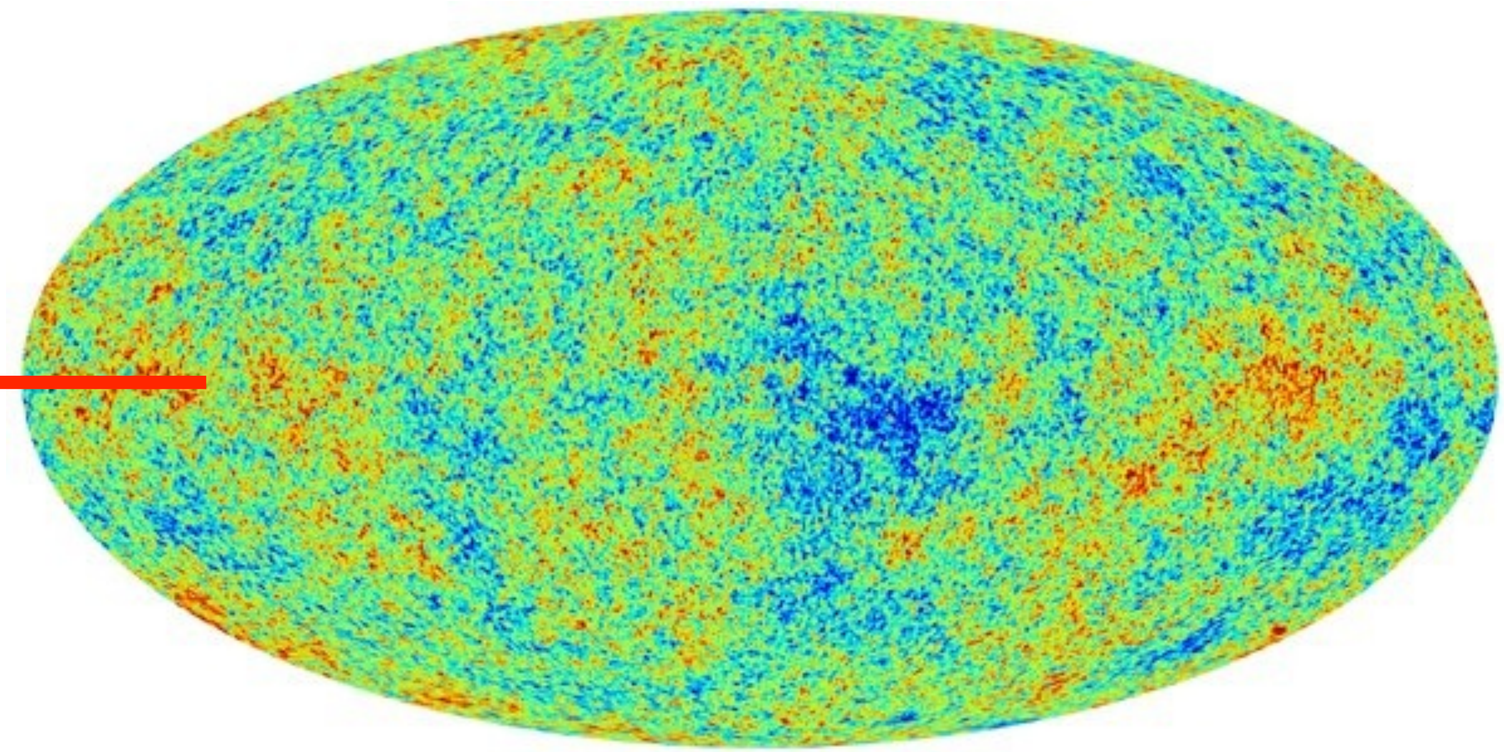
- The galaxy distribution as a random field
- Galaxy clustering observables: power spectrum & 2PCF
- Initial conditions: the matter power spectrum at recombination
- Linear Eulerian Perturbation Theory
- Non-Gaussianity and higher-order correlation functions
- Nonlinear PT and the power spectrum at one-loop
- Effective Field Theory of Large-Scale Structure

Lecture 2

- Galaxy bias
- Baryonic Acoustic Oscillations & Infrared Resummation
- Redshift-Space Distortions
- Stochastic contributions
- Recent analyses of the BOSS survey
- *Neutrinos*
- *Non-Gaussianity & Higher-Order Statistics*
- *Primordial non-Gaussianity*

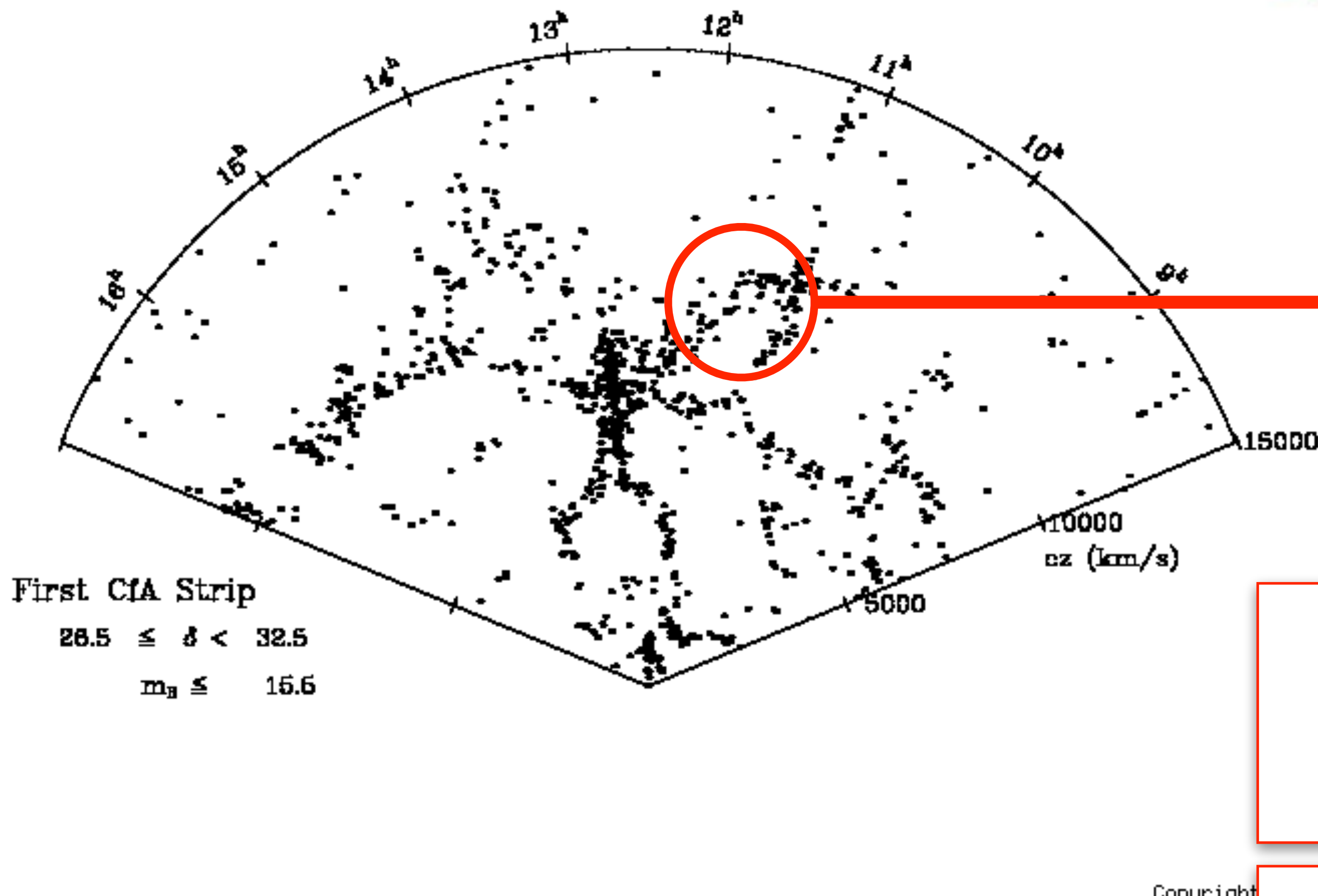
Cosmological Random Fields

Cosmological perturbations



CMB temperature fluctuations

$$T(\hat{n})$$



$$n_g(\vec{x})$$

number density of galaxies

We can only study the **statistical properties** of cosmological perturbations

Mathematically, these are **random fields**

Random fields

If ϕ is a **random *variable*** with Probability Distribution Function (PDF) $\mathcal{P}(\phi)$
we can compute:

$$\langle\phi\rangle = \int d\phi\, \mathcal{P}(\phi)\, \phi$$ mean

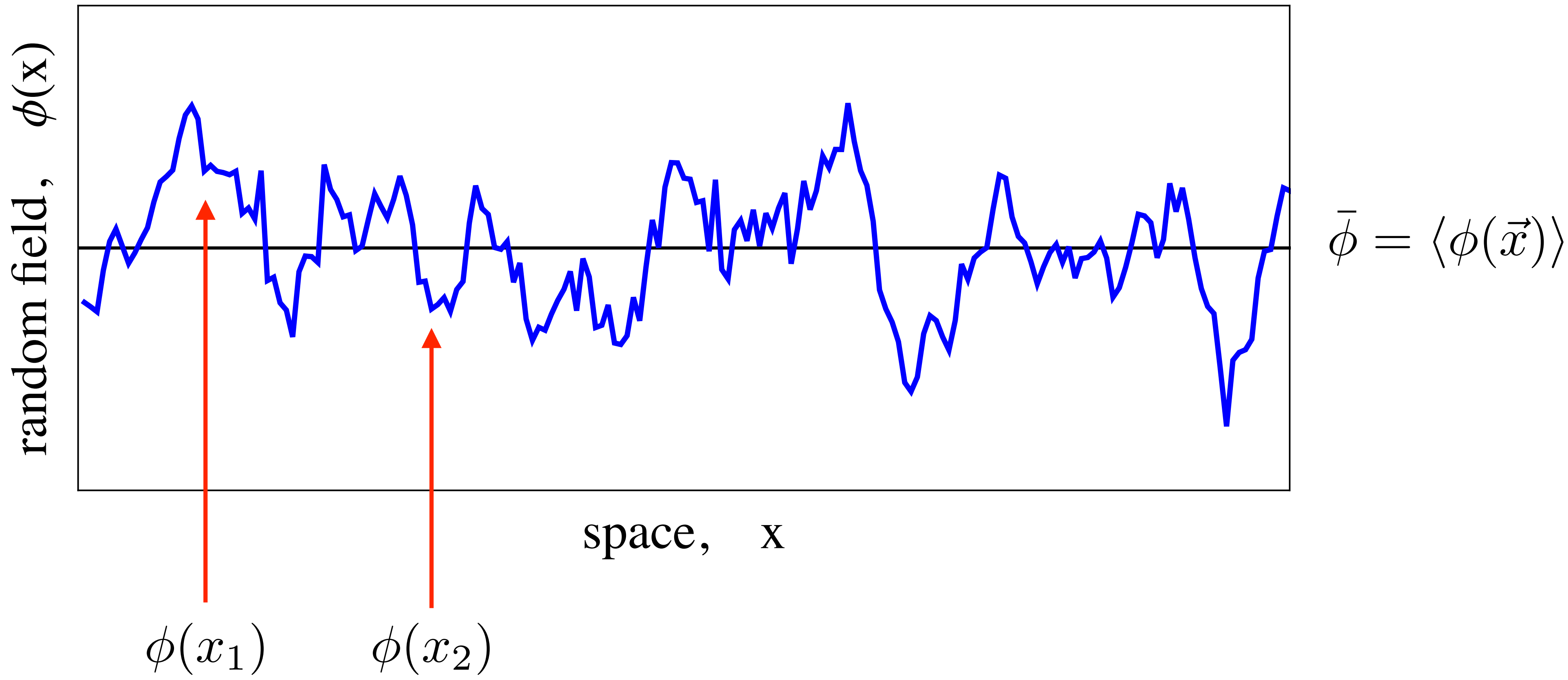
$$\langle\phi^2\rangle = \int d\phi\, \mathcal{P}(\phi)\, \phi^2$$ 2-nd-order moment

$$\langle\phi^n\rangle = \int d\phi\, \mathcal{P}(\phi)\, \phi^n$$ n-th-order moment

$$\sigma_\phi^2 = \langle\phi^2\rangle - \langle\phi\rangle^2$$ variance

Random fields

If $\phi(\vec{x})$ is a **random *field*** we can also compute **correlation functions**



two-point function $\langle \phi(x_1)\phi(x_2) \rangle = \langle \phi(x_1) \rangle \langle \phi(x_2) \rangle + \langle \phi(x_1)\phi(x_2) \rangle_c$

three-point function
$$\langle \phi(x_1)\phi(x_2)\phi(x_3) \rangle = \langle \phi(x_1) \rangle \langle \phi(x_2) \rangle \langle \phi(x_3) \rangle + \\ + \langle \phi(x_1)\phi(x_2) \rangle_c \langle \phi(x_3) \rangle + \mathrm{perm.} + \\ + \langle \phi(x_1)\phi(x_2)\phi(x_3) \rangle_c$$

...

n-point function $\langle \phi(x_1)\phi(x_2)\ldots\phi(x_n) \rangle$

The distribution of galaxies

The galaxy number density and its perturbations as random fields

$$\boxed{n_g(\vec{x}) \equiv \bar{n}_g \left[1 + \delta_g(\vec{x})\right]}$$

galaxy number density

mean galaxy number

$$\boxed{\delta_g(\vec{x}) \equiv \frac{n_g(\vec{x}) - \bar{n}_g}{\bar{n}_g}}$$

galaxy overdensity
or density contrast

N.B. $\langle \delta_g(\vec{x}) \rangle \equiv 0$

$\delta_g(\vec{x}) \geq -1$

Matter, peculiar velocities, gravitational potential

We will write the equations of motions for *perturbations* and as a function of comoving coordinates $\overrightarrow{x}$ and conformal time $d\tau = dt/a(t)$.

$$\boxed{\rho(\vec{x},\tau) = \bar{\rho}(\tau)[1+\delta(\vec{x},\tau)]}$$

$\delta(\vec{x},\tau)$ **matter perturbations**

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$\delta(\vec{x},\tau)$ **matter perturbations**

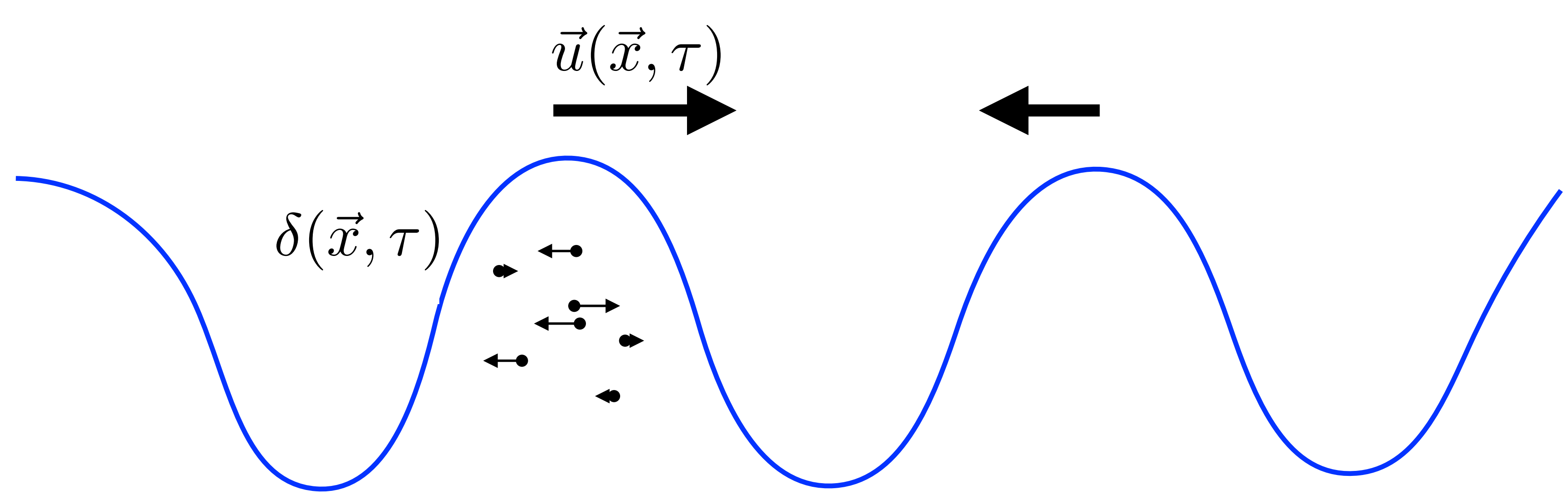
Velocities have a competent due to the Hubble expansion and one due to peculiar motion

$$\vec{r} = a(t)\,\vec{x} \qquad \vec{v} \equiv \frac{d\vec{r}}{dt} = \frac{da}{dt}\vec{x} + a\frac{d\vec{x}}{dt} = H(\tau)\,\vec{r}(\tau) + \vec{u}(\vec{x},\tau) \qquad \mathcal{H} \equiv \frac{1}{a}\frac{da}{d\tau} = a\,H$$

Hubble flow

$$\boxed{\vec{v}(\vec{x},\tau) = \mathcal{H}(\tau)\,\vec{x}(\tau) + \vec{u}(\vec{x},\tau)}$$

$\vec{u}(\vec{x},\tau)$ **peculiar velocities**



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$\vec{u}(\vec{x},\tau)$ **peculiar velocities**

Matter perturbations are related to perturbations in the gravitational potential via **Poisson equation**

$$\boxed{\nabla^2\Phi(\vec{x},t) = 4\pi G\,a^2\,\bar{\rho}\,\delta(\vec{x},t)}$$

in comoving coordinates

$\Phi(\vec{x},\tau)$ **gravitational potential**

Correlations Function

The galaxy two-point correlation function

What is the probability of finding two galaxies in the volume elements dV_1 and dV_2 ?

$$dP = dV_1\, dV_2\, \langle\, n_g(\vec{x}_1)\, n_g(\vec{x}_2)\, \rangle$$

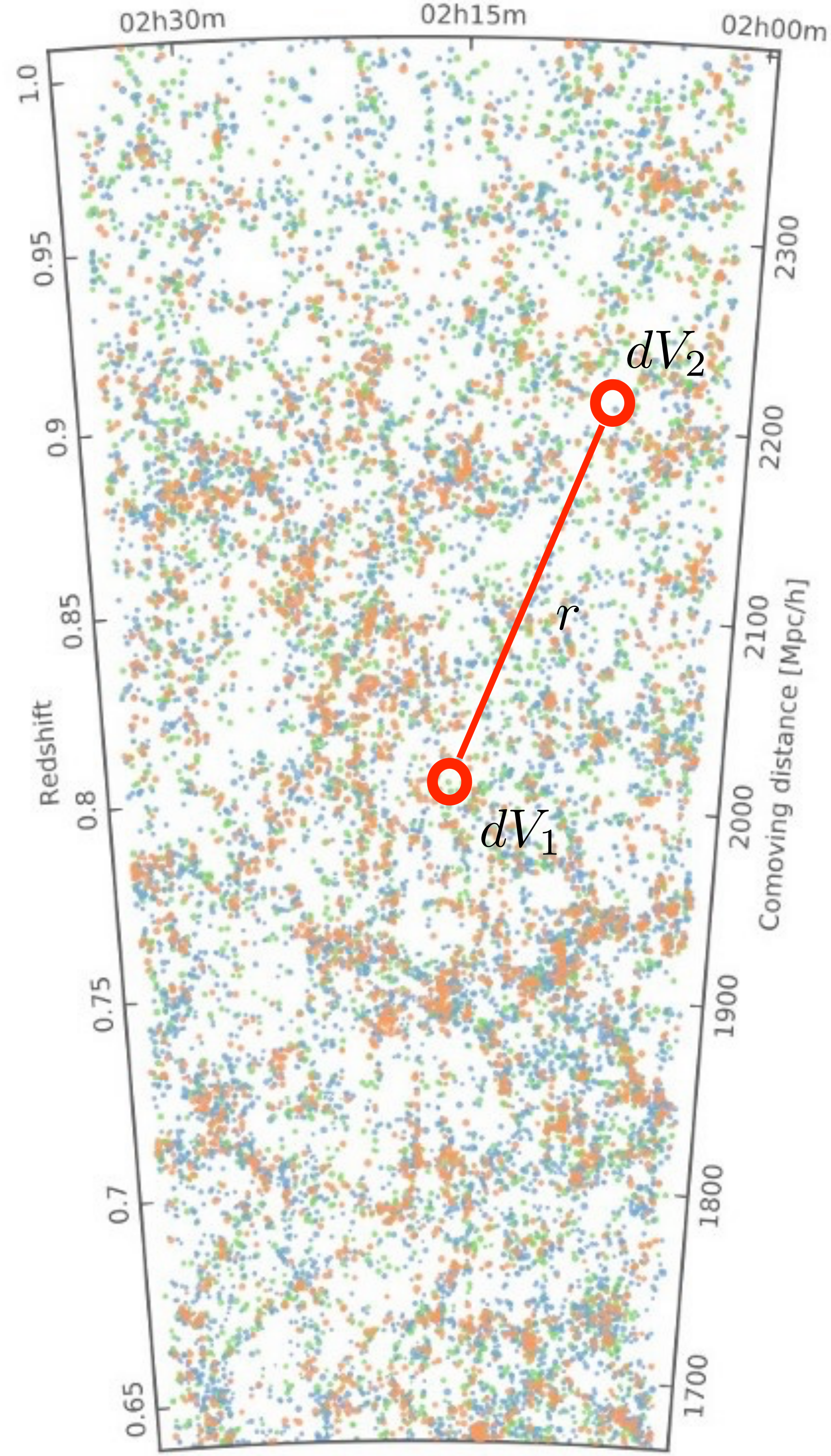
$$= dV_1\, dV_2\, \bar{n}_g^2\, [1 + \langle\, \delta_g(\vec{x}_1)\, \delta_g(\vec{x}_2)\, \rangle]$$

excess probability

We now make the assumption of ***statistical* homogeneity and isotropy**

$$\xi(|\vec{x}_1 - \vec{x}_2|) \equiv \langle\, \delta_g(\vec{x}_1)\, \delta_g(\vec{x}_2)\, \rangle$$

the two-point correlation function $\xi(r)$
only depends on the distance $r = |\vec{x}_1 - \vec{x}_2|$
between the two points



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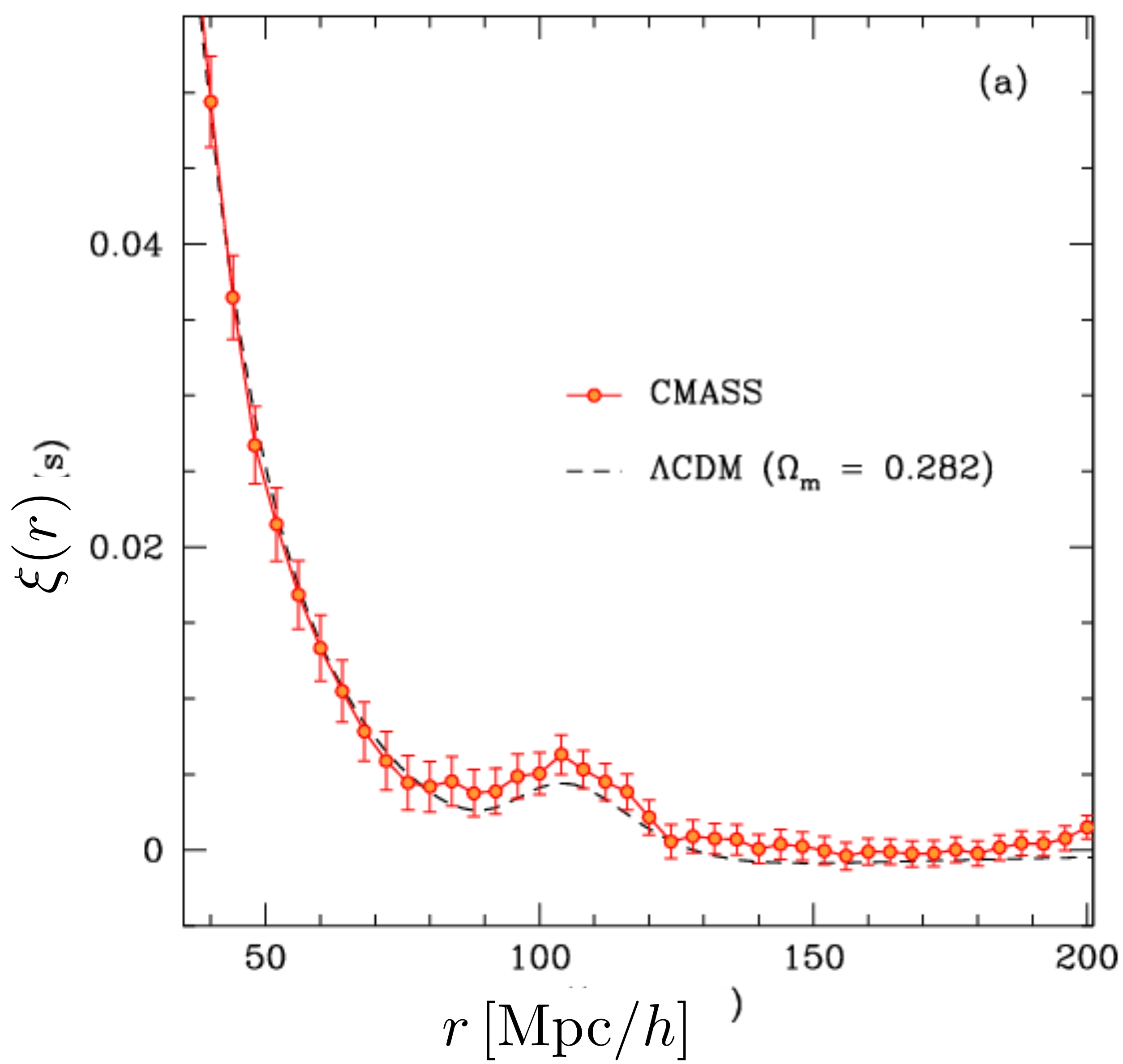
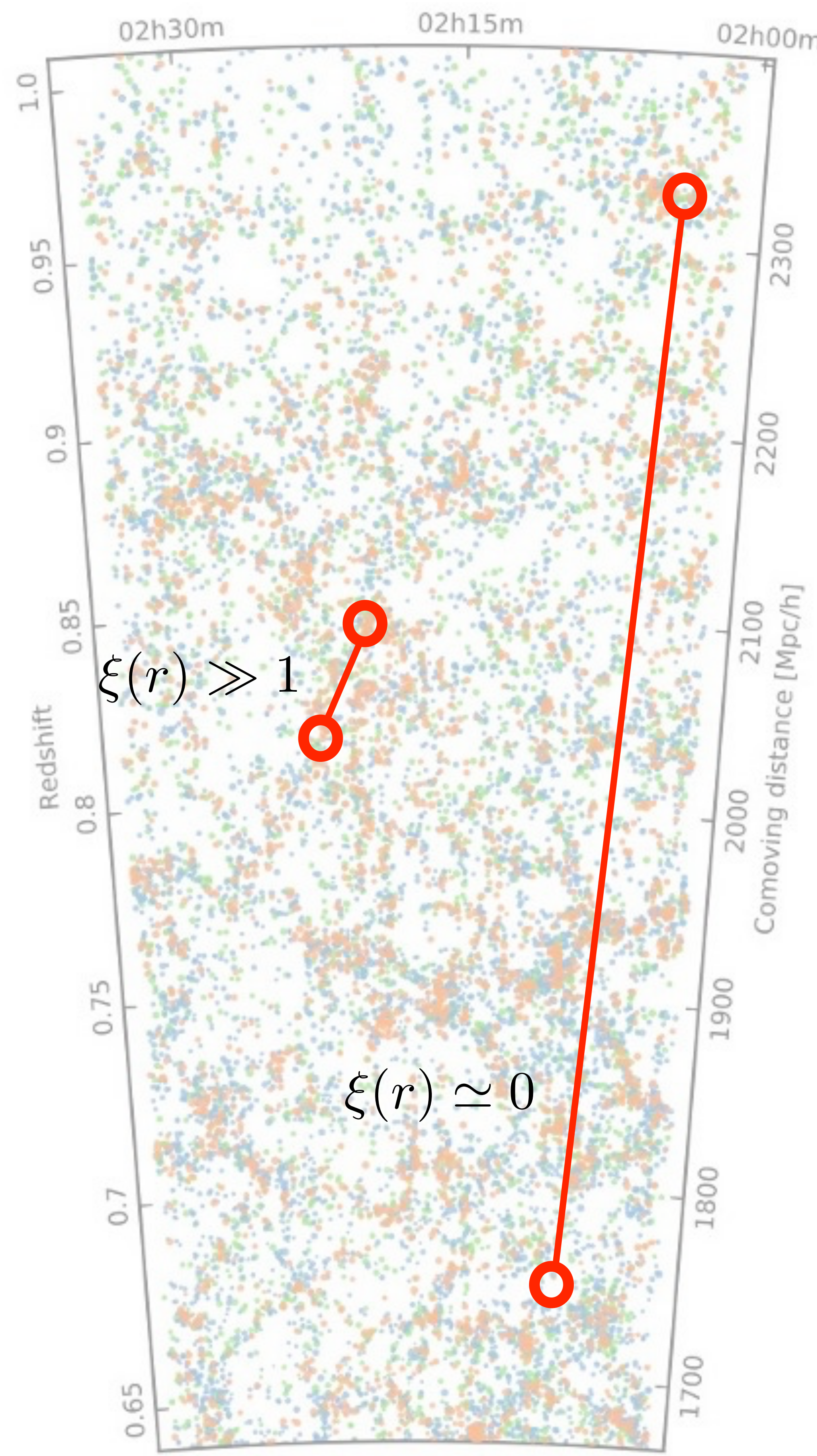




The galaxy three-point correlation function

Similarly I can ask the probability of finding three galaxies in the volume elements dV_1 , dV_2 and dV_3

$$dP = dV_1 dV_2 dV_3 \langle n_g(\vec{x}_1) n_g(\vec{x}_2) n_g(\vec{x}_3) \rangle$$

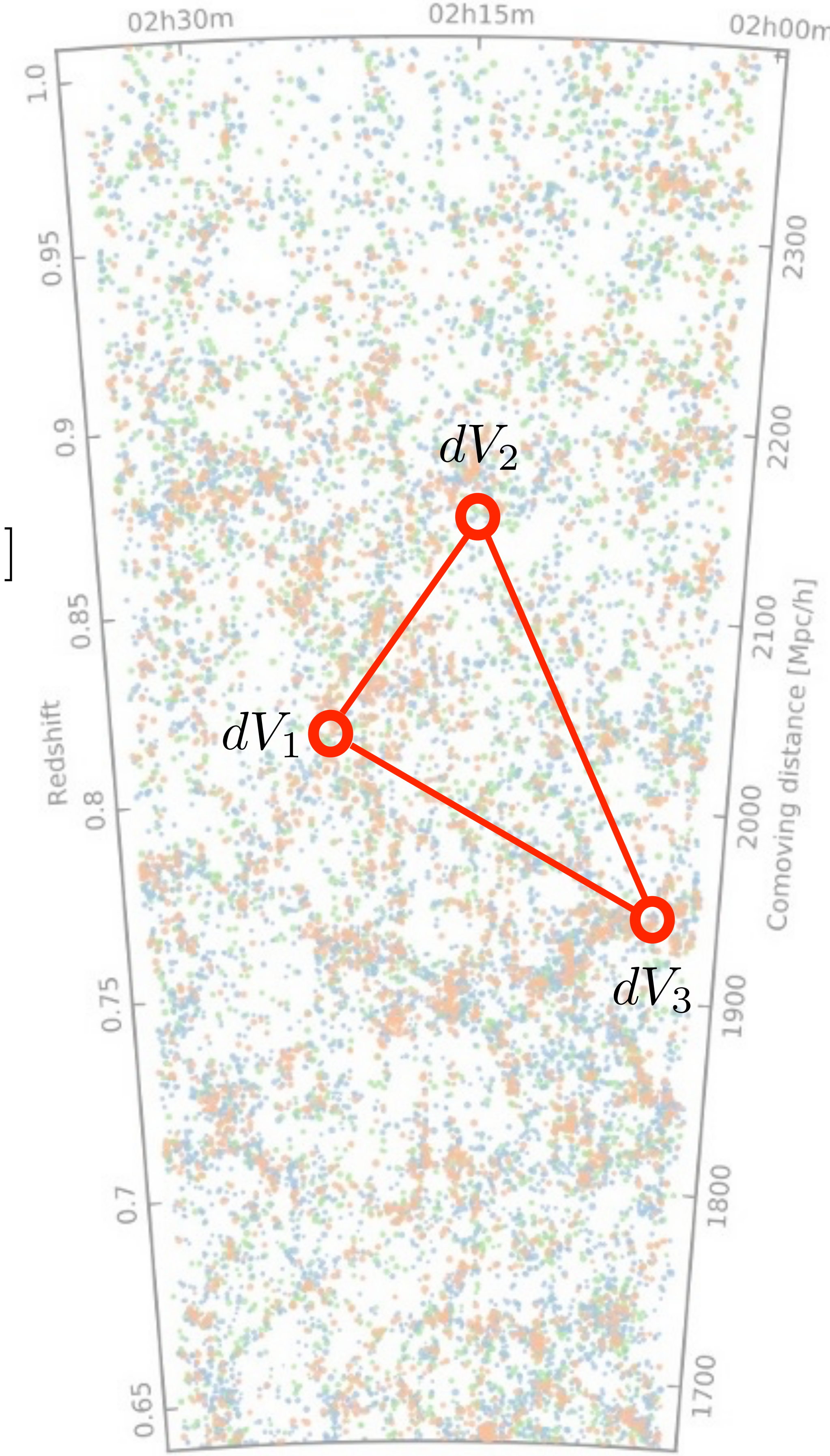
$$= dV_1\, dV_2\, dV_3\, \bar{n}_g^3\, [1 +$$

$$+ \xi(r_{12}) + \xi(r_{13}) + \xi(r_{23}) + \zeta(r_{12}, r_{13}, r_{23})]$$

excess probability

$$\zeta(r_{12}, r_{13}, r_{23}) \equiv \langle \delta_g(\vec{x}_1) \delta_g(\vec{x}_2) \delta_g(\vec{x}_3) \rangle$$

the 3-point correlation function represents the (excess) probability to find 3 galaxies forming a triangle of a given shape and size

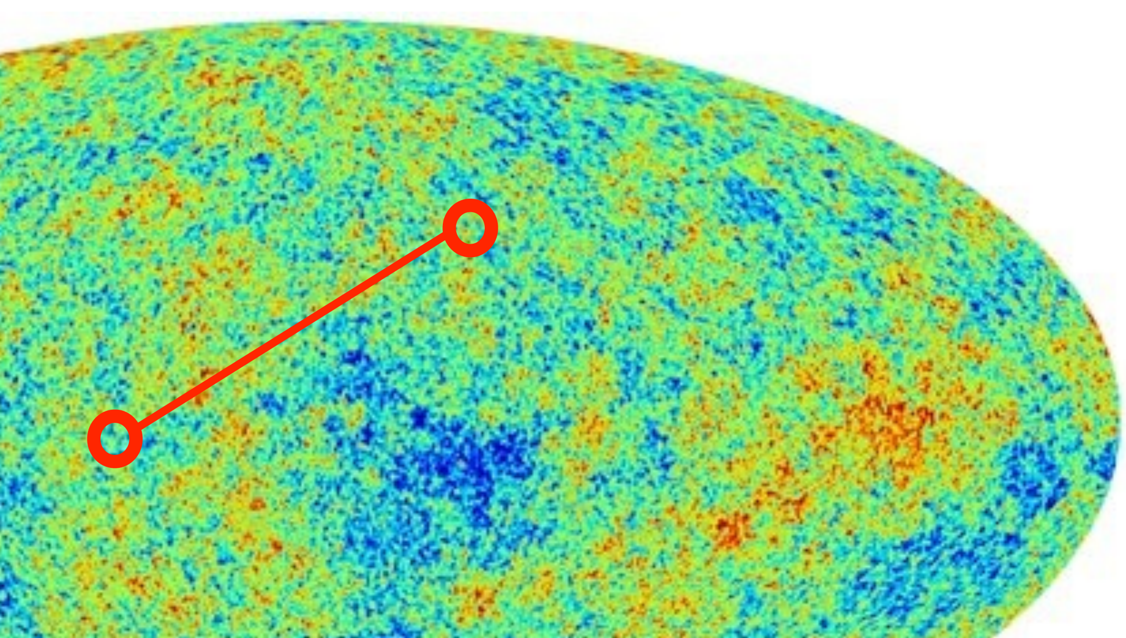


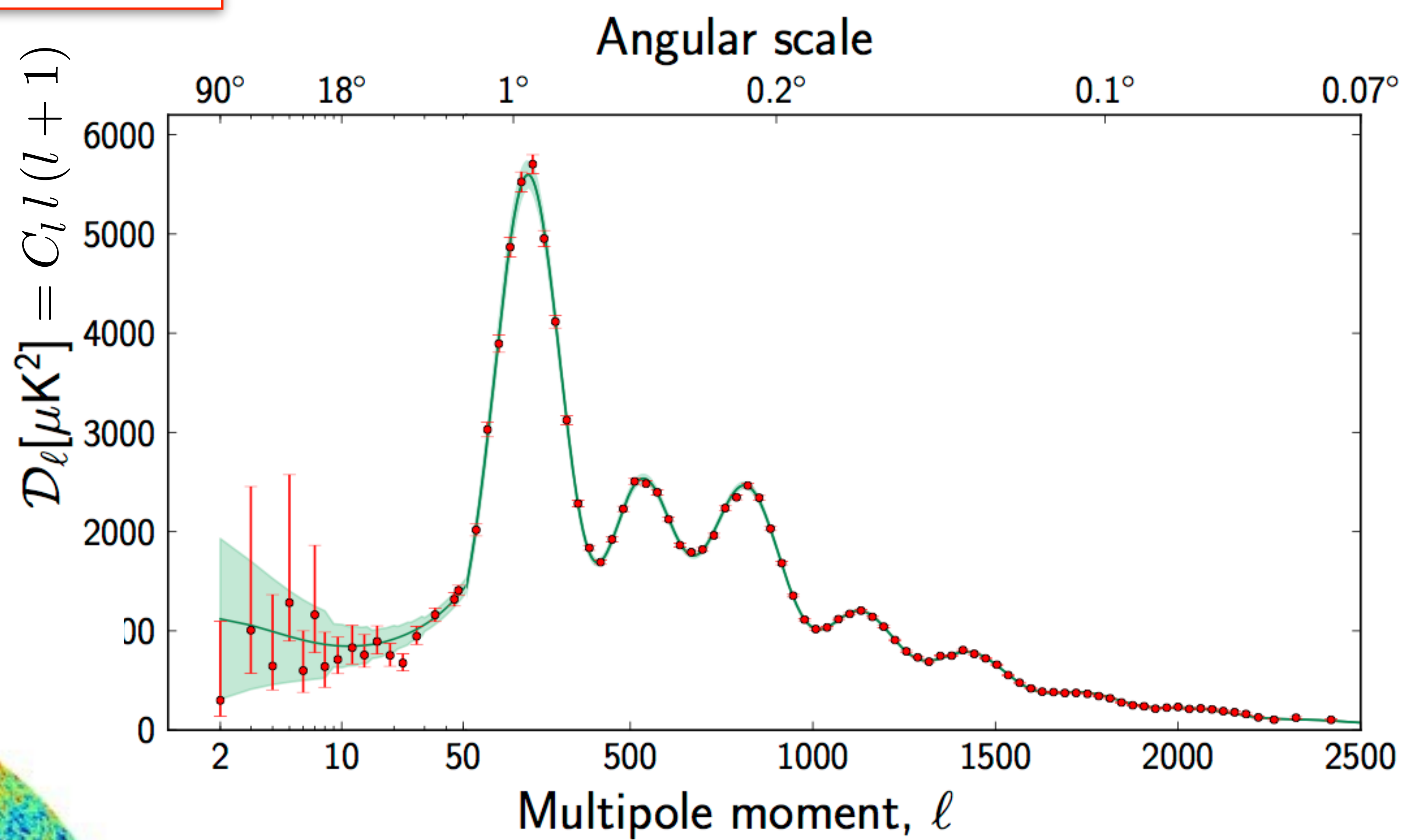
Gaussian and non-Gaussian random fields

The statistical properties of a Gaussian random field are completely characterised by its 2-point correlation function. All higher-order, *connected* correlation functions are vanishing

$$\delta_T(\hat{n}) \equiv \frac{T(\hat{n}) - \bar{T}}{\bar{T}}$$

$$\mathcal{P}[\delta_T(\hat{n})] = \frac{1}{\sqrt{2\pi\sigma_T^2}} e^{-\frac{1}{2}\frac{\delta_T^2}{\sigma_T^2}}$$



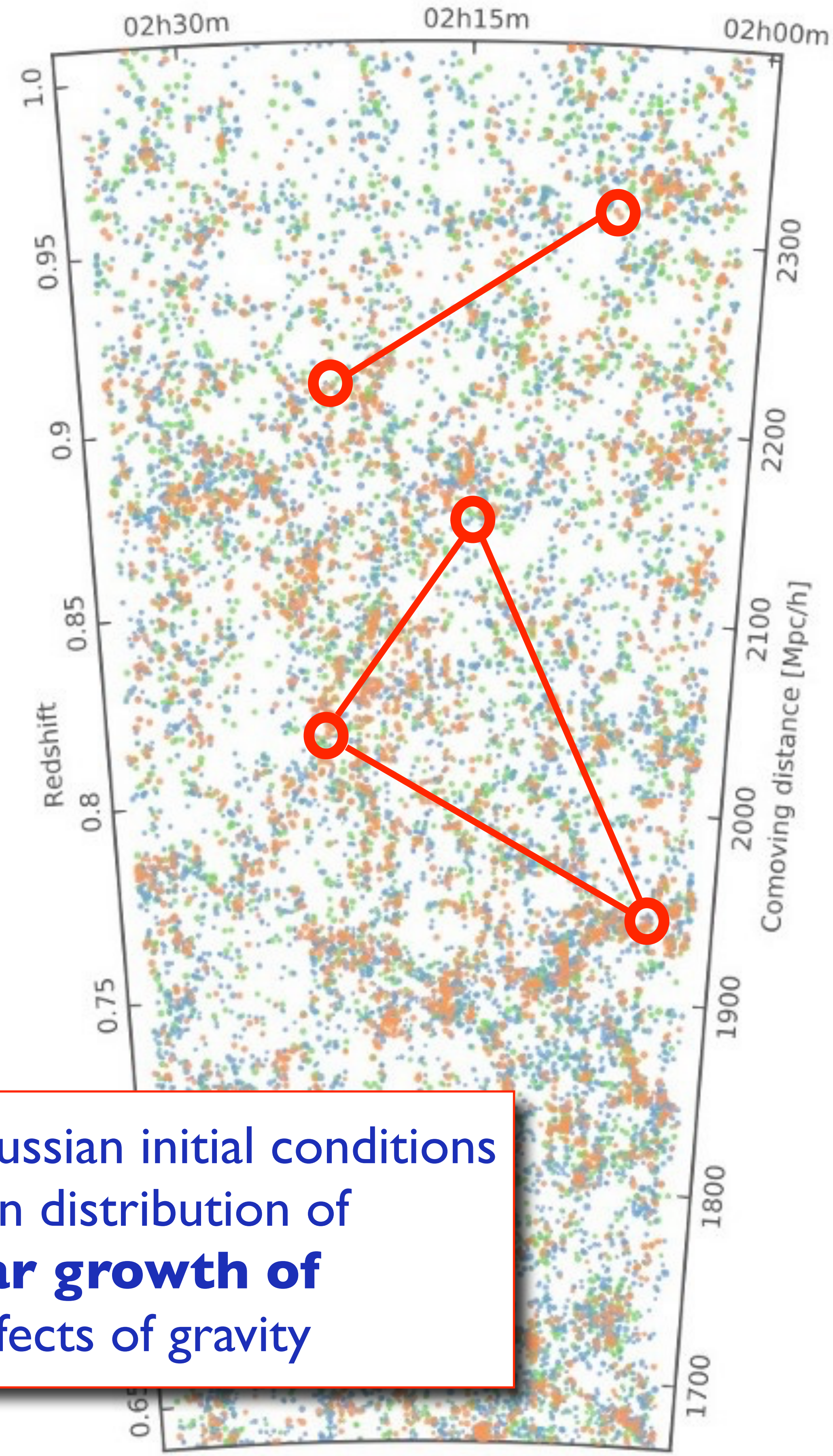


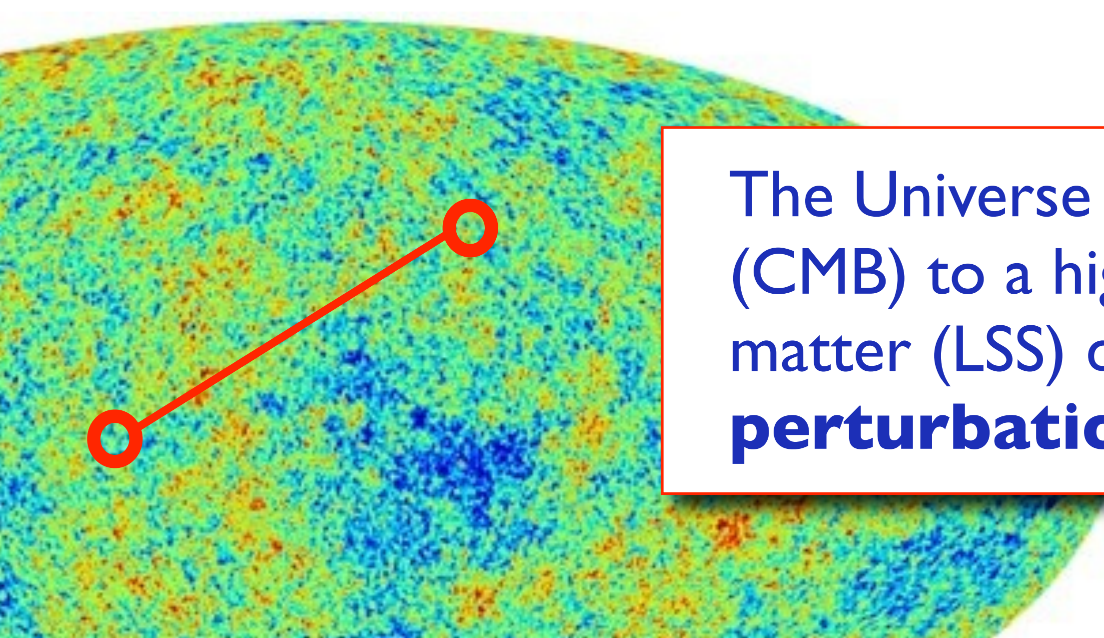
Perturbations in the CMB are one of the best examples of Gaussian random field

Gaussian and non-Gaussian random fields

The statistical properties of a Gaussian random field are completely characterised by its 2-point correlation function. All higher-order, *connected* correlation functions are vanishing

all other random fields are non-Gaussian!





The Universe evolves from Gaussian initial conditions (CMB) to a highly non-Gaussian distribution of matter (LSS) due to **nonlinear growth of perturbations** under the effects of gravity

Ergodic hypothesis

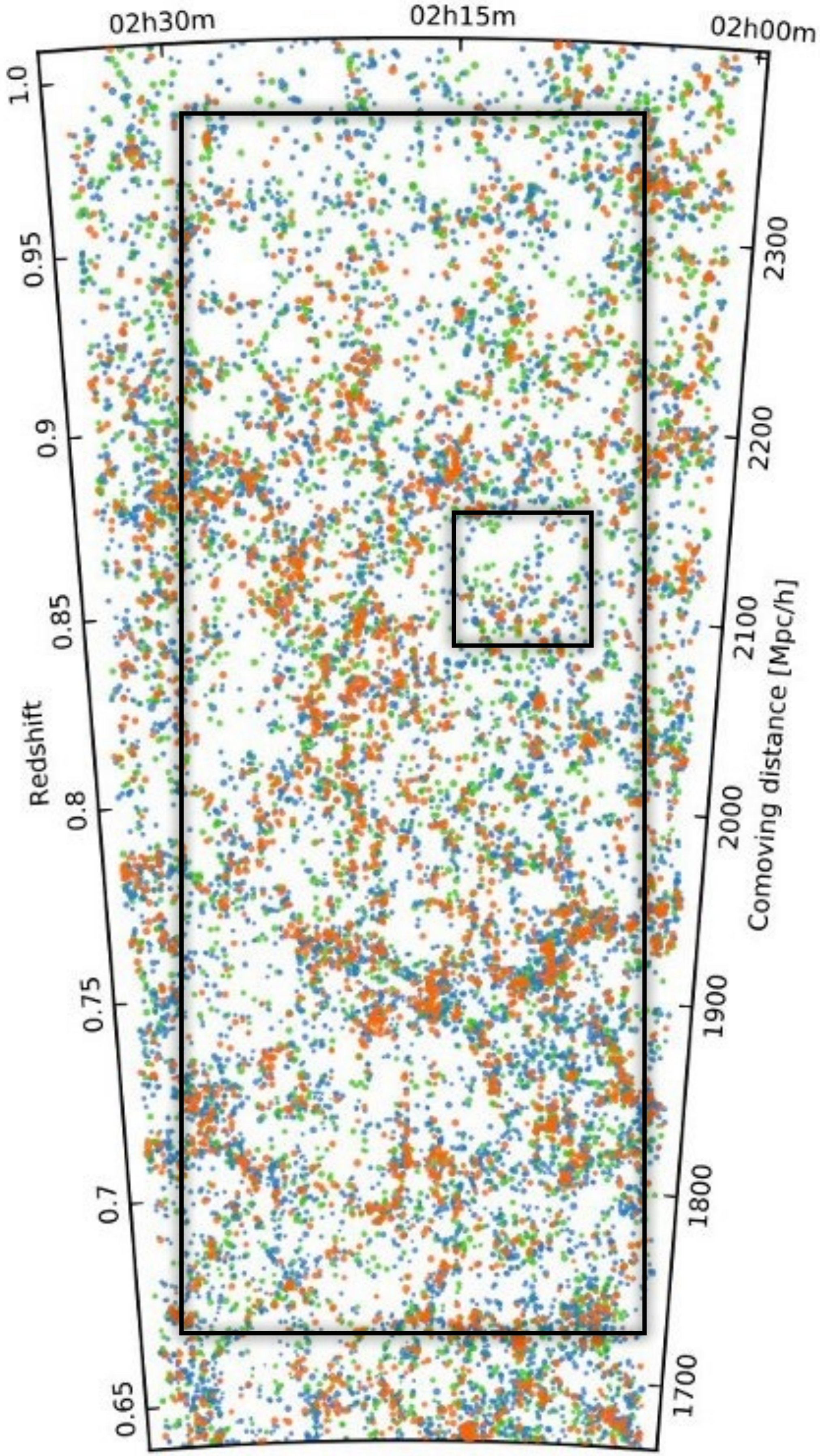
Expectation values, in principle, are to be intended as *ensemble averages*, i.e. averages over many "realisations of the Universe" …

… but we only have one Universe!

We assume the **ergodic hypothesis: ensemble averages are equal to spatial averages**

$$\langle \phi(\vec{x}) \rangle \equiv \int d\phi \, \phi \, \mathcal{P}(\phi) = \frac{1}{V} \int_V d^3x \, \phi(\vec{x})$$

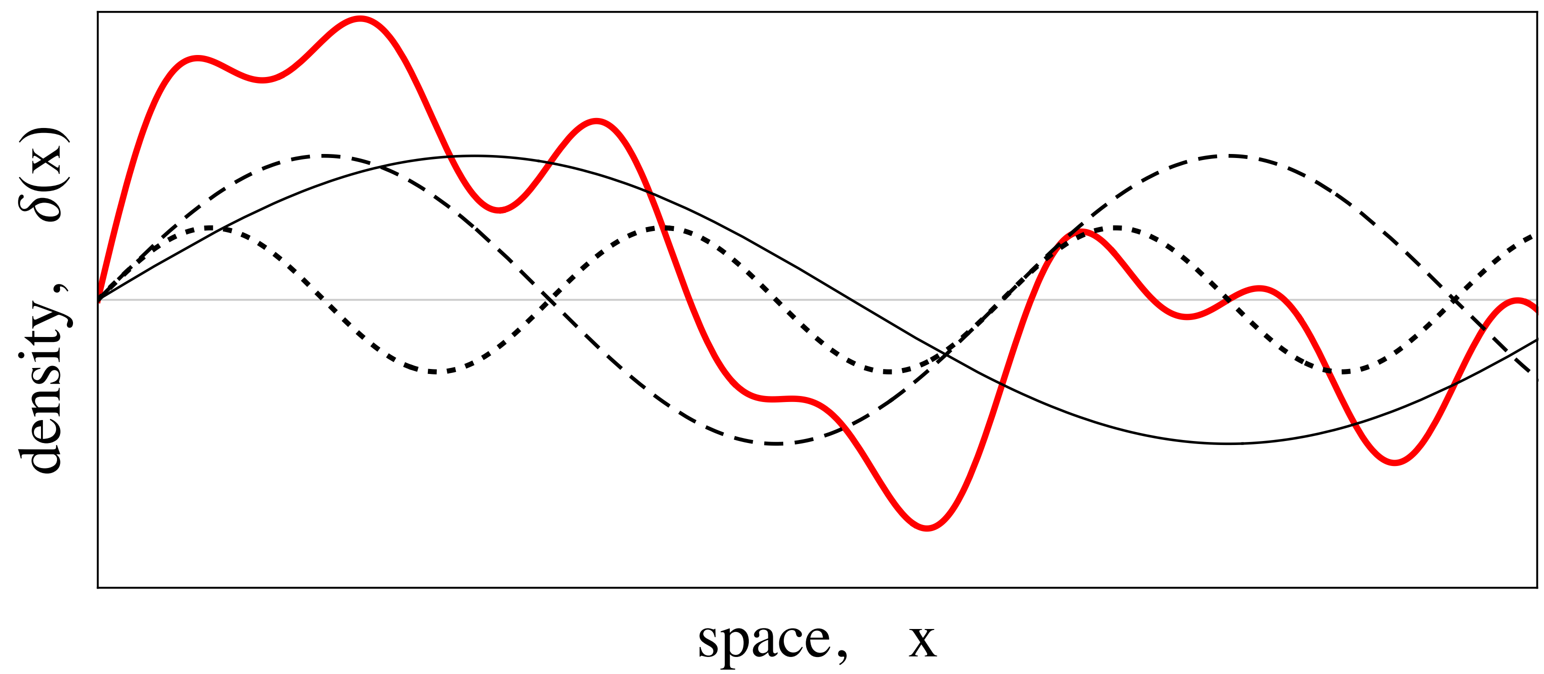
We should make sure, however, that the observed volume correspond to a "fair sample" of the Universe



Fourier space

Theoretical predictions for the matter correlation functions are performed in **Fourier space**

Fourier analysis naturally separates perturbations at different scales:



$$\delta_{\vec{k}} = \int \frac{d^3x}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}} \delta(\vec{x})$$

$$\delta(\vec{x}) = \int d^3k e^{i\vec{k}\cdot\vec{x}} \delta_{\vec{k}}$$

- Since $\delta(\vec{x})$ is a random field $\delta_{\vec{k}}$ is also a random field
- Since $\delta(\vec{x})$ is real $\delta^*_{\vec{k}} = \delta_{-\vec{k}}$

Fourier space: correlation functions

The 2-point function in Fourier space: the **power spectrum**

$$\langle \delta_{\vec{k}_1} \delta_{\vec{k}_2} \rangle = \delta_D(\vec{k}_1 + \vec{k}_2) \, P(k_1)$$

homogeneity & isotropy

$$P(k) = \int \frac{d^3x}{(2\pi)^3} e^{i \vec{k}\cdot\vec{x}} \xi(x)$$

The power spectrum is the *Fourier Transform* of the 2-point correlation function

The power spectrum is a measure of the amplitude of perturbations as a function of scale

$$\Delta(k) \equiv 4\pi \, k^3 \, P(k)$$

adimensional power spectrum

$$\sigma_\delta^2 \equiv \langle \delta^2(\vec{x}) \rangle = 4\pi \int dk \, k^2 \, P(k) = \int \frac{dk}{k} \Delta(k)$$

Fourier space: correlation functions

Higher-order correlation functions:

$$\langle \delta_{\vec{k}_1} \delta_{\vec{k}_2} \delta_{\vec{k}_3} \rangle \equiv \delta_D(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)\, B(k_1, k_2, k_3)$$

the **bispectrum**

$$\langle \delta_{\vec{k}_1} \delta_{\vec{k}_2} \delta_{\vec{k}_3} \delta_{\vec{k}_4} \rangle \equiv \delta_D(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4)\, T(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4)$$

the **trispectrum**

The bispectrum and trispectrum are the lowest-order correlation functions to characterise the *three-dimensional nature* of matter perturbations

Our goal

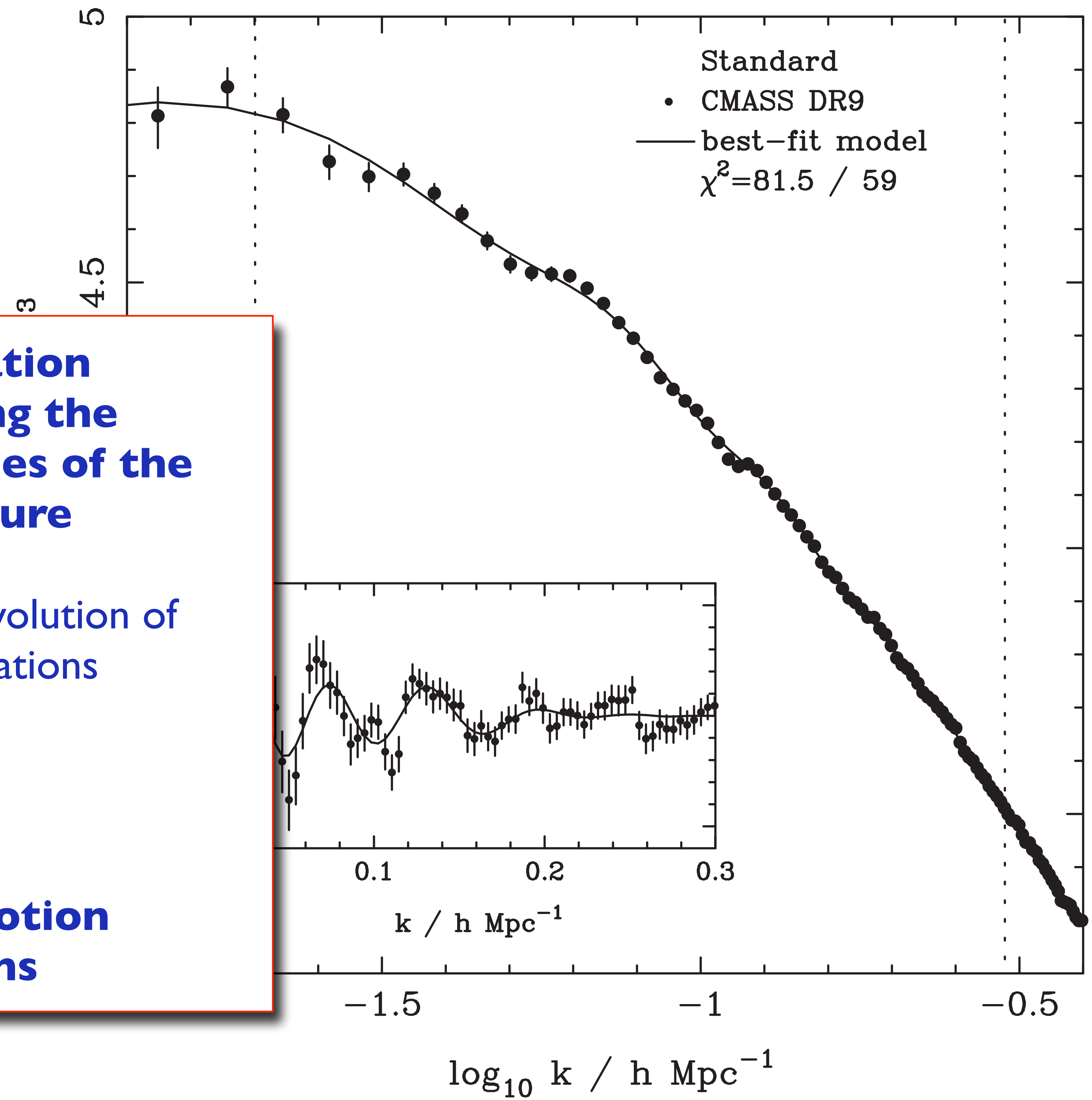
predict the correlation functions describing the statistical properties of the Large-Scale Structure

for this we study the evolution of matter density perturbations

$$\delta_{\vec{k}}(t)$$

We need:

1. **Equations of motion**
2. **Initial conditions**



Initial Conditions

Density Perturbations from Inflation

Inflation predicts the power spectrum of the primordial perturbations in the gravitational potential

$$\Delta_\Phi(k) \equiv 4\pi k^3 P_\Phi(k) \simeq \text{constant} \simeq (10^{-5})^2$$

Harrison-Zeldovich power spectrum

$$P_\Phi(k) = \underbrace{\frac{2}{9M_p^4}\frac{H^2V^2}{V'^2}}_{\text{amplitude}}\, \underbrace{k^{-4+n_s}}_{\text{scale-dependence}}\Big|_{aH=k}$$

amplitude

scale-dependence

spectral index: $n_s = 1 - 2M_p^2\left(\frac{V'}{V}\right)^2 + 2M_p^2\frac{V''}{V}$

$$\nabla^2\Phi = 4\pi G\, a^2\,\bar{\rho}\,\delta \longrightarrow P(k) \sim k^4\, P_\Phi(k) \sim C k^{n_s}$$

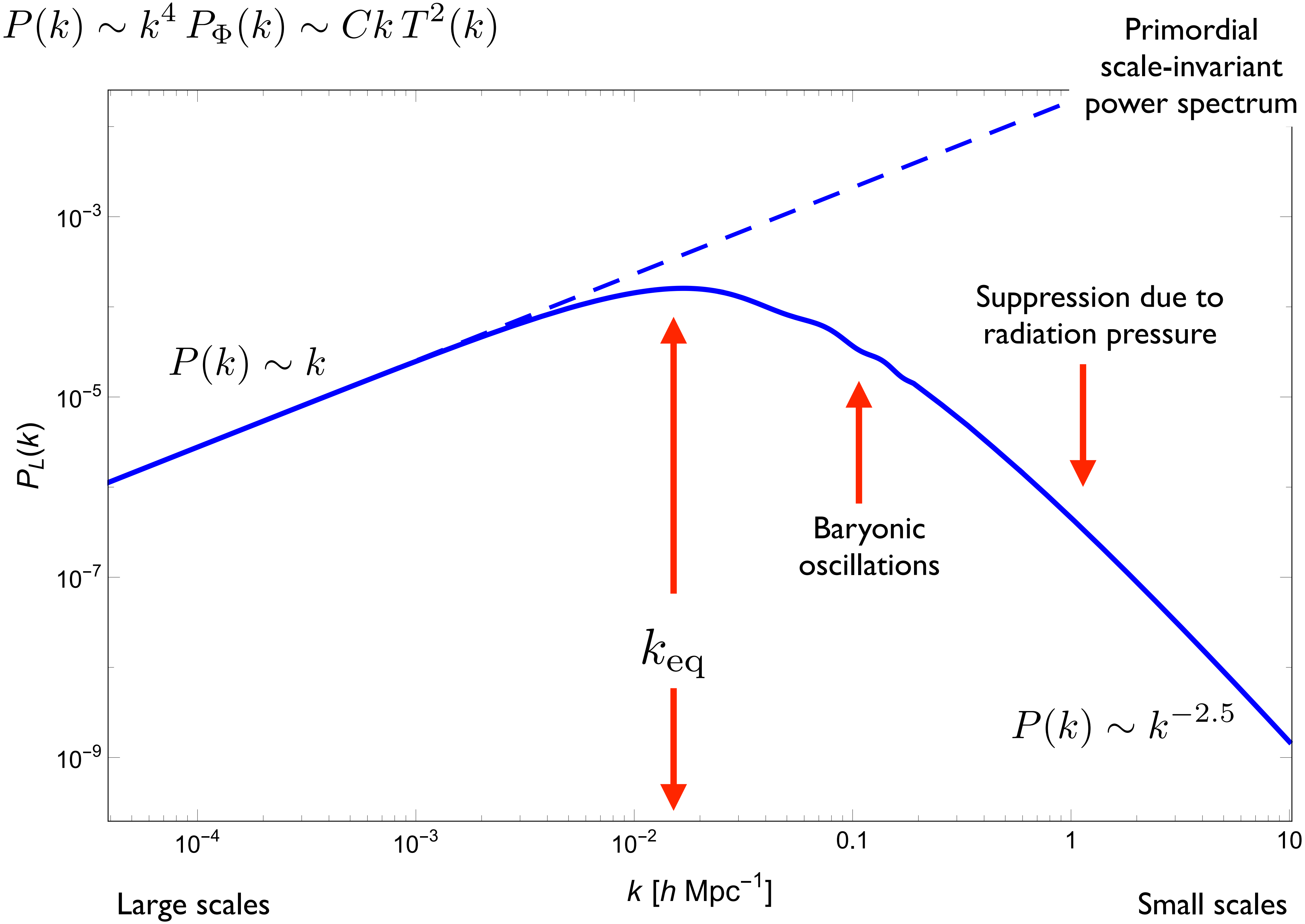
Poisson equation

matter power spectrum

The “initial” matter power spectrum

The linear matter power spectrum at recombination, $z \sim 1100$

$$P(k) \sim k^4\, P_\Phi(k) \sim C k\, T^2(k)$$



Linear Eulerian Perturbation Theory

Evolution of matter perturbations

We will consider now the following approximations for the evolution of matter perturbations:

1. **All matter is cold** (ignore the effects of baryons & neutrinos)

2. **Newtonian approximation:**
   $k \gg a\,H(a)$ scales much smaller than the horizon
   $v \ll c$ velocities much smaller than the speed of light

3. **Matter domination** (ignore effects of dark energy at late times)

Vlasov equation

PT review: Bernardeau *et al.* (2002)

Phase-space conservation for the particle number density $f(\tau, \vec{x}, \vec{p})$

$$\frac{df}{d\tau} \equiv \frac{\partial f}{\partial \tau} + \frac{\vec{p}}{am} \cdot \vec{\nabla} f - a\, m\, \vec{\nabla}\Phi \cdot \vec{\nabla}_p f = 0$$

Comoving coordinates $\vec{x} = \vec{r}/a$ and conformal time τ

$$\int d^3p\, f(\tau, \vec{x}, \vec{p}) = \rho(\tau, \vec{x})$$ density

$$\int d^3p\, \frac{\vec{p}}{am} f(\tau, \vec{x}, \vec{p}) = \rho(\tau, \vec{x})\, \vec{u}(\tau, \vec{x})$$ peculiar velocity field, $\vec{u}$

$$\int d^3p\, \frac{p_i p_j}{a^2 m^2} f(\tau, \vec{x}, \vec{p}) = \rho(\tau, \vec{x})\, u_i(\tau, \vec{x})\, u_j(\tau, \vec{x}) + \sigma_{ij}(\tau, \vec{x})$$

stress-tensor $$\sigma_{ij} = -P\delta^K_{ij} + \eta\left(\nabla_i u_j + \nabla_j u_i - \frac{2}{3}\delta^K_{ij}\vec{\nabla}\cdot\vec{u}\right) + \zeta\delta^K_{ij}\vec{\nabla}\cdot\vec{u}$$

pressure viscosity

Vlasov equation

Phase-space conservation for the particle number density $f(\tau, \vec{x}, \vec{p})$

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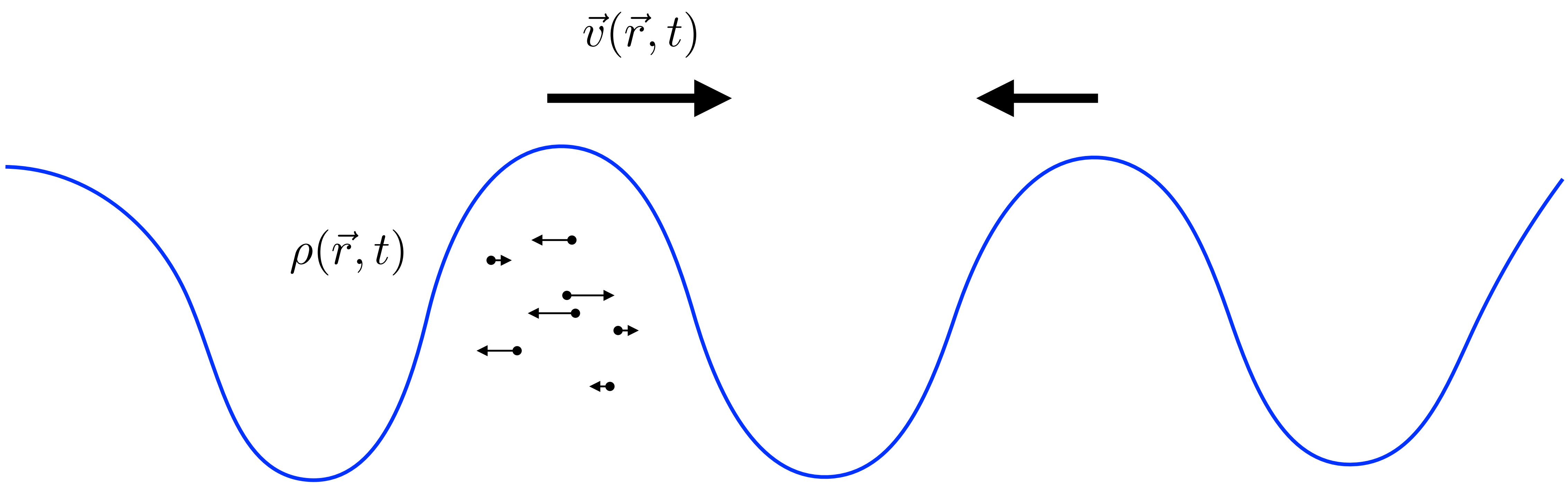
$\sigma_{ij} = 0$ **Single-stream approximation**

→ Closed set of equations for density and velocity

Single-stream approximation

for **Cold** Dark Matter we can ignore the
thermal motion of individual particles,
and study the evolution of **perturbations**



Fluid equations for the perturbations

Phase-space conservation for the particle number density $f(\tau, \vec{x}, \vec{p})$

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Comoving coordinates $\vec{x} = \vec{r}/a$ and conformal time τ

$$\int d^3p \, \frac{df}{d\tau} = 0 \quad \rightarrow \quad \boxed{\frac{\partial \delta}{\partial \tau} + \vec{\nabla} \cdot [(1+\delta)\,\vec{u}] = 0}$$

continuity equation
(conservation of mass)

$$\int d^3p \, \frac{p_i}{am} \frac{df}{d\tau} = 0 \quad \rightarrow \quad \boxed{\frac{\partial \vec{u}}{\partial \tau} + \mathcal{H}\vec{u} + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\vec{\nabla}\Phi}$$

Euler equation
(conservation of momentum)

$\sigma_{ij} = 0$ **Single-stream approximation**

Fluid equations

Phase-space conservation for the particle number density $f(\tau, \overrightarrow{x}, \overrightarrow{p})$

$$\frac{df}{d\tau} \equiv \frac{\partial f}{\partial \tau} + \frac{\vec{p}}{am} \cdot \vec{\nabla} f - a\,m\,\vec{\nabla}\Phi \cdot \vec{\nabla}_p f = 0$$

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Euler equation
(conservation of momentum)

$$+ \quad \boxed{\nabla^2 \Phi = 4\pi\, G\, a^2\, \bar{\rho}\, \delta}$$

Poisson equation

3 equations &
3 unknowns:
$\rho, \overrightarrow{u}, \Phi$

Linear equations for the perturbations

$$\frac{\partial \delta}{\partial \tau} + \vec{\nabla} \cdot [(1+\delta)\, \vec{u}] = 0$$

continuity equation

$$\frac{\partial \vec{u}}{\partial \tau} + \mathcal{H}\vec{u} + (\vec{u} \cdot \vec{\nabla})\, \vec{u} = -\vec{\nabla}\Phi$$

Euler equation

$$\nabla^2 \Phi = \frac{3}{2}\, \mathcal{H}^2 \delta$$

Poisson equation

Linear equations for the perturbations

$$\frac{\partial \delta}{\partial \tau} + \vec{\nabla} \cdot [(1 + \cancel{\delta)} \vec{u}] = 0$$

continuity equation

$$\frac{\partial \vec{u}}{\partial \tau} + \mathcal{H}\vec{u} + \cancel{(\vec{u} \cdot \vec{\nabla})\,\vec{u}} = -\vec{\nabla}\Phi$$

Euler equation

$$\nabla^2 \Phi = \frac{3}{2}\,\mathcal{H}^2 \delta$$

Poisson equation

Linear equations for the perturbations

$$\frac{\partial \delta}{\partial \tau} + \vec{\nabla} \cdot [(1 + \cancel{\delta)} \vec{u}] = 0$$

continuity equation

$$\vec{\nabla} \cdot \left(\frac{\partial \vec{u}}{\partial \tau} + \mathcal{H} \vec{u} + \cancel{(\vec{u} \cdot \vec{\nabla}) \vec{u}} = -\vec{\nabla} \Phi \right)$$

Euler equation

$$\nabla^2 \Phi = \frac{3}{2} \mathcal{H}^2 \delta$$

Poisson equation

then introducing the **velocity divergence** $\theta(\vec{x}, \tau) \equiv \vec{\nabla} \cdot \vec{u}(\vec{x}, \tau)$

Linear equations for the perturbations

$$\frac{\partial \delta}{\partial \tau} + \theta = 0$$ **continuity equation**

$$\frac{\partial \theta}{\partial \tau} + \mathcal{H}\theta + \frac{3}{2}\mathcal{H}^2\,\delta = 0$$ **Euler's equation**

$$\boxed{\frac{\partial^2 \delta}{\partial \tau^2} + \underbrace{\mathcal{H}\frac{\partial \delta}{\partial \tau}}_{\text{friction}} - \underbrace{\frac{3}{2}\mathcal{H}^2\,\delta}_{\text{gravity}} = 0}$$ 2nd order equation

where (for a flat, matter-dominated Universe) $\mathcal{H} = \frac{1}{a}\frac{da}{d\tau} = \frac{2}{\tau}$

Linear growth of perturbations

$$\frac{\partial^2 \delta_{\vec{k}}}{\partial \tau^2} + \mathcal{H}\frac{\partial \delta_{\vec{k}}}{\partial \tau} - \frac{3}{2}\mathcal{H}^2\, \delta_{\vec{k}} = 0$$

2nd order equation in Fourier space

Linear growth of perturbations

$$\frac{\partial^2 \delta_{\vec{k}}}{\partial \tau^2} + \mathcal{H} \frac{\partial \delta_{\vec{k}}}{\partial \tau} - \frac{3}{2} \mathcal{H}^2 \, \delta_{\vec{k}} = 0$$

2nd order equation in Fourier space

Look for a separable solution like $\delta_{\vec{k}}(\tau) = D(\tau) \, A_{\vec{k}}$ $D(\tau)$ **growth factor**

$$\rightarrow \begin{cases} D_+(a) \sim a & \text{growing mode} \\ D_-(a) \sim a^{-3/2} & \text{decaying mode} \end{cases}$$

$$\boxed{\begin{aligned} \delta_{\vec{k}}(a) &= A_{\vec{k}} \, a + B_{\vec{k}} \, a^{-3/2} \\ \theta_{\vec{k}}(a) &= -\frac{\partial \delta_{\vec{k}}}{\partial \tau} = -\mathcal{H} \left(A_{\vec{k}} \, a - \frac{3}{2} \, B_{\vec{k}} \, a^{-3/2} \right) \end{aligned}}$$

Linear growth of perturbations

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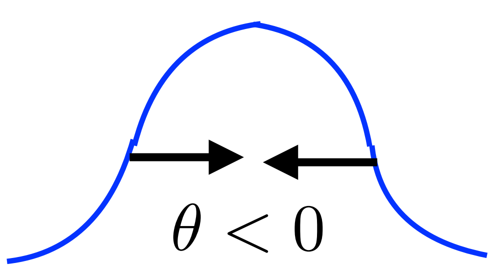
growing mode

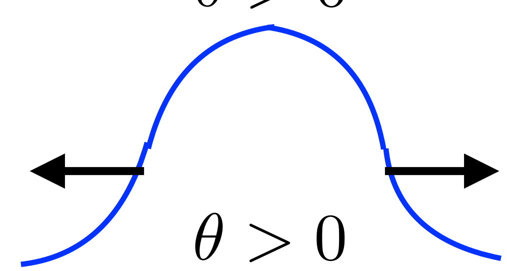
$\delta > 0$



$A_{\vec{k}} \neq 0 \quad B_{\vec{k}} = 0$

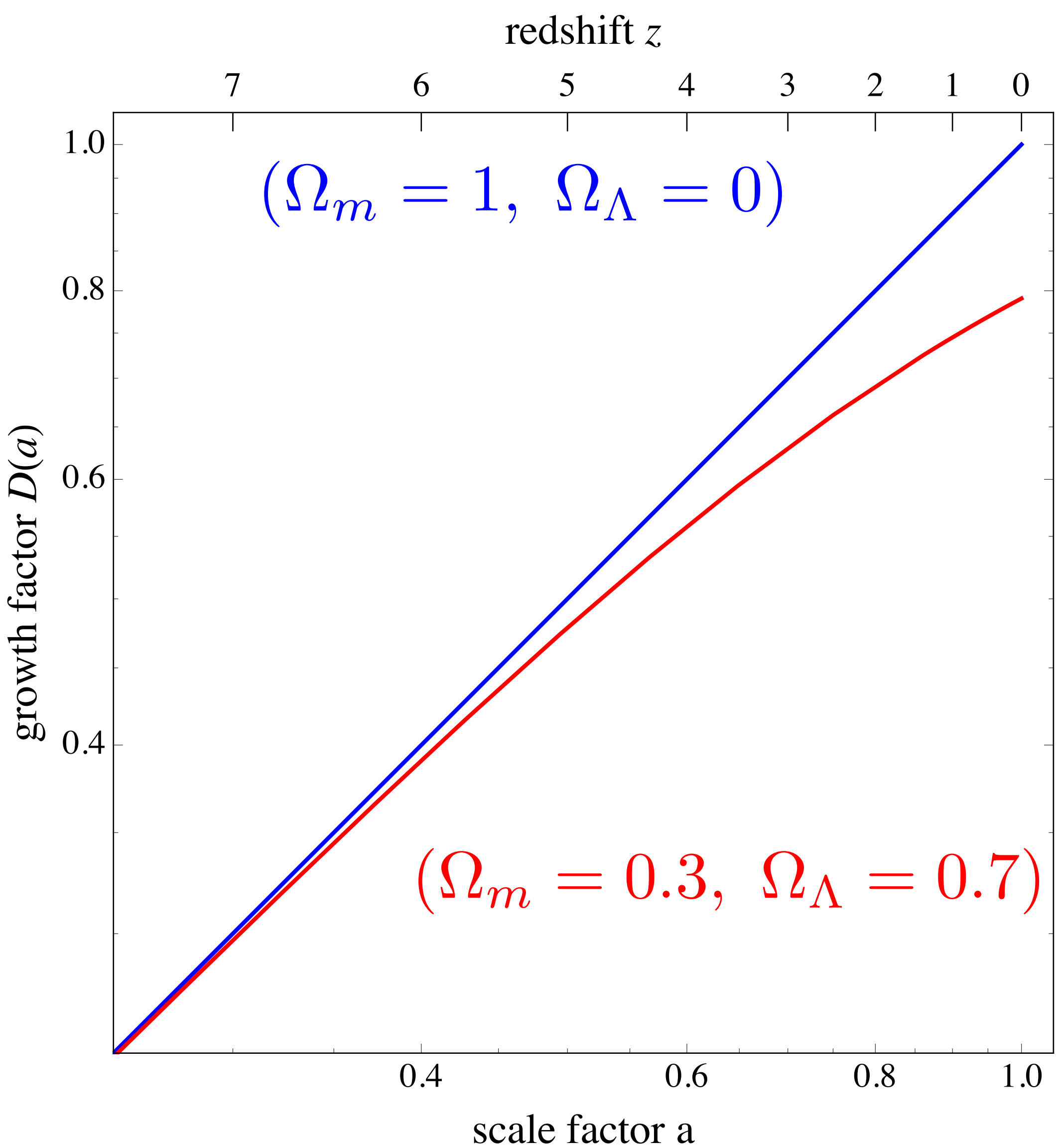
decaying mode

$\delta > 0$



$A_{\vec{k}} = 0 \quad B_{\vec{k}} \neq 0$

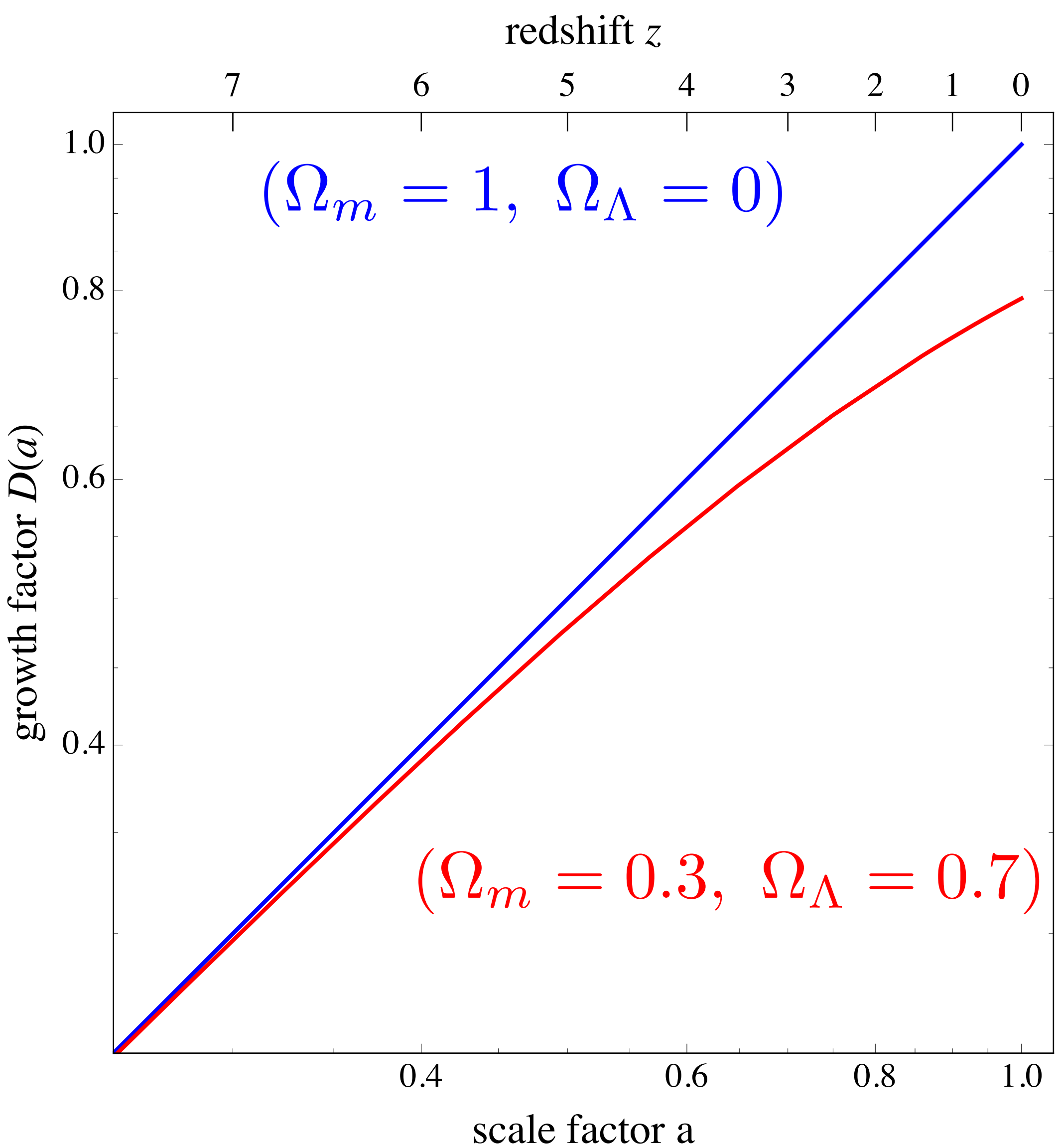
Linear growth in a ΛCDM cosmology



$$D_+(a) = \frac{5}{2} H_0^2 \,\Omega_{m,0}\, H(a) \int_0^a \frac{da'}{[a' H(a')]^3}$$

exact solution for the growth factor

Linear growth in a ΛCDM cosmology

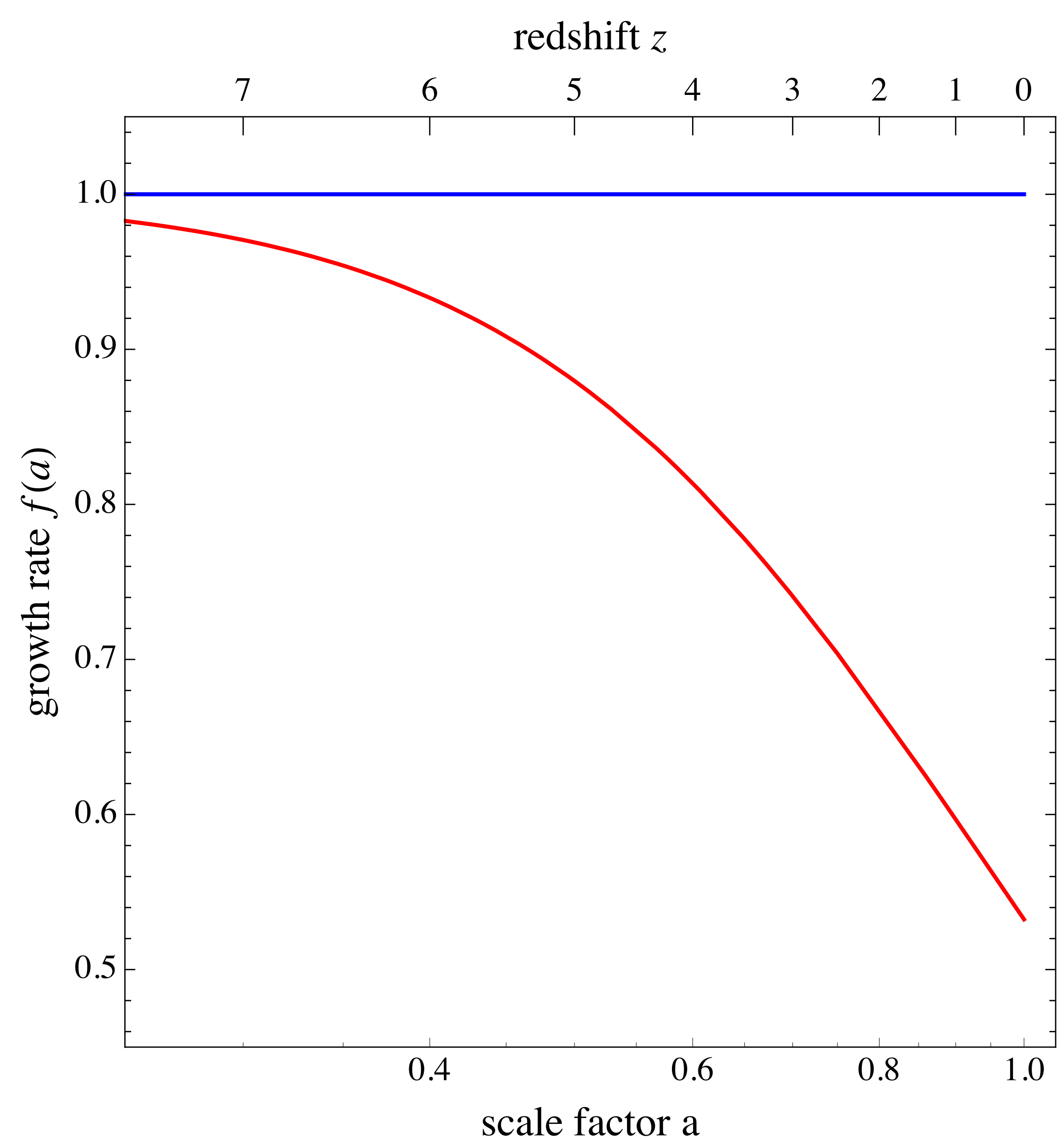


$$D_+(a) = \frac{5}{2} H_0^2 \, \Omega_{m,0} \, H(a) \int_0^a \frac{da'}{[a' H(a')]^3}$$

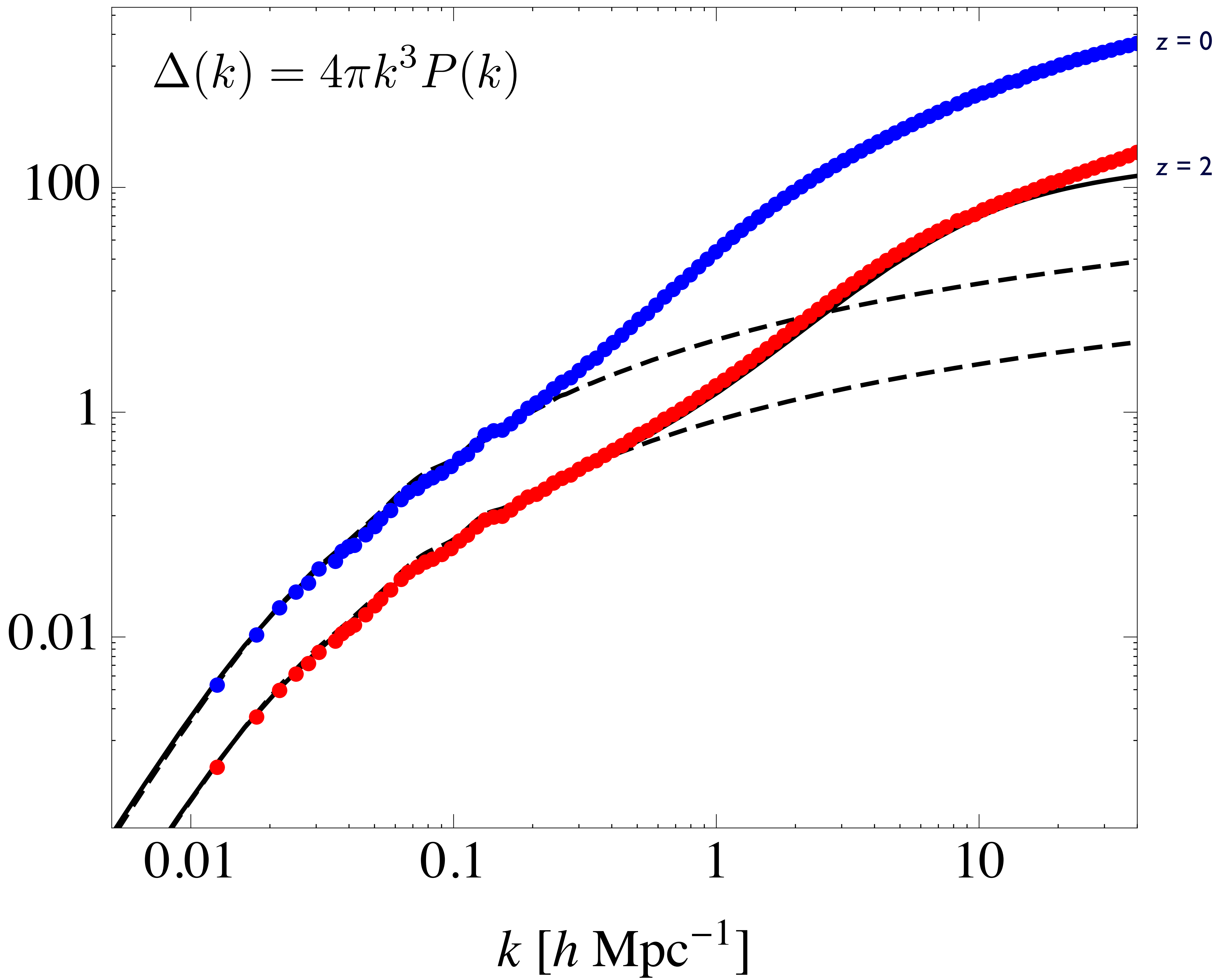
exact solution for the growth factor

growth *rate*

$$f \equiv \frac{d \ln D}{d \ln a} \simeq \Omega_m^\gamma(a)$$



Nonlinear growth of matter perturbations

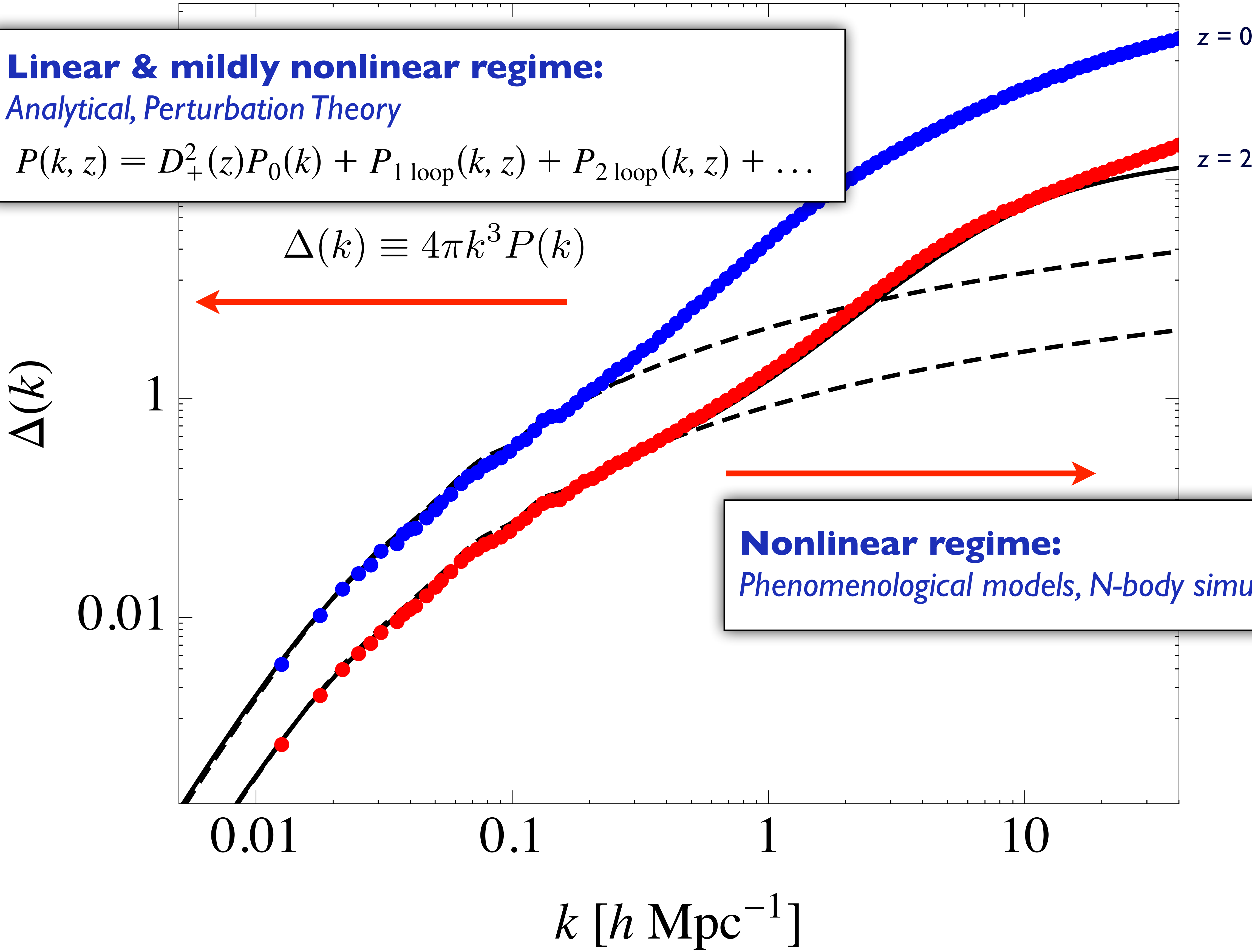


Nonlinear growth of matter perturbations

Linear & mildly nonlinear regime:
Analytical, Perturbation Theory

$$P(k,z) = D_+^2(z)P_0(k) + P_{1\,\text{loop}}(k,z) + P_{2\,\text{loop}}(k,z) + \ldots$$

$$\Delta(k) \equiv 4\pi k^3 P(k)$$



Nonlinear regime:
Phenomenological models, N-body simulations

Nonlinear Perturbation Theory

Back to the Equations of Motion $(\Omega_m = 1)$

Assuming **CDM as ideal fluid** we need the following equations:

$$\boxed{\frac{\partial \delta}{\partial \tau} + \theta + \vec{\nabla} \cdot (\delta \vec{u}) = 0}$$

continuity equation

$$\boxed{\frac{\partial \theta}{\partial \tau} + \mathcal{H}\theta + \vec{\nabla} \cdot [(\vec{u} \cdot \vec{\nabla})\vec{u}] = -\frac{3}{2}\mathcal{H}^2 \delta}$$

Euler equation

$$\eta = \ln a \qquad \vec{U} = -\frac{\vec{u}}{\mathcal{H}} \qquad \Theta = -\frac{\theta}{\mathcal{H}} \qquad \rightarrow$$

$$\frac{\partial \delta}{\partial \eta} + \Theta = -\vec{\nabla} \cdot (\delta \vec{U})$$

$$\frac{\partial \Theta}{\partial \eta} + \Theta + \frac{3}{2}\delta = -\vec{\nabla} \cdot [(\vec{U} \cdot \vec{\nabla})\vec{U}]$$

Nonlinear solutions in SPT ($\Omega_m = 1$)

We can rewrite things a bit …

$$\eta = \ln a \qquad \vec{U} = -\frac{\vec{u}}{\mathcal{H}} \qquad \Theta = -\frac{\theta}{\mathcal{H}}$$

$$\frac{\partial \delta}{\partial \eta} + \Theta = -\vec{\nabla} \cdot (\delta \vec{U})$$

continuity equation

$$\frac{\partial \Theta}{\partial \eta} + \Theta + \frac{3}{2}\delta = -\vec{\nabla} \cdot [(\vec{U} \cdot \vec{\nabla})\vec{U}]$$

Euler equation

In Fourier space …

$$\frac{\partial \delta_{\vec{k}}}{\partial \eta} + \Theta_{\vec{k}} = -\int d^3k_1 d^3k_2 \delta_D(\vec{k} - \vec{k}_{12}) \frac{\vec{k}_{12} \cdot \vec{k}_2}{k_2^2} \delta_{\vec{k}_1} \Theta_{\vec{k}_2}$$

$$\frac{\partial \Theta_{\vec{k}}}{\partial \eta} + \Theta_{\vec{k}} + \frac{3}{2}\delta_{\vec{k}} = -\int d^3k_1 d^3k_2\, \delta_D(\vec{k} - \vec{k}_{12})\, \frac{(\vec{k}_1 \cdot \vec{k}_2) k_{12}^2}{2k_1^2 k_2^2}\, \Theta_{\vec{k}_1} \Theta_{\vec{k}_2}$$

Nonlinear solutions in SPT ($\Omega_m = 1$)

Now we can look for *perturbative* solutions of the form

$$\delta_{\vec{k}} = \delta^{(1)}_{\vec{k}} + \delta^{(2)}_{\vec{k}} + \ldots$$

linear solution ($\sim a$) quadratic correction ($\sim a^2$)

$$\delta^{(2)}_{\vec{k}} = \int d^3q \, F_2(\vec{k}-\vec{q},\vec{q}) \, \delta^{(1)}_{\vec{k}-\vec{q}} \, \delta^{(1)}_{\vec{q}}$$

Then we can match order by order …

$$\frac{\partial \delta^{(1)}_{\vec{k}}}{\partial \eta} + \Theta^{(1)}_{\vec{k}} = 0$$

$$\frac{\partial \Theta^{(1)}_{\vec{k}}}{\partial \eta} + \Theta^{(1)}_{\vec{k}} + \frac{3}{2}\delta^{(1)}_{\vec{k}} = 0$$

$\longrightarrow$ linear solution

Nonlinear solutions in SPT ($\Omega_m = 1$)

Now we can look for *perturbative* solutions of the form

$$\delta_{\vec{k}} = \underbrace{\delta^{(1)}_{\vec{k}}}_{\text{linear solution } (\sim a)} + \underbrace{\delta^{(2)}_{\vec{k}}}_{\text{quadratic correction } (\sim a^2)} + \ldots$$

$$\delta^{(2)}_{\vec{k}} = \int d^3q \, F_2(\vec{k}-\vec{q}, \vec{q}) \, \delta^{(1)}_{\vec{k}-\vec{q}} \, \delta^{(1)}_{\vec{q}}$$

Then we can match order by order …

$$\frac{\partial \delta^{(2)}_{\vec{k}}}{\partial \eta} + \Theta^{(2)}_{\vec{k}} = -\int d^3k_1 d^3k_2 \delta_D(\vec{k}-\vec{k}_{12}) \frac{\vec{k}_{12}\cdot\vec{k}_2}{k_2^2} \delta^{(1)}_{\vec{k}_1} \Theta^{(1)}_{\vec{k}_2}$$

$$\underbrace{\frac{\partial \Theta^{(2)}_{\vec{k}}}{\partial \eta}}_{\sim a^2} + \underbrace{\Theta^{(2)}_{\vec{k}}}_{\sim a^2} + \underbrace{\frac{3}{2}\delta^{(2)}_{\vec{k}}}_{\sim a^2} = -\int d^3k_1 d^3k_2 \, \delta_D(\vec{k}-\vec{k}_{12}) \, \frac{(\vec{k}_1\cdot\vec{k}_2)k_{12}^2}{2k_1^2 k_2^2} \, \underbrace{\Theta^{(1)}_{\vec{k}_1}}_{\sim a} \underbrace{\Theta^{(1)}_{\vec{k}_2}}_{\sim a}$$

$$\rightarrow \quad F_2(\vec{k}_1, \vec{k}_2) = \frac{2}{7} + \frac{1}{2}\frac{(\vec{k}_1\cdot\vec{k}_2)}{k_1 k_2}\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right) + \frac{5}{7}\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{k_1^2 k_2^2}$$

Nonlinear solutions in SPT (ΛCDM)

Poisson equation now reads

$$\nabla^2 \Phi = \frac{3}{2} \mathcal{H}^2 \, \Omega_m \, \delta$$

NB: *time-dependent* $\Omega_m(a)$

and, using the linear growth factor as time variable and defining now $\Theta \equiv -\,\theta/(f\,\mathscr{H})$, $f = d\ln D/d\ln a$ being the growth rate

the Euler equation becomes

$$\frac{\partial \Theta_{\vec{k}}}{\partial \eta} + \left(\frac{3}{2} \frac{\Omega_m}{f^2} - 1 \right) \Theta_{\vec{k}} + \frac{3}{2} \frac{\Omega_m}{f^2} \delta_{\vec{k}} = - \int \delta_D(\ldots) \, \frac{(\vec{k}_1 \cdot \vec{k}_2) k_{12}^2}{2 k_1^2 k_2^2} \, \Theta \Theta$$

… not separable anymore! However, it happens that $f \simeq \Omega_m^{0.55}(a)$, and so

$$\frac{\Omega_m}{f^2} \simeq 1$$

$$\delta_{\vec{k}} = \underset{\sim D(a)}{\delta_{\vec{k}}^{(1)}} + \underset{\sim D^2(a)}{\delta_{\vec{k}}^{(2)}} + \ldots$$

and the kernels derived for EdS ($\Omega_m = 1$) are still a very good approximation

Non-Gaussianity from nonlinear evolution

From the perturbative solution for the matter density we obtain a perturbative solution for the matter 3-point function, or, in Fourier-space, the bispectrum

$$\langle\delta\delta\delta\rangle = \langle\delta^{(1)}\delta^{(1)}\delta^{(1)}\rangle + \langle\delta^{(1)}\delta^{(1)}\delta^{(2)}\rangle + \ldots$$

loop corrections

= 0 for Gaussian initial conditions

non-zero bispectrum *induced by gravity*

The leading order (tree-level) expression is

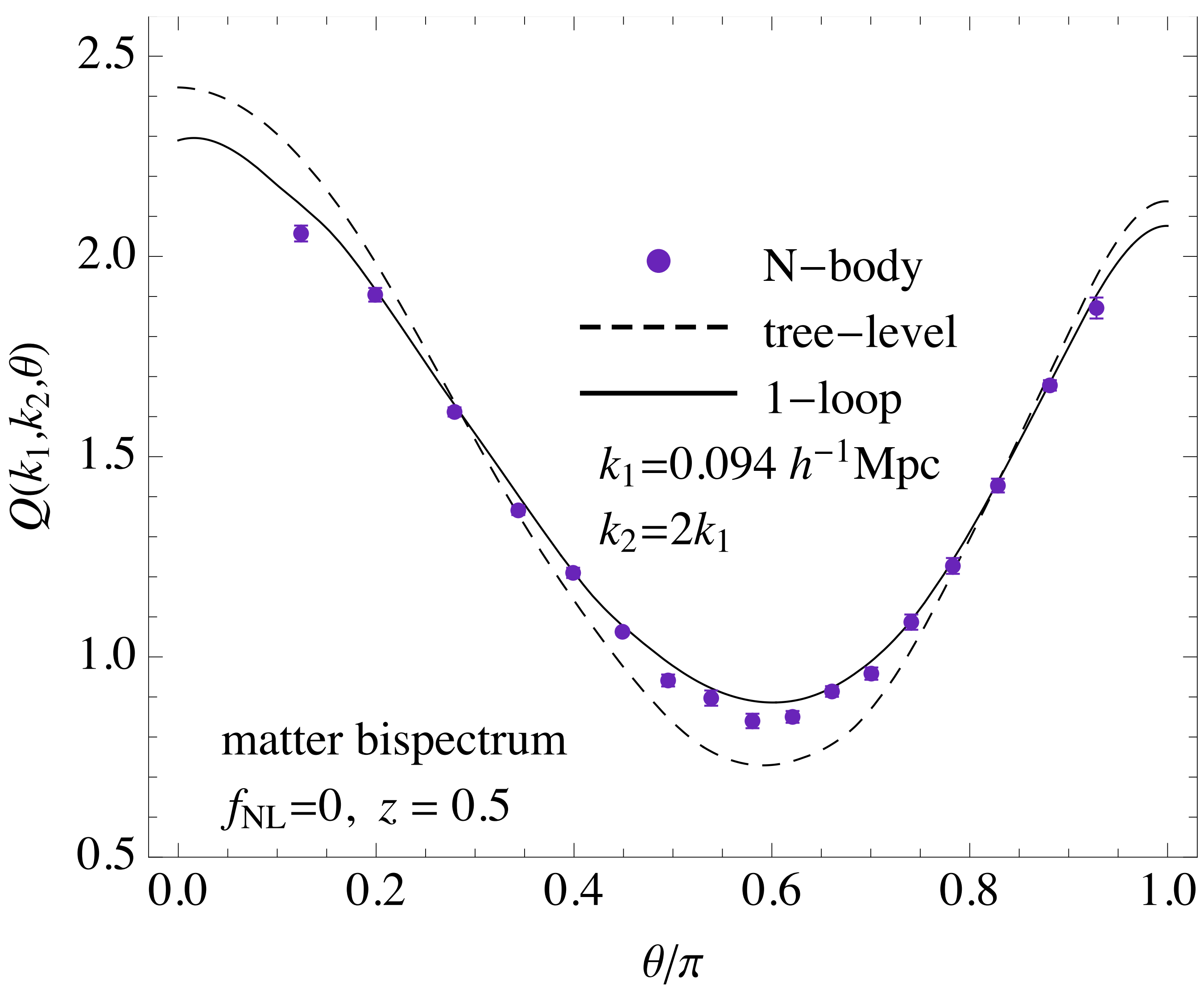
$$B(k_1, k_2, k_3) = F_2(\vec{k}_1, \vec{k}_2)\, P_L(k_1)\, P_L(k_2) + 2 \text{ perm.}$$

A (very specific) non-Gaussianity *is induced* by the nonlinear evolution

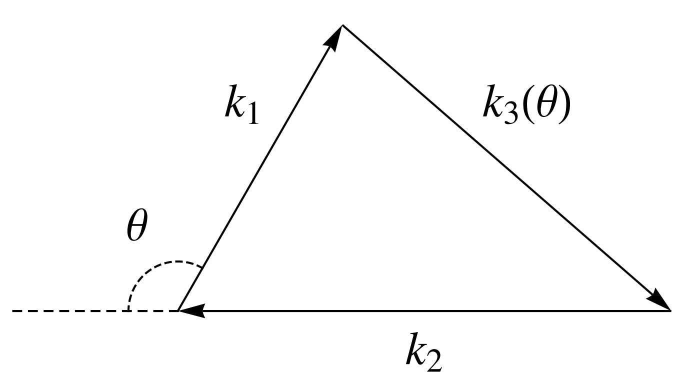
The matter bispectrum

A reduced bispectrum:

$$Q(k_1, k_2, k_3) = \frac{B(k_1, k_2, k_3)}{P(k_1)P(k_2) + P(k_1)P(k_3) + P(k_2)P(k_3)}$$

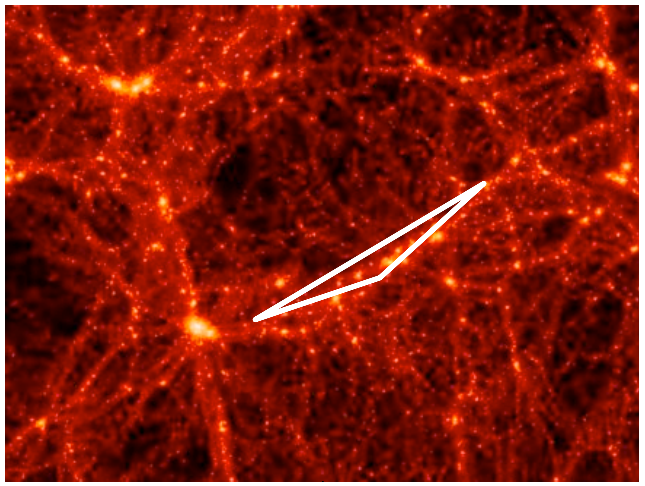


Plot of the reduced bispectrum with **fixed** k_1 and k_2 as a function of the **angle** between the two wavenumbers

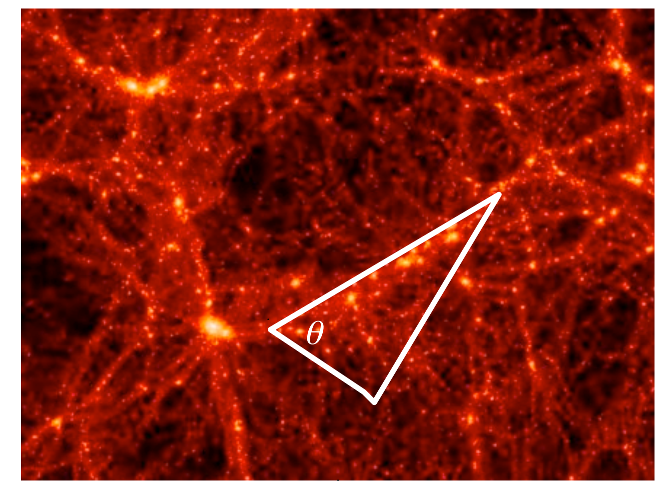


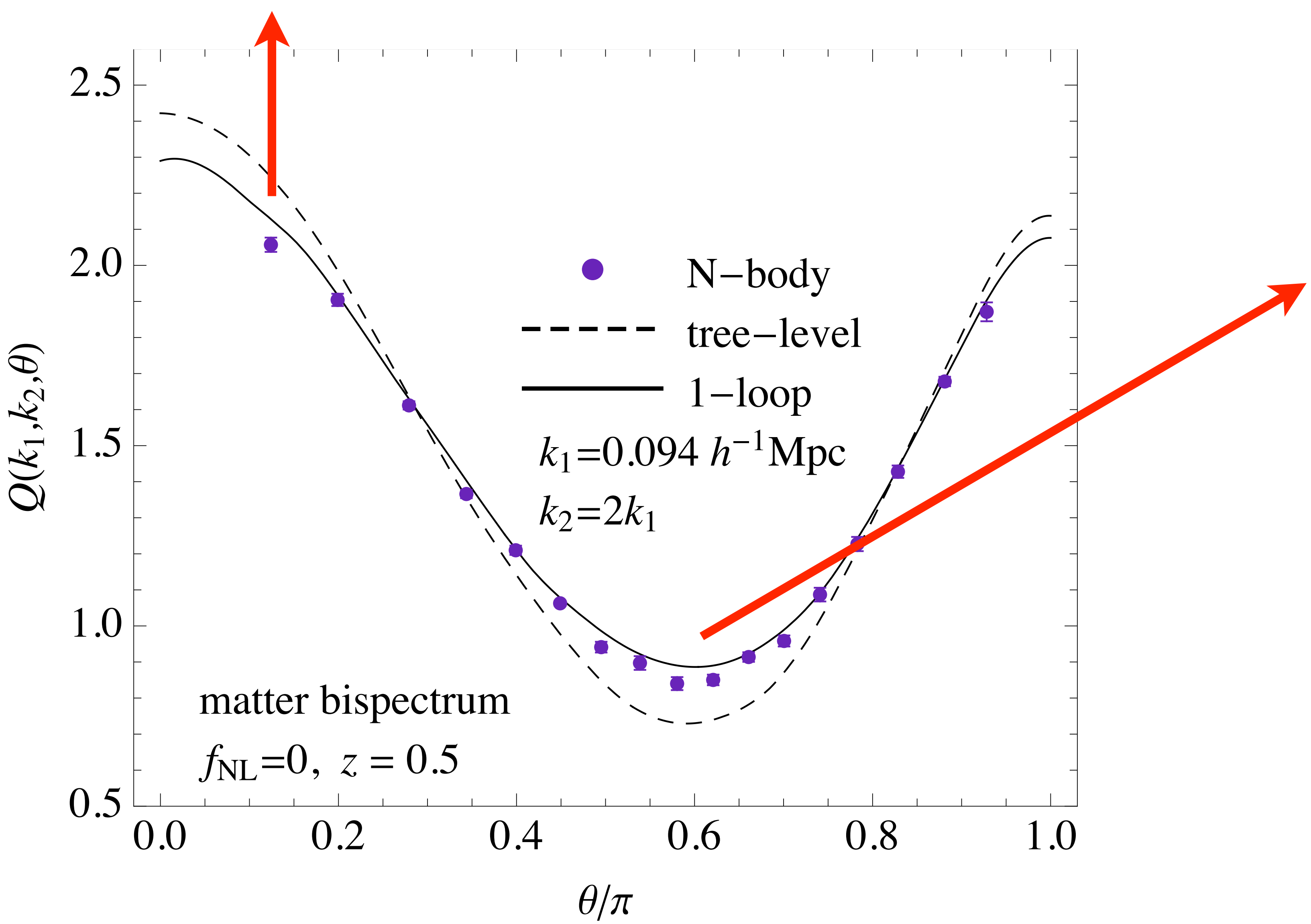
The matter bispectrum

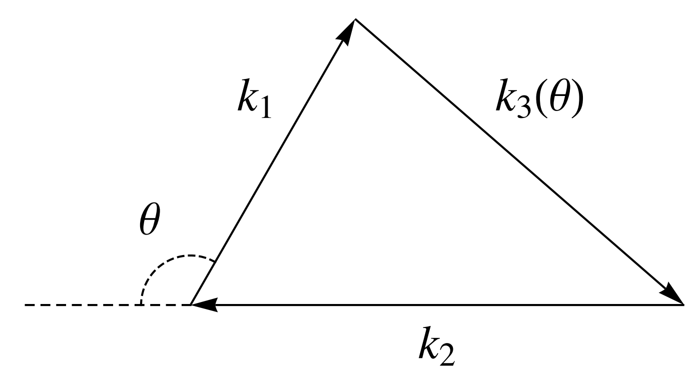
We can predict (quantitatively) the non-Gaussianity we recognise (qualitatively) in the LSS



$$Q(k_1, k_2, k_3) = \frac{B(k_1, k_2, k_3)}{P(k_1)P(k_2) + P(k_1)P(k_3) + P(k_2)P(k_3)}$$







The nonlinear Power Spectrum in SPT

Again, from the perturbative solution for the matter density we obtain a perturbative solution for nonlinear matter power spectrum

$$\delta_D(\vec{k}_{12})P(k) \equiv \langle \delta_{\vec{k}_1}\delta_{\vec{k}_2}\rangle =$$

$$\langle \delta^{(1)}_{\vec{k}_1}\delta^{(1)}_{\vec{k}_2}\rangle + \cancel{\langle \delta^{(1)}_{\vec{k}_1}\delta^{(2)}_{\vec{k}_2}\rangle + \text{perm.}} + \langle \delta^{(2)}_{\vec{k}_1}\delta^{(2)}_{\vec{k}_2}\rangle + \langle \delta^{(1)}_{\vec{k}_1}\delta^{(3)}_{\vec{k}_2}\rangle + \text{perm.} + \mathcal{O}(\delta_L^5)$$

Linear power spectrum $P_L(k)$

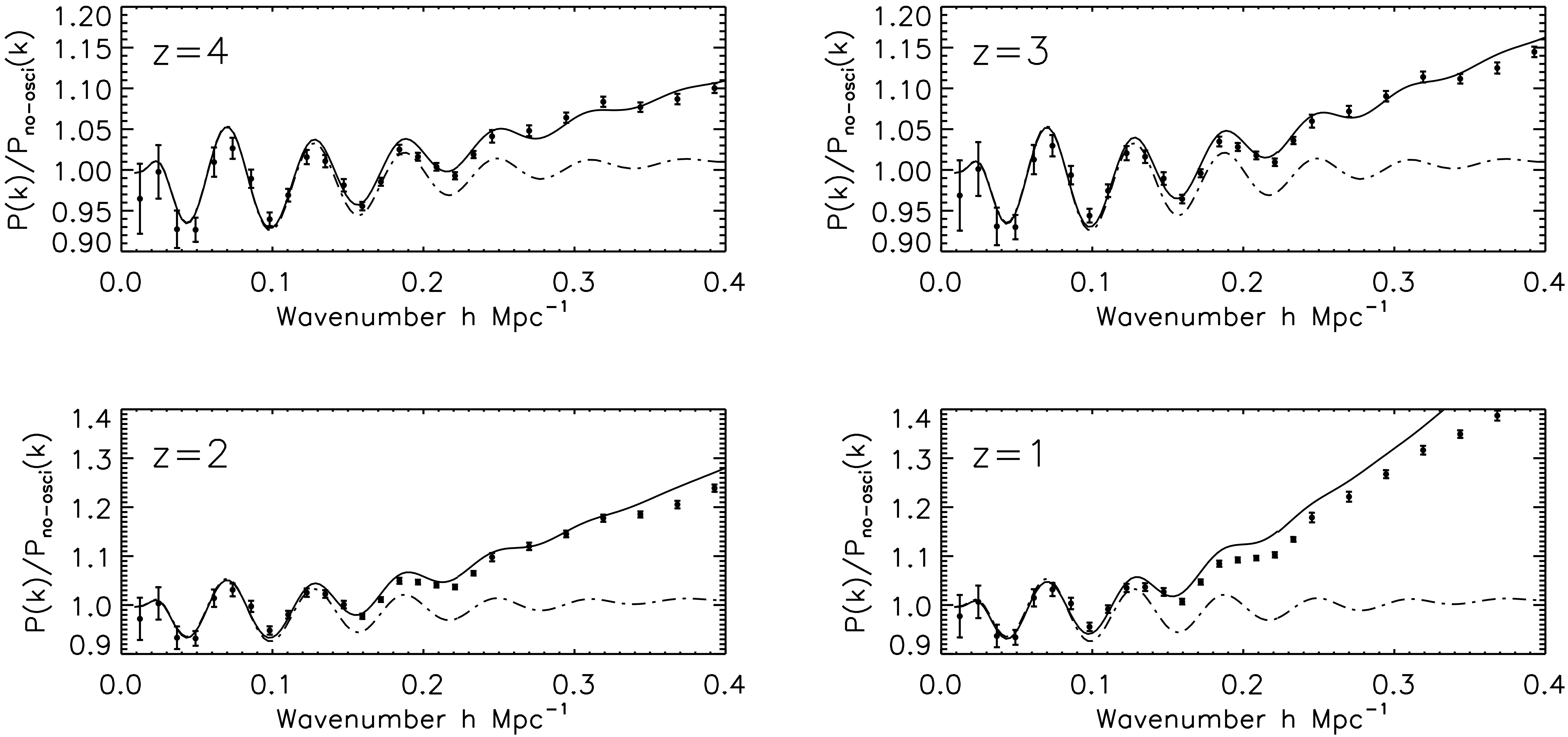
$\sim \langle \delta^{(1)}\delta^{(1)}\delta^{(1)}\rangle = 0$ for Gaussian initial conditions

$P_{22}(k)$ and $P_{13}(k)$ one-loop corrections

$$P_{22}(k) = 2\int d^3q\, F_2(\vec{q},\vec{k}-\vec{q})\, P_L(q)\, P_L(|\vec{k}-\vec{q}|)$$

$$P_{13}(k) = 6\, P_L(k)\int d^3q\, F_3(\vec{k},\vec{q},\vec{k}-\vec{q})\, P_L(q)$$

The matter power spectrum at one-loop



Jeong & Komatsu (2006)

Some problems with Standard PT

- No small parameters (unlike QED)
- The expansion is ill-defined
- The convergence of the loop integrals is accidental …

$$P_{22}(k) = 2\int d^3q\, F_2(\vec{q}, \vec{k}-\vec{q})\, P_L(q)\, P_L(|\vec{k}-\vec{q}|)$$

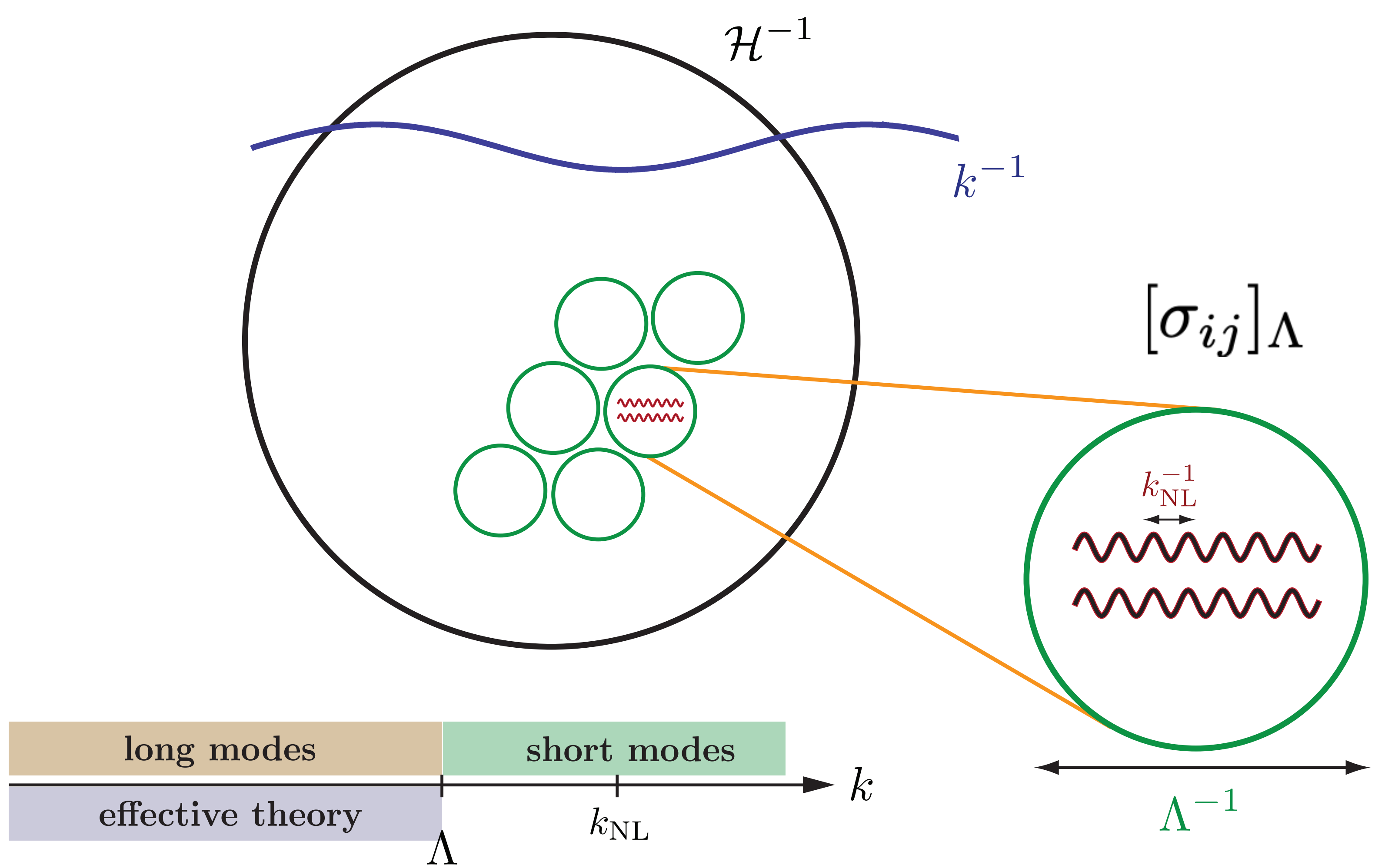
$$P_{13}(k) = 6\, P_L(k) \int d^3q\, F_3(\vec{k}, \vec{q}, \vec{k}-\vec{q})\, P_L(q)$$

Effective Field Theory of Large-Scale Structure

We still have the problem of how to deal with the small scale dynamics, or, more precisely, the *effect of small scales on large-scale perturbations*

$$\delta = \delta_l + \delta_s$$

$$\delta_l(\vec{x}) = \int d^3y\, W_\Lambda(|\vec{x} - \vec{y}|)\, \delta(\vec{y})$$



Even assuming a vanishing stress-tensor, $\sigma_{ij} = 0$ (as we did in the single-stream approximation), small-scale dynamics induces an ***effective stress tensor***, affecting the large-scale perturbations

Baumann, Nicolas, Senatore & Zaldarriaga (2010)
Carrasco, Hertzberg & Senatore (2012)

Effective Field Theory of Large-Scale Structure

We can expect an additional term in Euler equation

$$\frac{\partial\theta}{\partial\tau} + \mathcal{H}\theta + \vec{\nabla}\cdot[(\vec{u}\cdot\vec{\nabla})\vec{u}] = -\frac{3}{2}\mathcal{H}^2\delta \boxed{- \frac{1}{\bar{\rho}}\nabla_i\nabla_j\langle[\sigma_{ij}]_\Lambda\rangle}$$

with the effective stress-tensor depending on large-scale fluctuations

$$\langle[\sigma_{ij}]_\Lambda\rangle = \langle[\sigma_{ij}]_\Lambda\rangle_0 + \left.\frac{\partial\langle[\sigma_{ij}]_\Lambda\rangle}{\partial\delta_l}\right|_{\delta_l=0}\delta_l + \mathcal{O}(\delta_l^2)$$

FT

$$\xrightarrow{\text{FT}} \quad -\frac{1}{\bar{\rho}}k_i k_j\langle[\sigma_{ij}]_\Lambda\rangle \supset k^2\left(c_s^2\delta_l + c_v^2\frac{\theta_l}{f\mathcal{H}}\right) = k^2\, c_0\, \delta_l^{(1)}$$

our nonlinear solution for the matter density becomes

$$\delta_l = \delta_l^{(1)} + \delta_l^{(2)} + \delta_l^{(3)} + c_0\, k^2\, \delta_l^{(1)} + \ldots$$

with c_0 a free parameter …

The one-loop power spectrum in the EFTofLSS

The 2-point correlator gains a new contribution

$$\langle \delta_l \delta_l \rangle \supset \langle \delta_l^{(1)} \, c_0 \, k^2 \, \delta_l^{(1)} \rangle \sim c_0 \, k^2 \, P_L(k)$$

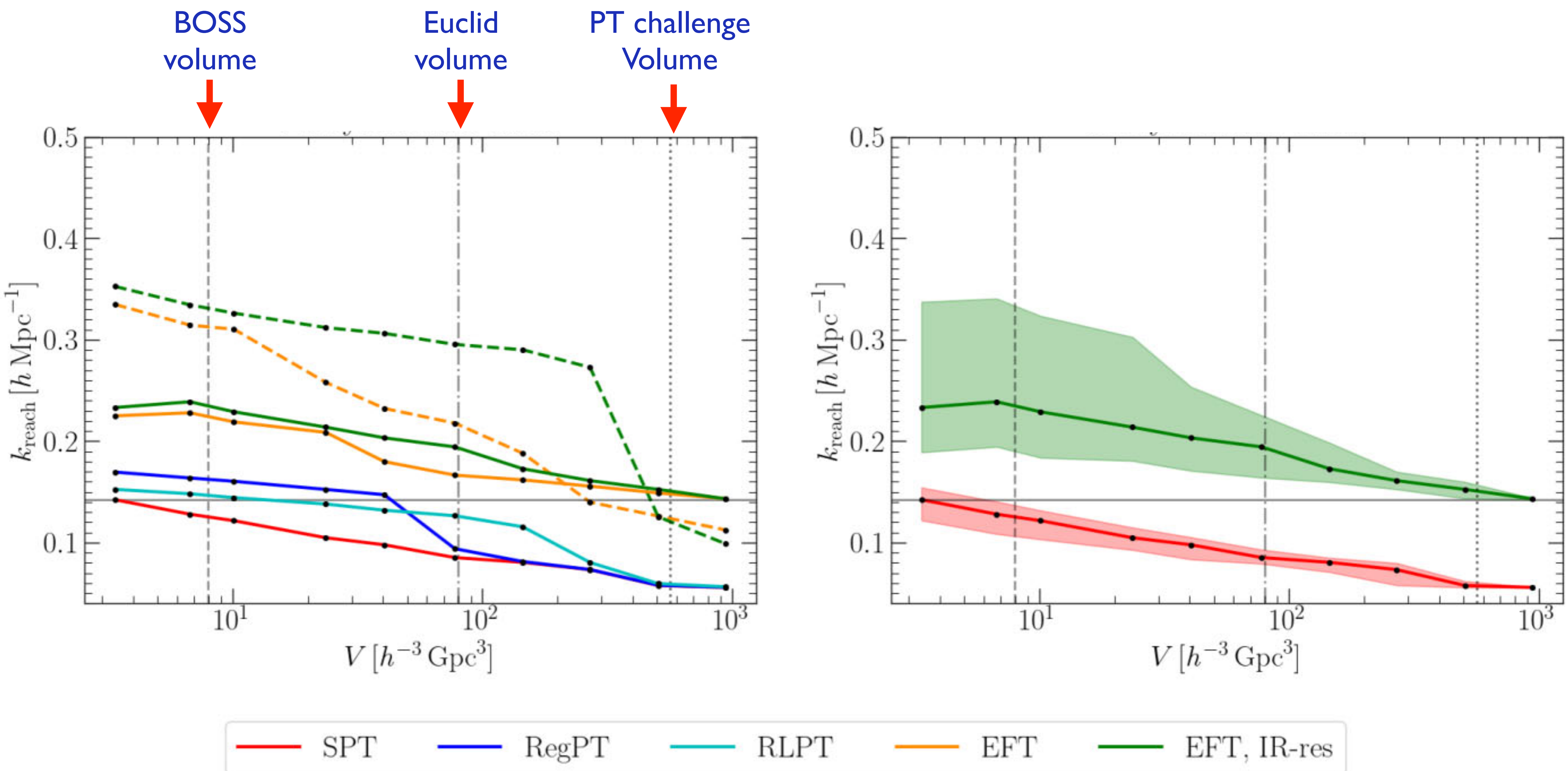
A counterterm regularising the one-loop integrals

$$P(k) = P_L(k) + P_{22}(k) + P_{13}(k) + c_0 \, k^2 \, P_L(k) + \mathcal{O}(\delta_l^6)$$

$$\int_0^\infty = \int_0^k + \int_k^\infty \quad \longrightarrow \quad P_{22}^{\mathrm{UV}} + P_{13}^{\mathrm{UV}} \simeq P_{13}^{\mathrm{UV}} \simeq \frac{16}{23} P_L(k) k^2 \int_k^\infty \frac{q}{2\pi^2} P_L(q)$$

The value of c_0 ensures the convergence of the integrals. In practice this is a nuisance parameters to be fixed in the comparison with data or simulations

The reach of PT models



Alkhanishvili *et al.* (2022)