Bispectrum and finite volume effects
window-convolution and wide-angle effects

– Future Cosmology School, Cargèse 2023 –

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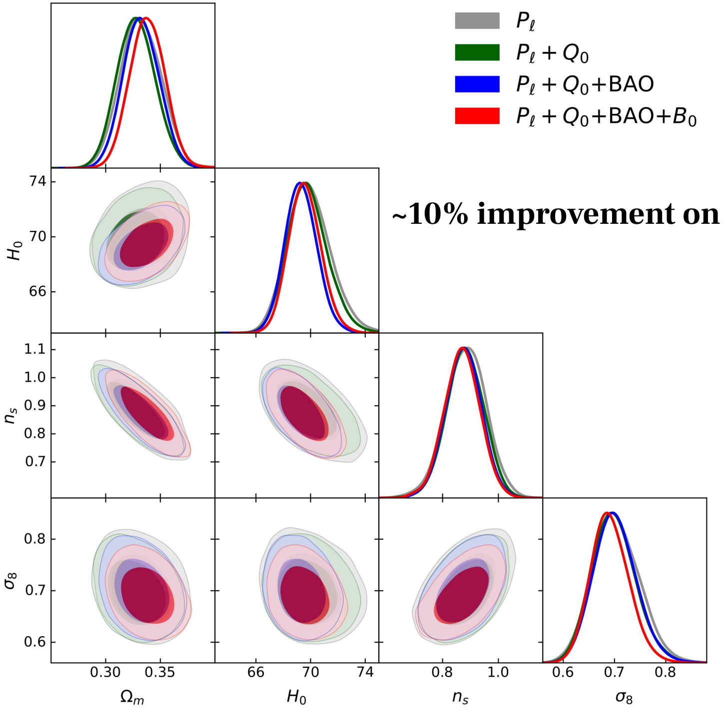
Including the bispectrum is useful (ex: BOSS DR12)

Constraint on cosmological params:

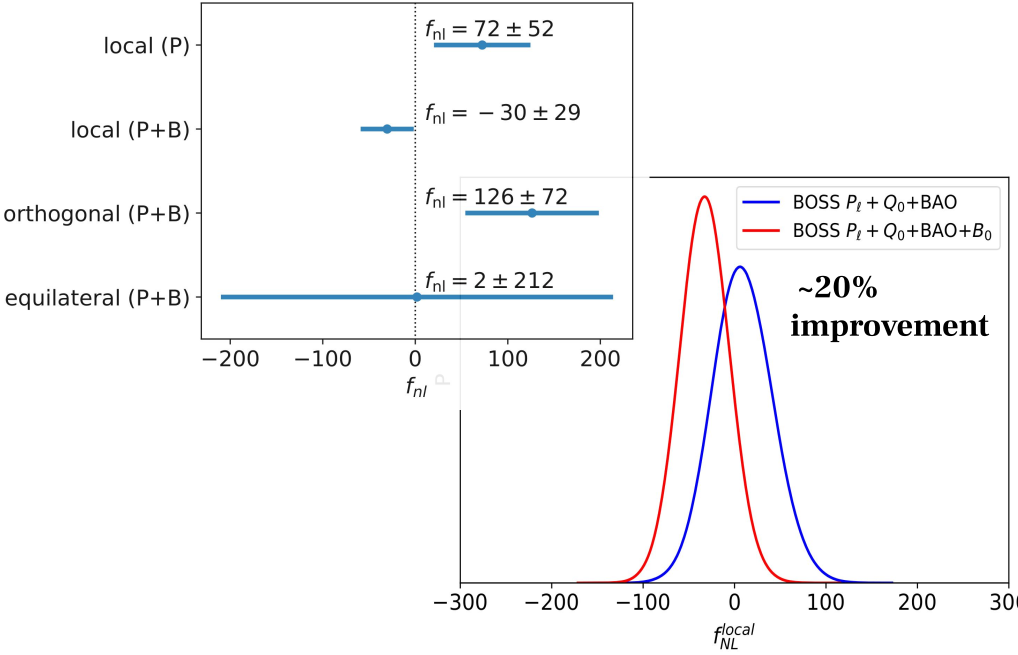


~10% improvement on σ_8

Philcox&Ivanov21 (also: D'Amico+19)

also one-loop bispectrum: Philcox+22, D'Amico+22b
also including bispectrum multipoles: D'Amico+22b, Ivanov+23

Constraint on primordial non-Gaussianity:



~20% improvement

Cabass+22b, D'Amico+22a

non-local PNG: Cabass+22a, bispectrum is **necessary**

The estimator

Scoccimarro estimator Scoccimarro15

FFT-based, optimal on small-scale

$$\hat{B}_L(k_1,k_2,k_3) = \frac{2L+1}{V_B V} \prod_{i=1}^{3} \left[\int_{k_i} d^3q_i \int_V d^3x_i e^{-i\mathbf{q}_i\cdot\mathbf{x}_i} \right] \delta_D(\mathbf{q}_{123}) \tilde{\delta}(\mathbf{x}_1)\tilde{\delta}(\mathbf{x}_2)\tilde{\delta}(\mathbf{x}_3)\mathcal{L}_L(\hat{\mathbf{q}}_1\cdot\hat{\mathbf{x}}_3)$$

$$\int_{k_1} d^3q_1 \equiv \int_{|k_1-\Delta k/2|\leq k_1 \leq |k_1+\Delta k/2|} d^3q_1 \qquad V_B \equiv \int_{k_1} d^3q_1 \int_{k_2} d^3q_2 \int_{k_3} d^3q_3 \delta_D(\mathbf{q}_{123})$$

Estimator is biased on large scale

Scoccimarro estimator Scoccimarro15

$$\hat{B}_L(k_1,k_2,k_3) = \frac{2L+1}{V_B V} \prod_{i=1}^{3} \left[\int_{k_i} d^3 q_i \int_V d^3 x_i e^{-i\mathbf{q}_i \cdot \mathbf{x}_i} \right] \delta_D(\mathbf{q}_{123}) \tilde{\delta}(\mathbf{x}_1) \tilde{\delta}(\mathbf{x}_2) \tilde{\delta}(\mathbf{x}_3) \mathcal{L}_L(\hat{\mathbf{q}}_1 \cdot \hat{\mathbf{x}}_3)$$

$$\langle \hat{B}_L \rangle \neq B_L$$

… but, the estimator is biased on large scale

ex: binning effect

Scoccimarro estimator Scoccimarro15

$$\hat{B}_L(k_1,k_2,k_3) = \frac{2L+1}{V_B V} \boxed{\prod_{i=1}^{3}\left[\int_{k_i} d^3q_i\right.} \left.\int_V d^3x_i e^{-i\mathbf{q}_i\cdot\mathbf{x}_i}\right] \delta_D(\mathbf{q}_{123})\tilde{\delta}(\mathbf{x}_1)\tilde{\delta}(\mathbf{x}_2)\tilde{\delta}(\mathbf{x}_3)\mathcal{L}_L(\hat{\mathbf{q}}_1\cdot\hat{\mathbf{x}}_3)$$

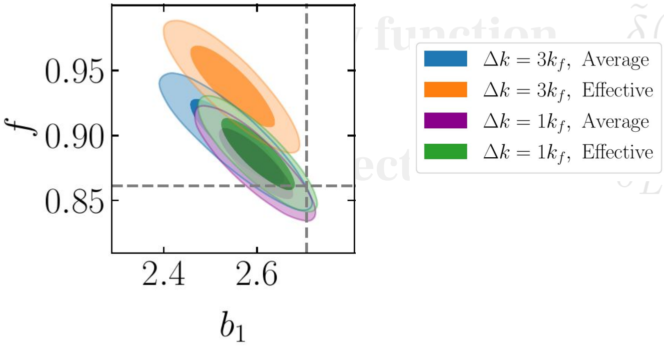
1. binning effect

$$\frac{1}{V_B}\prod_{i=1}^{3}\left[\int_{k_i} d^3q_i\right] \delta_D(\mathbf{q}_{123})$$

on periodic boxes ...

function $\tilde{\delta}(\mathbf{x}) = W(\mathbf{x})\delta(\mathbf{x})$

ect $\tilde{\delta}_L(\mathbf{q}) = \int d^3x\, \tilde{\delta}(\mathbf{x})\mathcal{L}_L(\mathbf{q}\cdot\mathbf{x})e^{-i\mathbf{q}\cdot\mathbf{x}}$



Rizzo, Moretti,Pardede+22

Additionally …

Scoccimarro estimator Scoccimarro15

$$\hat{B}_L(k_1,k_2,k_3) = \frac{2L+1}{V_B V}\prod_{i=1}^{3}\left[\int_{k_i} d^3q_i \int_V d^3x_i e^{-i\mathbf{q}_i\cdot\mathbf{x}_i}\right] \delta_D(\mathbf{q}_{123})\boxed{\tilde{\delta}(\mathbf{x}_1)}\tilde{\delta}(\mathbf{x}_2)\tilde{\delta}(\mathbf{x}_3)\boxed{\mathcal{L}_L(\hat{\mathbf{q}}_1\cdot\hat{\mathbf{x}}_3)}$$

1. binning effect $\frac{1}{V_B}\prod_{i=1}^{3}\left[\int_{k_i} d^3q_i\right] \delta_D(\mathbf{q}_{123})$

on real surveys …

2. window function* $\tilde{\delta}(\mathbf{x}) = W(\mathbf{x})\delta(\mathbf{x})$

3. LOS effect** $\tilde{\delta}_L(\mathbf{q}) \equiv \int d^3x\, \tilde{\delta}(\mathbf{x})\mathcal{L}_L(\mathbf{q}\cdot\mathbf{x})e^{-i\mathbf{q}\cdot\mathbf{x}}$

*window-free estimator Tegmark97, has been revived recently: Philcox20, Philcox21

**see also Milad Noorikuhani&Scoccimaro22

Survey window effects in the bispectrum

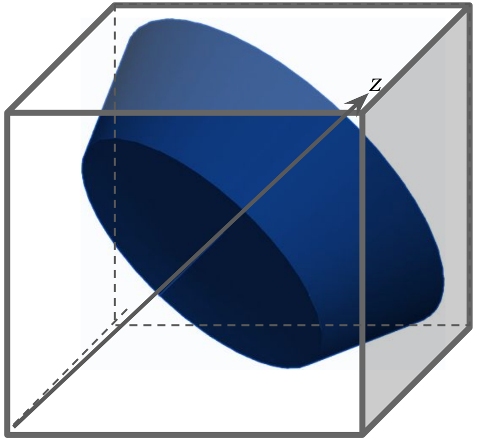
$$\hat{B}_L(k_1,k_2,k_3) = \frac{2L+1}{V_B V}\prod_{i=1}^{3}\left[\int_{k_i} d^3q_i \int_V d^3x_i e^{-i\mathbf{q}_i\cdot\mathbf{x}_i}\right] \delta_D(\mathbf{q}_{123})\tilde{\delta}(\mathbf{x}_1)\tilde{\delta}(\mathbf{x}_2)\tilde{\delta}(\mathbf{x}_3)\mathcal{L}_L(\hat{\mathbf{q}}_1\cdot\hat{\mathbf{x}}_3)$$

2. window function $\tilde{\delta}(\mathbf{x}) = W(\mathbf{x})\delta(\mathbf{x})$

Pardede, Rizzo, Biagetti, Castorina, Sefusatti, Monaco (arXiv: 2203.04174)

The window function

$$\hat{B}_L(k_1,k_2,k_3) = \frac{2L+1}{V_B V}\prod_{i=1}^{3}\left[\int_{k_i} d^3q_i \int_V d^3x_i e^{-i\mathbf{q}_i\cdot\mathbf{x}_i}\right] \delta_D(\mathbf{q}_{123})\tilde{\delta}(\mathbf{x}_1)\tilde{\delta}(\mathbf{x}_2)\tilde{\delta}(\mathbf{x}_3)\mathcal{L}_L(\hat{\mathbf{q}}_1\cdot\hat{\mathbf{x}}_3)$$



Baumgart, Fry 1991

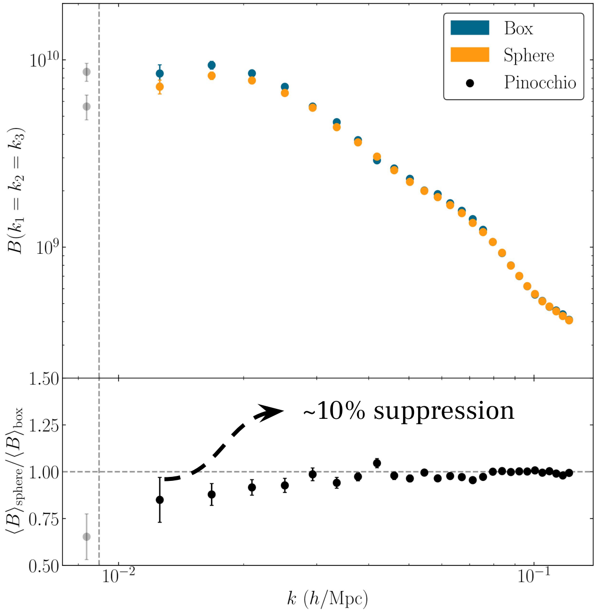
$$\tilde{\delta}(\mathbf{x}) = \underbrace{W(\mathbf{x})}_{\text{window function}}\delta(\mathbf{x})$$

$$\Rightarrow \tilde{\delta}(\mathbf{k}) = \int \frac{d^3k'}{(2\pi)^3} W(\mathbf{k}-\mathbf{k}')\delta(\mathbf{k}')$$

… in the bispectrum

$$\hat{B}_L(k_1,k_2,k_3) = \frac{2L+1}{V_B V}\prod_{i=1}^{3}\left[\int_{k_i} d^3q_i \int_V d^3x_i e^{-i\mathbf{q}_i\cdot\mathbf{x}_i}\right] \delta_D(\mathbf{q}_{123})\boxed{\tilde{\delta}(\mathbf{x}_1)}\tilde{\delta}(\mathbf{x}_2)\tilde{\delta}(\mathbf{x}_3)\mathcal{L}_L(\hat{\mathbf{q}}_1\cdot\hat{\mathbf{x}}_3)$$

Equilateral configurations



NOTE: huge total volume ~ 3500 (Gpc/h)^3

(schematically, for the monopole)

$$\tilde{B}(\vec{k}_1,\vec{k}_2) = \int \frac{d^3p_1}{(2\pi)^3}\int \frac{d^3p_2}{(2\pi)^3} B_W(\vec{k}_1-\vec{p}_1,\vec{k}_2-\vec{p}_2)B(\vec{p}_1,\vec{p}_2)$$

- 6D integral
- Time ~ **hours**/evaluation
- Not feasible for likelihood analysis

As a matrix multiplication

2DFFTLog

$$\tilde{B}_\ell[T_i] = \sum_{j,\ell'} \mathcal{M}_{\ell\ell'}[T_i, T'_j] B_{\ell'}[T'_j]$$

Mixing matrix
Computable via (2D) FFTLog
e.g. `2D-FFTLog` (Fang+20)
of the window 3PCF multipoles

Bispectrum
Function of three sides (k_1, k_2, k_3)

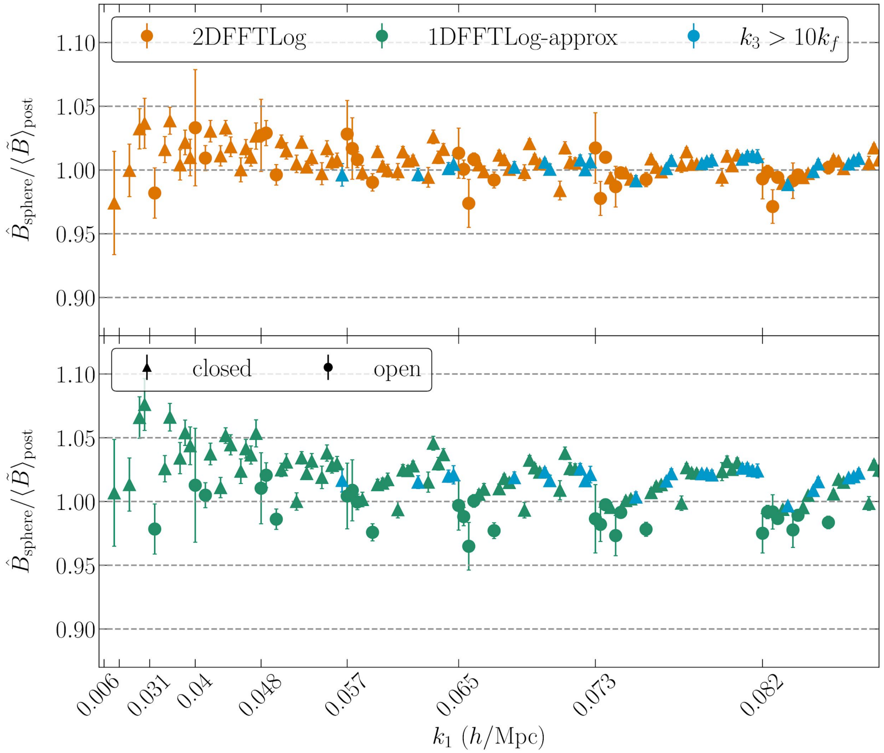
Each evaluation
~ **2 seconds**

⇒ comparable to a typical Boltzmann solver call

see also D. Alkanishvili+23 (using neural network)

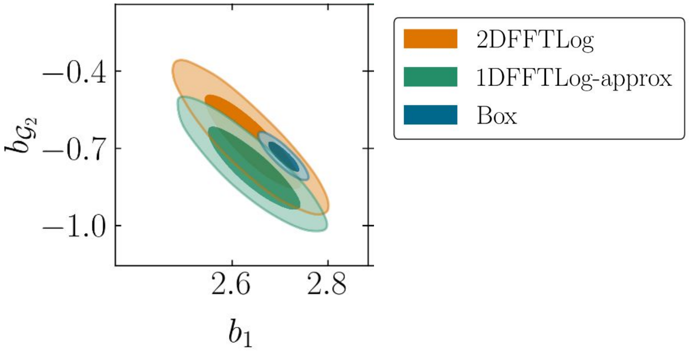
Spherical window convolution in real-space

Fit on **Pinocchio** mocks



volume ≈ 3500 [Gpc/h]^3

Analysis on **Minerva** data



volume ≈ 6 times the largest Euclid redshift bin

Wide-angle effects in the bispectrum

$$\hat{B}_L(k_1, k_2, k_3) = \frac{2L+1}{V_B V} \prod_{i=1}^{3} \left[\int_{k_i} d^3 q_i \int_V d^3 x_i e^{-i\mathbf{q}_i \cdot \mathbf{x}_i} \right] \delta_D(\mathbf{q}_{123}) \tilde{\delta}(\mathbf{x}_1) \tilde{\delta}(\mathbf{x}_2) \tilde{\delta}(\mathbf{x}_3) \boxed{\mathcal{L}_L(\hat{\mathbf{q}}_1 \cdot \hat{\mathbf{x}}_3)}$$

3. LOS effect

$$\tilde{\delta}_L(\mathbf{q}) \equiv \int d^3x \; \tilde{\delta}(\mathbf{x}) \mathcal{L}_L(\mathbf{q} \cdot \mathbf{x}) e^{-i\mathbf{q}\cdot\mathbf{x}}$$

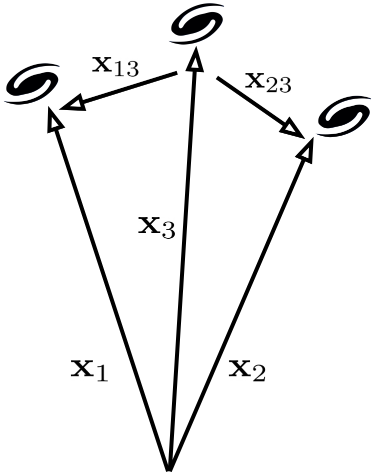
Pardede, Di Dio, Castorina (arXiv: 2302.12789)

Beyond flat-sky formulation for the bispectrum

$$\hat{B}_L(k_1,k_2,k_3) = \frac{2L+1}{V_B V}\prod_{i=1}^{3}\left[\int_{k_i} d^3q_i \int_V d^3x_i e^{-i\mathbf{q}_i\cdot\mathbf{x}_i}\right] \delta_D(\mathbf{q}_{123})\tilde{\delta}(\mathbf{x}_1)\tilde{\delta}(\mathbf{x}_2)\tilde{\delta}(\mathbf{x}_3)\boxed{\mathcal{L}_L(\hat{\mathbf{q}}_1\cdot\hat{\mathbf{x}}_3)}$$

need to go to the 3PCF first

$$\langle\delta(\mathbf{x}_1)\delta(\mathbf{x}_2)\delta(\mathbf{x}_3)\rangle = \int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3} e^{i\mathbf{p}_1\cdot\mathbf{x}_{13}} e^{i\mathbf{p}_2\cdot\mathbf{x}_{23}} \mathcal{F}(\mathbf{p}_1,\mathbf{p}_2,\hat{\mathbf{x}}_1,\hat{\mathbf{x}}_2,\hat{\mathbf{x}}_3)$$



expand perturbatively

$$\mathcal{F}(\mathbf{p}_1,\mathbf{p}_2,\hat{\mathbf{x}}_1,\hat{\mathbf{x}}_2,\hat{\mathbf{x}}_3) = \sum_{ij} \mathcal{F}^{(ij)}(\mathbf{p}_1,\mathbf{p}_2,\hat{\mathbf{x}}_{13},\hat{\mathbf{x}}_{23},\hat{\mathbf{x}}_3)\left(\frac{x_{13}}{x_3}\right)^i\left(\frac{x_{23}}{x_3}\right)^j$$

↓ to go back to Fourier space

$$\sum_{\ell_1\ell_2\cdots} \mathcal{F}^{(ij)}_{\ell_1\ell_2\cdots}(p_1,p_2)\mathcal{L}_{\ell_1}(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2)\mathcal{L}_{\ell_2}(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{x}}_3)\mathcal{L}_{\ell_3}(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{x}}_3)\cdots$$

(for all possible dot product combinations)

… another matrix multiplication

$$\hat{B}_L(k_1,k_2,k_3) = \frac{2L+1}{V_B V}\prod_{i=1}^{3}\left[\int_{k_i} d^3q_i \int_V d^3x_i e^{-i\mathbf{q}_i\cdot\mathbf{x}_i}\right] \delta_D(\mathbf{q}_{123})\tilde{\delta}(\mathbf{x}_1)\tilde{\delta}(\mathbf{x}_2)\tilde{\delta}(\mathbf{x}_3)\boxed{\mathcal{L}_L(\hat{\mathbf{q}}_1\cdot\hat{\mathbf{x}}_3)}$$

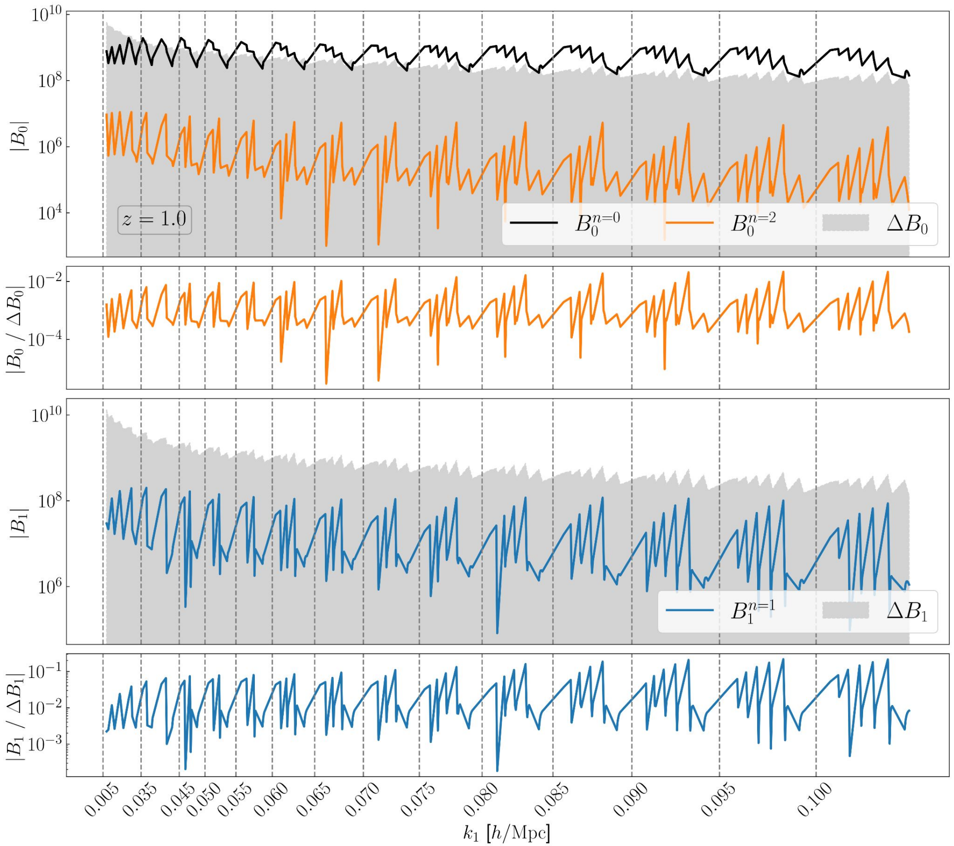
$$\Downarrow$$

(schematically)

$$B_L(k_1,k_2,k_3) = \sum_{i+j\le n,\ell_1,\ell_2,\cdots} B^{(n)}_{L,\ell_1,\ell_2,\ldots}(\mathbf{k}_1,\mathbf{k}_2)\mathcal{M}^{(n)}_{L,\ell_1,\ell_2,\cdots}\left(\frac{1}{k_1x_3}\right)^i\left(\frac{1}{k_2x_3}\right)^j$$

The wide-angle effects, how big?

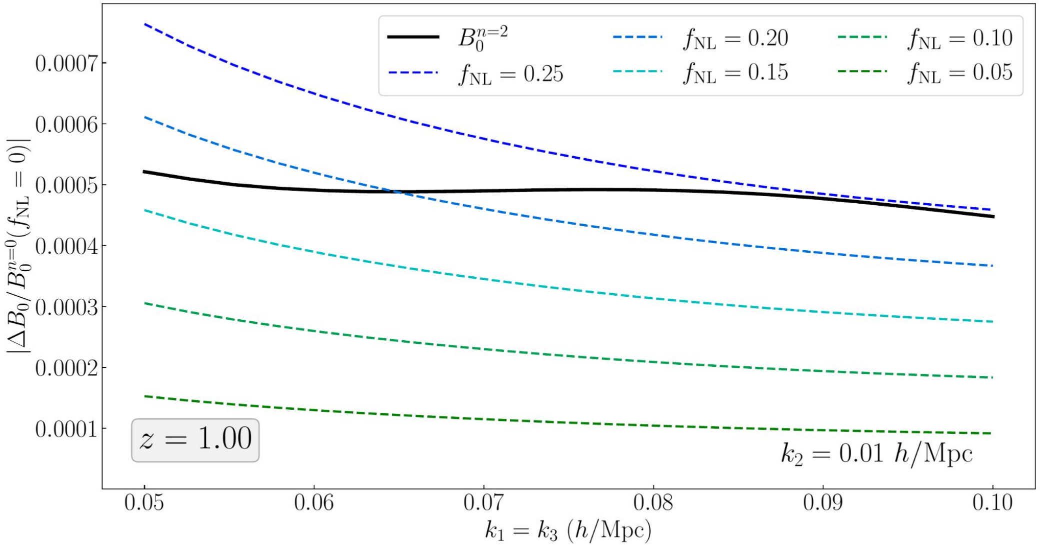
- Monopole correction only generated at second order ~ 0.1%
- Dipole correction ~ 1% of the flat-sky monopole
- S/N of dipole ~ $V/(8\ \mathrm{Gpc}^3\ h^{-3})$



comparison with Gaussian variance: $V = 8\ (\mathrm{Gpc}/h)^3$, n_g~$6.10^{-4}\ (h/\mathrm{Mpc})^3$ @ $z = 1$

Monopole correction ~ small local f_{NL}

- Monopole correction scales as k^{-2}
- Can mimic a local PNG signal



Summary

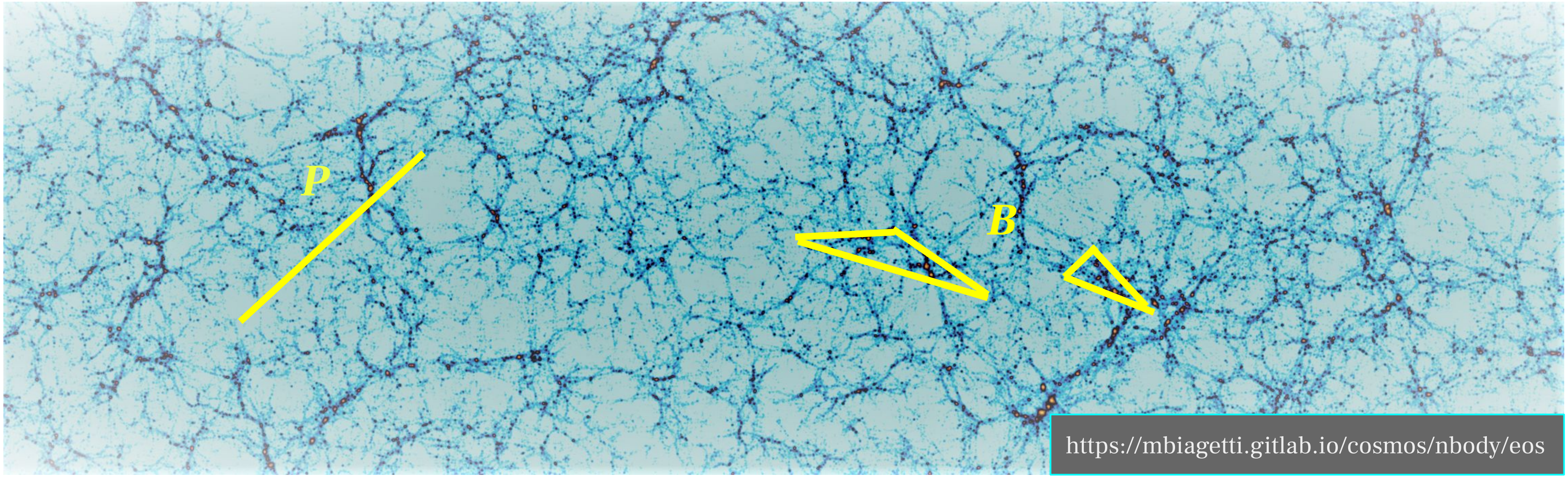
1. Including bispectrum multipoles analysis is useful but come with extra modelling complexity, ex: survey window effects and wide-angle effects
2. We gave an efficient formulation for bispectrum window convolution – which is tested in an ideal case of spherical window convolution in real space
3. Wide-angle corrections have a sizable effect to the bispectrum multipoles and can mimic actual physical signal, ex: ~0.1 local f_{NL}

Thank you!

-Extras-

Bispectrum captures the non-Gaussianity

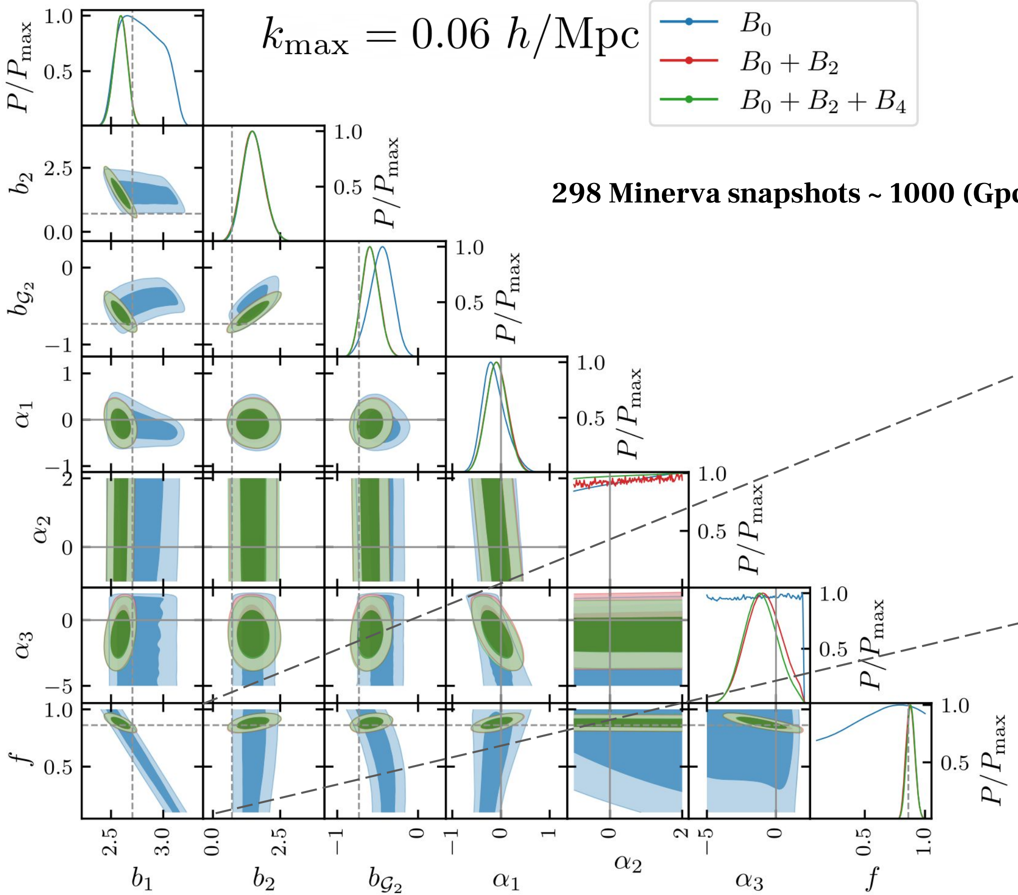
Power Spectrum (P) + **Bispectrum** (B) + …



$$\langle \delta(\mathbf{k}_1)\delta(\mathbf{k}_2)\delta(\mathbf{k}_3)\rangle = (2\pi)^3\delta_D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)B(\mathbf{k}_1,\mathbf{k}_2)$$

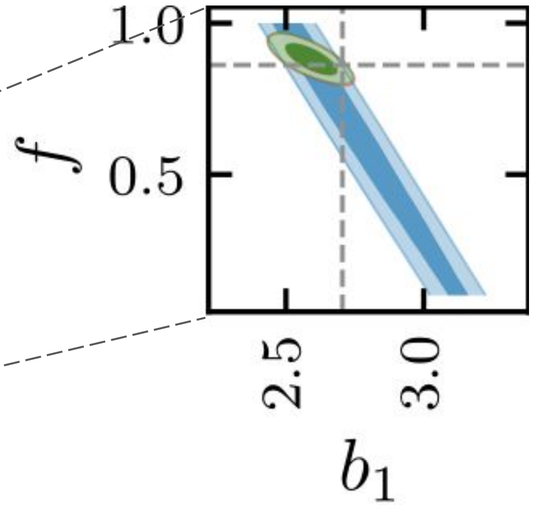
Inclusion of the bispectrum multipoles

$k_{\max} = 0.06\ h/\mathrm{Mpc}$



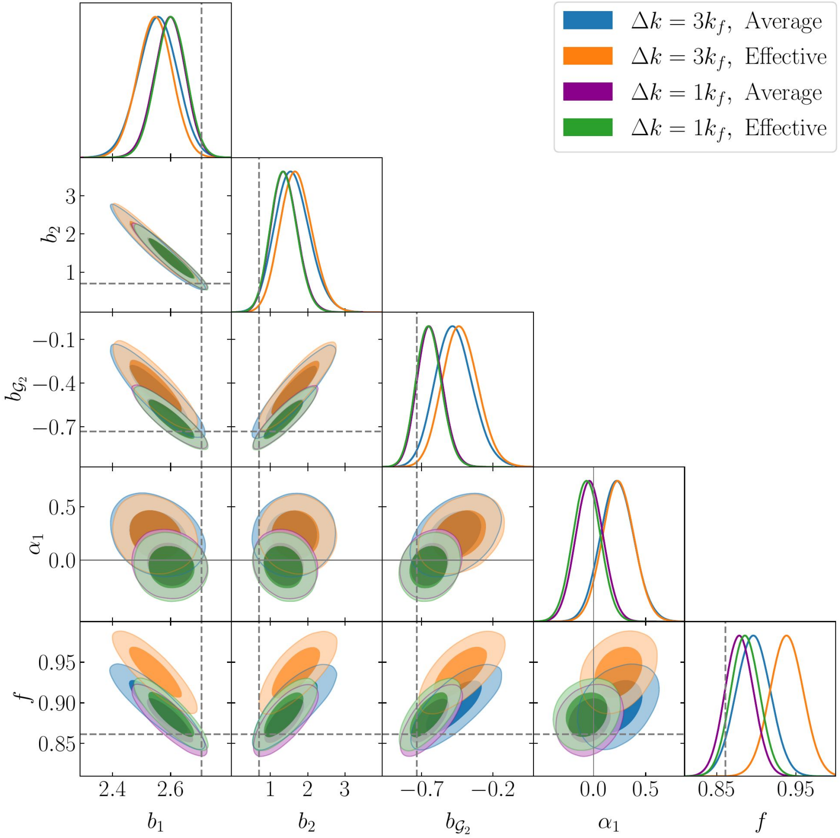
298 Minerva snapshots ~ 1000 (Gpc/*h*)^3

B_0
\+ ***B_2*** (significant information)
\+ ***B_4*** (negligible information)



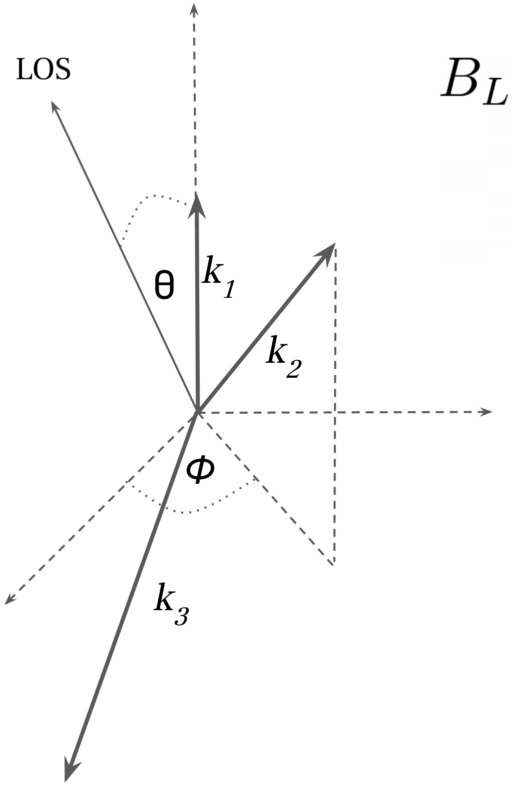
Rizzo, Moretti, Pardede+22
ΛCDM P_ℓ+B_ℓ: Moretti, Rizzo, Pardede+ (in prep.)

The binning effect



Rizzo, Moretti, Pardede+22

The bispectrum multipoles



$$B_L(k_1, k_2, k_3) = \frac{2L+1}{4\pi} \int d\cos\theta \int d\phi B(k_1, k_2, k_3, \theta, \phi)\, \mathcal{L}_L(\cos\theta)\,.$$

PT (perturbation theory) model

angles w.r.t line of sight

- galaxies are not in their rest frame
- $m \neq 0$ contains negligible information Gagrani+16
- **tree-level: only even multipoles exist B_0, B_2, B_4, …**

Tree-level bispectrum

$$B(\mathbf{k}_1, \mathbf{k}_2, \hat{\mathbf{x}}) = B^{(\mathrm{det})}(\mathbf{k}_1, \mathbf{k}_2, \hat{\mathbf{x}}) + B^{(\mathrm{stoch})}(\mathbf{k}_1, \mathbf{k}_2, \hat{\mathbf{x}})$$

$$B^{(\mathrm{det})}(\mathbf{k}_1, \mathbf{k}_2, \hat{\mathbf{x}}) = 2Z_2(\mathbf{k}_1, \mathbf{k}_2, \hat{\mathbf{x}})Z_1(\mathbf{k}_1, \hat{\mathbf{x}})Z_1(\mathbf{k}_2, \hat{\mathbf{x}})P_L(k_1)P_L(k_2) + \mathrm{cyc.}$$

$$B^{(\mathrm{stoch})}(\mathbf{k}_1, \mathbf{k}_2, \hat{\mathbf{x}}) = \frac{1}{\bar{n}}[(1+\alpha_1)b_1 + (1+\alpha_3)f(\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{x}})^2]Z_1(\mathbf{k}_1, \hat{\mathbf{x}})P_L(k_1) + \mathrm{cyc.} + \frac{1+\alpha_2}{\bar{n}^2}$$

$$Z_1(\mathbf{k}, \hat{\mathbf{x}}) = b_1 + f(\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{x}})^2$$

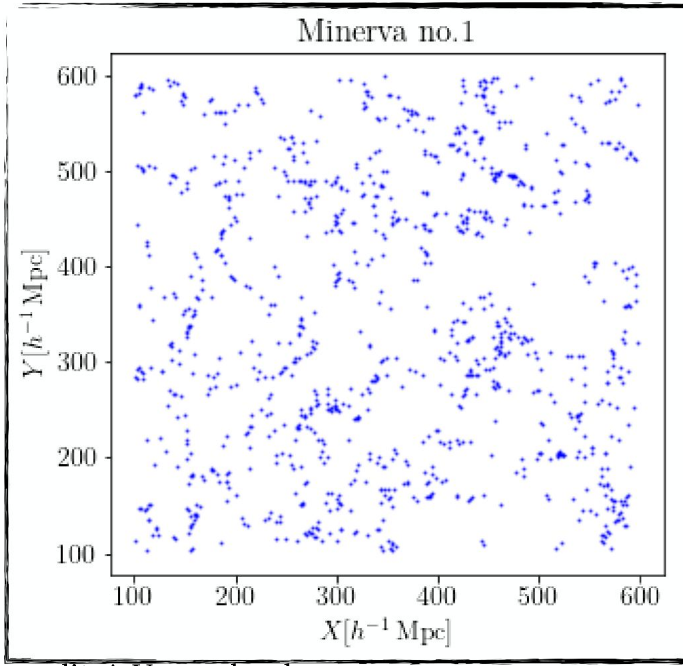
$$Z_2(\mathbf{k}_1, \mathbf{k}_2, \hat{\mathbf{x}}) = \frac{b_2}{2} + b_1 F_2(\mathbf{k}_1, \mathbf{k}_2) + b_{\mathcal{G}_2} S(\mathbf{k}_1, \mathbf{k}_2)$$

$$f(\hat{\mathbf{k}}_{12} \cdot \hat{\mathbf{x}})^2 G(\mathbf{k}_1, \mathbf{k}_2) + \frac{f(\mathbf{k}_{12} \cdot \hat{\mathbf{x}})}{2}\left[\frac{\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{x}}}{k_1} Z_1(\mathbf{k}_2, \hat{\mathbf{x}}) + \frac{\hat{\mathbf{k}}_2 \cdot \hat{\mathbf{x}}}{k_2} Z_1(\mathbf{k}_1, \hat{\mathbf{x}})\right]$$

$$\mathbf{k}_{12} \equiv \mathbf{k}_1 + \mathbf{k}_2$$

The bispectrum multipoles: test on simulations

1. 298 **Minerva** (N-body) Grieb+16
2. 10000 **Pinocchio** (3LPT) Monaco+02



credit: A.Veropalumbo

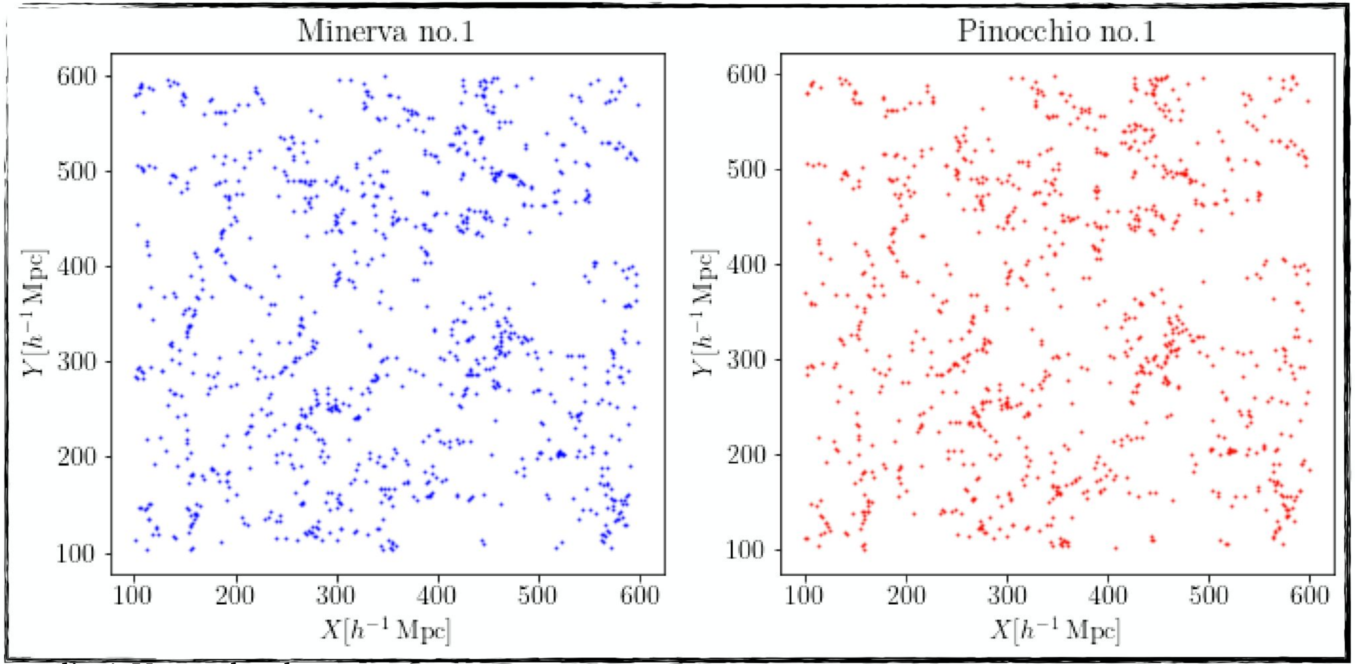
Rizzo, Moretti, Pardede+ (arXiv: 2204.13628)

@$z = 1$
ΛCDM cosmology
$L_{box} = 1500$ Mpc/h
$V_{eff} \simeq 1000\ (\mathrm{Gpc}/h)^3$
$\simeq$ 2x volume in PT-challenge Nishimichi+20

… the numerical covariance

1. 298 **Minerva** (N-body) Grieb+16
2. 10000 **Pinocchio** (3LPT) Monaco+02

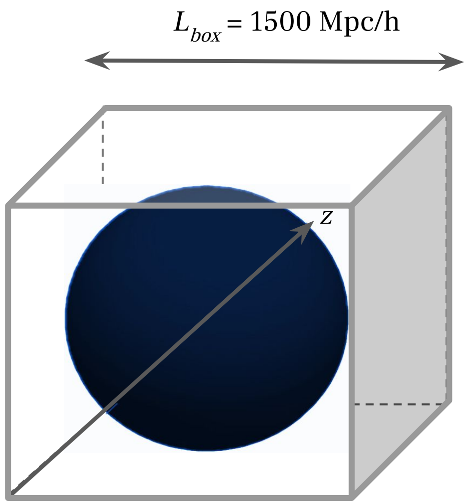
- approx. based on Lagrangian pert. theory
- relatively fast and accurate → provide a robust estimate of the covariance



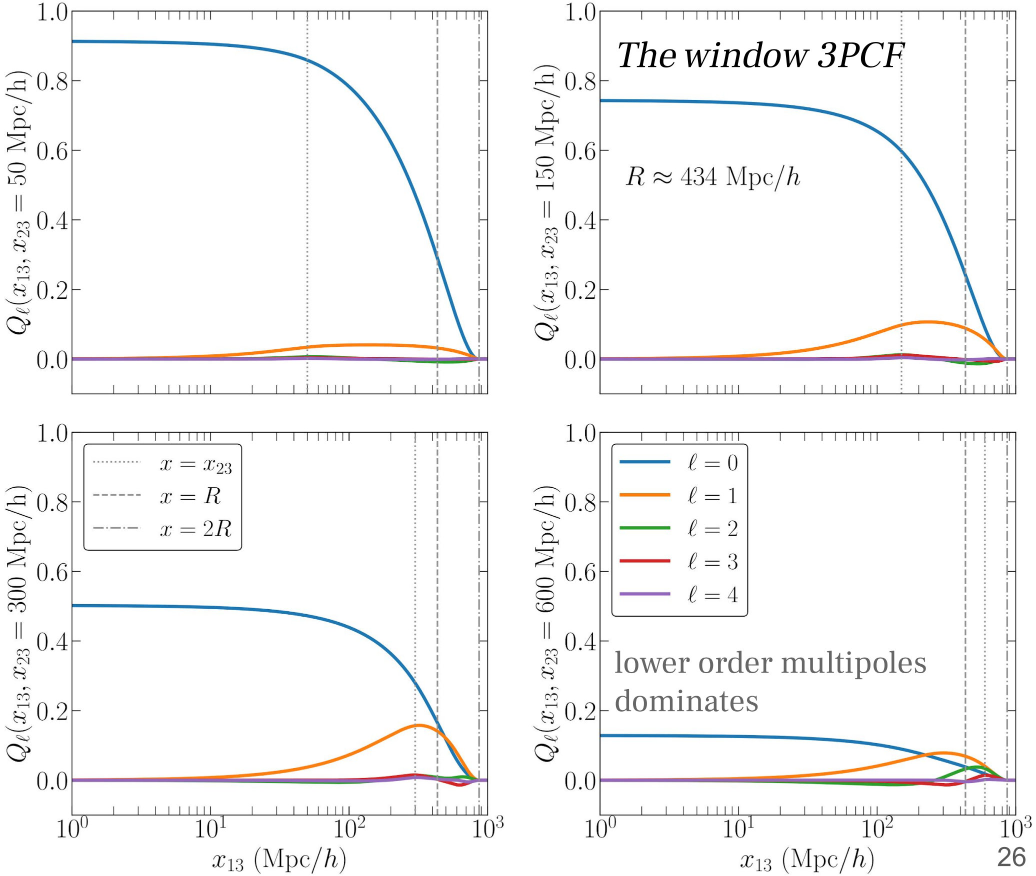
credit: A.Veropalumbo

Spherical window convolution in real-space

Sphere catalogue:
Minerva/Pinocchio carved on a sphere of $R \sim 434$ Mpc/h



Total vol = 700^3 (Mpc/h)$]^3$



An approximation

$$\tilde{B}[P_L] \simeq B[\tilde{P}_L]$$

1DFFTLog-approx

$$\tilde{B}(\mathbf{k}_1, \mathbf{k}_2) \simeq Z(\mathbf{k}_1, \mathbf{k}_2)\tilde{P}_L(k_1)\tilde{P}_L(k_2) + \text{cyc.}$$

- Reduced to power spectrum-window convolution
  see e.g. Wilson+15, Castorina+17, d'Amico+19
- BOSS DR 11/12 Gil-Marin+14a, b and +16a, b
- Recently used in d'Amico +19,+22
- Doesn't work for squeezed triangles

Computed via (1D) FFTLog

First few triangles

Fit on **Pinocchio** mocks

volume $\approx 3500\ [\mathrm{Gpc}/h]^3$

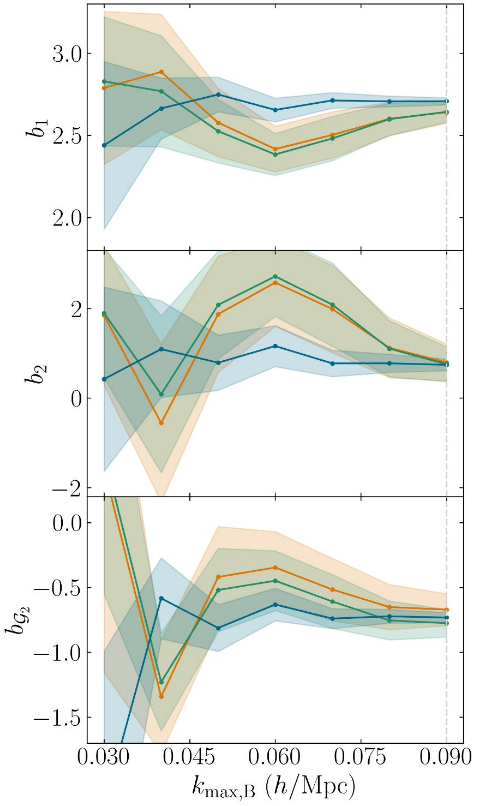
Minerva volume:
consistent within 1-sigma

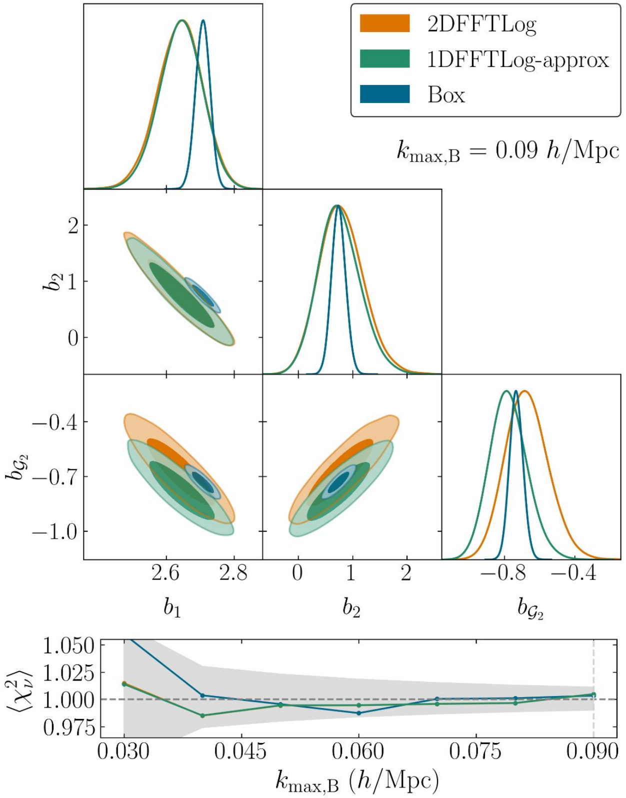
Recovering bias parameters

Analysis on **Minerva** data

≈ 1/6 times volume in Nishimichi+20

≈ 6 times $z \in [1.5, 1.8]$ *Euclid* volume

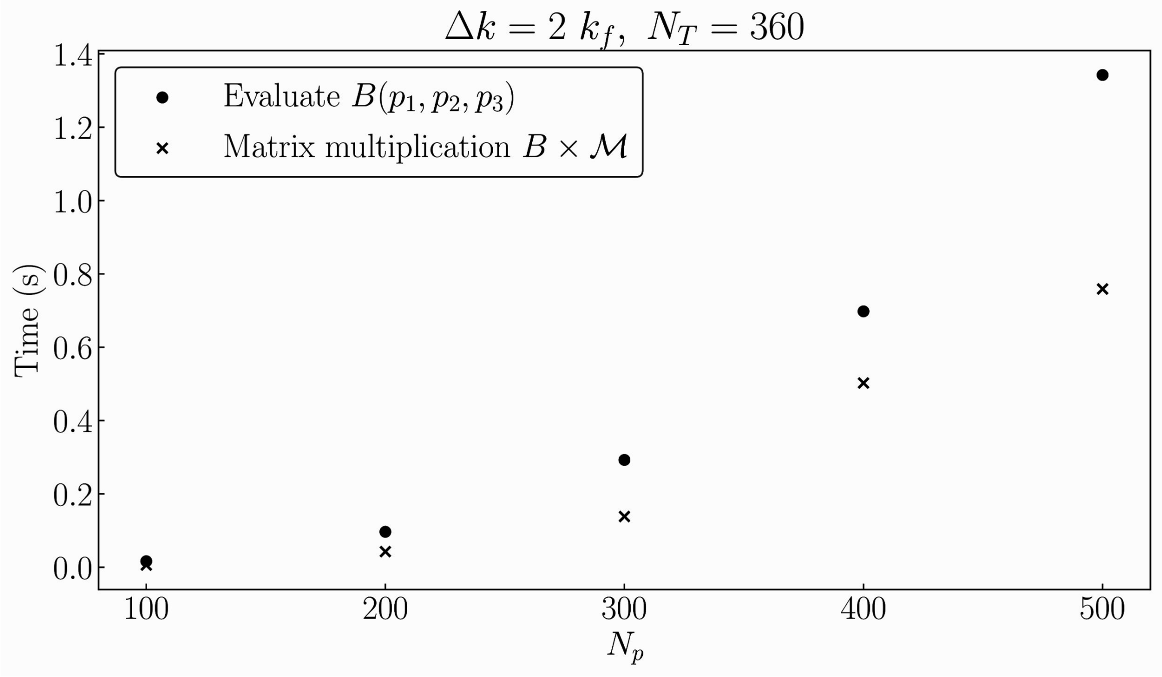




Window convolution computation time

Takes ~ **2 seconds**

$\Rightarrow$ comparable to a typical Boltzmann solver call



Exact bispectrum window convolution

Taking $\langle \hat{B}_L \rangle = \tilde{B}_L$

$$\tilde{B}_L(k_1, k_2, k_3) = \frac{2L+1}{V_B} \int_{k_1} d^3 q_1 \int_{k_2} d^3 q_2 \int_{k_3} d^3 q_3 \, \delta_D(\vec{q}_{123})$$
$$\times \int d^3 x_3 \int d^3 x_{13} \int d^3 x_{23} \, e^{-i\vec{q}_1 \cdot \vec{x}_{13}} e^{-i\vec{q}_2 \cdot \vec{x}_{23}} \zeta(\vec{x}_{13}, \vec{x}_{23}, \hat{x}_3)$$
$$\times W(\vec{x}_1) \, W(\vec{x}_2) \, W(\vec{x}_3) \, \mathcal{L}_L(\hat{q}_1 \cdot \hat{x}_3)$$

Need to: systematically reduce the angular integration

… the final expression

Some form of integral between
the unconvolved bisp. and the mixing matrix

$$\tilde{B}_L(k_1, k_2, k_3) = \int \frac{dp_1}{2\pi^2} p_1^2 \int \frac{dp_2}{2\pi^2} p_2^2 \int \frac{dp_3}{2\pi^2} p_3^2 \sum_{L'M'} B_{L'M'}(p_1, p_2, p_3) \times \sum_{\ell} \underbrace{I_{\ell\ell 0}(p_1, p_2, p_3) \mathcal{Q}^{L}_{L', -M', \ell}(k_1, k_2, k_3; p_1, p_2)}_{\text{the mixing matrix}}$$

the mixing matrix

Window convolution ~ matrix mult.

One part of the mixing matrix is a known function

$$\tilde{B}_L(k_1,k_2,k_3) = \int \frac{dp_1}{2\pi^2}p_1^2 \int \frac{dp_2}{2\pi^2}p_2^2 \int \frac{dp_3}{2\pi^2}p_3^2 \sum_{L'M'} B_{L'M'}(p_1,p_2,p_3)$$

$$\times \sum_{\ell} \boxed{I_{\ell\ell 0}(p_1,p_2,p_3)}\, \mathcal{Q}^{L}_{L',-M',\ell}(k_1,k_2,k_3;p_1,p_2)$$

enforce the triangle condition

$$\tilde{B}_\ell[T_i] = \sum_{j,\ell'} \mathcal{M}_{\ell\ell'}[T_i,T'_j]B_{\ell'}[T'_j]$$

$$I_{\ell\ell 0}(x,y,z) = (-1)^\ell \frac{\pi^2}{xyz}\theta(1-\hat{x}\cdot\hat{y})\theta(1+\hat{x}\cdot\hat{y})\mathcal{L}_\ell(\hat{x}\cdot\hat{y})$$

Contribution from the the window 3PCF

The other contribution is coming from the random catalogue

$$\tilde{B}_L(k_1,k_2,k_3) = \int \frac{dp_1}{2\pi^2}p_1^2 \int \frac{dp_2}{2\pi^2}p_2^2 \int \frac{dp_3}{2\pi^2}p_3^2 \sum_{L'M'} B_{L'M'}(p_1,p_2,p_3)$$

$$\times \sum_{\ell} I_{\ell\ell 0}(p_1,p_2,p_3)\, \boxed{\mathcal{Q}^L_{L',-M',\ell}(k_1,k_2,k_3;p_1,p_2)}$$

require several steps of computations

How to compute the 3PCF contribution?

$W(\mathbf{x})$ ← the random catalogue

3PCF of the window e.g. by direct-counting →

$$Q^{L}_{L'M'\ell\ell'\ell_1\ell_2}(x_{13}, x_{23})$$

(2D) Hankel transform e.g. 2D-FFTLog (Fang+20) →

$$\mathcal{W}^{L}_{L'M'\ell\ell'\ell_1\ell_2}(q_1, q_2, p_1, p_2)$$

linear combination →

$$\mathcal{Q}^{L}_{L'M'\ell}(k_1, k_2, k_3; p_1, p_2)$$

The window 3PCF - **measurement**

$$Q^{L}_{L'M'\ell\ell'\ell_1\ell_2}(x_{13},x_{23}) \equiv (-1)^{M'} \sum_{\tilde{\ell}_1,\tilde{\ell}_2} \sum_{\substack{M,m_1,m_2\\ m,m',\tilde{m}_1,\tilde{m}_2}} 4\pi i^{\ell'-\ell+\ell_2-\ell_1}\, \mathcal{G}^{Mm_1m_2}_{L\ell_1\ell_2}\, \mathcal{G}^{M'mm'}_{L'\ell\ell'}\, \mathcal{G}^{m_1m'\tilde{m}_1}_{\ell_1\ell'\tilde{\ell}_1}\, \mathcal{G}^{m_2m\tilde{m}_2}_{\ell_2\ell\tilde{\ell}_2}$$

$$\times \int d^3x_3 \int \frac{d^2\hat{x}_{13}}{4\pi} \int \frac{d^2\hat{x}_{23}}{4\pi} Y^*_{LM}(\hat{x}_3) Y_{\tilde{\ell}_1\tilde{m}_1}(\hat{x}_{13}) Y_{\tilde{\ell}_2\tilde{m}_2}(\hat{x}_{23})$$
$$\times W(\vec{x}_3+\vec{x}_{13}) W(\vec{x}_3+\vec{x}_{23}) W(\vec{x}_3).$$

Computed via e.g. direct counting, FFT-based, etc.

The window 3PCF – **Hankel transf.**

Combination of two dimensional Hankel transforms

$$\mathcal{W}^{L}_{L'M'\ell\ell'\ell_1\ell_2}(q_1,q_2;p_1,p_2) \equiv (4\pi)^2 \int dx_{13}\, x_{13}^2 \int dx_{23}\, x_{23}^2\, j_{\ell'}(p_1 x_{13})\, j_{\ell}(p_2 x_{23}) \times \left[j_{\ell_1}(q_1 x_{13})\, j_{\ell_2}(q_2 x_{23})\, Q^{L}_{L'M'\ell\ell'\ell_1\ell_2}(x_{13},x_{23}) \right],$$

A two dimensional Hankel transform

e.g. `2DFFTLog` Fang+20

Window 3PCF multipoles

The window 3PCF – **binning**

How to handle the binning operator?

$$\mathcal{Q}^{L}_{L'M'\ell}(k_1,k_2,k_3;p_1,p_2) \simeq \sum_{\ell_1,\ell_2,\ell'} 16\pi^2 \frac{I_{\ell_2\ell_2 0}(k_1,k_2,k_3)}{I_{000}(k_1,k_2,k_3)} \mathcal{W}^{L}_{L'M'\ell\ell'\ell_1\ell_2}(k_1,k_2;p_1,p_2)\,.$$

- Evaluated at the center of the bin
- Bin numerically later

The window 3PCF - **FFT-based**

$$Q^{L}_{L'M'\ell\ell'\ell_1\ell_2}(x_{13},x_{23}) = (-1)^{M'} \sum_{\substack{M,m_1,m_2\\ m,m'}} \sum_{\substack{\tilde{\ell}_1,\tilde{\ell}_2\\ \tilde{m}_1,\tilde{m}_2}} 4\pi i^{\ell'-\ell+\ell_2-\ell_1}\ \mathcal{G}^{Mm_1m_2}_{L\ell_1\ell_2}\mathcal{G}^{M'mm'}_{L'\ell\ell'}\,\mathcal{G}^{\tilde{m}_1m_1m'}_{\tilde{\ell}_1\ell_1\ell'}\,\mathcal{G}^{\tilde{m}_2m_2m}_{\tilde{\ell}_2\ell_2\ell}$$

$$\times \int d^3x_3\ W_{\tilde{\ell}_1\tilde{m}_1}(\vec{x}_3;x_{13})W_{\tilde{\ell}_2\tilde{m}_2}(\vec{x}_3;x_{23})W_{LM}(\vec{x}_3)$$

$$W_{\ell m}(\vec{x}_3;x_{ij}) \equiv\ i^{\ell}\int \frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{x}_3}j_\ell(qx_{ij})Y_{\ell m}(\hat{q})W(\vec{q})$$

$$W_{LM}(\vec{x}_3) \equiv\ W(\vec{x_3})Y^*_{LM}(\vec{x}_3)\,.$$

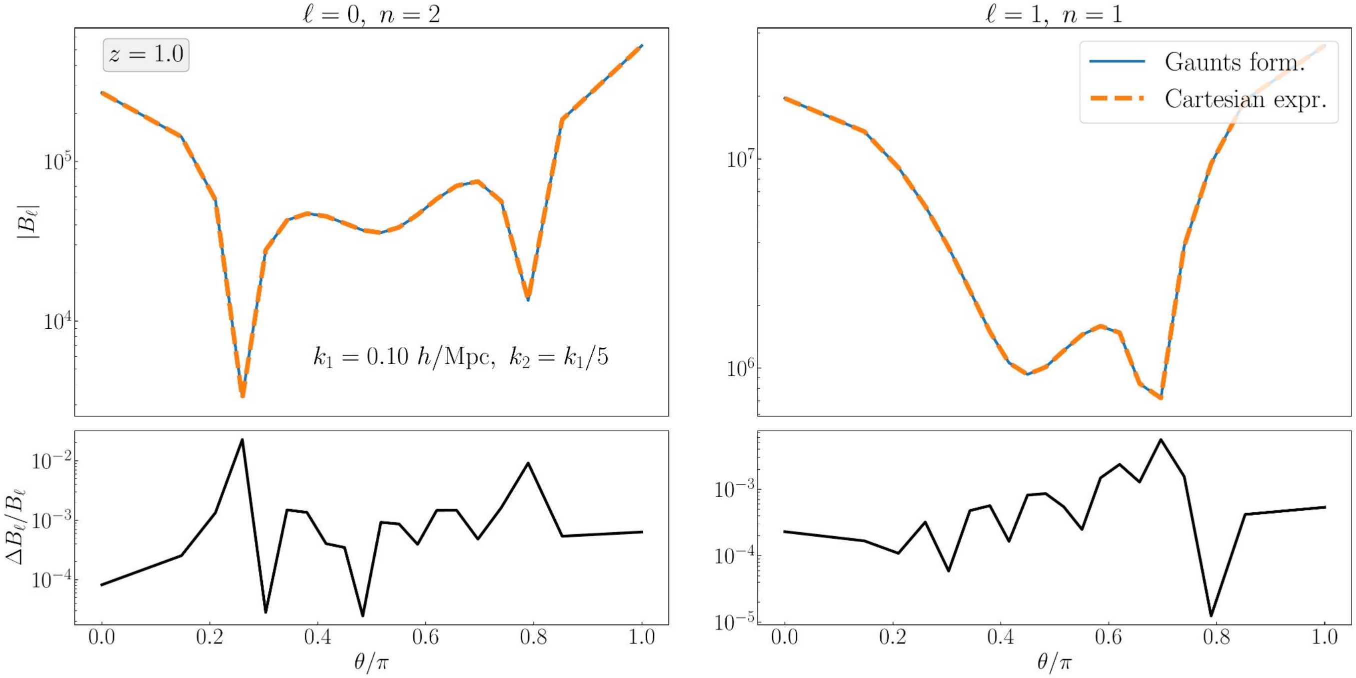
Default parameters

N_p	ℓ_{max}	ℓ'_{max}	P_{min} [h Mpc^{-1}]	p_{max} [h Mpc^{-1}]
512	30	2	10^{-5}	0.5

Wide-angle formulation for the bispectrum

$$
\begin{aligned}
\langle B_L(k_1,k_2,k_3)\rangle = &\\
&\sum_{ij}\sum_{\ell_6+\ell_7+\ell_8\le i}\sum_{\ell_9+\ell_{10}+\ell_{11}\le j}\sum_{\substack{\ell_1,\ell_2,\ell_3,\ell_4\\ \ell_5,\ell_{12},\ell_{13},\ell_{14}}} i^{\ell_1+\ell_2-\ell_{12}-\ell_{13}}\\
&\times\Bigg[\frac{1}{(2\pi)^6}\int\frac{dx_3x_3^2}{V}\int dx_{13}x_{13}^2\int dx_{23}x_{23}^2\int dp_1p_1^2\int dp_2p_2^2\\
&\quad j_{\ell_1}(p_1x_{13})j_{\ell_2}(p_2x_{23})j_{\ell_{12}}(k_1x_{13})j_{\ell_{13}}(k_2x_{23})\mathcal{F}^{(ij)}_{\ell_3\ell_4\ell_5\ell_6\ell_7\ell_8\ell_9\ell_{10}\ell_{11}}(p_1,p_2)\left(\frac{x_{13}}{x_3}\right)^i\left(\frac{x_{23}}{x_3}\right)^j\Bigg]\\
&\times\Bigg[\frac{(4\pi)^{14}}{N_{\ell_3\ell_4\ell_5}N_{\ell_6\ell_7\ell_8}N_{\ell_9\ell_{10}\ell_{11}}}\sum_{m,m_i}\sum_{LM}(-1)^{m_3+m_8+m_{11}+M}\mathcal{G}^{m_4mM}_{\ell_4\ell L}\\
&\quad\mathcal{G}^{m_1m_3m_4m_6m_9}_{\ell_1\ell_3\ell_4\ell_6\ell_9}\mathcal{G}^{m_2,-m_3m_5m_7m_{10}}_{\ell_2\ell_3\ell_5\ell_7\ell_{10}}\mathcal{G}^{m_1m_{12}m_6m_7,-m_8}_{\ell_1\ell_{12}\ell_6\ell_7\ell_8}\mathcal{G}^{m_2m_{13}m_9m_{10},-m_{11}}_{\ell_2\ell_{13}\ell_9\ell_{10}\ell_{11}}\mathcal{G}^{-Mm_5m_8m_{11}}_{L\ell_5\ell_8\ell_{11}}\\
&\quad\mathcal{G}^{mm_{12}m_{14}}_{\ell\ell_{12}\ell_{14}}Y_{\ell_{14}m_{14}}(\hat{\mathbf{k}}_1)Y^*_{\ell_{13}m_{13}}(\hat{\mathbf{k}}_2)\Bigg]
\end{aligned}
$$

Gaunt vs Cartesian formulation



Convergence test