$f(R)$ THEORIES OF GRAVITY

AND DARK ENERGY

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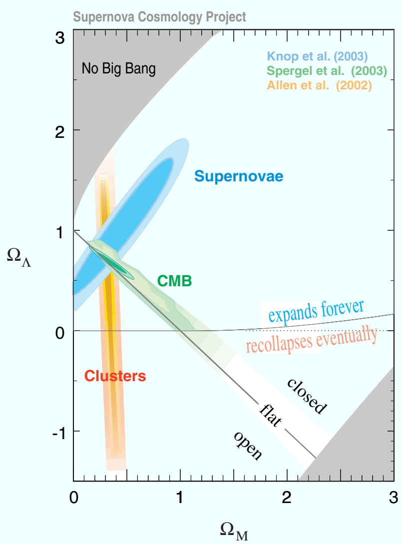
INTRODUCTION

The observed universe is well represented by a Friedmann-Lemaître spacetime
the scale factor of which started to accelerate recently



$$ds^2 = g_{ij}dx^i dx^j = -dt^2 + a^2(t)d\sigma_{\rm k}^2$$

$$G_{ij} = \kappa T_{ij}^{\rm total} \quad ; \quad D_j T^{ij} = 0$$

$$H^2 + \frac{\rm k}{a^2} = \frac{\kappa}{3}\rho^{\rm total} \; ; \quad H = \frac{1}{a}\frac{da}{dt}$$

$$\left(\frac{H}{H_0}\right)^2 = \Omega^0_{\rm rad}\left(\frac{a_0}{a}\right)^4 + \Omega^0_{\rm mat}\left(\frac{a_0}{a}\right)^3 + \Omega^0_{\rm k}\left(\frac{a_0}{a}\right)^2 + \Omega^0_\Lambda$$

$$H_0 \approx 70 \; ; \; \Omega^0_{\rm rad} \approx 10^{-4} \; ; \; \Omega^0_{\rm mat} \approx 0.3 \; , \; \Omega^0_{\rm k} \approx 0$$

$\Omega^0_\Lambda \approx 0.7$: Dark Energy

$$\Longrightarrow \quad \kappa T^\Lambda_{ij} = -\Lambda g_{ij} \; , \; a(t) \to e^{\sqrt{\Lambda/3}\,t} \; , \; \Lambda = 3\Omega^0_\Lambda H_0^2$$

Origin of this acceleration

- An artefact of the averaging process ?

  $G_{ij}(<g_{kl}>) = \kappa <T_{ij}>$ instead of $<G_{ij}(g_{kl})>= \kappa <T_{ij}>$
  $<g_{kl}>\neq g_{kl}$ (See Rasanen)

- Exotic matter ? $(\rho_\Lambda + p_\Lambda \approx 0)$

  – Chaplygin gas : $p\rho = -A$, (See Moschella)
  – Quintessence : R.G. plus φ with $V \propto 1/\varphi^n$ (See Martin)

- "Modified" gravity ?

  – Λ : (See Bernardeau)
  – MOND (See Combes, Esposito-Farese)
  – "Branes" : (See Deffayet)
  – $f(R)$ lagrangian, instead of Hilbert's R (simpler than $f({R^i}_{jkl})$!)
  C.D.T.T. (2003), Capozziello *et al.* (2003)

Outline of the Lecture

1. $f(R)$ theories as scalar-tensor theories of gravity

– gravity is described by a "graviton" (g_{ij}) and a "scalaron" ($s \sim R$)
– coupling of s to matter : minimal *vs* "detuned" coupling
– Jordan *vs* Einstein frames or : choosing a "physical" spacetime

2. $f(R)$ cosmological models of dark energy
– late time accelerating phase (CDTT, 2003 *et seq.*)
– "s-MDE" era problem (Amendola *et al.* 2006, *et seq.*)
– Viable cosmological models of $f(R)$ DE

3. $f(R)$ gravity and local tests
– Equivalent to $\omega = 0$ Brans-Dicke theory ? (Chiba, 2003, *et seq.*)
– Giving the scalaron a mass : "Chameleon" effect (Khoury *et al.* 2003)

4. Can "detuning" help ?

1. $f(R)$ theories are scalar-tensor theories of gravity

- d.o.f. : gravity is described by a "graviton" and a "scalaron"

$$\bar{S}[g_{ij}] = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\, f(R) + S_{\rm m}(\Psi\ ; g_{ij}) \qquad (\kappa = 8\pi G_*)$$

(Weyl 1918, Pauli 1919, Eddington, 1924)

Metric variation yields a 4th order diff eqn for g_{ij} :

$$f'(R)\, G_{ij} + \frac{1}{2}(Rf' - f)g_{ij} + g_{ij}D^2 f' - D_{ij}f' = \kappa T_{ij} \quad (\Rightarrow \quad D_j T^{ij} = 0)$$

The trace :

$$3D^2 f' + (Rf' - 2f) = \kappa T$$

is a (2nd order) eom for R (or $f'(R)$), the "scalaron" (Starobinski, 1980)

Remark : "Palatini" variations yield different eom (Vollick, 2003 *et seq.*)

- Isolating the scalaron and coupling it to matter

Introduce a "Helmholtz" lagrangian :

$$\bar{S}[g_{ij}, s] = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,[f'(s)R - (sf'(s) - f(s)] + S_{\rm m}(\Psi\ ;\tilde{g}_{ij} = e^{2C(s)}g_{ij})$$

hence TWO second order differential equations of motion :

$$f'(s)G_{ij} + \frac{1}{2}g_{ij}(sf'(s) - f(s)) + g_{ij}D^2 f'(s) - D_{ij}f'(s) = \kappa T_{ij}$$

$$s = R - 2\kappa C'(s)T/f''(s) \qquad (\Rightarrow D_j T^j_i = T C'(s)\partial_i s)$$

$C(s) = 0$: standard $f(R)$ gravity ; $s = R$, same eom as before.

$C(s) \neq 0$: "detuned" $f(R)$ gravity (ND, Sasaki, Sendouda, 2007)

- Jordan *vs* Einstein frame description of $f(R)$ gravity

The "Jordan frame" is the spacetime, $\tilde{\mathcal{M}}$, with metric $\tilde{g}_{ij} = e^{2C} g_{ij}$ to which matter is minimally coupled (that is : $\tilde{D}_j \tilde{T}^{ij} = 0$, *e.g.* $\tilde{\rho} \propto 1/\tilde{a}^3$).

In this frame the action is a Brans-Dicke type action (up to a divergence)

$$\tilde{S}[\tilde{g}_{ij}, \Phi] = \frac{1}{2\kappa}\int d^4x\sqrt{-\tilde{g}}\left[\Phi\tilde{R} - \frac{\omega(\Phi)}{\Phi}(\tilde{\partial}^2\Phi) - 2U(\Phi)\right] + S_{\rm m}[\Psi; \tilde{g}_{ij}]$$

where $\quad \Phi(s) = f'(s)e^{-2C(s)} \quad , \qquad U(s) = \frac{1}{2}(sf'(s) - f(s))e^{-2C(s)}$

and $\qquad \omega(s) = -\frac{3K(s)(K(s)-2)}{2(K(s)-1)^2} \quad$ with $\quad K(s) = \frac{d\ C}{d\ln\sqrt{f'}}$.

For standard $f(R)$ gravity, $C(s) = 0$; the Jordan frame is the original one. And $\omega = 0$; if $U \approx 0$, $f(R)$ gravity is ruled out since $\omega > 40000$ (Cassini)

The "Einstein frame" is the spacetime, $\mathcal{M}^*$, the metric of which, $g^*_{ij} = e^{-2k}\tilde{g}_{ij}$, makes the action for $f(R)$ gravity look like Einstein's :

$$S^*[g^*_{ij}, \varphi] = \frac{2}{\kappa}\int d^4x\sqrt{-g^*}\left[\frac{R^*}{4} - \frac{1}{2}(\partial^*\varphi)^2 - V(\varphi)\right] + S_{\rm m}[\Psi; \tilde{g}_{ij} = e^{2k(\varphi)}g^*_{ij}]$$

where $\varphi(s) = \sqrt{3}\ln\sqrt{f'(s)}\ ,\ V(s) = \frac{sf'(s)-f(s)}{4f'^2(s)} \quad , \quad e^{2k(s)} = \frac{e^{2C(s)}}{f'(s)}$

$\tilde{\mathcal{M}} \neq \mathcal{M}^*$ unless $\tilde{g}_{ij} = g^*_{ij}$, that is, $C(s) = \ln\sqrt{f'(s)}$,
(Magnano-Sokolewski, 1993, 2007)

$f(R)$ gravity and coupled quintessence

Ellis *et al.* (1989), Damour-Nordvedt-Polyakov (1993), Wetterich (1995), Amendola (1999), Copeland *et al.* (2006),...

• Jordan *vs* Einstein frames : choosing does matter

Mathematically the actions for (g_{ij}, s), $(\tilde{g}_{ij}, \Phi)$ or (g^*_{ij}, φ) are equivalent.

If we decide that $\mathcal{M}^*$ with line element $ds_*^2 = -dt^2 + a_*^2(t)d\vec{x}^2$ represents physical spacetime, then t is cosmic time
but if we decide that it is $\tilde{\mathcal{M}}$ with line element $d\tilde{s}^2 = e^{2k(\varphi)}ds_*^2$ then cosmic time is $\tilde{t}$ such that $d\tilde{t} = e^{k(\varphi)}dt$.

In all frames test particles fall the same way (WEP), but only in the Jordan frame do they follow geodesics
Only in the Jordan frame can the EEP be enforced : $\tilde{D}_j\tilde{T}_i^j = 0 \;\Rightarrow\; \partial_j\tilde{T}_i^j \approx 0$ in local inertial frame ($\sqrt{-\tilde{g}}\tilde{T}_{ij} = -2\delta S_{\rm m}/\delta\tilde{g}^{ij}$)

the example of Nordström's theory, 1913 : $\frac{du^i}{d\tau} = -\frac{1}{1+\phi}(\partial^i\phi + u^iu^j\partial_j\phi)$
reformulated by Einstein-Fokker, 1914 : $\frac{\tilde{D}u^i}{d\tau} = 0$ with $\tilde{g}_{ij} = (1+\phi)^2\eta_{ij}$

2. $f(R)$ cosmological models of Dark Energy

• The Carroll-Duvvuri-Trodden-Turner
and Capozziello-Carloni-Troisi proposal (2003)

$$f(R) = R - \frac{\mu^{2(1+n)}}{R^n} \; (n > 0) \; ; \quad \mu^2 \sim 10^{-33}\mathrm{eV} \quad \text{or} \quad \mu^2 = \frac{1}{\ell^2} \text{ with } \ell \sim H_0^{-1}$$

Late time Einstein frame Friedmann equations when matter has become negligible (φ large > 2, say) :

$$3H_*^2 - \dot{\varphi}^2 - 2V(\varphi) \approx 0, \quad \ddot{\varphi} + 3H_*^2\dot{\varphi} + \frac{dV}{d\varphi} \approx 0 \quad \text{with} \quad V(\varphi) \propto e^{-\frac{(n+2)\varphi}{2\sqrt{3}(n+1)}}$$

Solution : $a_*(t) \propto t^q \quad (q \to 3 \; , \; w^*_{\mathrm{DE}} \to -0.77$ for large n$)$, $\varphi \sim \sqrt{3}\, p \ln t$

Jordan frame scale factor : $d\tilde{s}^2 = t^{-2p} ds_*^2 = -d\tilde{t}^2 + \tilde{a}^2(\tilde{t}) dx^2$

hence : $\tilde{a}(\tilde{t}) \propto \tilde{t}^{\frac{2}{3(1+\tilde{w}_{\mathrm{DE}})}}$
with $\tilde{w}_{\mathrm{DE}} = -1 + \frac{2(n+2)}{3(n+1)(2n+1)} \to -1$ for large n (2,3,4 is enough)

• A first flaw

Amendola, Polarski, Tsujikawa *et al.*, 2003-2007

Friedmann equations when matter dominates over DE ($\rho_* = e^{-4\varphi/\sqrt{3}}\tilde{\rho}$) :

$$3H_*^2 - \dot{\varphi}^2 \approx \kappa\rho_* \;, \quad \ddot{\varphi} + 3H_*^2\dot{\varphi} \approx \frac{\kappa\rho_*}{2\sqrt{3}} \;, \quad \dot{\rho}_* + H_*\rho_* = -\frac{\dot{\varphi}}{\sqrt{3}}\rho_*$$

Hence : $\tilde{a}(\tilde{t}) \propto \tilde{t}^{1/2}$ instead of $\tilde{t}^{2/3}$ (for $\varphi \approx 0$, V is negligible, not $\dot{\varphi}$)

the CDTT model is ruled out

• Conditions for a standard matter era followed by late acceleration

Introduce the dynamical variables : $x_1 = -\frac{\dot{f}'}{Hf'}$, $x_2 = -\frac{f}{6Hf'}$, $x_3 = -\frac{R}{6H^2}$

Define $r(R) = -\frac{Rf'}{f}$ and $m(R) = \frac{Rf''}{f'}$ (Copeland *et al.* 1997, 2006)

Write the Friedmann equations as $\frac{\dot{x}_3}{H} = -\frac{x_1 x_3}{m} - 2x_3(x_3 - 2)$ *etc* ;
Find the fixed points such that $\dot{x}_i = 0$: $P = (x_1(m), x_2(m), x_3(m))$

The "phase space trajectory" $m = m(r)$ can connect a (saddle) matter point

$$P_{\rm M} = (r \approx -1, m \approx 0+) \quad \text{with} \quad dm/dr|_{-1} > -1$$

to a stable fixed point corresponding to

– either exponential acceleration, $P_{\rm S} \in r = -2$, if $0 < m(-2) \leq 1$

– or $\tilde{a} \propto \tilde{t}^r, r > 1$, $P_{\rm A} \in m = -(1+r)$, if $\frac{dm}{dr} < -1$ and $\frac{\sqrt{3}-1}{2} < m < 1$

Example :

$$f(R) = R \pm \frac{\mu^{4+n}}{R^n} \quad \text{with} \quad -1 < n < 0$$

Developments (Capoziello-Tsujikawa, 2007) :

– $w_{\rm DE} < -1$ for $z < z_b$ "crossing of the phantom boundary"

– $w_{\rm DE}$ diverges at $z = z_c$, z_b and $z_c \to \infty$ for $n \to -1$

3. $f(R)$ gravity and local tests

- An ninety year old mistake

CDTT, 2003 : “Many solar system tests of gravity theory depend on the Schwarzschild solution, which Birkhoff’s theorem ensures is the unique, static, spherically symmetric solution (...) It is clear that astrophysical tests of gravity will be unaffected by the modification we have made.”

Weyl (1918), Pauli (1919) and Eddington (1924) had said the same...

Buchdahl (1961), Whitt (1984), Mignemi-Wiltshire (1992) :
Schwarzschild metric is the black hole solution of $R + \sum a_n R^n$ theories.

Pechlaner-Sexl (1966), Havas (1977) : $f(R)$ field equations are fourth order differential equations ; they possess extra-runaway solutions : there is no Birkhoff theorem ; the solution outside an extended source is not the Schwarzschild metric and depends on its equation of state.

- One-scale $f(R)$ models of DE are $\omega = 0$ Brans-Dicke theory

Chiba (2003), Olmo (2005), Erickcek et al (2006), Navarro-Acoleyen (2006),...

The short answer :

Recall that $f(R)$-gravity is of Brans-Dicke type with $\omega = 0$ and a potential

$$U = \frac{1}{2}(Rf'(R) - f(R))$$

Teyssandier-Tourrenc, 1983

Can U be neglected when studying gravity in the solar system ? Yes

Indeed $f(R) = R - H_0^4/R$, yields $U = \mathcal{O}(H_0^2) = \mathcal{O}(1/\ell^2)$ with $\ell \gg L_{\rm SS}$.

Therefore, see $e.g$ Will or Damour-Esposito Farese, ω must be large to comply with solar system observations : $\omega > 40000$ (Cassini).

Hence, all one-scale $f(R)$ models of dark energy are ruled out.

The details :

The equations of motion are :

$$D_*^2\varphi - \frac{dV}{d\varphi} = \frac{4\pi G_*}{\sqrt{3}} T^* \ , \ G_{ij}^* - 2\partial_i\varphi\partial_j\varphi + g_{ij}^* \left[(\partial^*\varphi)^2 + 2V(\varphi) \right] = 8\pi G_* T_{ij}^*$$

with $\quad g_{ij}^* = f'\tilde{g}_{ij} \quad , \quad \rho_* = \tilde{\rho}/f'^2 \quad , \quad f' = e^{2\varphi/\sqrt{3}}$

Linearize : $\varphi = \varphi_c + \varphi_1$, $g_{ij}^* = f'_c(\eta_{ij} + h_{ij})$ with $V_c = \mathcal{O}(\kappa\rho_c) = \mathcal{O}(1/\ell^2)$.

scalaron : $\triangle\varphi_1 - m^2\varphi_1 \approx -\frac{4\pi G_{\rm eff}}{\sqrt{3}}\tilde{\rho}_\odot \qquad (\triangle_* = \triangle/f'_c \ , \ G_{\rm eff} = G_*/f'_c)$

$m^2 = \frac{f'_c d^2V}{d\varphi^2}|_c = \mathcal{O}(1/\ell^2)$ whereas $\triangle = \mathcal{O}(1/L_{\rm SS}^2)$ hence $\varphi_1 \approx \frac{G_{\rm eff}\tilde{M}_\odot}{\sqrt{3}r}$

metric : $ds_*^2/f'_c \approx (1 - 2G_{\rm eff}\tilde{M}/r)dt^2 + (1 + 2G_{\rm eff}\tilde{M}/r)d\vec{x}^2$

$$d\tilde{s}^2 = -\left(1 - \frac{2\tilde{G}\tilde{M}}{r}\right)dt^2 + \left(1 + \frac{2\gamma\tilde{G}\tilde{M}}{r}\right)d\vec{x}^2 \ , \ \tilde{G} = \frac{4G_{\rm eff}}{3} \ , \quad \gamma = 1/2$$

A scalar curvature "locked" at its cosmological, small, value :

Equation of motion for the scalar curvature (or scalaron, that is, φ) :

$$3D^2 f' + (Rf' - 2f) = \kappa T \qquad \text{(not } R = \kappa T \text{ as in GR !)}$$

Linearize : $R = R_c + R_1$ so that $D^2 R_1 - m^2 R_1 = \frac{\kappa T}{3 f''_c}$, $m^2 = \frac{(f' - Rf'')|_c}{3 f''_c}$

Solve with source being, say, a constant density star—or numerically,
Multamaki-Vilja, Kanulainen et al, (2007)
and find that, if $m r_\odot \ll 1$, then $R \approx R_c$ outside and inside the star.

**

On the other hand, if $m r_\odot \gg 1$, find Schwarzschild (hence $\gamma = 1$) :

$$d\tilde{s}^2 \approx -\left(1 - \frac{2GM_\odot(1-\epsilon)}{r}\right) dt^2 + \left(1 + \frac{2GM_\odot(1+\epsilon)}{r}\right) d\vec{x}^2 \ , \ \epsilon = \frac{e^{-m(r-r_\odot)}}{2m^2 r_\odot^2} \approx 0$$

but... $R \approx -\kappa T \gg R_c$ inside the star : linear approximation breaks down ?

• Summary

– the original CDTT model $f(R) = H_0^2(RH_0^{-2} - 1/(RH_0^{-2})^n)$ failed : final acceleration but no matter era

– various one-scale cosmologically viable models had been proposed, e.g. :
$f(R) = H_0^2(RH_0^{-2} + (RH_0^{-2})^n)$ with $0 < n < 1$

– all badly failed to comply with local gravity constraints because the scalar curvature is locked at its cosmological value $R = \mathcal{O}(H_0^2) = \mathcal{O}(\kappa\rho_{\rm cosmo})$ everywhere, even inside the Sun where, instead, GR gives $R = \mathcal{O}(\kappa\rho_{\odot})$

– however, if the scalaron could be given a heavy mass, the linear approximation (violated in that limit) indicates that the gravity field of the Sun would be the same as in GR

– hence : look for two-scale $f(R)$ models and solve the full non-linear scalaron eom

• Conventional wisdom...

e.g. Starobinski, 2007

is based on the linearised scalaron eom on the background $\frac{dV}{d\varphi}|_{\rm SS} \propto \kappa\rho_\odot$:

$$\triangle\varphi_1 - m^2\varphi_1 \approx 0 \quad \text{with} \quad m^2 = \frac{f' - Rf''}{3f''}$$

If, for $R \gg 1/\ell^2$, *e.g.* $R \sim \kappa\rho_\odot$: $f' \to 1$ and $m^2_{\rm SS} \approx \frac{1}{3f''|_{\rm SS}}$ is positive and large ($(mL)_{\rm SS} \gg 1$) then deviations from Einstein's GR are small.

CDTT model : $m^2_{\rm SS}$ is large and negative (Dolgov-Kawasaki, 2003)

• ...and the "Chameleon" effect

Khoury-Weltman (2003)
Navarro-Acoleyen (2006), Capozziello-Tsujikawa (2007)

or : how can the scalar curvature become equal to the local matter density

Chameleon details

Once again, look at scalaron eom : $D_*^2\varphi - \frac{dV}{d\varphi} = \frac{\kappa}{2\sqrt{3}}T_*$

In $\approx$ flat background : $\triangle_*\varphi = \frac{dV_{\rm eff}}{d\varphi}$ with $\frac{dV_{\rm eff}}{d\varphi} = \frac{dV}{d\varphi} - \frac{\kappa\rho_*}{2\sqrt{3}}$

Contrarily to V, $V_{\rm eff}$ may have minima for $\varphi = \varphi_\odot$ inside the Sun and for $\varphi = \varphi_e$ outside. One looks for $\varphi|_{e,\odot} = \varphi(R_{e,\odot})$ such that $R_{e,\odot} \approx \kappa\rho_{e,\odot}$.

Solution outside : $\varphi = \varphi_e + C\frac{r_\odot}{r}$ when $m_e r_\odot \ll 1$ with $m_e^2 = \frac{d^2V}{d\varphi^2}|_e$

Solution inside : $\varphi \approx \varphi_\odot$ up to $r = r_1$, then interpolation to φ outside.

if $r_1 \approx r_\odot$, then $C = 3\beta C_{\rm lin}$ with $\beta = \frac{\varphi_e - \varphi_\odot}{GM_\odot/r_\odot} \approx \frac{\varphi_e}{GM_\odot/r_\odot}$

One needs $\beta < 10^{-5}$ to have $\gamma - 1 < 10^{-5}$ (Cassini)

- A new family of $f(R)$ models of dark energy

Hu-Sawicki (2007), Starobinski (2007), Odintsov et al (2008), ...

$$f(R) = R + \lambda R_c \left[\frac{1}{(1+R^2/R_c^2)^n} - 1\right],\ etc$$

$\frac{dV}{d\varphi} - \frac{\kappa\rho_e}{2\sqrt{3}} = 0$ gives $\varphi_e = \mathcal{O}\left(\rho_c/\rho_e\right)^{2n+1}$

Now, $\beta \approx \frac{\varphi_e}{GM_\odot/r_\odot} \approx 10^6\varphi_e < 10^{-5}$; hence $\varphi_e < 10^{-11}$

For $\rho_e = \rho_{\rm galaxy} = 10^5\rho_c$ then $\varphi_e = \mathcal{O}(10^{-5(2n+1)})$ and hence the models evade Local Gravity Constraints as soon as $n > 1/2$

In a nutshell : the Chameleon effect conforts conventional wisdom (heavy "local" scalaron mass to evade LGC) : it relies on (1) "locking" φ, that is, the scalar curvature, on its Einstein value inside the Sun, and (2) on a local environment much denser than the asymptotic, cosmological, value.

- An uncontrolable scalaron instability ?

The new family of models yield cosmological scale factors which are undistinguishable from Λ-CDM until after the end of the matter era and tend to a de Sitter regime $a \propto e^{H\tilde{t}}$.

... However, because the mass of the scalaron is chosen to be high in high matter density environment (in order to comply with Local gravity Constraints), the cosmological perturbations of the scalaron diverge in the early matter era unless their amplitude is tuned to a very small value.

Starobinski (2007), Tsujikawa (2007)

4. “Detuned” $f(R)$ gravity and Dark Energy

- Coupling matter to the scalaron

Recall the Jordan-frame action of $f(R)$ gravity whose scalaron is coupled to matter :

$$\tilde{S}[\tilde{g}_{ij},\Phi] = \frac{1}{2\kappa}\int d^4x\sqrt{-\tilde{g}}\left[\Phi\tilde{R} - \frac{\omega(\Phi)}{\Phi}(\tilde{\partial}^2\Phi) - 2U(\Phi)\right] + S_{\rm m}[\Psi;\tilde{g}_{ij}]$$

where $\quad \Phi(s) = f'(s)e^{-2C(s)} \quad , \qquad U(s) = \frac{1}{2}(sf'(s) - f(s))e^{-2C(s)}$

and $\qquad \omega(s) = -\frac{3K(s)(K(s)-2)}{2(K(s)-1)^2} \quad$ with $\quad K(s) = \frac{d\ C}{d\ln\sqrt{f'}}$.

“detuning” allows for a non-vanishing Brans-Dicke function ω.

Hence the programme : play with the coupling function K and try to salvage one-scale $f(R)$ models of dark energy.

- Equations of motion

The Einstein frame ($g^*_{ij} = f e^{-2C} \tilde{g}_{ij}$) equations of motion are :

$$G^*_{ij} - 2\partial_i\varphi\partial_j\varphi + g^*_{ij}\left[(\partial^*\varphi)^2 + 2V(\varphi)\right] = \kappa T^*_{ij}$$

where $\varphi(s) = \sqrt{3}\ln\sqrt{f'(s)}\ ,\ V(s) = \frac{sf'(s)-f(s)}{4f'^2(s)}$

and $D^2_*\varphi - \frac{dV}{d\varphi} = \frac{\kappa(1-K)}{2\sqrt{3}}T^*$ with $K(s) = \frac{d\ C}{d\ln\sqrt{f'}}$

Bianchi identity : $D^*_j T^{*j}_i = -\frac{(1-K)}{\sqrt{3}}T^*\partial_i\varphi$

If $K = 1$, the Einstein and Jordan frame coalesce, matter decouples from the scalaron, and the theory reduces to Einstein GR minimally coupled to matter and a "quintessence" scalar field with a potentiel $V(\varphi)$.

Important remark : $\tilde{R} = e^{-2C}s - 6\tilde{D}^2C + 6(\tilde{\partial}C)^2 + \frac{Ke^{2C}}{f'}\kappa\tilde{T} \quad \neq s$

- **Conclusion**

– 2003-2007 : five years of trial and errors to conciliate $f(R)$ gravity with cosmology and local gravity constraints

– MANY failed attempts (which reduce to Brans-Dicke with $\Omega = 0$; or are unstable)

– Chameleon mechanism and "detuning" : two means to try and save the models