On quantum cosmological models and their limitations

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Fundamental Issues of the Standard Cosmological Model

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Outline

1. Affine quantization
   (coherent states, affine group, phase space representation, lower symbols, . . .)
2. Quantized cosmological models
   (repulsive term, FRW, Bianchi I, . . .)

3. Time Problem
   (canonical vs coordinate transformation, spectra, semiclassical trajectories, . . .)

Basic quantization scheme

The minimal setting:

$(X, d\mu)$ - a measurable space, $\mathcal{H}$ - Hilbert space,
$X \ni x \mapsto \psi_x \in \mathcal{H}$. Let

$$\int_X |\psi_x\rangle\langle\psi_x| d\mu = 1,$$

Then define

$$f(x) \mapsto A_f := \int f(x)|\psi_x\rangle\langle\psi_x| d\mu$$

(in a weak sense). Overcomplete basis.

Affine group

$$\boxed{X = (q,\ p) \in \mathbb{R}_+ \times \mathbb{R}, \quad \omega = \mathrm{d}p \wedge \mathrm{d}q}$$

$X = (q, p)$ is viewed as the affine group $\mathrm{Aff}_+(\mathbb{R})$ of the real line:

$$(q, p) \cdot (q_0, p_0) = \left(qq_0, \frac{p_0}{q} + p \right), \ q \in \mathbb{R}_+^*, \ p \in \mathbb{R},$$

$$e = (1, 0), \quad (q, p)^{-1} = (q^{-1}, -qp)$$

q - dilation, p - translation.

$$\omega \quad \text{is left-invariant wrt } \mathrm{Aff}_+(\mathbb{R})$$

Affine quantization

UIR of $\mathrm{Aff}_+(\mathbb{R})$ in $\mathcal{H} = L^2(\mathbb{R}_+^*, dx)$

$$\boxed{U(q,p)\psi(x) = e^{i\mathfrak{p}x}\frac{1}{\sqrt{\mathfrak{q}}}\psi(x/\mathfrak{q})}$$

Given a <u>normalized</u> vector $\psi_0 \in \mathcal{H}$, a continuous family of unit vectors are defined as

$$|q,p\rangle = U(q,p)|\psi_0\rangle\,, \quad \langle x|q,p\rangle = e^{i\mathfrak{p}x}\frac{1}{\sqrt{\mathfrak{q}}}\psi_0(x/\mathfrak{q})\,.$$

The resolution of unity guaranteed via Schur's Lemma

$$\int_X \frac{\mathrm{d}q\mathrm{d}p}{2\pi a_P}|q,p\rangle\langle q,p| = c_{-1}\cdot 1$$

Quantization:

$$\boxed{f \mapsto A_f = \int_X \frac{\mathrm{d}q\mathrm{d}p}{2\pi c_{-1}}\, f(q,p)\,|q,p\rangle\langle q,p|}$$

Results of quantization

The quantization of coordinate functions reads:

$$q \mapsto A_q = \frac{c_0(\psi_0)}{c_{-1}(\psi_0)} Q \,, \quad p \mapsto A_p = P \,,$$

where Q and P are position and momentum operators (on $\mathbb{R}_+^*$).

The quantization of the kinetic term reads:

$$p^2 \mapsto P^2 + \frac{K(\psi_0)}{Q^2} + \frac{J(\psi_0)}{2}(PQ^{-1} + Q^{-1}P) \,,$$

for $K \geqslant 3/4$ the above is self-adjoint. We fix

$$\psi_0(x) = \frac{1}{\sqrt{2x\, K_0(\nu)}} e^{-\frac{\nu}{4}\left(\frac{K_1(\nu)}{K_2(\nu)}x + \frac{K_2(\nu)}{K_1(\nu)x}\right)}$$

Phase space representation

For $|\phi\rangle$ one has the phase space representation:

$$|\phi\rangle \mapsto \Phi(q,p) = \langle q,p|\phi\rangle/\sqrt{2\pi c_{-1}}$$

with associated probability distribution:

$$|\phi\rangle \mapsto \rho_\phi(q,p) = \frac{1}{2\pi c_{-1}}|\langle q,p|\phi\rangle|^2$$

Having found the (energy) eigenstates of a quantum Hamiltonian H, one computes the time evolution

$$\rho_\phi(q,p,t) := \frac{1}{2\pi c_{-1}}|\langle q,p|e^{-iHt}|\phi\rangle|^2$$

for any state ϕ.

Semiclassical description

A lower symbol of an observable $f(q,p)$:

$$\check{f}(q,p) := \langle q,p|A_f|q,p\rangle = \int_{\Pi_+} \frac{\mathrm{d}p'\mathrm{d}q'}{2\pi c_{-1}} |\langle q,p|q',p'\rangle|^2 f(q',p')$$

provides a semiclassical description of A_f.

For example:

$$\check{q} = q$$
$$\check{p} = p$$
$$\check{p^2} = p^2 + 2a_P^2 K(\nu)\frac{1}{q^2}$$

Singularity model: FRW

Gravity+perfect fluid:

$$S := S_g + S_f = \int R\sqrt{-g}\ d^4x + \int \sqrt{g}\ \mathrm{p}(\mu, S) d^4x$$

with $\mu = \mu(\phi, \alpha, \beta, \theta, S)$. Follow ADM decomposition:

$$ds^2 = -N^2 dt^2 + q_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

$+$ hamiltonian analysis $+$ symmetry imposition:

$$H := H_g + H_f = N\left(-\frac{\tilde{p}_a^2}{24\tilde{a}} - 6\tilde{k}\tilde{a} + \frac{p^T}{\tilde{a}^{3w}}\right) \approx 0$$

where $\mathrm{p} = w\rho$,

$$\tilde{k} = V_0^{2/3}k, \quad \tilde{a} := aV_0^{\frac{1}{3}}, \quad \tilde{p}_a := -12\frac{\dot{a}a}{N}V_0^{\frac{2}{3}}$$

$$(\tilde{a}, \quad \tilde{p}_a) \in R_+ \times R, (T, p^T) \in R \times R_+$$

Reduced phase space

Fix N so that:

$$N\left(-\frac{\tilde{p}_a^2}{24\tilde{a}} - 6\tilde{k}\tilde{a} + \frac{p^T}{\tilde{a}^{3w}}\right) = -p^T + \frac{\tilde{p}_a^2}{24}\tilde{a}^{3w-1} + 6\tilde{k}\tilde{a}^{3w+1}$$

Solve the constraint for p_T:

$$h_T := \frac{\tilde{p}_a^2}{24}\tilde{a}^{3w-1} + 6\tilde{k}\tilde{a}^{3w+1}, \quad (\tilde{a}, \tilde{p}, T)$$

Perform a canonical transformation:

$$(\tilde{a}, \tilde{p}) \mapsto (q, p)$$

to obtain

$$h_T = \alpha(w)p^2 + 6\tilde{k}q^{\mu(w)}, \quad q > 0$$

$k = 1, w = \frac{1}{3}$

A harmonic oscillator on the half-line.

$$h_T = \frac{1}{24} p^2 + 6\tilde{k} q^2$$

Quantum hamiltonian

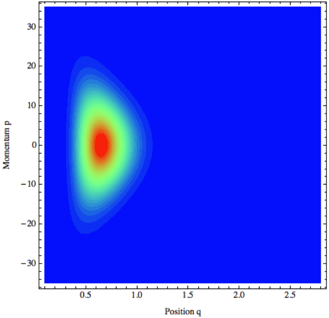
$$\boxed{A_{h_T} = \tfrac{1}{24} P^2 + \tfrac{1}{24} \tfrac{K(\nu)}{Q^2} + 6\tilde{k} M(\nu) Q^2}$$

Semiclassical point of view: a displacement of the equilibrium point of the potential $q_e^4 = \frac{1}{144} \frac{K(\nu)}{M(\nu)}$. The singularity $Q = 0$ is 'shielded' by the infinite potential.

Ground state ϕ_0

Probability distribution:

$$|\phi_0\rangle \mapsto \rho_{\phi_0}(q,p) = \frac{1}{2\pi c_{-1}} |\langle q, p | \phi_0 \rangle|^2$$



Dynamics

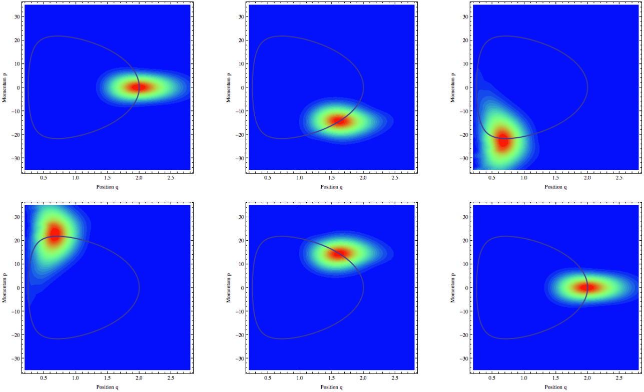


Figure: *Phase space distributions* $\rho_{\,|\,q_0, p_0, \tau\,\rangle}(p, q)$ *at different times equally spaced (from top left to bottom right). Increasing values of the function are encoded by the colors from blue to red.*

Averaged distribution

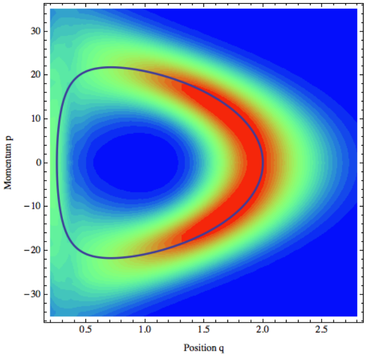


Figure: *Time average phase space distribution evolving following the Hamiltonian. Increasing values of the function are encoded by the colors from blue to red.*

Semiclassical constraint

Lower symbol $\rightarrow$ Modified Friedmann equation:

$$H^2 + c^2 a_P^2 (1-w)^2 \frac{\nu}{128} \frac{1}{V^2} + \frac{kc^2}{a^2} = \frac{8\pi G}{3c^2} \rho$$

$*$ repulsive potential is of quantum geometrical origin
$*$ singularity resolution confirmed: $\frac{1}{V^2} \sim a^{-6}$ grows faster than $\rho \sim a^{-3(1+w)}$ for $a \rightarrow 0$
$*$ meaningful for compact universes only
$*$ departure from EFE: $|V\theta| \lesssim \frac{3}{8\sqrt{2}} a_P (1-w)\sqrt{\nu}$

Anisotropic singularity

The constraint:

$$H := H_g + H_m = Nq^{\mu}\left(\frac{1}{24}\left(-\alpha^2 p^2 + q^{-2}(p_+^2 + p_-^2)\right) + p_T\right) \approx 0$$

is solved for p_T

$$h = \frac{\alpha^2}{24}p^2 - \frac{1}{24}(p_+^2 + p_-^2)q^{-2}$$

$$(q,p) \in \mathbb{R}_+^* \times \mathbb{R} \qquad (\beta^{\pm}, p_{\pm}) \in \mathbb{R}^4$$

Note:

$$\boxed{h > 0}$$

Quantization:

$$A_h := \frac{\alpha^2}{24}P^2 + a_P^2\frac{\alpha^2}{24}\frac{K}{Q^2} - \frac{1}{24}\frac{c_{-3}}{c_{-1}}(P_+^2 + P_-^2)Q^{-2}$$

Positivity constraint

We quantize $\theta(h)h$:

$$A_{\theta(h)h} = A_h - A_{\theta(-h)h}$$

and find the non-local part:

$$\langle x|A_{\theta(-h)h}|x'\rangle =$$

$$\frac{2}{\pi K_0(\nu)c_{-1}}\left(\frac{k\mathrm{Re}[K_0(2\gamma)]}{\sqrt{xx'}(x-x')^2} - \frac{\xi\mathrm{Im}[\gamma K_1(2\gamma)]}{\sqrt{xx'}(x-x')^3\frac{\nu}{4}(\frac{1}{x}+\frac{1}{x'})}\right)$$

$$\text{with } \gamma = \frac{\nu}{4}\sqrt{(\frac{1}{x}+\frac{1}{x'})(x+x'-i\frac{4}{\nu}\frac{k}{\xi}(x-x'))}$$

We go for semiclassical description:

$$\langle q,p|A_{\theta(h)h}|q,p\rangle$$

Result

$$\langle A_{\theta(h)h} \rangle_{q,p} \approx \frac{1}{q^2} \left(p^2 q^2 + A(\nu) - B(\nu) k^2 + a_\nu(k^2) \cdot b_\nu(k^2, p^2 q^2) \right)$$

where

$$a_\nu(k^2) = -A(\nu) \frac{\lambda_1(\nu) k^2}{1 + \lambda_1(\nu) k^2} + B(\nu) k^2,$$

$$b_\nu(k^2, p^2 q^2) = \frac{1 + \lambda_1(\nu) k^2}{1 + \lambda_1(\nu) k^2 + \lambda_2(\nu) p^2 q^2}$$

Solution reads:

$$q(T) = \sqrt{\frac{1}{\tilde{h}} \frac{\alpha^2}{24} \left(4\tilde{h}^2 T^2 + A(\nu) - B(\nu) \frac{k^2}{\alpha^2} + a_\nu \left(\frac{k^2}{\alpha^2} \right) \cdot b_\nu \left(\frac{k^2}{\alpha^2}, 4\tilde{h}^2 T^2 \right) \right)}$$

and hence

$$V_{min} = \left(\frac{A(\nu)}{1 + \lambda_1 \frac{k^2}{\alpha^2}} \frac{\alpha^2}{24\tilde{h}} \right)^{\frac{3}{2\alpha}}$$

Big Bounce

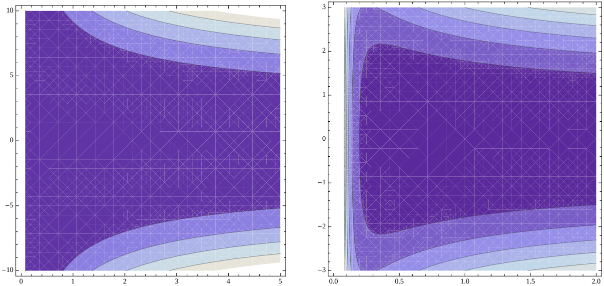


Figure: *On the left: classical trajectories diverging in the vicinity of the singularity. On the right: semiclassical trajectories (lower symbol of physical hamiltonian) exhibiting smooth transition from collapsing to expanding branch.*

Summary

Affine quantization offers a singularity resolution and:

1. "extra" quantum terms
2. adjustable
3. CCR
4. self-adjointness
5. small volume/high curvature quantum effects
6. semiclassical description with the repulsive potential
7. quantize functions and distributions

Can we trust quantum cosmological models?

imposing symmetry + quantizating $\neq$ quantizating + imposing symmetry
hamiltonian constraint system:
Dirac vs RPS
Hilbert space problem vs multiple choice problem (for a given ST foliation)

Solving the constraint

Hamiltonian constraint:

$$H = P + h(Q, q^r, p_s) \approx 0$$

E.o.m.:

$$\dot{q}^r = \{p^r, H\} = \{q^r, h(Q, q^r, p_s)\},$$

$$\dot{p}^s = \{p^s, H\} = \{p^s, h(Q, q^r, p_s)\},$$

$$\dot{Q} = 1.$$

Reduced framework:

$$h(Q, q^r, p_s) = \textit{true hamiltonian}$$

$$Q = \textit{clock variable}$$

$$(q^r, p_s) = \textit{physical phase space}$$

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Canonical transformation:

$$(P, Q, p_s, q^r) \mapsto (\tilde{P}, \tilde{Q}, \tilde{p}_s, \tilde{q}^r)$$

and new constraint function

$$\tilde{H} = \tilde{P} + \tilde{h}(\tilde{Q}, \tilde{p}_s, \tilde{q}^r) \approx H$$

Another reduced framework:

$$\tilde{h}(\tilde{Q}, \tilde{q}^r, \tilde{p}_s) = \textit{true hamiltonian}$$

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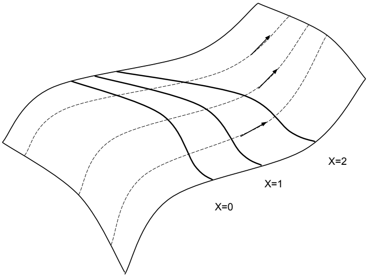
Basic fact: Reduced frameworks are not canonically equivalent

Constraint surface

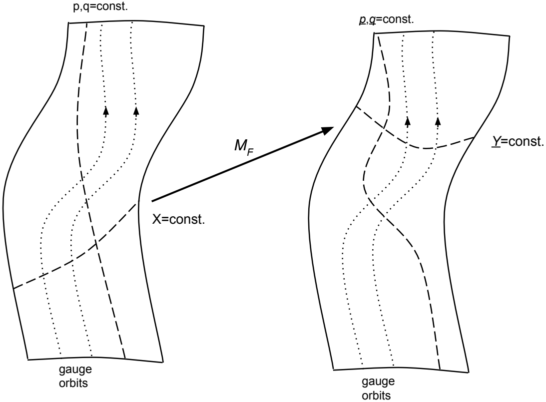
$H = 0$:



$\omega|_{H=0} = E^*\omega$ - degenrate two-form, $v_H = \{\cdot, H\}$ - hamiltonian vector field, $X : \mathcal{C} \mapsto \mathbb{R}$ - time function such that $v_H(X) \neq 0$

Active transformations in $\mathcal{C}$

<u>Let</u> (X, q^r, p_s) and $(Y, \tilde{q}^r, \tilde{p}_s)$ be two deparametrizations of a given hamiltonian constraint system and h and $\tilde{h}$ be the respective true hamiltonians.



<u>For</u> a given $F : \mathbb{R} \mapsto \mathbb{R}$, there exists a unique, invertible mapping $\mathcal{M}_F : (X, q^r, p_s) \mapsto (Y, \tilde{q}^r, \tilde{p}_s)$ such that $X \mapsto Y = F(X)$ and $\pi \circ \mathcal{M}_F = \pi$, where π is the projection generated by the gauge transformation. Moreover, the mapping satisfies: i) for every value of X, $\mathcal{M}_F|_X : (q^r, p_s) \mapsto (\tilde{q}^r, \tilde{p}_s)$ is canonical, ii) $dh = d\mathcal{M}_F^*(\tilde{h}) - i_{\bar{X}}\mathcal{M}_F^*(d\tilde{q}^r d\tilde{p}_r)$

Quantum discrepancy

Quantize $\mathcal{M}_F$:

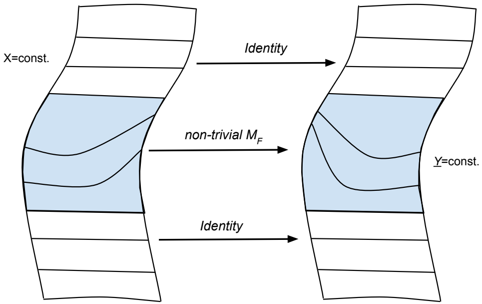
$$\mathbb{M}: \ \mathcal{H}_X \mapsto \mathcal{H}_Y, \quad \hat{O}(X) \mapsto \hat{O}(Y) = \mathbb{M}\hat{O}(X)\mathbb{M}^\dagger$$

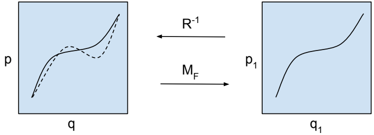
$$\hat{h}_X \mapsto \hat{h}_Y = \mathbb{M}(Y)\left(\frac{dX}{dY}\,\hat{h}_X + i\,\frac{d\mathbb{M}^\dagger(Y)}{dY}\mathbb{M}(Y)\right)\mathbb{M}^\dagger(Y)$$

Physically, $\mathcal{R} : (X, q^r, p_s) \mapsto (Y, \tilde{q}^r, \tilde{p}_s)$:

$$\hat{O}(X) \mapsto \hat{O}(Y) = \widehat{\mathcal{R}}\left[O(X)\right]$$

Assume $Y(X) = X, \ \frac{d\mathbb{M}(Y)}{dY} = 0$:





Example: semiclassical description

Consider a constraint surface

$$(q, p, X) \in \mathbb{R}^3 \quad \{q, p\} = 1 \quad \mathcal{O} = \mathcal{O}(X, q, p)$$

Use the Schrödinger representation

$$q \mapsto \hat{q} := x, \quad p \mapsto \hat{p} := \frac{1}{i}\frac{\partial}{\partial x}$$

Pick a gaussian state (in the carrier space of the Dirac algebra's representation)

$$\psi(x) = \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}}$$

Examine dynamical observable X^2*.*

$$\boxed{X \to t = X + q^k}$$

$$\langle X^m \rangle_t = \int_{-\infty}^{\infty} \pi^{-\frac{1}{2}} e^{-x^2} (t - x^k)^m dx = \sum_{l=0}^{m} (-1)^l t^{m-l} \frac{m!(lk)!}{l!(m-l)! 2^{\frac{lk}{2}} (\frac{lk}{2})!} \epsilon_{kl}$$

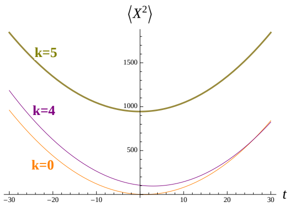
. . . semiclassical description

We have $\langle X^2 \rangle_X = X^2$,

$$\langle X^2 \rangle_t = \begin{cases} t^2 - \frac{2k!}{2^{\frac{k}{2}} (\frac{k}{2})!} t + \frac{(2k)!}{2^k k!} & \text{if } k \text{ even} \\ t^2 + \frac{(2k)!}{2^k k!} & \text{if } k \text{ odd.} \end{cases}$$

$$\begin{aligned} \min \langle X^2 \rangle_X &= 0 \quad \text{for time } X, \\ \min \langle X^2 \rangle_t &= \frac{(2k)!}{2^k k!} - \left(\frac{k!}{2^{\frac{k}{2}} (\frac{k}{2})!} \right)^2 \quad \text{for } k \text{ even}, \\ \min \langle X^2 \rangle_t &= \frac{(2k)!}{2^k k!} \quad \text{for } k \text{ odd}, \end{aligned}$$

One may lift the minimum as much as one wishes by increasing k.



Example: spectrum

- *Consider:*

$$\mathbb{R}^3 \ni (q, p, X) \mapsto (p_1, q_1, Y) \in \mathbb{R}^3$$

and the coordinate transformation $\mathcal{R}$:

$$(q_1, p_1, Y) := (X, p, X - q)$$

- *Define $\mathcal{M}$:*

$$(q_1, p_1, Y) = (q + X, p, X)$$

- *Apply the Schrödinger representation:*

$$\hat{q} := x, \quad \hat{p} := \frac{1}{i}\frac{\partial}{\partial x}, \quad \hat{q}_1 := x_1, \quad \hat{p}_1 := \frac{1}{i}\frac{\partial}{\partial x_1}$$

$$\mathbb{M} : L^2(\mathbb{R}, \mathrm{d}x) \ni \psi(x) \mapsto \psi(x_1 - Y) \in L^2(\mathbb{R}, \mathrm{d}x_1)$$

$$\hat{h}_1 = -i\mathbb{M}\frac{d\mathbb{M}^\dagger}{dY} = \hat{p}_1$$

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Example:

$$O = p^2 + X^2$$

$$\mathcal{R}[O] = p_1^2 + q_1^2$$

Spectra:

$$sp(\hat{O})\Big|_{X=const} = (X^2, \infty)$$

$$sp(\hat{O})\Big|_{Y=const} = 2n + 1, \;\; n \in \mathbb{Z}$$

Conclusions

- *Affine quantization is an attractive proposal in quantum cosmology. It resolves the singularity problem in anisotropic models. But. . .*
- *Time problem makes it difficult to interpret the results. Different choices of time lead to different quantum scenarios, even if quantization is unique!*
- *Whenever one talks about spectra of geometrical operators, critical values of energy density or volume at the big bounce etc, one must always refer to the choice of time function*
- *The problem is independent of the quantization method: Dirac's, rps . . .*
- *To apply to cosmology and incorporate the cosmological perturbations, one makes use of some effective background geometry. The process of particle production in the early universe is going to be influenced by the choice of time function there!*