Viability of $f(R)$ gravity: a quick review

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Workshop: Fundamental Issues of the Standard Cosmological Model,
21-27 September 2014

- Papers: L.Jaime,L.Patiño, M.S:
- On cosmology:
- arXiv:1206.1642
- arXiv:1211.0015
- PRD vol. 87, 024029 (2013)
- PRD vol. 89, 084010 (2014)
- On compact objects: PRD vol. 83,024039 (2011)

Outline

Summary

Most of issues that I'll be tackling in this talk are not new. Most of them started since the discovery of the accelerated expansion of the Universe in 1998. Here I'll review one of the most popular alternatives to explain this phenomena and which consist in modifying gravity in the most simple way, without introducing new fields and while respecting most of the basic tenets of Einstein's GR. This alternative is termed $f(R)$ gravity, a particular case being the paradigmatic $f_{GR}(R) = R - 2\Lambda$, (i.e. Einstein's theory + the *infamous* cosmological constant). Suitable modifications of $f_{GR}(R)$ without the Λ term may produce an adequate accelerated expansion with a "dark-energy equation of state" $\omega \approx -1$, but which varies in cosmic time; an interesting possibility that can be tested in a near future. Nevertheless, modifying one of the most successful theories in physics comes with a high price: many of the usual GR predictions might be spoiled (the field equations are different), thus, for every specific proposal $f(R) \neq f_{GR}(R)$ all the gravitational test must be repeated. Here I'll try to summarize until what extent this alternative theories can be viable in several astrophysical and cosmological scenarios.

Motivation

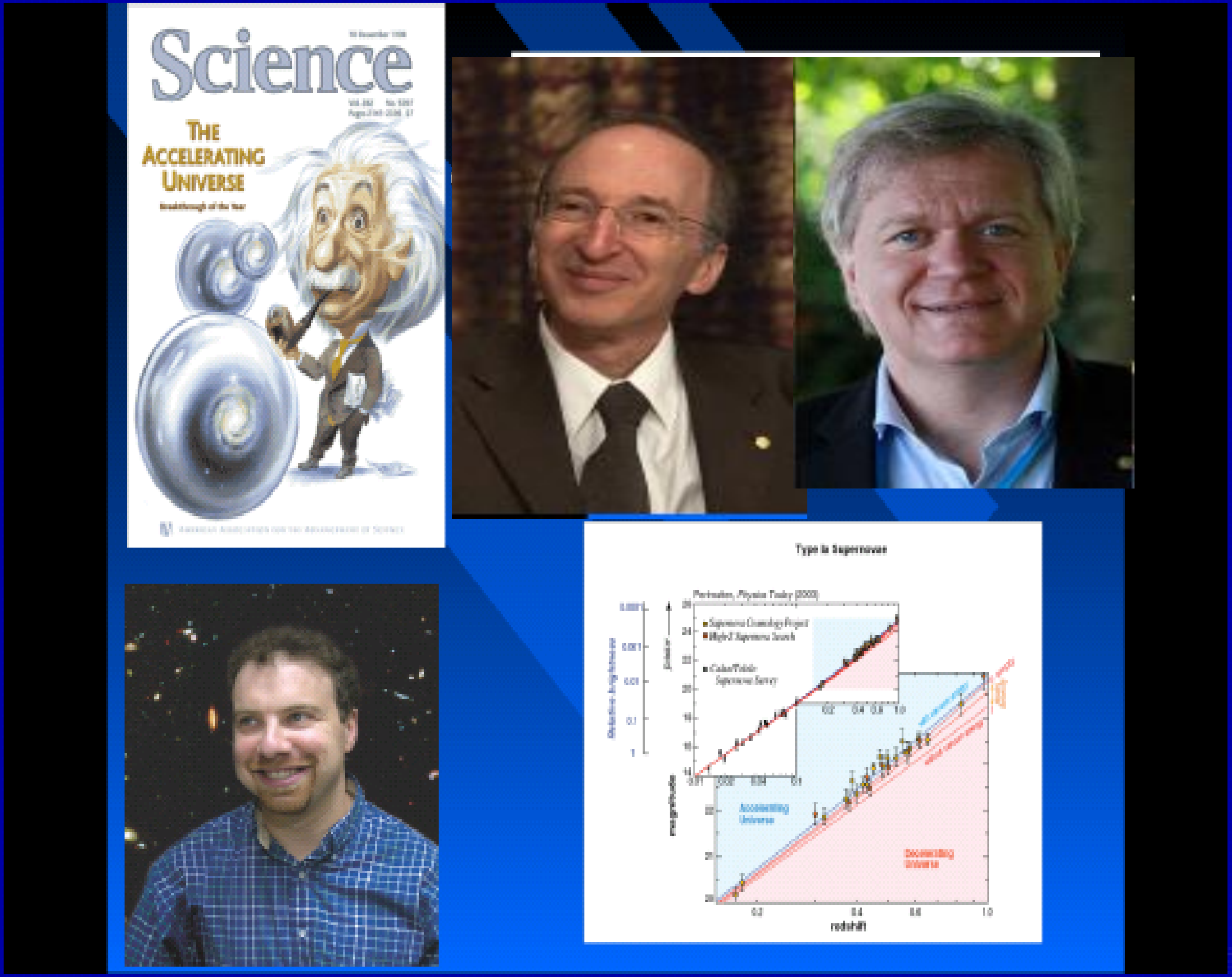
– ΛDM paradigm within GR: the simplest and perhaps most successful cosmological model.

– Alternative Theories of Gravity: try to "replace" Dark (matter-energy) components. This is just one among several possibilities (e.g. inhomogeneous models within GR). More complicated, but it's a worth exploring possibility (I skip the heuristic and philosophyical arguments about the "problems" of Λ. But if you want a thorough and recent review on the subject see: E. Bianchi & C .Rovelli arXiv:1002.3966 and J. Martin: arXiv:1205.3365.)

– $f(R)$ metric theories of gravity: a possible explanation for the accelerated expansion of the Universe as opposed to the Cosmological Constant. (As far as we know, DM must be considered, otherwise it seems impossible to recover the rest of cosmological observations.). These alternative theories of gravity (like others) allows for an "EOS of (geometric) dark energy" that varies in cosmic time, unlike Λ.

SNIa DATA

$D_L(z) = cH_0^{-1}(z+1)\int_0^z \frac{dz'}{H(z')}$ (for $k=0$), $\mu = 5\log(D_L/Mpc) + 25$.



$f(R)$ METRIC GRAVITY

$$S[g_{ab},\psi]=\int\frac{f(R)}{2\kappa}\sqrt{-g}\,d^4x+S_{\rm matt}[g_{ab},\psi]\;, \qquad (1)$$

where R =Ricci scalar, $f(R)$ is a C^3 but otherwise arbitrary function of R, $\kappa := 8\pi G_0$, and ψ represents collectively the matter fields – ordinary and DM – (here $c=1$).

Varying the action Eq. (1) with respect to the metric yields

$$f_R R_{ab}-\frac{1}{2}fg_{ab}-\left(\nabla_a\nabla_b-g_{ab}\Box\right)f_R=\kappa T_{ab}\;, \qquad (2)$$

where $f_R := \partial_R f$, $\Box = g^{ab}\nabla_a\nabla_b$, and T_{ab} is the EMT of matter.

Theorem (Exercise)

Take ∇^a on both sides of Eq. (2) and prove that the EMT of matter is conserved:

$$\nabla^a T_{ab}=0\;. \qquad (3)$$

When expanding the derivative ∇ acting on f_R we obtain

$$G_{ab} = \frac{1}{f_R}\Big\{ f_{RR}\nabla_a\nabla_b R + f_{RRR}(\nabla_a R)(\nabla_b R) - g_{ab}\left[\frac{1}{2}(Rf_R - f) + f_{RR}\Box R + f_{RRR}(\nabla R)^2\right]\Big\} + \frac{\kappa}{f_R}T_{ab} \; , \tag{4}$$

where $(\nabla R)^2 := g^{ab}(\nabla_a R)(\nabla_b R)$.

When taking the trace we obtain the following EOM for R:

$$\Box R = \frac{1}{3f_{RR}}\left[\kappa T - 3f_{RRR}(\nabla R)^2 + 2f - Rf_R\right] \; , \tag{5}$$

where $T := T^a{}_a$. Using this equation to replace $\Box R$ in (4), the latter becomes

$$G_{ab} = \frac{1}{f_R}\left[f_{RR}\nabla_a\nabla_b R + f_{RRR}(\nabla_a R)(\nabla_b R) - \frac{g_{ab}}{6}\Big(Rf_R + f + 2\kappa T\Big) + \kappa T_{ab}\right] . \tag{6}$$

We shall be dealing with equations (5) and (6), and treat the theory as a system of second order coupled PDE for the Ricci scalar R and the metric g_{ab}, respectively.

Exercise: take $f(R) = R - 2\Lambda$ and show that Eqs. (6) and (5) reduce respectively to the GR+Λ theory:

$$G_{ab} + g_{ab}\Lambda = \kappa T_{ab} \; , \tag{7}$$

$$R = 4\Lambda - \kappa T \; . \tag{8}$$

GR vs. Alternative Theories of Gravity

The basic axioms of GR are kept in $f(R)$ gravity:

1. The spacetime is a 4-dimensional differential manifold endowed with a Lorentzian metric (M, g_{ab}).
2. Gravitation is described geometrically in terms of the Riemann tensor $R_{abcd} \neq 0$ ($R_{abcd} = 0$ only when the spacetieme is globally flat).
3. The theory should be covariant (diffeomorphism invariant).
4. The equivalence principle holds: test particles move on geodesics of the metric g_{ab}. The laws of physics (those compatible with special relativity) are still valid locally.
5. The only quantity pertaining to spacetime that should appear in the laws of physics is the metric (*minimal coupling*).
6. Assume the usual Levi-Civita connection (no torsion and the theory is metric compatible $\nabla_a g_{ab} = 0$).
7. The field equations should be linear in the second derivatives (quasilinear PDE).

$f(R)$ theories keep all this axioms except "7": only fulfilled when $f(R) = R - 2\Lambda$.

$f(R)$ GRAVITY (BRIEF HISTORICAL REMARKS)

– Non-linear Lagragians in R, R_{ab}, and R_{abcd} date back since the years that followed GR (H. Weyl, 1921; K. Lanczos, 1938; Buchdahl 1970). They were analyzed much later in different contexts. For instance, in cosmology ...

– 1979 (A. Starobinsky), as models for *inflation.*: $f(R) = R - aR^2$.

– 1982 (R. Kerner) as a "cosmological model without singularity". Remark: Several $f(R)$ models considered today are very similar to those considered in that paper.

– 1986 (J.P. Durisseau & R. Kerner) as a "reconstruction of inflationary model".

– As mentioned before, the discovery of the accelerated expansion of the Universe renewed the interest in this kind of models. The first ones proposed within the specific goal of producing an accelerated expansion were: Cappozziello (2002), Cappozziello et al. (2003), Carroll et al. (2004,2005).

– Since 2003 a boom of papers analyzing $f(R)$ gravity in all possible scenarios have appeared in the literature: perhaps more than 1000 papers ! ($\approx 2/week$).

$f(R)$ GRAVITY CAN MIMIC Λ

Notice that in vacuum, $R = R_1 = const.$ is a solution of

$$\Box R = \frac{1}{3f_{RR}}\left[\kappa T^{=0} - 3f_{RRR}(\nabla R)^2 + 2f - Rf_R\right] , \tag{8}$$

provided R is a root of $V'(R) = 2f - Rf_R$ (i.e. $2f(R_1) = R_1 f_R(R_1)$), assuming for instance $f_{RR}(R_1) \neq 0$.

That is, R_1 is a critical point (e.g. maximum or minimum) of the "potential" $V(R)$.

In such an instance, the field equation

$$G_{ab} = \frac{1}{f_R}\left[f_{RR}\nabla_a\nabla_b R + f_{RRR}(\nabla_a R)(\nabla_b R) - \frac{g_{ab}}{6}\left(Rf_R + f + 2\kappa T^{=0}\right) + \kappa T_{ab}^{=0}\right] . \tag{9}$$

reduces to

$$G_{ab} = -g_{ab}\frac{R_1}{4} \qquad \text{(in vacuum)} . \tag{10}$$

$f(R)$ theory behaves like GR with an *effective cosmological constant* $\Lambda_{eff} = R_1/4$!

The fact that the theory can admit this solution for R allows one to find non-trivial solutions that asymptotically (past,future, or spatial infinity) match a De Sitter solution, which in turn can explain several cosmological observations.

Gravitational tests

Important tests for any gravitational theory:

- **Cosmology:** SNIa data, age of the Universe, nucleosynthesis, perturbed FRW (CMB), etc.
- **Solar system:** classical tests: Does $f(R)$ gravity really pass those tests or not ?
- **Strong gravity:** binary pulsar, neutron stars (mass vs. radius, EOS), BH's etc.
- **Formal issues:** Cauchy problem, singularity theorems, BH's: existence and uniqueness, etc.

FRW COSMOLOGY

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] , \qquad (11)$$

where $k = \pm 1, 0$. When obtaining numerical solutions we shall focus only on the "flat" case $k = 0$.

$$H^2 + \frac{k}{a^2} + \frac{1}{f_R}\left[f_{RR}H\dot{R} - \frac{1}{6}(f_R R - f)\right] = \frac{\kappa\rho}{3f_R} , \qquad (12)$$

$$\ddot{a}/a = \dot{H} + H^2 = \underbrace{\frac{1}{f_R}\left(f_{RR}H\dot{R} + \frac{f}{6} - \frac{\kappa\rho}{3}\right)}_{R_1/12 - \Lambda_{\rm eff}/3 \text{ when } \rho\to 0 \text{ and if } R\to R_1} , \qquad (13)$$

where $H = \dot{a}/a$ is the Hubble expansion. From Eq. (4) we find

$$\ddot{R} = -3H\dot{R} - \frac{1}{3f_{RR}}\left[3f_{RRR}(\dot{R})^2 + \underbrace{2f - f_R R}_{V'(R)} - \kappa(\rho - 3p)\right] . \qquad (14)$$

If $R(t)$ reaches R_1 of the potential $V(R)$ (with vanishing $\dot{R}$, and $\ddot{R}$), in the far future where the matter contributions $\rho, p \ll \rho_{\rm crit}$ and $R \approx R_1$ today, then $\dot{H} + H^2 = \ddot{a}/a \approx R_1/12 = \Lambda_{\rm eff}/3 > 0$ if $\Lambda_{\rm eff} > 0$. Thus $\ddot{a} > 0 \rightarrow$ Accelerated expansion !!. This what happens precisely when solving the full equations numerically taking into account all the matter terms.

Now, the expresion for the Ricci scalar is given by

$$R = 6\left(\dot{H} + 2H^2 + \frac{k}{a^2}\right) \quad . \tag{15}$$

Note that by using Eqs. (12) and (13) in Eq. (15) we obtain an identity $R \equiv R$, which shows the consistency of the equations (c.f. the SSS case) !

T_{ab} of matter is a mixture of three kinds of perfect fluids: baryons, radiation and dark matter, in a epoch where they don't interact with each other except gravitationally. Then for each matter component the EMT conserves separately and $\nabla_a T_i^{ab} = 0$ ($i = 1, 3 \rightarrow$ baryons, radiation, DM) leads to

$$\dot{\rho}_i = -3H\left(\rho_i + p_i\right) \quad . \tag{16}$$

The total energy-density is $\rho = \sum_i \rho_i = -T^t_{\ t}$ and since $p_{\rm bar,DM} = 0$, and $p_{\rm rad} = \rho_{\rm rad}/3$ then $T = T_{\rm bar} + T_{\rm DM} = -(\rho_{\rm bar} + \rho_{\rm DM})$. Then Eq. (16) integrates

$$\rho = \frac{\rho^0_{\rm bar} + \rho^0_{\rm DM}}{(a/a_0)^3} + \frac{\rho^0_{\rm rad}}{(a/a_0)^4} \quad , \tag{17}$$

where the knotted densities are the densities today. Here $a_0 = a(t_0)$, t_0 is present cosmic time. The differential equations will depend explicitly on $a(t)$ via the matter terms.

EQUATION OF STATE (EOS) OF GDE (1ST PART)

In the ΛCDM model the equation of state $\omega_\Lambda = p_\Lambda/\rho_\Lambda = -1$. We shall define an EOS for the modified gravity contribution given by (for $f(R) \neq R$)

$$\omega_X = \frac{p_X}{\rho_X} \ , \tag{18}$$

where ρ_X is defined from the modified Friedmann equation, so that it reads $H^2 = \frac{\kappa}{3}\left(\rho + \rho_X\right) = \frac{\kappa \rho_{\rm tot}}{3}$, which leads to

$$\rho_X = \frac{1}{\kappa f_R}\left\{\frac{1}{2}\left(f_R R - f\right) - 3 f_{RR} H \dot{R} + \kappa \rho \left(1 - f_R\right)\right\} \ , \tag{19}$$

In a similar way we define p_X, so that the dynamic equation for H reads

$$\dot{H} + H^2 = -\frac{\kappa}{6}\left\{\rho + \rho_X + 3\left(p_{\rm rad} + p_X\right)\right\} = -\frac{\kappa \rho_{\rm tot}}{6}\left\{1 + 3\omega_{\rm tot}\right\} \ , \tag{20}$$

where $\omega_{\rm tot} = p_{\rm tot}/\rho_{\rm tot}$. From this latter, we obtain

$$p_X = -\frac{1}{3\kappa f_R}\left[\frac{1}{2}\left(f_R R + f\right) + 3 f_{RR} H \dot{R} - \kappa\left(\rho - 3 p_{\rm rad} f_R\right)\right] \tag{21}$$

Which $f(R)$?

Among the infinite a priori possible choices of $f(R)$ (restricted by $f_R > 0$ so as to $G_{\rm eff} = G_0/f_R > 0$ and $f_{RR} > 0$, stable perturbations around a background), how to choose ?

– Simplicity $\rightarrow f(R) = R - 2\Lambda$. But we don't want this. We want something with $\omega_X(t)$ such that today $\omega_x \approx -1$.

– Ingeeniring, trial and error, handcraft, reconstruction, ...

– Is there any new physical principle that single out an $f(R)$ different from $f_{GR}(R)$, that match all the tested gravitational observations and yet provide new and "unexpected" predictions ? Ans. Maybe.

Specific $f(R)$ models

Given a specific $f(R)$, we integrate the differential equations forward from past to future with suitable "initial conditions". We have considered three specific $f(R)$ models which have become very popular in the literature

- Miranda *et. al.* model (PRL 102, 221101, 2009)

$$f(R)_{\rm MJW} = R - \beta R_* \ln\left(1 + \frac{R}{R_*}\right) \ . \tag{22}$$

  We used $\beta = 2$ and $R_* = H_0^2$.

- Starobinsky model (JETP Lett. 86, 157 2007)

$$f(R)_{\rm St} = R + \lambda R_S \left[\left(1 + \frac{R^2}{R_S^2}\right)^{-q} - 1\right] \ . \tag{23}$$

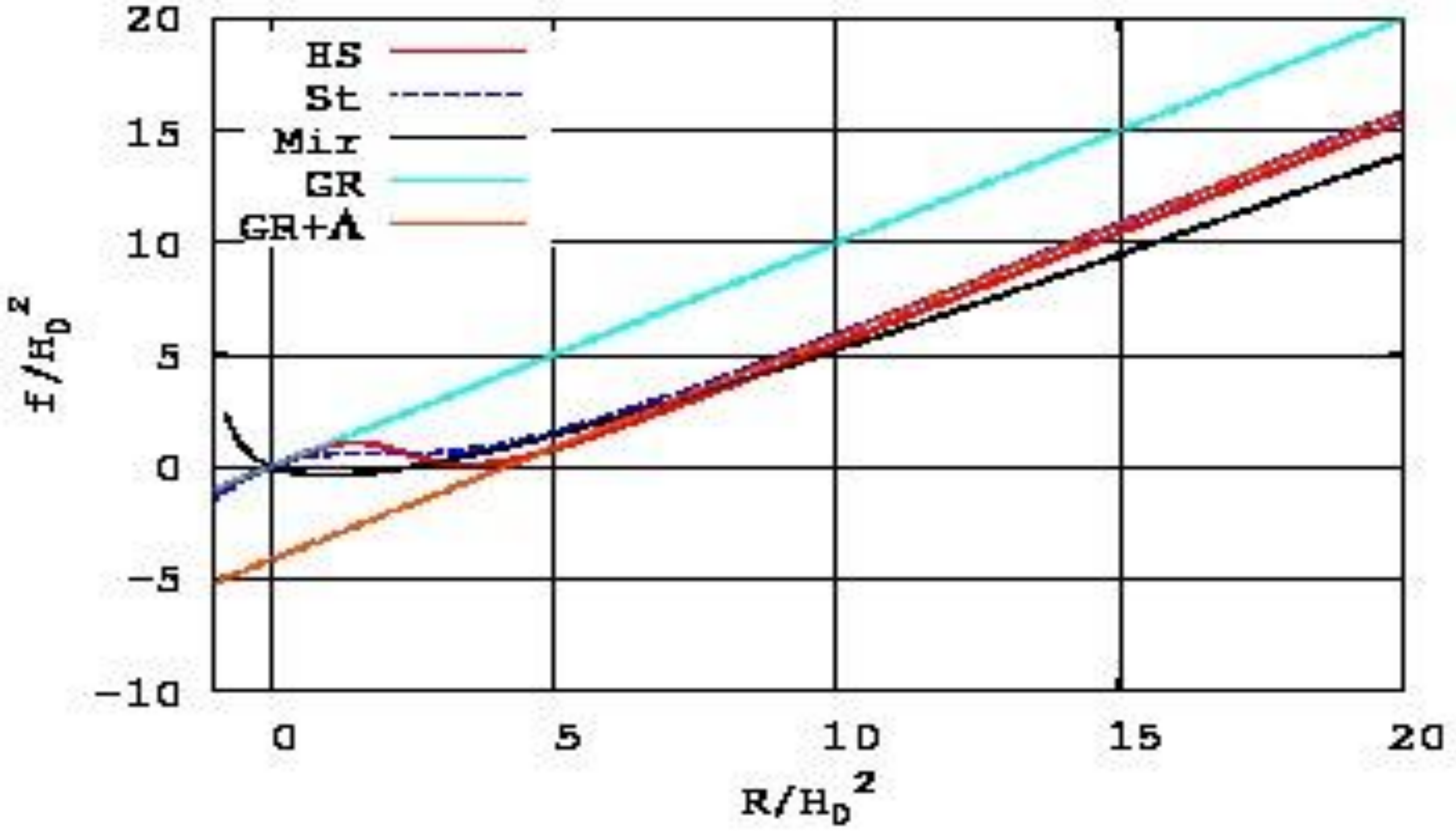
  We take $q = 2$ and $\lambda = 1$, $R_S \approx 4.17 H_0^2$.

- Hu & Sawicky model (PRD 76, 064004, 2007)

$$f(R)_{HS} = R - m^2 \frac{c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1} \ . \tag{24}$$

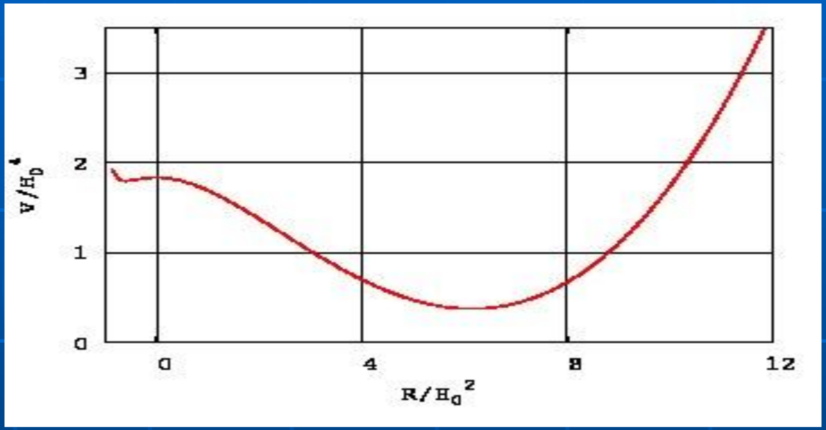
  We take $n = 4$, $m^2 \approx 0.24 H_0^2$, $c_1 \approx 1.25 \times 10^{-3}$ and $c_2 \approx 6.56 \times 10^{-5}$.

$f(R)$ Models

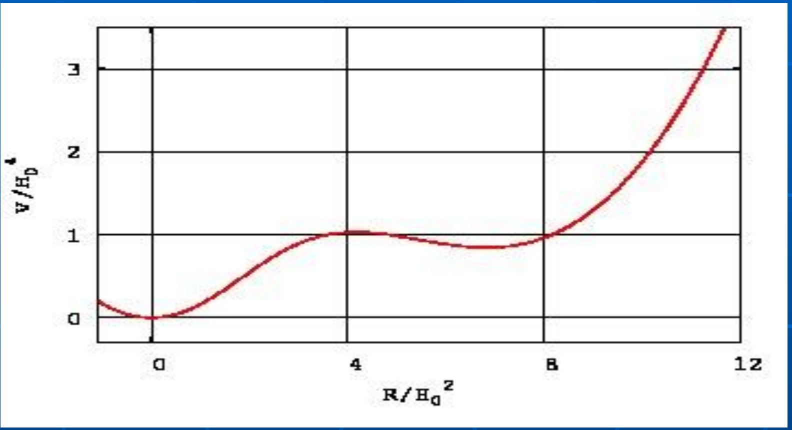


Potentials $V(R) = -Rf(R)/3 + \int^R f(x)dx$ such that $V'(R) = \frac{1}{3}(2f - Rf_R)$. At the extrema of $V(R)$ (notably at the global minimum) the de Sitter "point" is reached where the models behave as a GR plus $\Lambda_{eff} = R_1/4$, where $V'(R_1) = 0$. The specific cosmological models interpolate between a large R (at early time) and near the nontrivial minimum $R \neq 0$ at present time.

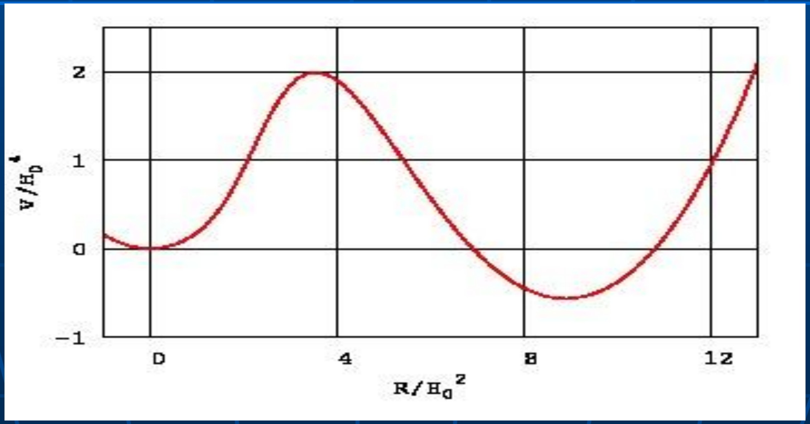
$f(R)_{MJW}$



$f(R)_{St}$



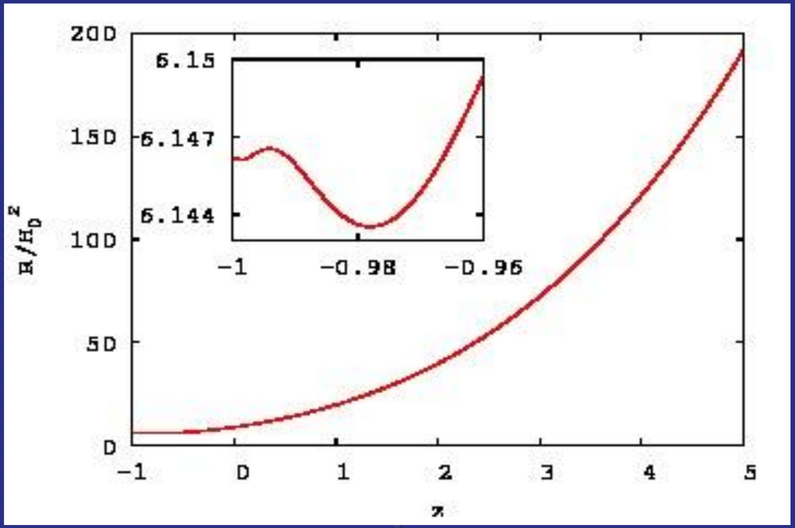
$f(R)_{HS}$



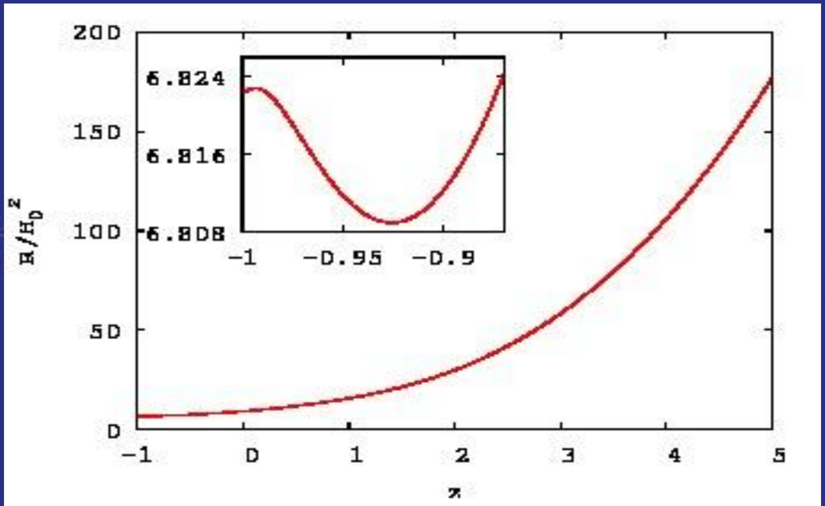
Numerical Results

Plot $\frac{R}{H_0^2}$ vs z, $z = \frac{\omega_e}{\omega_d} - 1 = \frac{\omega(t)}{\omega_0} - 1 = \frac{a_0}{a(t)} - 1$

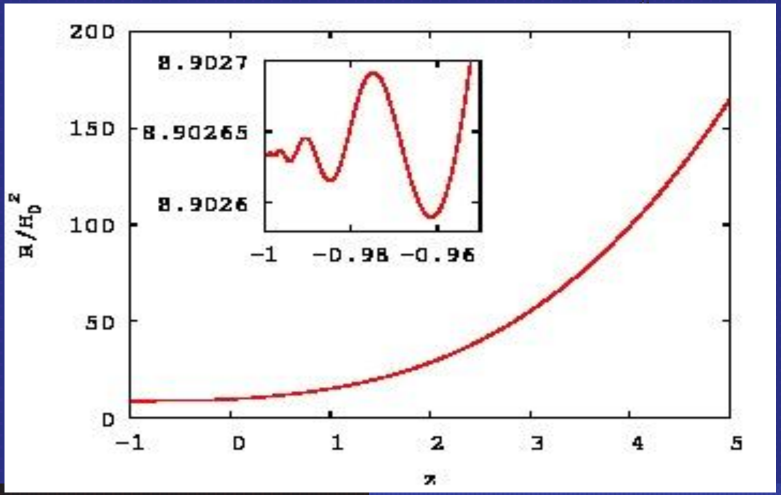
$f(R)_{MJW}$



$f(R)_{St}$

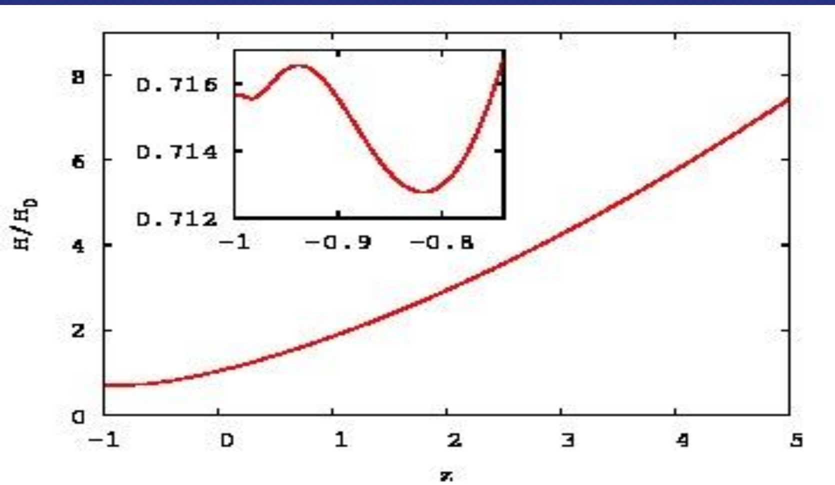


$f(R)_{HS}$

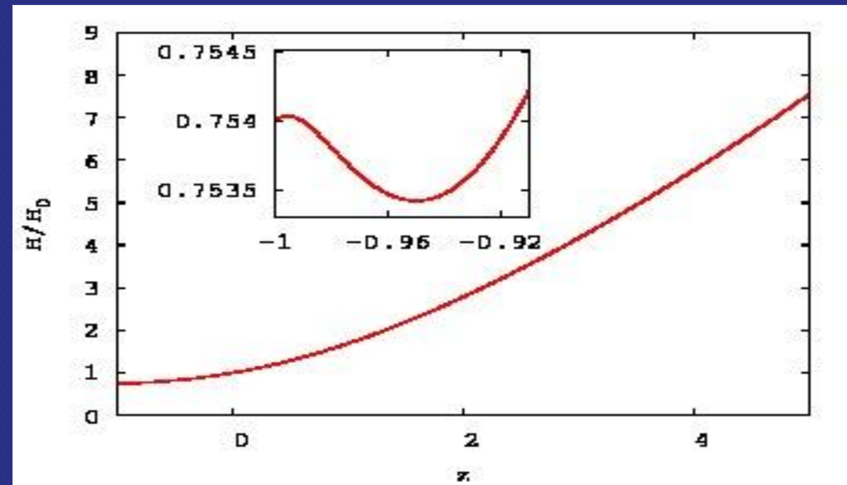


Plot $\frac{H}{H_0}$ vs z

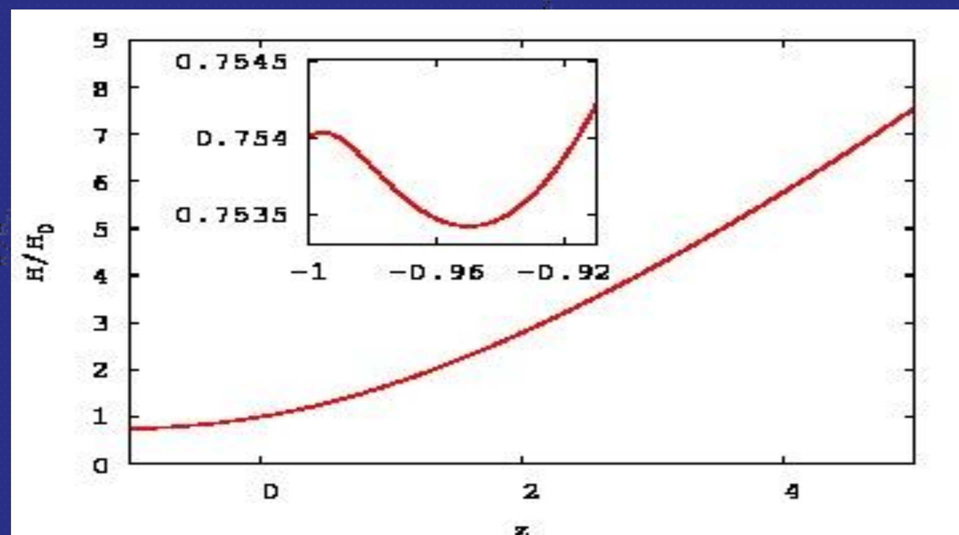
$f(R)_{MJW}$



$f(R)_{St}$



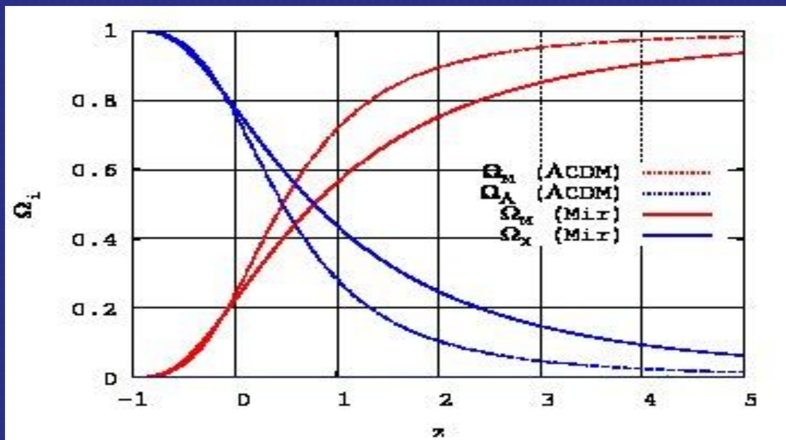
$f(R)_{HS}$



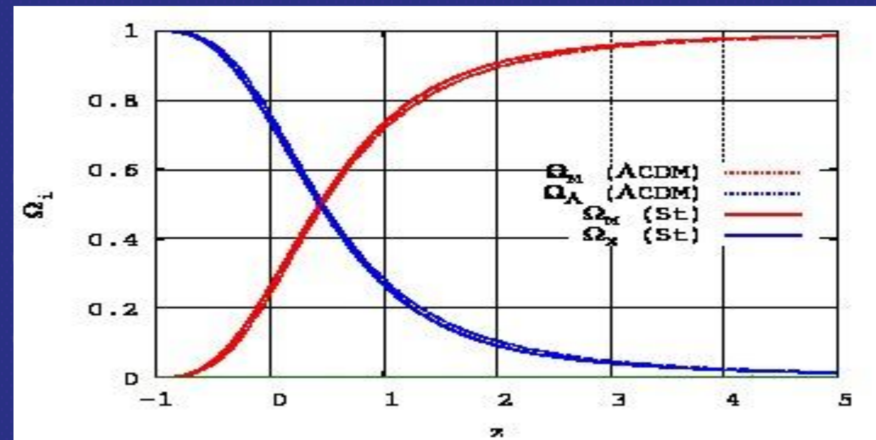
Plot Ω_i vs z

$\Omega_i = \kappa\rho_i/(3H^2)$, $i = \mathrm{matt}, \mathrm{rad}, X$; $\mathrm{matt} = baryons + DM$.

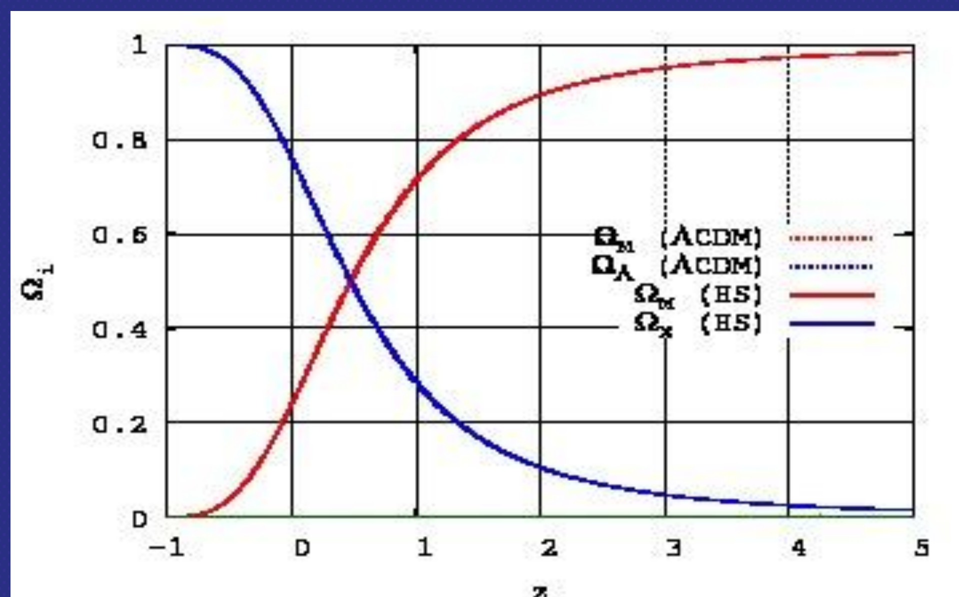
$f(R)_{MJW}$



$f(R)_{St}$

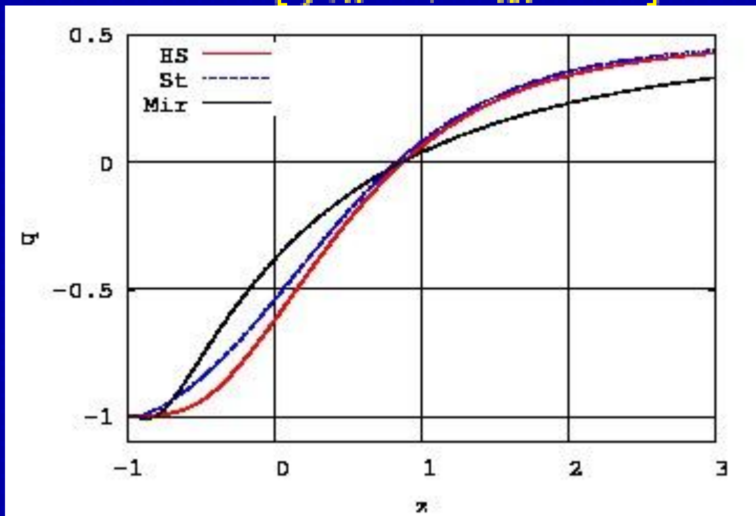


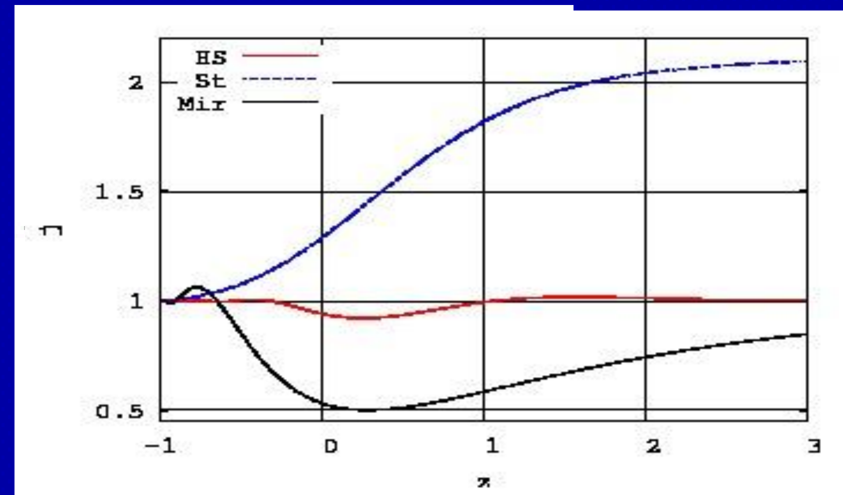
$f(R)_{HS}$



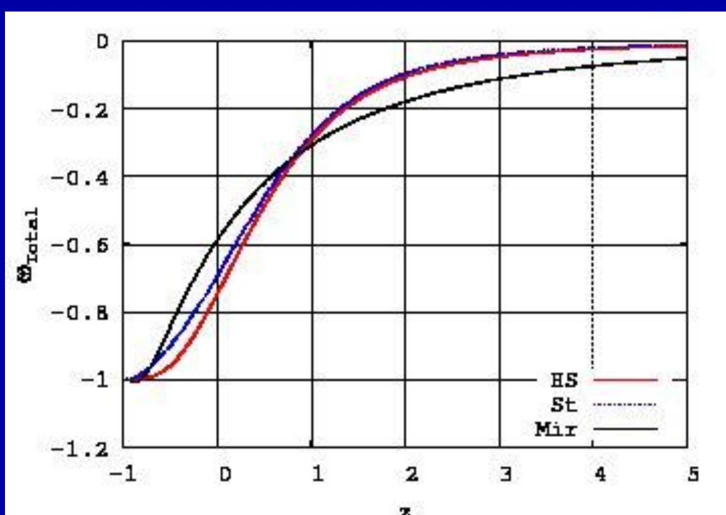
Deceleration parameter: $q := -\frac{\ddot{a}}{aH^2} = -\frac{H^2+\dot{H}}{H^2} = 1 - \frac{R}{6H^2} = \frac{1}{2}(1+3\omega_{tot})$.

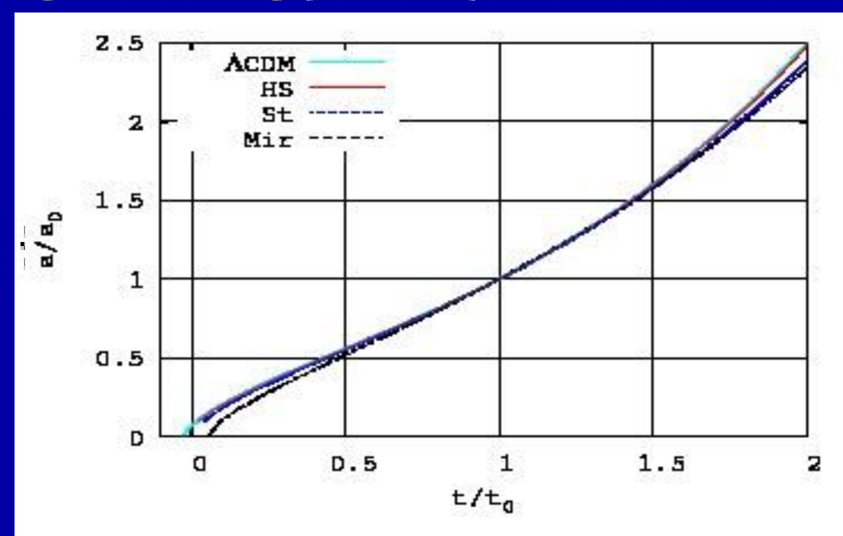
$\omega_{tot} = -\frac{1}{3}\left[\frac{\frac{1}{2}(f_R R+f)+3f_{RR}H\dot{R}-\kappa\rho}{\frac{1}{2}(f_R R-f)-3f_{RR}H\dot{R}+\kappa\rho}\right]$. Jerk: $j := \frac{\dddot{a}}{aH^3} = \frac{\ddot{R}}{6H^3} - \frac{\dot{H}}{H^2} + 1 = \frac{\dot{R}}{6H^3} + q + 2$



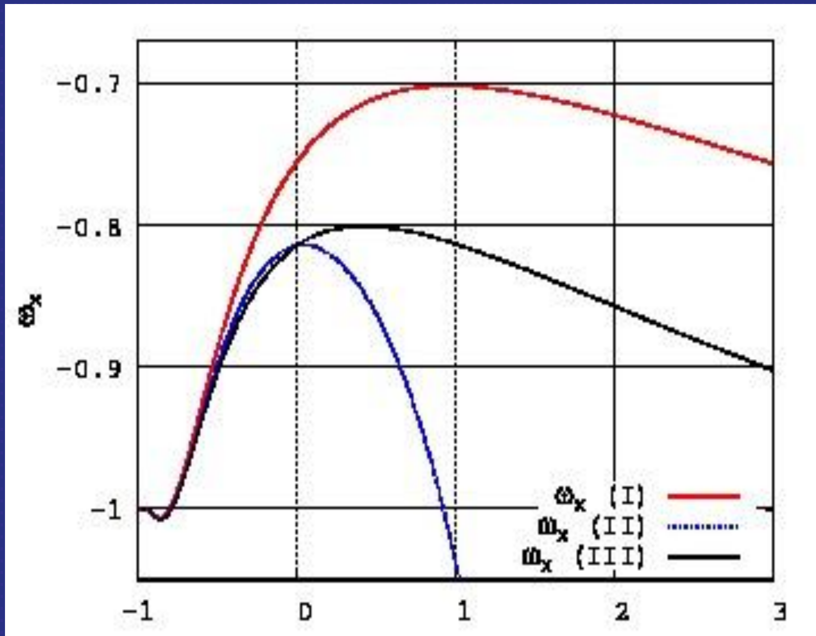


The age of the Universe: $\sim t_0 = H_0^{-1} \approx 9.78h^{-1} \times 10^9$y $\sim 13.97 \times 10^9$y (with $h = 0.7$)

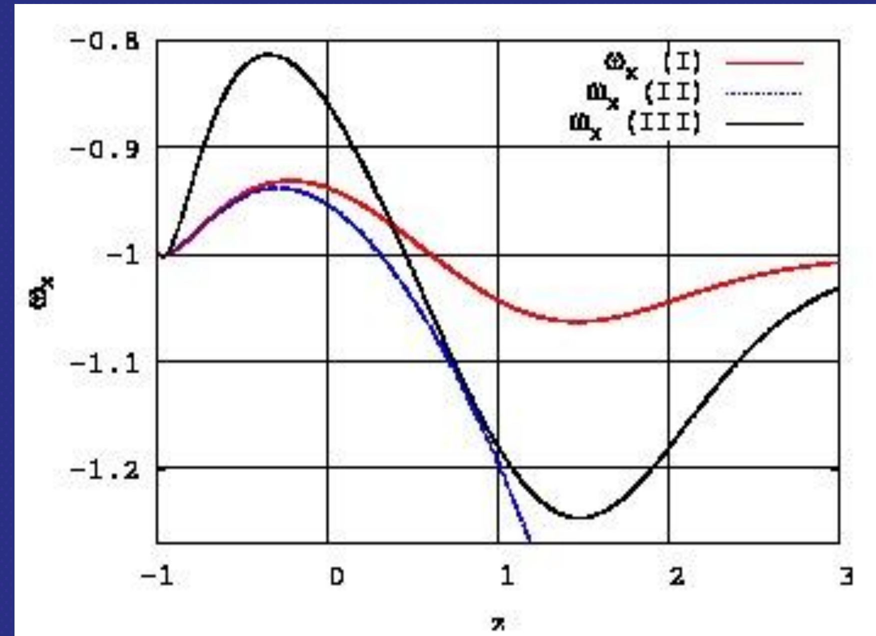




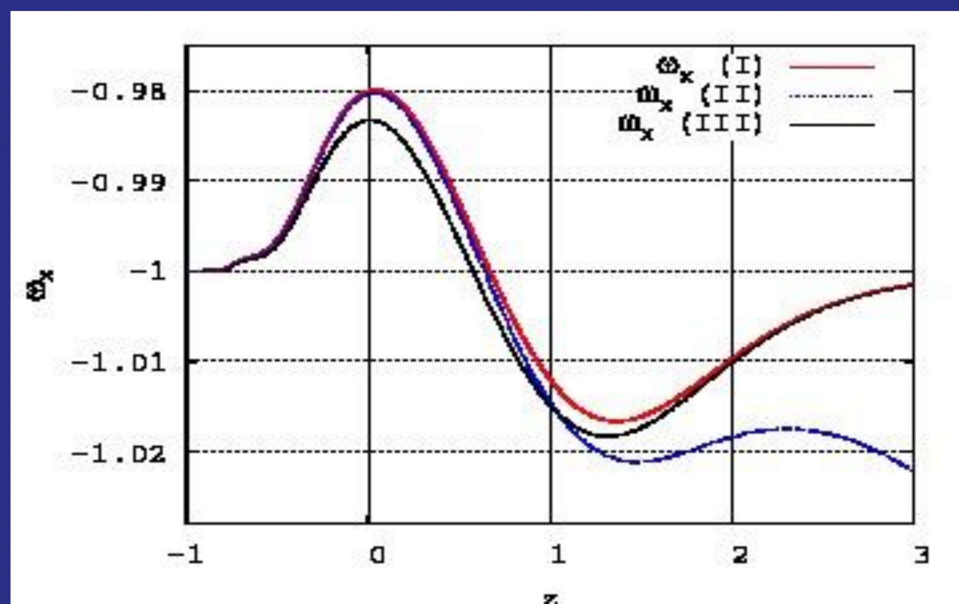
$f(R)_{MJW}$



$f(R)_{St}$

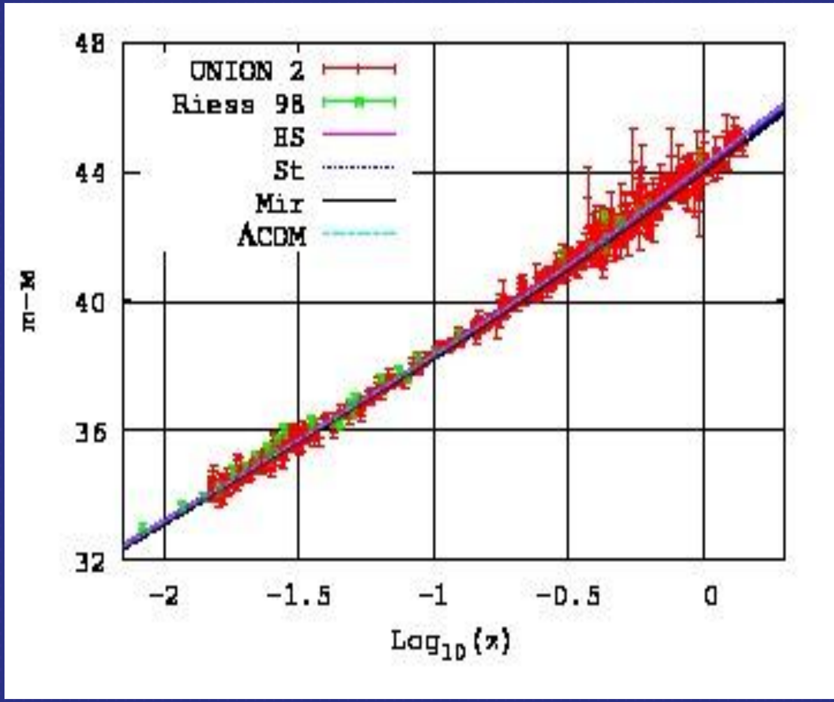


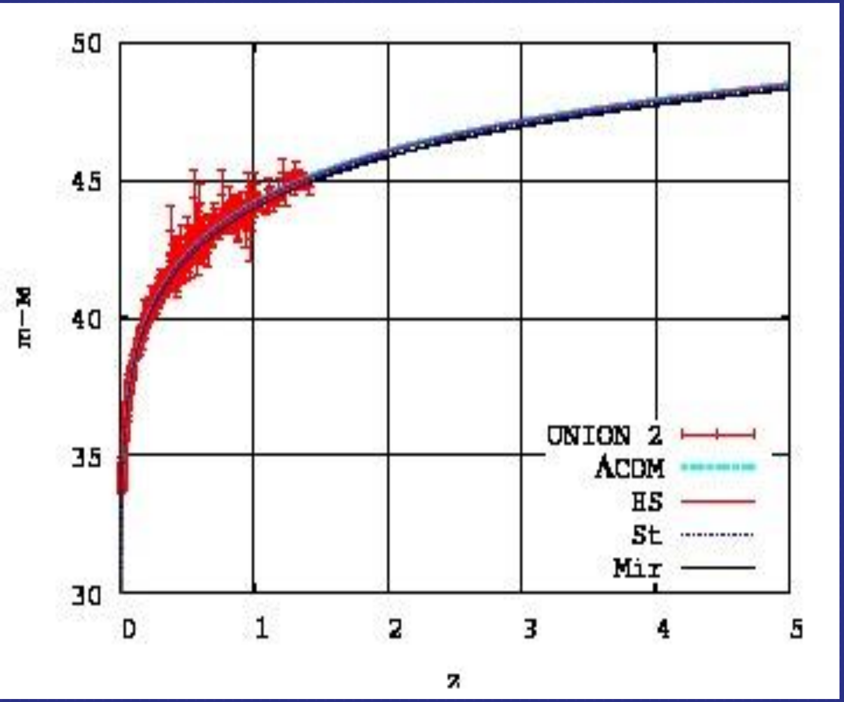
$f(R)_{HS}$



Luminosity distance and SNIa data confrontation ($k = 0$): $d_L^{\text{flat}} = \frac{\zeta(\bar{a})}{\bar{a}}$, where $\zeta = c\, H_0^{-1} \int_{\bar{a}}^{1} \frac{d\bar{a}^*}{\bar{a}^{*\,2} H(\bar{a}^*)}$, $z = \frac{1}{\bar{a}} - 1$.

The luminous distance in log-scale (modulus distance) is given by $\mu := m - M = 5\log_{10}(d_L^{\text{flat}}/\text{Mpc}) + 25$.





The EOS of geometric dark energy – ambiguities– (2nd part) see PRD **89**, 084010, 2014

For cosmological applications it is sometimes useful to write the $f(R)$ field equations as "Einstein" field equations with a total-effective EMT: $G_{ab} = \kappa T_{ab}^{\rm tot}$

$$\kappa T_{ab}^{\rm tot} = \frac{1}{f_R}\left[f_{RR}\nabla_a\nabla_b R + f_{RRR}(\nabla_a R)(\nabla_b R) - \frac{g_{ab}}{6}\left(Rf_R + f + 2\kappa T\right) + \kappa T_{ab}\right] .$$

Because the EMT of matter T_{ab} itself is mixed in a non-trivial way with $f(R)$ factors, thus there is non canonical way of defining the EMT of "geometric dark energy":

$$\tilde{T}_{ab}^{X}(A,B) := AT_{ab}^{\rm tot} - BT_{ab} \ , \qquad (25)$$

Depending on the values adopted for the scalars A and B (see next slide) the defintion of the GDE EMT changes from author to author.

Alternatively, one can write $\tilde{T}^X_{ab}(A,B)$ in terms of purely geometrical quantities:

$$\tilde{T}^X_{ab}(A,B) = \kappa^{-1}\left(AG_{ab} - B\mho_{ab}\right) \ . \tag{26}$$

where

$$\mho_{ab} = f_R G_{ab} - f_{RR}\nabla_a\nabla_b R - f_{RRR}(\nabla_a R)(\nabla_b R) + g_{ab}\left[\frac{1}{2}(Rf_R - f) + f_{RR}\Box R + f_{RRR}(\nabla R)^2\right]$$

EMT of GDE	energy-density, pressure and EOS of GDE
$\tilde{T}^{X}_{ab}(A,B) := A T^{\rm tot}_{ab} - B T_{ab}$	$\tilde{\rho}_X = \frac{A}{\kappa f_R}\left[\frac{1}{2}(f_R R - f) - 3 f_{RR} H\dot{R} + \kappa\rho\left(1 - \frac{B f_R}{A}\right)\right]$ $\tilde{p}_X = -\frac{A}{3\kappa f_R}\left[\frac{1}{2}(f_R R + f) + 3 f_{RR} H\dot{R} - \kappa\left(\rho - 3 p_{\rm rad}\frac{B f_R}{A}\right)\right]$ $\tilde{\omega}_X = \frac{\tilde{p}_X}{\tilde{\rho}_X}$

Definition	A	B	EMT
I $(T^{X}_{ab}, \rho_X, p_X, \omega_X)$	1	1	T^{X}_{ab} conserved
II $(T^{II,X}_{ab}, \rho^{II}_X, p^{II}_X, \omega^{II}_X)$	f^0_R	1	$T^{II,X}_{ab}$ conserved
III $(T^{III,X}_{ab}, \rho^{III}_X, p^{III}_X, \omega^{III}_X)$	f_R	1	$T^{III,X}_{ab}$ not conserved
IV $(T^{IV,X}_{ab}, \rho^{IV}_X, p^{IV}_X, \omega^{IV}_X)$	1	f^{-1}_R	$T^{IV,X}_{ab}$ not conserved (conserved only in vacuum)

Now, the X–EMT of the Definitions I and II are conserved ($\nabla^a T^X_{ab} = 0$) because $T^{\rm tot}_{ab}$ is conserved (due to the Bianchi identities→ $G_{ab} = \kappa T^{\rm tot}_{ab}$, $\tilde{T}^X_{ab}(A,B) := A T^{\rm tot}_{ab} - B T_{ab}$) and the EMT T_{ab} of matter alone is also conserved (Exercise). In particular, for cosmology, Definitions I and II yield

$$\dot{\rho}^i_X + 3H\left(\rho^i_X + p^i_X\right) = 0 \ . \tag{27}$$

(for $i = I, II$).

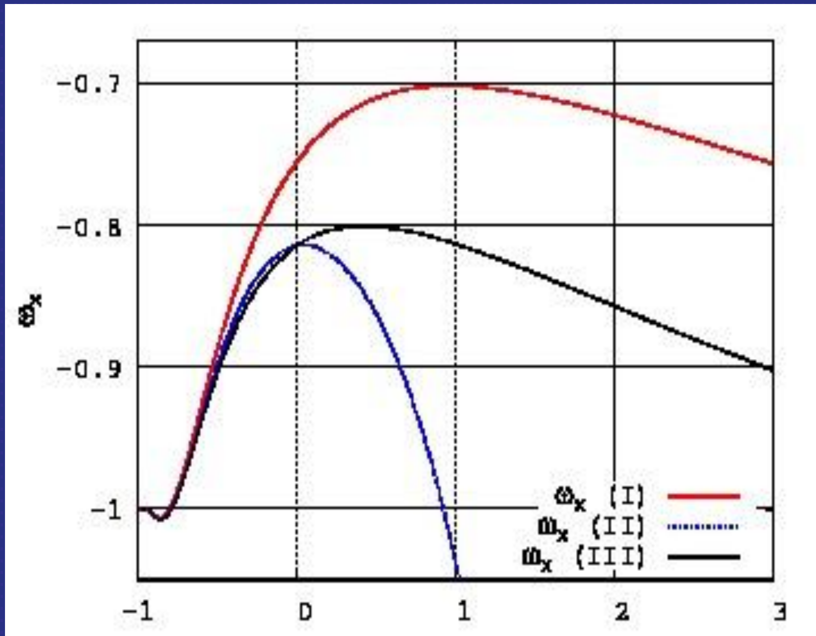
A corollary is that the EMT of Definitions III and IV are not conserved: therefore

$$\dot{\rho}^i_X + 3H\left(\rho^i_X + p^i_X\right) \neq 0 \ . \tag{28}$$

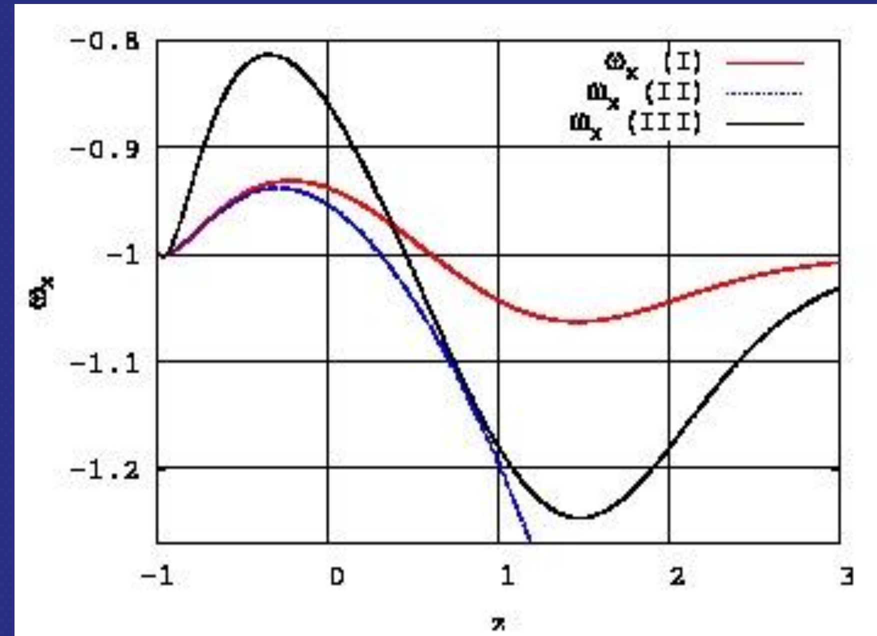
(for $i = III, IV$).

This is rather unpleasant (in my opinion). Yet several authors have considered them. Furthermore, despite that the EMT of Definition II is conserved, the associated EOS in cosmology turns to be ill defined because it diverges as I will show in a moment.

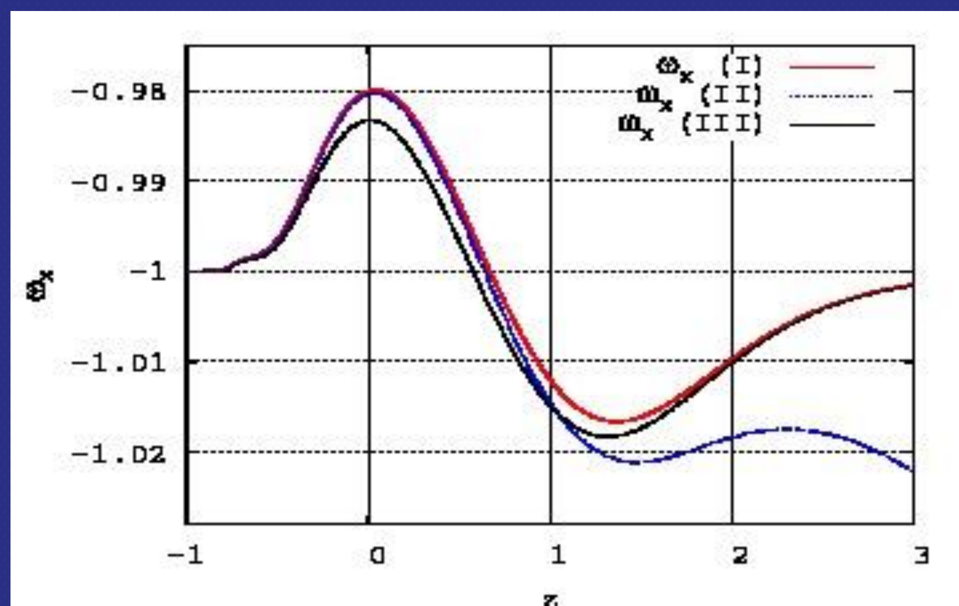
$f(R)_{MJW}$



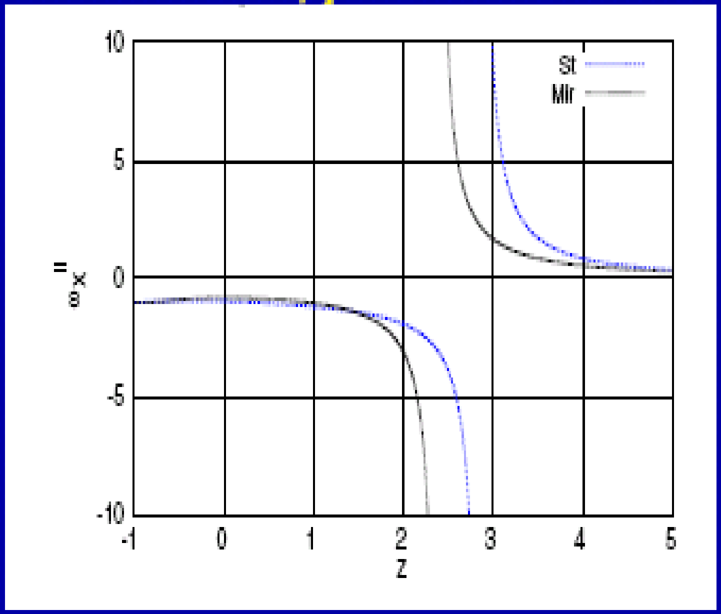
$f(R)_{St}$



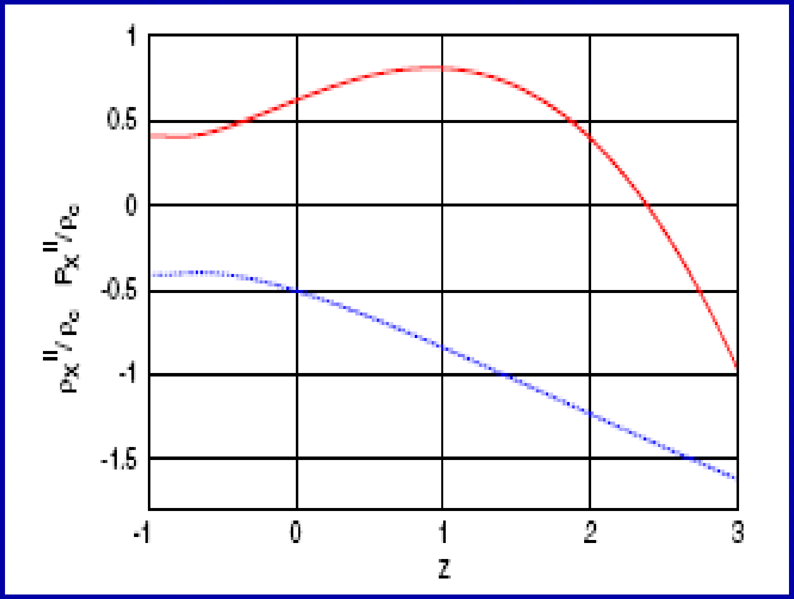
$f(R)_{HS}$



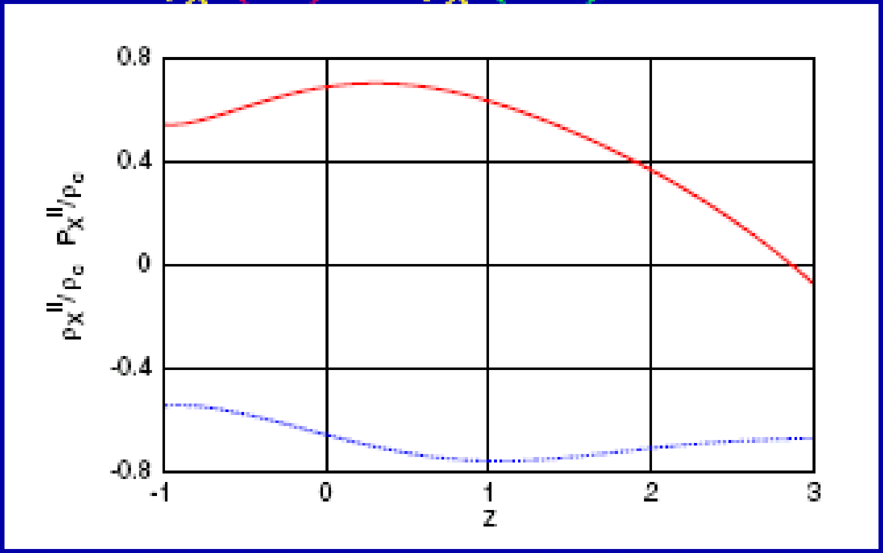
EOS II $\omega_X^{II} = \frac{p_X^{II}}{\rho_X^{II}}$: MJWQ and St



ρ_X^{II} (red) and p_X^{II} (blue) II: MJWQ



ρ_X^{II} (red) and p_X^{II} (blue) II: St



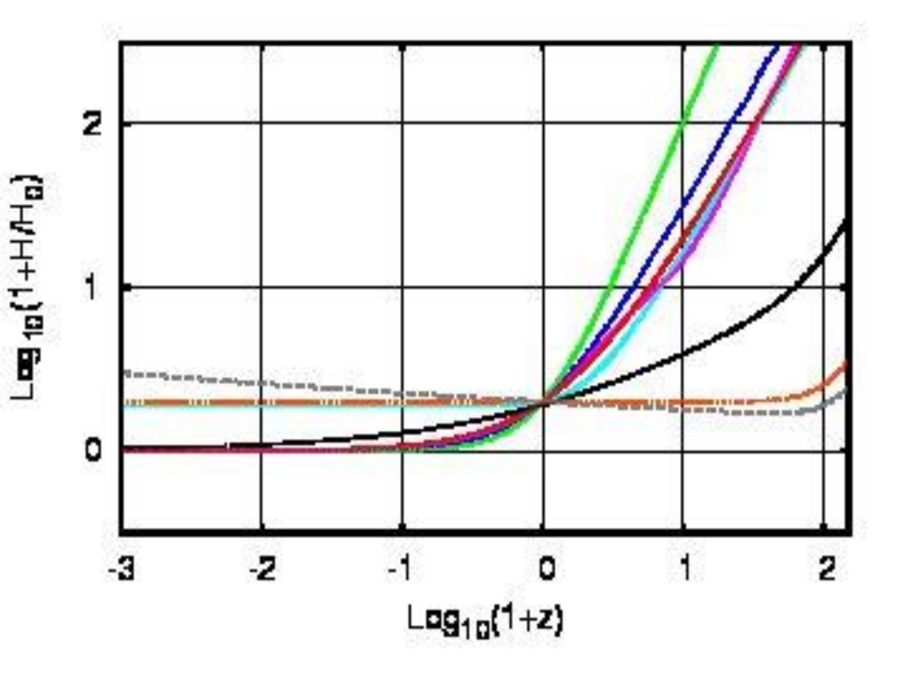
Other $f(R)$ models

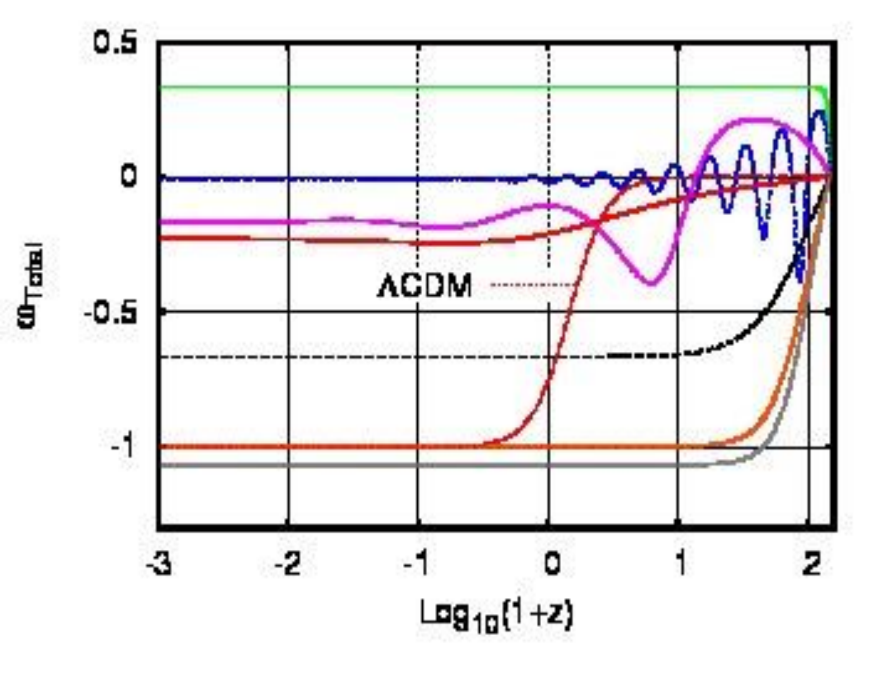
The prototype model $f(R) = \lambda R_n (R/R_n)^n$ (where $\lambda R_n = const. = \alpha_n H_0^2$, the dimensionless constant α_n is some kind of "normalization factor" which is fixed so as that for all the models, we have that $H = H_0$ today, when integrating from the matter domination epoch to the future) was one of the first to be analyzed so that it produced a late accelerated expansion. Recently it was the object of debate between several authors (S. Capozziello et al., PLB 639, 135, 2006; PLB 664, 135, 2008; GRG 40, 357, 2008; Carloni et al., CQG 22, 4839, 2005; GRG 41, 1757, 2009) and the results of L. Amendola et al. (PRL 98, 131302, 2007; PRD 75, 083504, 2007; IJMPD 16, 1555, 2007). The orange group claimed that this kind of models were ruled out because wheter the produced a late time acceleration but an inadequate matter domination epoch or the opposite. The green group criticized their analysis on two grounds: 1) They resorted to the scalar-tensor approach, which the Capozziello et al. group raised "doubts"; 2) The phase-space (dynamical system) analysis was "incomplete" (Carloni et al. group).

As concerns the first criticisms Amendola et al. repeated the analysis in the original frame and recovered the same conclusions. They have not address the second criticism.

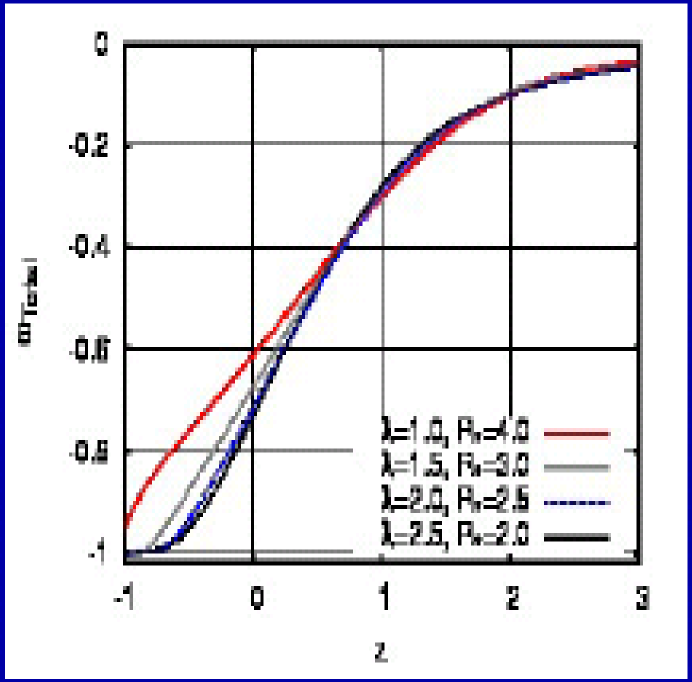
We have performed a full numerical analysis based upond the equations presented before, and we confirmed the same findings of Amendola et al., namely, these models appeared to be ruled out (L. Jaime et al., PRD 87, 024029, 2013).

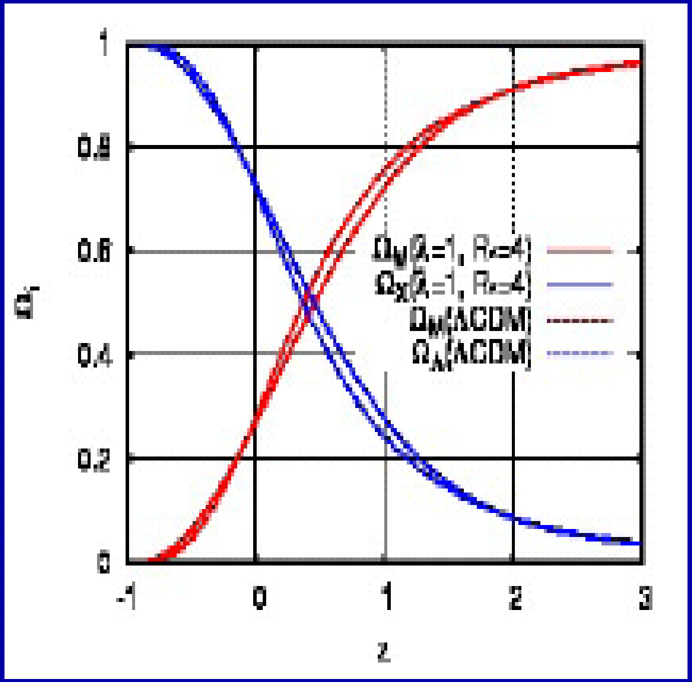
$q := -\frac{\ddot{a}}{aH^2} = -\frac{H^2+\dot{H}}{H^2} = 1 - \frac{R}{6H^2} = \frac{1}{2}\left(1 + 3\omega_{\rm tot}\right)$. So if $\omega_{\rm tot} < -1/3$ the Universe start accelerating. The figure on the right summarized our findings:





We have also analyzed the so called *exponential gravity* model $f(R) = R_*[\tilde{R} - \lambda(1 - e^{-\tilde{R}})]$, where $\tilde{R} := R/R_*$ and $R_* \sim H_0^2$ (see arXiv:1211.0015: Proc. 100 years after Einstein in Prague). This model seems to be also cosmologically viable:





This model have been studied in more detail (perturbations) by Linder (PRD 80, 123528, 2009) who showed that is a potentially viable model.

The following models have been ruled out in one way or another (cosmology, solar system, etc.):

$$f(R) = R - \frac{\mu^4}{R}\,, \quad (29)$$

$$f(R) = R - \frac{\mu_1^4}{R} + \mu_2^4 R\,, \quad (30)$$

$$f(R) = \alpha R^{-n}\,, \quad (31)$$

$$f(R) = R + \alpha R^{-n}\,,\ \text{(possibly viable for } \alpha < 0, n \approx 1) \quad (32)$$

$$f(R) = R^p e^{qR}\,, \quad (33)$$

$$f(R) = R^p (\log \alpha R)^2\,,\ \text{(might succed for } p = 1, q > 0, q \neq 1) \quad (34)$$

$$f(R) = R^p e^{q/R}\,, \quad (35)$$

$$f(R) = R + \alpha R^2\,, \quad (36)$$

$f(R)$ GRAVITY AND THE CMB

In order to analyze the angular anisotropies in the CMB, and all its accompanied features, within the framework of $f(R)$ gravity, a linear perturbation analysis similar to the one of GR has to be performed. In practice, everything is more-less the same, except that instead of having an EMT of matter in the r.h.s. of the Einstein equations, one has an effective EMT that includes the geometrical parts due to the modifications of gravity. So, the perturbation procedure proceeds as follows:

$$g_{ab} = g^0_{ab} + \delta g_{ab} \;\;,\;\; \delta g_{ab} \ll g^0_{ab} \tag{33}$$

$$\phi = \phi^0 + \delta\phi \;\;,\;\; \delta\phi \ll \phi^0 \tag{34}$$

$$T_{ab} = T^0_{ab} + \delta T_{ab} \;\;,\;\; \delta T_{ab} \ll T^0_{ab} \;\;, \tag{35}$$

where g^0_{ab} stands for the unperturbed FRW metric, and δg_{ab} is the metric perturbation which will describe the inhomogeneities and anisotropies associated with the perturbed spacetime. Here ϕ is any scalar associated with $f(R)$ gravity, like R, f_R, f_{RR}, and f_{RRR}; finally the last equation describe the pertubed EMT of matter (baryons, photons and DM, as in GR). This analysis is not new and dates back since the Starobinsky (1981) analysis of inflation and the Mukhanov *et al.* formalism (Phys. Rep. 215, 1992). One obtains then (modulo gauges) a set of field equations for the perturbation δg_{ab} and the scalar field δR or δf_R.

Many articles treat $f(R)$ gravity like an example of a scalar-tensor theory. In that instance, the formalism of perturbations have been developed in the past. More recently, this formalism has been revisited in many articles (e.g. Hu & Sawicky, PRD, **76**, 104030, 2007; Pogosian & Silvestri, PRD; **76**, 023503, 2008) specifically for $f(R)$ gravity. Indeed, a so called *Parametrized post-Friedmannian framework* for modified gravity, was devised in the previous papers, allowing to parametrize the deviations with respect to GR-cosmology independently of the metric theory at hand. This framework is reminiscent of the *Parametrized Post-Newtonian formalism* intended to parametrize the deviations of GR with respect to other metric theories of gravity, but within the context of the solar system experiments and binary pulsar.

So, for instance, when considering only scalar metric pertubations in the Newtonian gauge around a FRW metric with Euclidean (flat) 3-slices one has

$$ds^2 = -(1+2\Psi)dt^2 + a^2(t)(1-2\Phi)\delta_{ij}dx^i dx^j \quad . \tag{36}$$

In the early Universe in pure GR, one has

$$\Phi = \Psi \tag{37}$$

since $\delta T^i{}_j \approx 0$ (for $i \neq j$). However, in $f(R)$ gravity, the corresponding components diagonal components of δT_{ab}^{eff} (which includes the modifications of gravity) are not zero, then $\Phi \neq \Psi$.

So one of the PPF quantities is

$$\gamma = \frac{\Phi}{\Psi} \ . \tag{43}$$

which, like in the PPN formalism, parametrize the deviations with respect to GR. The fact that in modified gravity $\gamma \neq 1$, affects the primordial (plateau) Sachs-Wolfe effect (small ℓ: large angular scales) , which is related to the CMB temperature anisotropies produced by the gravitational shifts of light when the latter traverses well potentials producen by the inhomogeneities of matter.

$$\left.\frac{\delta T}{T}\right|_{t_e}^{t_d} = \Phi(\vec{x_e}, t_e) - \Phi(\vec{x_d}, t_d) + \int_{t_e}^{t_d} \frac{\partial[\Phi(\vec{x}(t), t) + \Psi(\vec{x}(t), t)]}{\partial t} dt$$

where t_e = time at recombination (last scattering surface) and t_d = today
The term $\Phi(\vec{x_d}, t_d)$ gives an isotropic contribution around the observer (i.e. the probe), while the temperature anisotropies at different points of the last scattering surface $\frac{\delta T}{T}|_{t_e}$ combined with the correponding gravitational potential $\Phi(\vec{x_e}, t_e)$ gives the known term of $\Phi(\vec{x_e}, t_e)/3$. The last term corresponds to the ISW (see Merlin & Salgado, GRG 43, 2701, 2011 for a simple and geometrical derivation)

$f(R)$ models vs CMB

From "Cosmological constraints on $f(R)$ accelerating models", Y.S. Song, H. Peiris, and W. Hu, PRD vol.76, 063517 (2007)

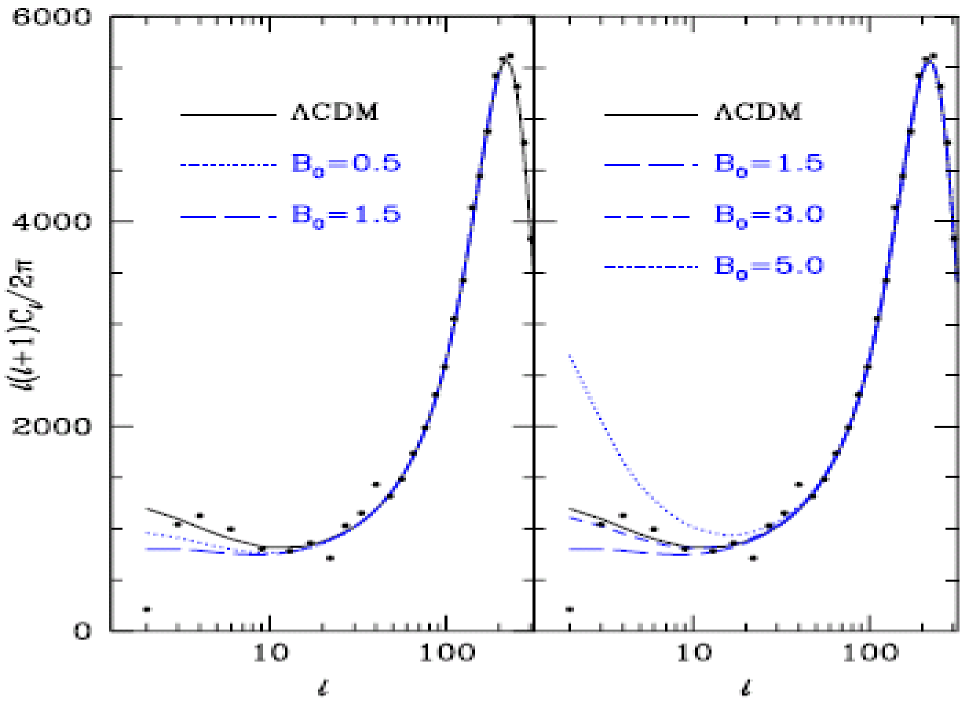


FIG. 2: CMB angular power spectrum C_ℓ for $f(R)$ models with the Compton wavelength parameter $B_0 = 0$ (ΛCDM), 0.5, 1.5, 3.0, 5.0. As B_0 increases, the ISW contributions to the low multipoles decrease, change sign, and then increase. WMAP3 data with noise error bars are overplotted and rule out $B_0 \geq 4.3$ (95% CL).

$f(R)$ GRAVITY AND THE SOLAR SYSTEM TESTS

Solar system tests: weak field limit. Consider static and spherically symmetric perturbations ($|\phi|, |\psi| \ll 1$) around a De Sitter background:

$$ds^2 = -(1 - \phi - \Lambda_{eff} r^2)dt^2 + (1 + \psi - \Lambda_{eff} r^2)dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \quad (39)$$

In GR+Λ

$$\phi = 2M/r \quad (40)$$

$$\psi = 2M/r \quad (41)$$

$$\Lambda_{eff} = \Lambda \quad (42)$$

$$\gamma = \frac{\psi}{\phi} = 1 \quad (43)$$

where γ is one of the Post-Newtonian parameters. At solar system scales we can in fact neglect the term $\Lambda_{eff} r^2$. Now, in $f(R)$ gravity

$$\phi = 2M/r \quad (44)$$

$$\psi = 2\gamma M/r \quad (45)$$

$$\Lambda_{eff} = R_1/4 \quad (46)$$

$$\gamma \neq 1 \quad (47)$$

In fact γ depends on the the parameters of the theory $f(R)$ and on the global properties of the Sun, like $R_\odot$ and $M_\odot$. According to the observations (Cassini probe: Bertotti *et al.* Nature **425**, 2003, 474)

$$|\gamma - 1| \sim 10^{-5} \tag{48}$$

It turns out that (Faulkner et al., PRD **76**, 063505, 2007)

$$|\gamma - 1| = \frac{2\Delta}{3+\Delta} \tag{49}$$

where Δ is the so-called *thin shell parameter* which is related to the *chameleon*: (Khoury & Weltman, PRL **93**, 171104, 2004); PRD **69**, 044026, 2004) the scalar field degree of freedom f_R is supressed in regions of "high" density (the Sun) and at low density (cosmological scales) has noticeable effects, like the cosmic acceleration. This phenomenon is highly dependent on the contrast density between the central object and the surrounding environment and also on the details of the specific $f(R)$ theory. When the chamelon effect takes place, the scalar field f_R behaves like the electric potential within a conductor: inside the object $f_R \approx$ *const.* except within a thin shell $\delta R_\odot$ with $\Delta = \delta R_\odot / R_\odot \ll 1$, where the gradient of f_R is large (*screening effect* like within a conductor).

VOLUME 93, NUMBER 17 PHYSICAL REVIEW LETTERS week ending 22 OCTOBER 2004

Chameleon Fields: Awaiting Surprises for Tests of Gravity in Space

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(Received 10 September 2003; published 22 October 2004)

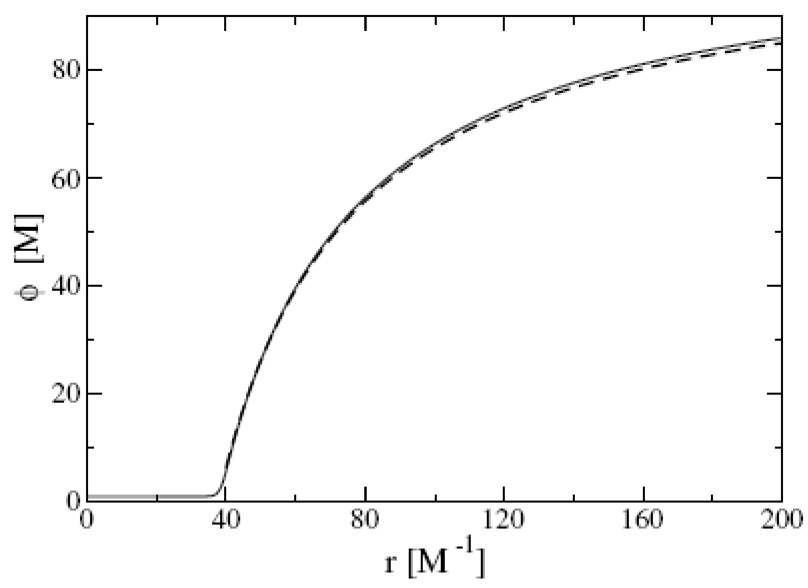

FIG. 2. Example of solution with thin shell.

Outside the object $f_R \propto M_\odot/r$. In this instance it is possible to satisfy the bound (Faulkner et al., PRD 76, 063505, 2007)

$$|\gamma - 1| = \frac{2\Delta}{3+\Delta} < 10^{-5} \tag{50}$$

if $\Delta = \delta R_\odot / R_\odot \ll 1$. The thin shell parameter depends on the two minima of the effective potential $V_{\rm eff}(f_R, \rho_{\rm in,out})$ whose respective values inside the extended object (e.g. the Sun) where and outside depend on $\rho_{\rm in}$ and $\rho_{\rm out}$ and the bulk properties of the object (e.g. $M_\odot, R_\odot$).
However, when the chameleon does not ensue, f_R behaves like the electric potential within a dielectric: it has important variations within the object and the "thin" shell disapears: $\Delta = \delta R_\odot / R_\odot \sim 1$ and therefore

$$|\gamma - 1| = \frac{2\Delta}{3+\Delta} \sim \frac{1}{2} >> 10^{-5} \tag{51}$$

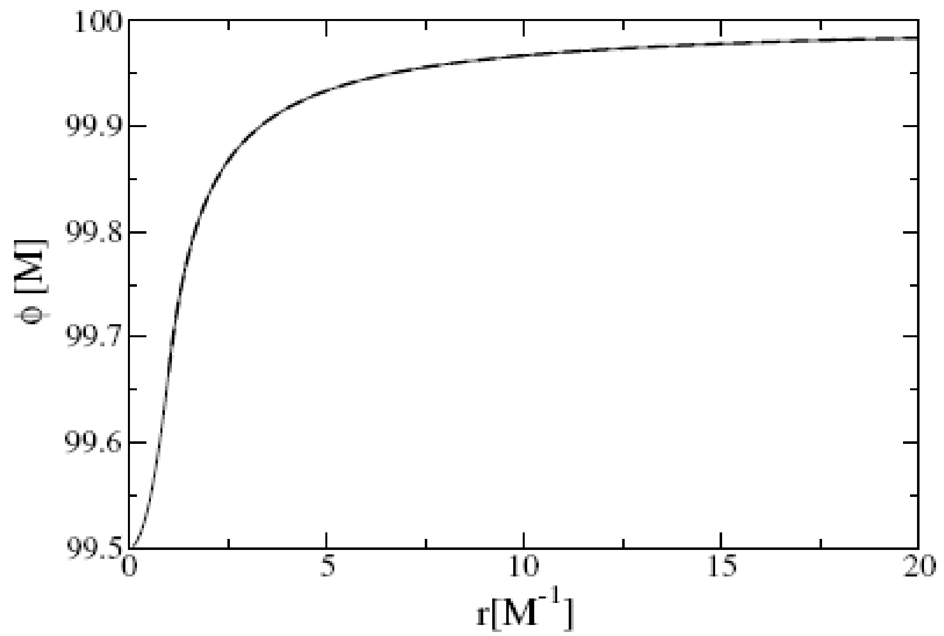

FIG. 3. Example of solution without thin shell.

Models of $f(R)$ Cosmic Acceleration that Evade Solar-System Tests

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(Dated: February 11, 2013)

In Fig. 8 we show $|\gamma - 1|$ for the same $n = 4$ models. The deviations peak at $\sim 10^{-15}$. Such deviations easily pass the stringent solar system tests of gravity from the Cassini mission [95]

$$|\gamma - 1| < 2.3 \times 10^{-5} \qquad (63)$$

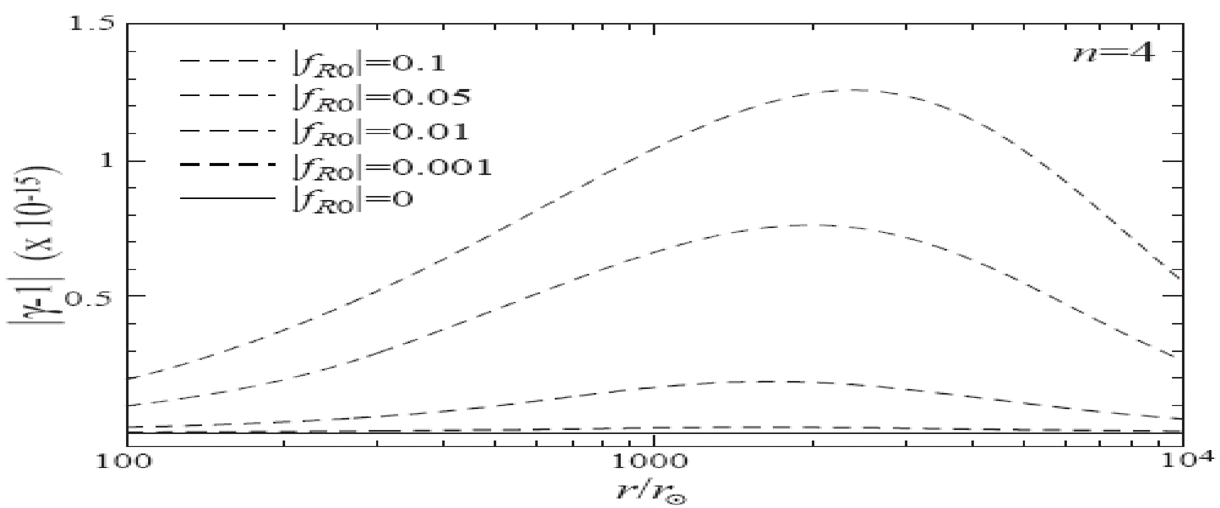


FIG. 8: Metric deviation parameter $|\gamma - 1|$ for $n = 4$ models and a series of cosmological field amplitudes f_{R0} with a galactic field that minimizes the potential. These deviations are unobservably small for the whole range of amplitudes.

Compact objects: tool to test further $f(R)$ theories

- In fact it was only recently that several authors have tried to construct relativistic extended objects in the framework of $f(R)$ gravity.
- In particular **Kobayashi and Maeda** (PRD 78, 064019, 2008; PRD 79, 024009, 2009) using the Starobinsky model $f(R) = R - \lambda R_* \left\{1 - \left[1 + (R/R_*)^2\right]^{-\beta}\right\}$ shown that such objects cannot be constructed because a curvature singularity developed within the object.
- Later **Babichev and Langlois** (PRD 80, 121501(R) 2009; gr-qc/0911.1297, 2010) reanalyzed the issue and concluded that KM results was a consequence of the use of an incompressible fluid, and that using a more realistic EOS (polytropes) such singularities were not found.
- However, **Upadhye and Hu** (PRD 80, 064002,2009) found that relativistic extended objects can indeed be constructed, but that the absence of singularities got nothing to do with the EOS, but rather with a "chameleon mechanism".

- The common feature of the aforementioned works is that authors used the above mapping to constuct such objects.
- Under such mapping the Ricci scalar has a behavior of the sort $R \sim 1/(\chi - \chi_0)$ where $\chi := \partial_R f$ and $\chi_0 = \text{const}$.
- The key point is to determine if the dynamics of χ leads it or not to the value $\chi = \chi_0$ within the spacetime generated by the relativistic object.
- Irrespective of the different results and confusing explanations obtained by those works we argue that their conclusions are rather questionable due to the fact that the above scalar-field variables are ill defined. To be more specific, the scalar-field potential used to study the dynamics of χ is not single valued and possesses pathological features. Since similar kind of singularities were also found in the cosmological setting (Frolov, PRL 101, 061103, 2008), it is then worrisome that the ill-defined potential play such a crucial role in those analyses (several authors have already criticized the use of such potentials).

We consider the following metric that allows us to describe SSS

$$ds^2 = -n(r)dt^2 + m(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) . \tag{91}$$

The field Eqs. then read

$$R'' = \frac{1}{3f_{RR}}\left[m(\kappa T + 2f - Rf_R) - 3f_{RRR}R'^2\right] + \left(\frac{m'}{2m} - \frac{n'}{2n} - \frac{2}{r}\right)R' . \tag{92}$$

(where $' := d/dr$). From the $t-t$, $r-r$, and $\theta-\theta$ of field Eqs. and after several non-trivial manipulations we found

$$\begin{aligned} m' &= \frac{m}{r(2f_R + rR'f_{RR})}\left\{2f_R(1-m) - 2mr^2\kappa T^t{}_t + \frac{mr^2}{3}(Rf_R + f + 2\kappa T) + \frac{rR'f_{RR}}{f_R}\left[\frac{mr^2}{3}(2Rf_R - f + \kappa T)\right.\right. \\ &\quad \left.\left. -\kappa mr^2(T^t{}_t + T^r{}_r) + 2(1-m)f_R + 2rR'f_{RR}\right]\right\} , \end{aligned} \tag{93}$$

$$n' = \frac{n}{r(2f_R + rR'f_{RR})}\left[mr^2(f - Rf_R + 2\kappa T^r{}_r) + 2f_R(m-1) - 4rR'f_{RR}\right] , \tag{94}$$

$$n'' = \frac{2nm}{f_R}\left[\kappa T^\theta{}_\theta - \frac{1}{6}(Rf_R + f + 2\kappa T) + \frac{R'}{rm}f_{RR}\right] + \frac{n}{2r}\left[2\left(\frac{m'}{m} - \frac{n'}{n}\right) + \frac{rn'}{n}\left(\frac{m'}{m} + \frac{n'}{n}\right)\right] . \tag{95}$$

Remarks of the above system of ODE's:

- Notice that Eqs. for n' and n'' are not independent. In fact, one has the freedom of using one or the other. Nevertheless, we have used both to check the consistency of our equations and the numerical code (solutions).
- Now, from the usual expression of R in terms of the Christoffel symbols one obtains,

$$R = \frac{1}{2r^2n^2m^2}\Big[4n^2m(m-1) + rnm'(4n+rn') - 2rnm(2n'+rn'') + r^2mn'^2\Big]\,. \tag{96}$$

As one can check by a direct calculation, that using the Eqs. for m', n', n'' in the above Eq., one finds an identity $R \equiv R$. This result confirms two things: 1) Our Eqs. are consistent and no elementary mistake was made in their derivation; 2) The previous expression for R does not provide any further information.

- When defining the first order variables $Q_n = n'$ and $Q_R := R'$, the above system of ODE's have the form $dy^i/dr = \mathcal{F}^i(r, y^i)$ where $y^i = (m, n, Q_n, R, Q_R)$ and therefore can be solved numerically. As far as we are aware, such a system has not been considered previously. These equations can be used to tackle several aspects of SSS spacetimes in $f(R)$ gravity.

- We observe that for $f(R) = R$ our system of ODE's reduce to the well known equations of GR for SSS spacetimes.

- Like in the general case, our system of ODE's in vacuum has the exact de Sitter solution $n(r) = m(r)^{-1} = 1 - \Lambda_{\rm eff} r^2/3$, $R = R_1 = {\rm const.}$ with $\Lambda_{\rm eff} = R_1/4$ and $R_1 = 2f(R_1)/f_R(R_1)$.

- We also need the matter equations $\nabla^a T_{ab} = 0$. So for $T_{ab} = (\rho + p)u_a u_b + g_{ab}p$, we get

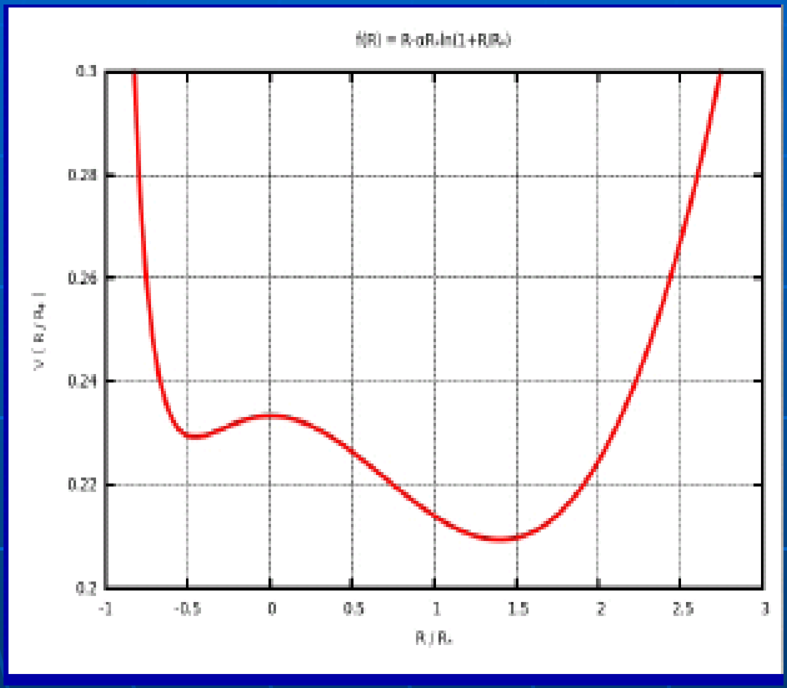
$$p' = -(\rho + p)n'/2n \tag{97}$$

This is the modified Tolman-Oppenheimer-Volkoff equation of hydrostatic equilibrium which is to be complemented by an EOS.

Numerical Results

- For simplicity we shall assume an incompressible fluid ($\rho = const.$)
- We integrate the equations numerically outwards from the origin $r = 0$ and impose regularity conditions at $r = 0$. We fix $p(0)$ (the central pressure) to a given value (this fixes one "star" configuration).
- We obtain $R(0)$ by a shooting method so that asymptotically the solution matches the de Sitter solution $R = R_1 = const.$ where R_1 is a critical point of the "potential" $V(R) = -Rf(R)/3 + \int^R f(x)dx$. That is, R_1 is a point where $dV(R)/dR = (2f - f_R R)/3$ vanishes.
- This potential is radically different from the scalar-field potential that arises under the STT map. Furthermore, $V(R)$ is as well defined as the function $f(R)$ itself.

- We used first the model $f(R) = R - \alpha R_* \ln(1 + R/R_*)$ (α, R_* are positive constants; R_* sets the scale) proposed by Miranda et. al. (PRL 102, 221101, 2009). Warning: for this model f_R is not positive definite in general (only if $\alpha < 1 + R/R_*$) but $f_{RR} > 0$. Note also that $f(R)$ is only well defined for $0 < 1 + R/R_*$. A priori there is no guarantee that solutions for R exist satisfying such conditions.
- Those authors mapped the theory to the STT counterpart. However, unlike the Starobinsky model (see below), in this case the resulting scalar-field potential turns to be single-valued. This is why we take it in order to compare (calibrate) directly with our method.
- They did not find any singularity within the object.
- Under our approach we associate to this $f(R)$ the potential $V(R) = \frac{R_*^2}{6}\left\{(1+\tilde{R})\left[\tilde{R} + (6\alpha - 1)\right] - 2\alpha(3 + 2\tilde{R})\ln\left[1 + \tilde{R}\right]\right\}$, where $\tilde{R} = R/R_*$. For $\alpha = 1.2$ (the value that Miranda et al. assumed) this potential has several critical points



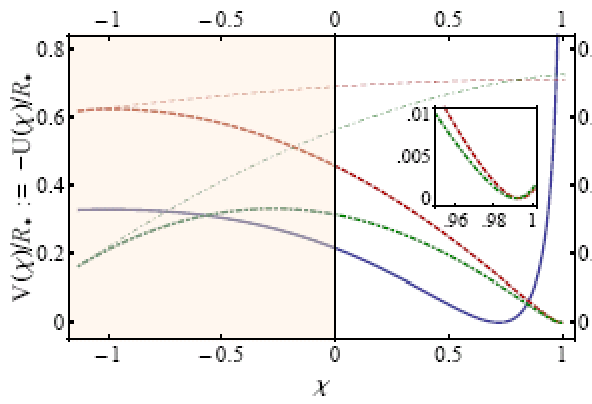
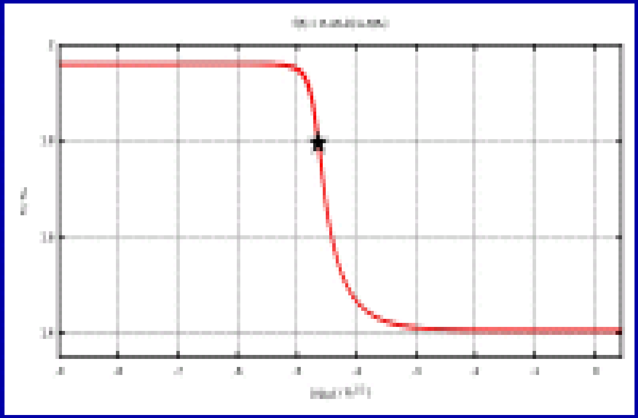
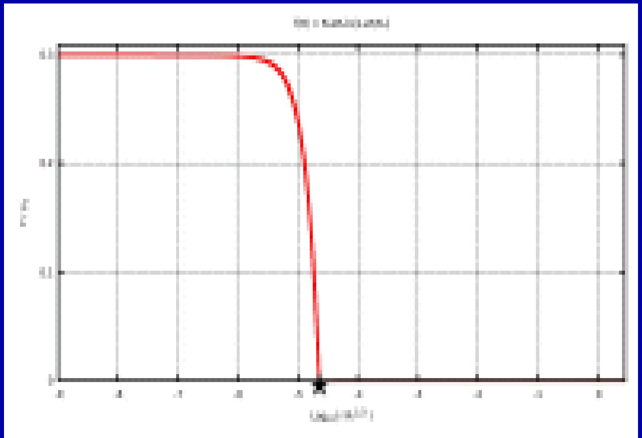
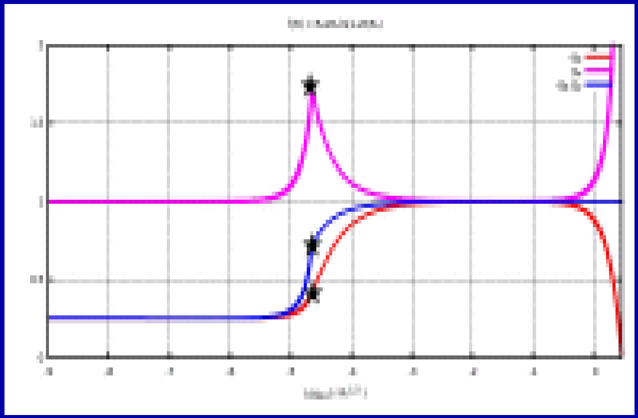
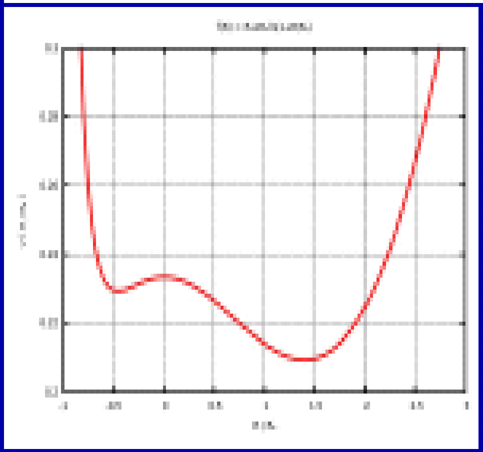


Figure 1: $V(\chi)/R_* := -U(\chi)/R_*$ for different models: ours, with $\alpha = 2$ (blue solid line), Starobinski's for {n=2, λ = 2} [see [9]] (red, dashed line) and Hu & Sawicki's for {n=2, $m^2 = 1$, $c_1/c_2 = 2$} [see [8]] (green dot-dashed line). The physically interesting region is $0 < \chi < 1$. For the multi-valued potentials only the lower lines are physical.

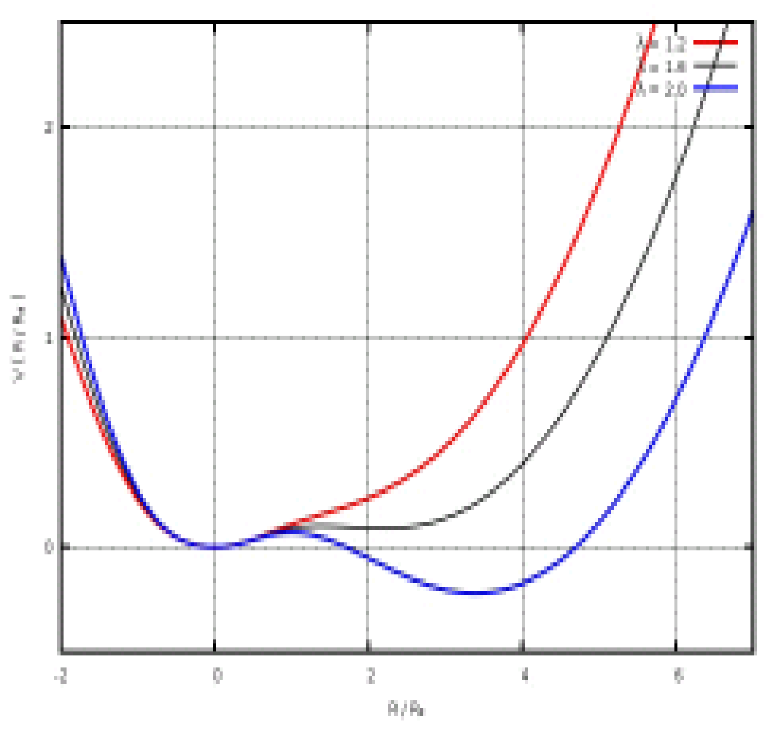
We have used our approach and found no singularities whatsoever in compact objects (like in Miranda et al.)

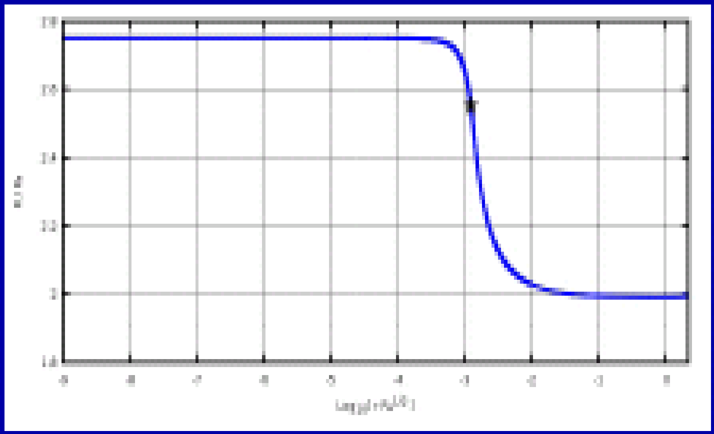
- We then used the Starobinsky model $f(R) = R - \lambda R_* \left\{1 - \left[1 + (R/R_*)^2\right]^{-\beta}\right\}$ with $\beta = 1$. The controversy about the existence of extended relativistic objects (or absence thereof) was originated using this model. As we saw, the STT approach gives rise to multivalued potentials.
- With our method, the potential is given by ($\tilde{R} = R/R_*$)

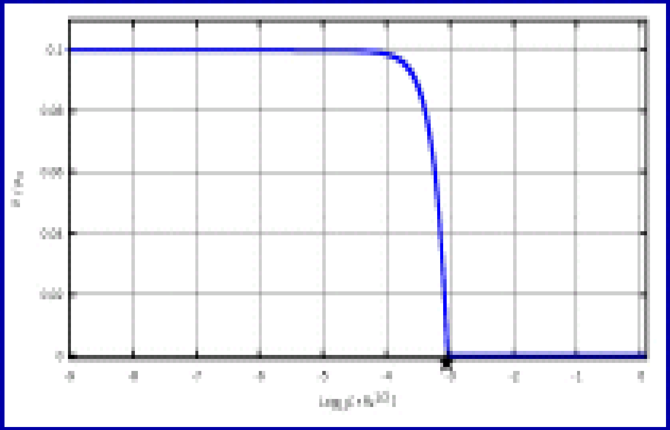
$$V(R) = \frac{R_*^2}{3}\left\{ \frac{\tilde{R}}{2}\left[\tilde{R} - 4\lambda - 2\lambda\left(1 + \tilde{R}^2\right)^{-1}\right] - 3\lambda \arctan\left(\tilde{R}\right) \right\}.$$

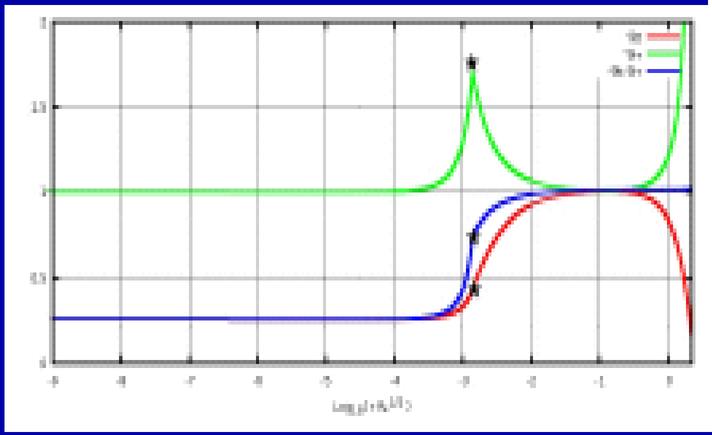
- This potential has a rich structure depending on the value of λ.

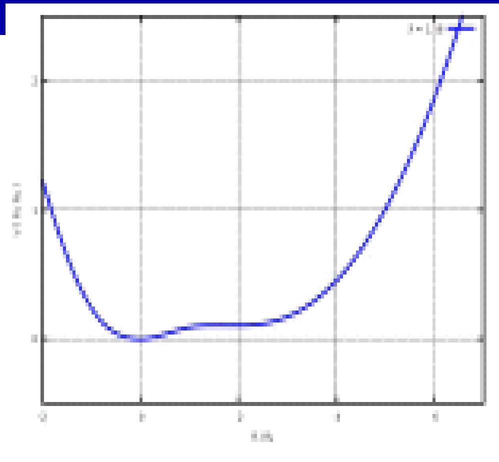


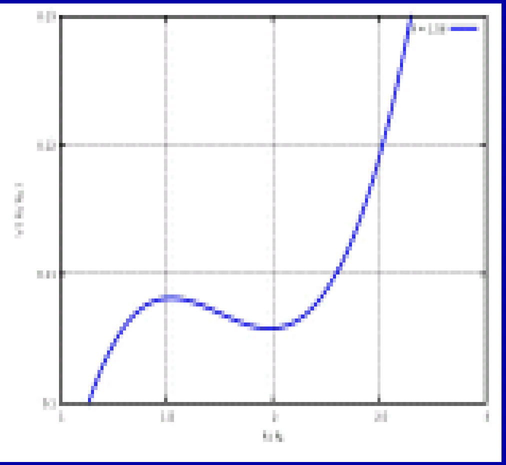
For the value $\lambda = 1.56$ we found the following solutions using a shooting method aiming to the local minimum. No singularities whatsoever where found.











Discussion:

Although we have not find singularities in the spacetime generated by this compact objects, there is a caveat in the above construction:

In almost all $f(R)$ models the dimension parameters like R_*, which settles the scale $\sim D^{-2}$, are chosen such that $R_*/G_0 \sim \tilde{\Lambda} \sim 10^{-29}\mathrm{g\,cm^{-3}}$ $(\tilde{\Lambda} = \Lambda/G_0)$. On the other hand, if one wants to build a realistic neutron star, the typical densities at the center are $\rho(0) \sim \rho_{\mathrm{nuc}} \sim 10^{14}\mathrm{g\,cm^{-3}}$. That is, there are around 43 orders of magnitude between the typical density within a neutron star and the average density of the Universe! This ratio between densities naturally appears in the equations since the parameters which define the specific $f(R)$ theory are of the order of $\tilde{\Lambda}$, while the appropriate dimension within neutron stars is ρ_{nuc}. So in units of ρ_{nuc}, the cosmological constant turns out to be ridiculously small, while in units of $\tilde{\Lambda}$, $\rho(r)$ and $p(r)$ turn out to be ridiculously large within the neutron star.

In other words, the scale of a neutron star is $\sim \mathrm{km}$ while the cosmological scales are $\sim 100\mathrm{Mpc}$. We, like the other authors, have not solved this technical problem...YET, and therefore have constructed "compact" objects which whether are realisitic but then $\tilde{\Lambda}$ is not. Or the opposite. In both cases the objects are compact and relativistic in the sense that $p \sim \rho$ and $G_0\mathcal{M}/\mathcal{R}$ is not far from $4/9$, where $\mathcal{M}$ is the "ADM" mass defined in asymptotically de Sitter spacetimes.

Conclusions

- $f(R)$ theories are alternative theories of gravity that can produce an accelerated expansion of the universe "without" the introduction of Λ. Some specific $f(R)$ models can pass several gravitational tests (e.g. the Solar System tests, cosmological). They have some predictions different from GR+ Λ (e.g. variable EOS of dark energy, new gravitational-wave modes – breathing mode –, different Sachs-Wolfe effect, ...)
- However, in my opinion they introduce more troubles than solutions. There is no fundamental principle that allows to single out one function $f(R)$. Simplicity favors: $f(R) = R - 2\Lambda$ (i.e. GR+ Λ). Time will tell if models different from GR will be taken seriously in the future.
- As concerns the EOS of dark-energy within $f(R)$ gravity ambiguities may arise. Be aware of them ! Putting aside this issue, further experiments will determine if such EOS varies in cosmic time or not (e.g BigBOSS–DESI–, EUCLID, PanSTARR, WFIRST, etc.).