Quantum Cosmological Perturbations of Multiple Fluids

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Workshop Fundamental Issues of the Standard Cosmological Model

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Outline

Motivation

- Find a consistent way to perform the quantization of coupled multiple components.
- Multifield inflation.
- Multiple fluids in the contracting branch of a bouncing model.

- Obtain the Hamiltonian for multiple-fluid perturbations plus metric perturbations.
- The only interaction considered between fluids is through gravity.
- Solving the Hamiltonian and momentum constrains leads naturally to gauge invariant variables.
- General quantization in a non-canonical form.
- Different field representations leads to (non)unitary evolution.
- Trivial field representation.
- Two fluids example.

Metric perturbation

Given the background metric $\bar{g}_{\mu\nu}$ and the perturbed metric $g_{\mu\nu}$ we define the perturbation tensor

$$\xi_{\mu\nu} \equiv g_{\mu\nu} - \bar{g}_{\mu\nu},$$

and the difference connection tensor

$$(\nabla_\mu - \bar{\nabla}_\mu)A_\nu = \mathcal{F}_{\mu\nu}{}^{\beta}A_\beta,$$
$$\mathcal{F}_{\alpha\beta\gamma} = -\frac{1}{2}\left(\bar{\nabla}_\alpha\xi_{\beta\gamma} + \bar{\nabla}_\beta\xi_{\gamma\alpha} - \bar{\nabla}_\gamma\xi_{\alpha\beta}\right).$$

- The use of the above variables greatly simplifies the calculations.
- The above variables are true tensors.
- Bar variables represent background quantities, all definitions are valid for both background and perturbed variables.

Second Order Lagrangian

Using the introduced variables we obtain the following second order Lagrangian,

$$\delta\mathcal{L}^{(2)}_{g\mathsf{k}} = \frac{\sqrt{-\bar{g}}}{2\kappa}\left[\mathcal{F}^{\mu\nu\gamma}\mathcal{F}_{\gamma(\mu\nu)} - \mathcal{F}_{a\,\mu}\mathcal{F}_{b}{}^{\mu}\right],$$
$$\delta\mathcal{L}^{(2)}_{g\mathsf{p}} = \frac{\sqrt{-\bar{g}}}{2\kappa}\left(\bar{G}_{\mu\nu} + \frac{\bar{g}_{\mu\nu}}{4}\bar{R}\right)\xi^{\mu}{}_{\alpha}\left(\xi^{\alpha\nu} - \frac{\bar{g}^{\alpha\nu}\xi}{2}\right),$$

where we have discarded the surface terms and $\kappa = 8\pi G/c^4$.

Background Foliation

To introduce a time direction we define a background foliation by the vector geodetic field $\bar{v}^{\mu}$. The same foliation is used to split the perturbed metric, however, the field must be normalized with respect to $g_{\mu\nu}$ giving v^{μ}. We define the time derivative by

$$\dot{T} = \bar{\gamma}\left[\pounds_{\bar{v}}T\right].$$

The hypersurfaces given by v^μ introduce:

- the projector $\gamma_{\mu\nu} = g_{\mu\nu} + v_\mu v_\nu$, and operator $\bar{\gamma}\left[A_\mu\right] = \bar{\gamma}_\mu{}^\nu A_\nu$,
- the extrinsic curvature $\nabla_\mu v_\nu = \mathcal{K}_{\mu\nu}$,
- the Expansion factor $\Theta = \mathcal{K}_\mu{}^\mu$ and shear $\sigma_{\mu\nu} \equiv \mathcal{K}_{\mu\nu} - \frac{\Theta}{3}\gamma_{\mu\nu}$,
- the unique covariant derivative D_μ compatible with $\gamma_{\mu\nu}$,
- the spatial curvature $\mathcal{R}_{\mu\nu\alpha}{}^\beta A_\beta \equiv [D_\mu D_\nu - D_\nu D_\mu]A_\alpha$.

Background Projection

With the background foliation we write the tensor $\xi_{\mu\nu}$ as

$$\xi_{\mu\nu} = 2\phi\bar{v}_\mu\bar{v}_\nu + 2B_{(\mu}\bar{v}_{\nu)} + 2C_{\mu\nu}, \qquad B_\mu\bar{v}^\mu = C_{\mu\nu}\bar{v}^\mu = 0.$$

And the scalar vector tensor (SVT) decomposition

$$\begin{aligned} B_\mu &= \bar{D}_\mu\mathcal{B} + \mathtt{B}_\mu, \\ C_{\mu\nu} &= \psi\gamma_{\mu\nu} - \bar{D}_\mu\bar{D}_\nu\mathcal{E} + \bar{D}_{(\mu}\mathtt{F}_{\nu)} + W_{\mu\nu}, \\ C_{\mu\nu} &= \psi^{\mathsf{t}}\gamma_{\mu\nu} - \bar{D}_{\langle\mu}\bar{D}_{\nu\rangle}\mathcal{E} + \bar{D}_{(\mu}\mathtt{F}_{\nu)} + W_{\mu\nu}, \end{aligned}$$

where $\bar{D}_\mu\mathtt{B}^\mu = \bar{D}_\mu\mathtt{F}^\mu = \bar{D}_\nu W_\mu{}^\nu = W_\mu{}^\mu = 0$.

Kinematic Perturbations

The advantage of the kinematic variables is that they are directly related to the tensors $\mathcal{F}_{\mu\nu}{}^{\gamma}$ and $\xi_{\mu\nu}$. For example,

$$\bar{\gamma}\left[\delta\mathcal{K}_{\mu\nu}\right] = -\phi\bar{\mathcal{K}}_{\mu\nu} + \bar{\gamma}\left[\mathcal{F}_{\mu\nu}{}^{\gamma}\bar{v}_{\gamma}\right].$$

Substituting in the Lagrangian one obtains

$$\begin{aligned}\delta\mathcal{L}_g^{(2)} = \frac{\sqrt{-\bar{g}}}{2\kappa}\Bigg\{ & \delta\mathcal{K}_{\mu}{}^{\nu}\delta\mathcal{K}_{\nu}{}^{\mu} - \delta\Theta^2 - C_{\mu}{}^{\nu}\delta\mathcal{R}_{\nu}{}^{\mu} + \left(\frac{C}{2} - \phi\right)\delta\mathcal{R} \\ & + 4C^{\mu\sigma}\bar{\sigma}_{\sigma}{}^{\gamma}B_{[\mu\|\gamma]} + 2\left(C_{\mu\nu\|\gamma} - C_{\|\mu}\bar{\gamma}_{\nu\gamma}\right)\bar{\sigma}^{\mu\nu}B^{\gamma} \\ & + 4C_{\sigma}{}^{\lambda}C_{\beta}{}^{\alpha}\left(\bar{\sigma}^{\sigma\beta}\bar{\sigma}_{\lambda\alpha} - \bar{\gamma}^{\sigma\beta}\bar{\sigma}_{\lambda}{}^{\gamma}\bar{\sigma}_{\gamma\alpha}\right) \\ & + \left[\left(\bar{\Theta}\phi + B^{\gamma}{}_{\|\gamma} - \delta\Theta\right)C_{\mu}{}^{\nu} - \left(\phi\bar{\mathcal{K}}_{\nu}{}^{\mu} + B_{\mu}{}^{\|\nu} - \delta\mathcal{K}_{\mu}{}^{\nu}\right)C\right]\bar{\sigma}_{\nu}{}^{\mu} \\ & + \bar{G}_{\bar{v}\bar{v}}\left(B_{\mu}B^{\mu} - \phi^2 - 2\phi C\right) + \bar{G}_{\bar{v}\mu}\left(4B_{\alpha}C^{\alpha\mu} - 2CB^{\mu}\right) \\ & + \bar{G}_{\mu\nu}\left(2C^{\mu\alpha}C_{\alpha}{}^{\nu} - C^{\mu\nu}C\right)\Bigg\}.\end{aligned}$$

Action Principle for a Thermodynamic Fluid

In the action principle for a thermodynamic fluid,[1] we write the specific enthalpy as the norm of a field, i.e., $\vartheta \equiv \sqrt{-\vartheta_\mu \vartheta_\nu g^{\mu\nu}}$. The field is defined by the following potentials and specific entropy $\vartheta_\mu = \nabla_\mu \varphi_1 + \varphi_2 \nabla_\mu \varphi_3 + \varphi_4 \nabla_\mu s$. The action for the fluid is

$$S_m = \int \mathrm{d}^4 x \mathcal{L}_m, \quad \text{with} \qquad \mathcal{L}_m = \sqrt{-g} p(\vartheta, s).$$

It is useful to express the matter field perturbations in terms of the velocity potential,

$$u_\mu = \bar{\gamma} \left[\frac{\delta \vartheta_\mu}{\bar{\vartheta}} \right] = \mathcal{V}_{\parallel\mu}, \quad \mathcal{V} \equiv \frac{\delta\varphi_1 + \bar{\varphi_2}\delta\varphi_3 + \bar{\varphi_4}\delta s}{\bar{\vartheta}},$$

and the gauge invariant variable

$$\Pi_{\mathcal{V}} = -\sqrt{-\bar{g}} \left(\delta\rho - \bar{\Theta}(\bar{\rho} + \bar{p})\mathcal{V} \right).$$

[1] B. Schutz, *Phys. Rev. D* 2.12 (1970), 2762.

Thermodynamic Fluid Lagrangian

Each fluid contributes with a Lagrangian in the form:

$$\frac{\delta\mathcal{L}^{(2)}_{\mathsf{m}i}}{\sqrt{-\bar{g}}} = \frac{\bar{c}^2_{si}\Pi^2_{\mathcal{V}i}}{2\sqrt{-\bar{g}}^2(\bar{\rho}_i+\bar{p}_i)} + \frac{(\bar{\rho}_i+\bar{p}_i)}{2}\mathcal{V}_i\bar{D}^2_K\mathcal{V}_i - \frac{3\kappa}{4}(\bar{\rho}_i+\bar{p}_i)(\bar{\rho}+\bar{p})\mathcal{V}^2_i + (\bar{\rho}_i+\bar{p}_i)\delta\Theta\mathcal{V}_i - \frac{\bar{\rho}_i}{2}\left(B_\gamma B^\gamma - \phi^2 - 2C\phi\right) - \frac{\bar{p}_i}{2}\left(2C_\gamma{}^\nu C_\nu{}^\gamma - C^2\right).$$

Combining with the gravitational part and discarding the terms proportional to the background equations of motion we obtain:

$$\frac{\delta\mathcal{L}^{(2,s)}}{\sqrt{-\bar{g}}} = \frac{3\bar{D}^2\Psi\bar{D}^2_K\Psi}{\kappa\bar{\Theta}^2} - \sum_i\frac{(\bar{\rho}_i+\bar{p}_i)}{3\kappa(\bar{\rho}+\bar{p})}\Xi^2_i + \sum_i\left[\frac{\bar{c}^2_{si}\Pi^2_{\mathcal{V}i}}{2\sqrt{-\bar{g}}^2(\bar{\rho}_i+\bar{p}_i)} + \frac{9(\bar{\rho}_i+\bar{p}_i)}{2\bar{\Theta}^2}\mathcal{U}_i\bar{D}^2_K\mathcal{U}_i\right],$$

where

$$\Psi \equiv \psi - \frac{\bar{\Theta}}{3}\delta\sigma^{\mathsf{s}}, \qquad \mathcal{U}_i \equiv \psi + \frac{\bar{\Theta}}{3}\mathcal{V}_i, \qquad \Xi_i \equiv \delta\Theta - \frac{3\kappa(\bar{\rho}+\bar{p})}{2}\mathcal{V}_i + \frac{3\delta\mathcal{R}}{4\bar{\Theta}}.$$

Constraints Reduction

A close inspection shows that the scalar Lagrangian:

- does not depend on time derivatives of the variables $(\phi,\ \mathcal{B})$,
- depends quadratically on the time derivatives of $(\psi,\ \mathcal{E},\ \mathcal{U}_i)$.

Using the Faddeev-Jackiv[2] we perform a Legendre transformation in the time derivatives variables

$$\delta\mathcal{L}^{(2,s)} = \bar{D}_K^2\Pi_{\mathcal{E}}\pounds_{\bar{v}}\bar{D}^2\mathcal{E} + \Pi_{\psi^{\mathsf{t}}}\dot{\psi}^{\mathsf{t}} + \sum_i \bar{D}_K^2\Pi_{\mathcal{U}i}\dot{\mathcal{U}}_i - \delta\mathcal{H}_c^{(2,s)},$$

The constrained Hamiltonian $\delta\mathcal{H}_c^{(2,s)}$:

$$\delta\mathcal{H}_c^{(2,s)} = \bar{D}_K^2\Pi_{\mathcal{E}}\pounds_{\bar{v}}\bar{D}^2\mathcal{E} + \Pi_{\psi^{\mathsf{t}}}\dot{\psi}^{\mathsf{t}} + \sum_i \bar{D}_K^2\Pi_{\mathcal{U}i}\dot{\mathcal{U}}_i - \sqrt{-\bar{g}}\frac{3\bar{D}^2\Psi\bar{D}_K^2\Psi}{\kappa\bar{\Theta}^2}$$
$$-\sum_i\left[\frac{\bar{c}_{si}^2\bar{\Theta}^2(\bar{D}_K^2\Pi_{\mathcal{U}i})^2}{18\sqrt{-\bar{g}}(\bar{\rho}_i+\bar{p}_i)} - \sqrt{-\bar{g}}\frac{(\bar{\rho}_i+\bar{p}_i)}{3\kappa(\bar{\rho}+\bar{p})}\Xi_i^2 + \sqrt{-\bar{g}}\frac{9(\bar{\rho}_i+\bar{p}_i)}{2\bar{\Theta}^2}\mathcal{U}_i\bar{D}_K^2\mathcal{U}_i\right].$$

[2] L. Faddeev and R. Jackiw, *Physical Review Letters* 60 (1988), 1692–1694.

Constraints Reduction

The variables in blue are the time derivatives written in terms of the momenta:

$$\pounds_{\bar{v}}[\bar{D}^2\mathcal{E}] = \bar{D}^2\mathcal{B} - \frac{3\bar{D}^2\psi}{\bar{\Theta}} + \frac{3\bar{D}^2\Psi}{\bar{\Theta}},$$

$$\dot{\psi}^{\mathsf{t}} = \frac{\kappa(\bar{\rho}+\bar{p})}{2}\mathcal{V} - \frac{\bar{\Theta}}{3}\phi - \frac{\bar{D}^2\mathcal{B}}{3} + \frac{\bar{D}^2_K\psi}{\bar{\Theta}} + \frac{\Xi}{3},$$

$$\dot{\mathcal{U}}_i = \frac{\dot{\bar{\Theta}}}{\bar{\Theta}}\mathcal{U}_i + \frac{\bar{c}^2_{si}\bar{\Theta}^2\bar{D}^2_K\Pi_{\mathcal{U}i}}{9\sqrt{-\bar{g}}(\bar{\rho}_i+\bar{p}_i)} + \frac{3\kappa(\bar{\rho}+\bar{p})}{2\bar{\Theta}}\mathcal{U} + \frac{\bar{D}^2\Psi}{\bar{\Theta}} + \frac{\Xi}{3},$$

where we introduced effective fluid variables

$$(\bar{\rho}+\bar{p})\mathcal{U} = \sum_i(\bar{\rho}_i+\bar{p}_i)\mathcal{U}_i, \quad (\bar{\rho}+\bar{p})\Xi = \sum_i(\bar{\rho}_i+\bar{p}_i)\Xi_i.$$

Therefore, the constrains are simply:

$$\bar{D}^2_K\Pi_{\mathcal{E}} = 0, \qquad \Pi_{\psi^{\mathsf{t}}} = 0.$$

Constraints Reduction

Using the expressions for the momenta:

$$\bar{D}_K^2 \Pi_{\mathcal{U}i} \equiv \frac{\partial \delta \mathcal{L}^{(2,s)}}{\partial \dot{\mathcal{U}}_i} = \frac{3\Pi_{\mathcal{V}i}}{\bar{\Theta}},$$

$$\bar{D}_K^2 \Pi_{\mathcal{E}} \equiv \frac{\partial \delta \mathcal{L}^{(2,s)}}{\partial \pounds_{\bar{v}}[\bar{D}^2 \mathcal{E}]} = \frac{2\sqrt{-\bar{g}}}{\kappa \bar{\Theta}} \bar{D}_K^2 \Psi - \sum_i \frac{\Pi_{\mathcal{U}i}}{3},$$

$$\Pi_{\psi^{\mathsf{t}}} \equiv \frac{\partial \delta \mathcal{L}^{(2,s)}}{\partial \dot{\psi}^{\mathsf{t}}} = -2\sqrt{-\bar{g}} \frac{\Xi}{\kappa} - \sum_i \Pi_{\mathcal{U}i},$$

then

$$\Xi = -\frac{\kappa}{2\sqrt{-\bar{g}}} \sum_i \bar{D}_K^2 \Pi_{\mathcal{U}i}, \quad \Psi = \frac{\kappa \Theta}{6\sqrt{-\bar{g}}} \Pi_{\mathcal{U}}, \quad \Pi_{\mathcal{U}} \equiv \sum_i \Pi_{\mathcal{U}i},$$

the equations above also imply

$$\bar{D}_K^2 \Psi = -\frac{\bar{\Theta}\Xi}{3}.$$

Adiabatic and Entropy Perturbations

The calculation above shows that:

- The constraints naturally lead to effective fluids variable.
- The physical degrees of freedom are $(\mathcal{U}_i,\ \Pi_{\mathcal{U}i})$.

We incorporate both notions writing the Lagrangian in terms of the effective fluid (curvature perturbation) variable:

$$\zeta \equiv \mathcal{U} - \frac{\overline{K}\bar{\Theta}\Pi_{\mathcal{U}}}{3\sqrt{-\bar{g}}(\bar{\rho}+\bar{p})},$$

and the gauge invariant energy density contrast

$$\delta_{\psi i} = 3\mathcal{U}_i - \frac{\bar{\Theta}\bar{D}_K^2\Pi_{\mathcal{U}i}}{3\sqrt{-\bar{g}}(\bar{\rho}_i+\bar{p}_i)} = \frac{\delta\rho_i}{\bar{\rho}_i+\bar{p}_i} + 3\psi.$$

Naturally we define

$$(\bar{\rho}+\bar{p})\delta_\psi = \sum_i(\bar{\rho}_i+\bar{p}_i)\delta_{\psi i}, \quad \tilde{\delta}_{\psi i} = \tilde{\delta}_{\psi i} - \delta_\psi, \quad \widetilde{\mathcal{U}}_i \equiv \mathcal{U}_i - \mathcal{U}.$$

Multiple Fluids Hamiltonian

Splitting the Lagrangian between adiabatic and entropy modes

$$\delta\mathcal{L}^{(2,s)} = \delta\mathcal{L}_a^{(2,s)} + \delta\mathcal{L}_s^{(2,s)}.$$

we have

$$\delta\mathcal{L}_a^{(2,s)} = \Pi_\zeta\dot{\zeta} - \delta\mathcal{H}_a^{(2,s)},$$

$$\delta\mathcal{L}_s^{(2,s)} = \sum_i \frac{3\sqrt{-\bar{g}}(\bar{\rho}_i + \bar{p}_i)\widetilde{\mathcal{U}}_i}{\bar{\Theta}}\dot{\tilde{\delta}}_{\psi i} - \delta\mathcal{H}_s^{(2,s)},$$

$$\delta\mathcal{H}_a^{(2,s)} = \frac{\bar{c}_s^2\bar{\Theta}^2\Pi_\zeta\bar{D}^2\bar{D}_K^{-2}\Pi_\zeta}{18\sqrt{-\bar{g}}(\bar{\rho}+\bar{p})} - \sqrt{-\bar{g}}\frac{9(\bar{\rho}+\bar{p})}{2\bar{\Theta}^2}\zeta\bar{D}_K^2\zeta,$$

$$\delta\mathcal{H}_s^{(2,s)} = \sum_i \Bigg[\frac{\sqrt{-\bar{g}}\bar{c}_{si}^2(\bar{\rho}_i + \bar{p}_i)\tilde{\delta}_{\psi i}^2}{2} - \sqrt{-\bar{g}}\frac{9(\bar{\rho}_i + \bar{p}_i)}{2\bar{\Theta}^2}\widetilde{\mathcal{U}}_i\bar{D}^2\widetilde{\mathcal{U}}_i - \frac{\bar{\Theta}\Pi_\zeta}{3(\bar{\rho}+\bar{p})}\bar{c}_{si}^2(\bar{\rho}_i + \bar{p}_i)\tilde{\delta}_{\psi i}\Bigg].$$

where we defined the momentum conjugated to ζ as $\Pi_\zeta \equiv \bar{D}_K^2\Pi_{\mathcal{U}}$.

Two Fluids Lagrangian

For two fluids the perturbations $\tilde{\delta}_{\psi i}$ and $\widetilde{\mathcal{U}}_i$ can be written in terms of

$$\Pi_Q \equiv \delta_{\psi 2} - \delta_{\psi 1}, \quad \text{and,} \quad Q = \frac{3\sqrt{-\bar{g}}\varpi}{\bar{\Theta}}(\mathcal{U}_1 - \mathcal{U}_2),$$

where the "reduced variable" is $\varpi \equiv (\bar{\rho}_1 + \bar{p}_1)(\bar{\rho}_2 + \bar{p}_2)/(\bar{\rho} + \bar{p})$. The Lagrangian/Hamiltonian then reads

$$\delta\mathcal{L}^{(2,s)} = \Pi_\zeta \zeta' + \Pi_Q Q' - \left[\frac{\Pi_\zeta^2}{2m_\zeta} + \frac{\Pi_Q^2}{2m_S} + \frac{m_\zeta \nu_\zeta^2 \zeta^2}{2} + \frac{m_S \nu_S^2 Q^2}{2} \right.$$
$$\left. + \frac{\bar{c}_n^2}{\bar{c}_s^2 \bar{c}_m^2} \frac{\Pi_\zeta \Pi_Q}{m_\zeta m_S N H \Delta_K}\right],$$

where we changed the time derivative introducing the lapse function N,

$$m_\zeta \equiv \frac{a^3(\bar{\rho} + \bar{p})}{N \bar{c}_s^2 \Delta_K H^2}, \qquad \nu_\zeta^2 \equiv -\frac{N^2 \bar{c}_s^2 \hat{D}^2}{a^2},$$
$$m_S \equiv \frac{1}{N a^3 \bar{c}_m^2 \varpi}, \qquad \nu_S^2 \equiv -\frac{N^2 \bar{c}_m^2 \hat{D}^2}{a^2}.$$

Symplectic structure

To perform the quantization we need the following structure:

- The phase space vector $\chi_a = (\varphi_1, \ldots, \varphi_n, \Pi_{\varphi 1}, \ldots, \Pi_{\varphi n})$.
- The Hamiltonian

$$\mathcal{H}(\chi) = \frac{\chi_a \mathcal{H}^{ab} \chi_b}{2},$$

  with $\mathcal{H}^{ab}$ being symmetric.
- The symplectic forms

$$\mathbb{S}_{ab} \doteq \mathsf{i} \begin{pmatrix} 0 & \mathbf{1}_{n\times n} \\ -\mathbf{1}_{n\times n} & 0 \end{pmatrix}, \quad \mathbb{S}^{ab} \doteq \mathsf{i} \begin{pmatrix} 0 & \mathbf{1}_{n\times n} \\ -\mathbf{1}_{n\times n} & 0 \end{pmatrix},$$

- The solutions thus satisfy $\mathsf{i}\pounds_{\bar{v}} \chi_a = \mathbb{S}_{ab} \mathcal{H}^{bc} \chi_c$.
- The product of two solutions χ and ϖ,

$$\mathbb{S}\left(\chi, \varpi\right) = \int_\Sigma \mathrm{d}^3 x \chi_a \varpi_b \mathbb{S}^{ab},$$

  is conserved, i.e., $\mathsf{i}\pounds_{\bar{v}}\left(\chi, \varpi\right) = 0$.

Complex Phase Space

We complexify the phase space and define the product

$$(\chi, \varpi) \equiv \mathbb{S}\,(\chi^*, \varpi)\,.$$

To probe the phase space we introduce the Laplacian eigenfunctions

$$\bar{D}^2\mathcal{Y}_q = -\lambda_q^2\mathcal{Y}_q, \quad \int_\Sigma \mathrm{d}^3x\mathcal{Y}_{q_1}\mathcal{Y}_{q_2} = \delta^3(q_1 - q_2).$$

We write the vector in the phase space for each mode as,

$$\begin{aligned} \mathsf{U}_{q,a} &\equiv \mathsf{T}_a\mathcal{Y}_q, \\ \mathsf{T}_a &= (\varphi_{q,1}, \ldots, \varphi_{q,n}, \Pi_{\varphi q,1}, \ldots, \Pi_{\varphi q,n}), \end{aligned}$$

where the functions T_a depend only on time $\bar{D}_\mu\mathsf{T}_a = 0$ and q.
The product of two solution will be given by

$$(\mathsf{U}_{q_1}, \mathsf{V}_{q_2}) = \mathsf{T}_a^*(q_1)\mathsf{W}_b(q_2)\mathbb{S}^{ab}\delta^3(q_1 - q_2),$$

where $\mathsf{V}_q = \mathsf{W}_a\mathcal{Y}_q$.

Finite Dimensional Phase Space

Now our problem is reduced to the product

$$\mathsf{T} \cdot \mathsf{W} = \mathsf{T}^*_a \mathbb{S}^{ab} \mathsf{W}_b,$$

defined in a finite $2n$-dimensional phase space for each mode q.
If for a given vector T_a its norm is positive

$$\mathsf{T} \cdot \mathsf{T} = \mathsf{T}^*_a \mathsf{T}_b \mathbb{S}^{ab} > 0,$$

the vector T^*_a will have the norm

$$\mathsf{T}^* \cdot \mathsf{T}^* = \mathsf{T}_a \mathsf{T}^*_b \mathbb{S}^{ab} = -\mathsf{T} \cdot \mathsf{T} < 0.$$

Thus we build normalized n-dimensional basis such that $e^i \cdot e^j = \delta^{ij}$, consequently

$$e^{i*} \cdot e^{j*} = -\delta^{ij}, \qquad e^{i*} \cdot e^j = e^i{}_a \mathbb{S}^{ab} e^j{}_b = 0.$$

Canonical Quantization

The Poisson bracket structure of our problem is given by

$$\{F_1,\ F_2\} = -\mathsf{i}\int_\Sigma \mathrm{d}^3x \frac{\delta F_1}{\delta\chi_a(x)}\mathbb{S}_{ab}\frac{\delta F_2}{\delta\chi_b(x)},$$

where F_1 and F_2 are two field functionals. Using the definitions above is easy to see that

$$\{\chi_a(x),\ \chi_b(x')\} = -\mathsf{i}\mathbb{S}_{ab}\delta^3(x-x'),$$

where we have, for example, $\{\varphi_1,\ \Pi_{\varphi 1}\} = \delta^3(x-x')$, as expected.
By the canonical quantization rules we promote the fields to Hermitian operators. The fields operators then satisfy the commutation relations,

$$[\hat{\chi}_a(x),\ \hat{\chi}_b(x')] = \mathbb{S}_{ab}\delta^3(x-x').$$

Creation and Annihilation Operators

First of all, we extend the product to the operators (and classical fields)

$$(\chi, \varpi) = \mathbb{S}\left(\chi^{\dagger}, \varpi\right).$$

Having the following properties:

- $\left[(\varpi, \hat{\chi})\,,\ (\vartheta, \hat{\chi})\right] = (\vartheta, \varpi^*)$,
- $\left[(\varpi, \hat{\chi})\,,\ (\vartheta, \hat{\chi})^{\dagger}\right] = (\varpi, \vartheta)$,
- $\left[(\varpi, \hat{\chi})^{\dagger}\,,\ (\vartheta, \hat{\chi})^{\dagger}\right] = (\vartheta^*, \varpi)$.

Given a orthonormal basis $\mathsf{U}^i_{q,a}$, the annihilation operators associated with this basis is

$$\mathsf{a}^i_q \equiv \left(\mathsf{U}^i_q, \hat{\chi}\right).$$

It follows directly from the definition and the properties above that

$$\left[\mathsf{a}^i_q,\ \mathsf{a}^{j\dagger}_{q'}\right] = \delta^{ij}\delta^3(q - q'), \qquad \left[\mathsf{a}^i_q,\ \mathsf{a}^j_{q'}\right] = 0 = \left[\mathsf{a}^{i\dagger}_q,\ \mathsf{a}^{j\dagger}_{q'}\right].$$

Representation Choice

- Each choice of basis $\mathsf{U}^i_{q,a}$ produces a different representation.
- The representation choice reduces to find a basis in the finite dimensional space of T^i.

Defining a vacuum at the instant t_1, $\mathsf{T}^i{}_a(t_1) = \mathsf{t}^i{}_a$, at a time t_2 we have

$$\mathsf{T}^i{}_a(t_2) = \alpha^i{}_j(t_2)\mathsf{t}^j{}_a + \beta^i{}_j(t_2)\mathsf{t}^j{}^*_a,$$

where the functions

$$\alpha^i{}_j(t) \equiv \mathsf{T}^i{}_a(t)\mathsf{t}_j{}^{a*}, \quad \beta^i{}_j(t) \equiv \mathsf{T}^i{}_a(t)\mathsf{t}_j{}^a,$$

satisfy $\alpha^i{}_j(t_1) = \delta^i{}_j$ and $\beta^i{}_j(t_1) = 0$.
Then, the annihilation and creation operators at t_2 can be written in terms of the same operators at t_1 as

$$\mathsf{a}^i_q(t_2) = \alpha^i{}^*_j(t_2)\mathsf{a}^j_q(t_1) - \beta^i{}^*_j(t_2)\mathsf{a}^{j\dagger}_q(t_1),$$
$$\mathsf{a}^{i\dagger}_q(t_2) = \alpha^i{}_j(t_2)\mathsf{a}^{j\dagger}_q(t_1) - \beta^i{}_j(t_2)\mathsf{a}^j_q(t_1).$$

Representation Choice

- The number operator $N^i_q(t) \equiv \mathsf{a}^{i\dagger}_q(t)\mathsf{a}^i_q(t)$ applied at $|0_{t_1}\rangle$ measures

$$\left\langle 0_{t_1}\middle|N^i_q(t)\middle|0_{t_1}\right\rangle = \delta^3(0)\int \mathrm{d}^3q \sum_j \left|\beta^i{}_j(t)\right|^2,$$

  where the $\delta^3(0)$ is volume of the spatial section.
- If the integral converges we have defined a unitary evolution.
- The time evolution of each matrix are

$$\mathsf{i}\dot{\alpha}^i{}_j = M^i{}_k\alpha^k{}_j - N^i{}_k\beta^{k*}_{\ \ j},$$
$$\mathsf{i}\dot{\beta}^i{}_j = M^i{}_k\beta^k{}_j - N^i{}_k\alpha^{k*}_{\ \ j},$$

  where we defined the following matrices

$$M^i{}_k \equiv \mathsf{T}^i{}_a\mathcal{H}^{ab}\mathsf{T}^*_{kb}, \qquad N^i{}_k \equiv \mathsf{T}^i{}_a\mathcal{H}^{ab}\mathsf{T}_{kb}.$$

- Note that the matrix $N_j{}^k$ control the mixing between $\alpha^i{}_j$ and $\beta^i{}_j$, for instance, if it is null then there is no particle creation.
- However, in general $\mathcal{H}_a{}^b$ will depend on time, thus, even if $N_j{}^k$ is null initially at t_1 nothing guarantees that it will remain null.

Hamiltonian Eigenvectors

- The matrix $\mathcal{H}_a{}^b \equiv \mathbb{S}_{ac}\mathcal{H}^{cb}$ works as an operator in the space of the solutions.
- This operator is Hermitian, given the product of the two vectors V_a and $\mathcal{H}_a{}^b U_b$,

$$V \cdot (\mathcal{H}U) = V_a^* \mathbb{S}^{ab}(\mathcal{H}_b{}^c U_c) = (\mathcal{H}_c{}^a V_a)^* \mathbb{S}^{cd} U_d = (\mathcal{H}V) \cdot U.$$

- Finally, if we assume that $\mathrm{t}^i{}_a$ are the eigenvectors of $\mathcal{H}_a{}^b(t_1)$, the condition of the time independent vacuum is simply

$$N^{ij}(t_1) = \mathrm{t}^i_a \mathbb{S}^{ab}\mathcal{H}_b{}^c \mathrm{t}^j{}_c\big|_{t_1} = \nu^j{}_k \mathrm{t}^i{}_a \mathbb{S}^{ab}\mathrm{t}^k{}_b\big|_{t_1} = 0,$$

  where $\mathcal{H}_a{}^b \mathrm{t}^i{}_b = \nu^i{}_j \mathrm{t}^j_a$ and $\nu^i{}_j$ is a diagonal real matrix containing the eigenvalues.
- This gives zero particle creation in first order expansion of $N^i{}_k(t)$.
- This choice does not guarantee that higher order terms will be zero or provides a convergent $\beta^i{}_j$.

Diagonal $\mathcal{H}^{ab}$

For a diagonal Hamiltonian

$$\mathcal{H}^{ab} \doteq \mathrm{diag}(m_1\nu_1^2, \ldots, m_n\nu_n^2, 1/m_i, \ldots, 1/m_n).$$

The eigenvectors are given by

$$V^i{}_a \doteq \left(\frac{1}{\sqrt{2m_i\nu_i}},\ -\mathrm{i}\sqrt{\frac{m_i\nu_i}{2}}\right).$$

- The canonical transformation in each field in the form

$$q_i \to \sqrt{\frac{m_i}{M_i}}q_i, \qquad p_i \to M_i\left(\sqrt{\frac{m_i}{M_i}}\right)' q_i + \sqrt{\frac{M_i}{m_i}}p_i.$$

- It changes the masses arbitrarily $m_i \to M_i$.
- The frequency, however, change as

$$\nu_i^2 \to W_i^2 = \nu_i^2 - \frac{(\sqrt{m_i})''}{\sqrt{m_i}} + \frac{(\sqrt{M_i})''}{\sqrt{M_i}}.$$

Diagonal $\mathcal{H}^{ab}$

In the new representation the eigenvectors are simply

$$V^i{}_a \doteq \left(\frac{1}{\sqrt{2M_iW_i}}, -\mathrm{i}\sqrt{\frac{M_iW_i}{2}}\right).$$

Then we can make two main choices for

$$M_iW_i = \sqrt{M_i^2\nu_i^2 - M_i^2\left(\frac{(\sqrt{m_i})''}{\sqrt{m_i}} - \frac{(\sqrt{M_i})''}{\sqrt{M_i}}\right)}.$$

- Constant $M_i^2\nu_i^2$, defines algebraically M_i for each field and it is possible to show that leads to a unitary evolution with particle creation.
- Constant M_iW_i, leads to a differential equation definition of W_i, i.e.,

$$W_i^2 + \frac{1}{2}\frac{W_i''}{W_i} - \frac{3}{4}\frac{W_i'^2}{W_i^2} = \nu_i^2 - \frac{(\sqrt{m_i})''}{\sqrt{m_i}}.$$

In this representation the eigenvectors are constant, and is possible to show that it leads to unitary evolution with zero particle creation.

Adiabatic Vacuum

- The last option leads naturally to the adiabatic vacuum, the differential equation can be approximated by a asymptotic series starting with
$$W_i^2 \approx \nu_i^2 - \frac{(\sqrt{m_i})''}{\sqrt{m_i}}.$$
- This is exactly what one would obtain from the WKB approximation of the field, for any field representation.
- This can be extended to non-diagonal Hamiltonians, and is there is a limit where the Hamiltonian is diagonal, we can show that there is also a representation with zero particles production.

To perform the quantization we the last steps and make the canonical transformation:

$$A = \sqrt{\frac{m_\zeta}{m_A}}\zeta, \quad P_A = \sqrt{\frac{m_A}{m_\zeta}}\Pi_\zeta + m_A\sqrt{\frac{m_\zeta}{m_A}}'\zeta,$$
$$B = \sqrt{\frac{m_S}{m_B}}Q, \quad P_B = \sqrt{\frac{m_B}{m_S}}\Pi_Q + m_B\sqrt{\frac{m_S}{m_B}}'Q,$$

With the new frequencies satisfying

$$\nu_A^2 + \frac{1}{2}\frac{\nu_A''}{\nu_A} - \frac{3}{4}\frac{\nu_A'^2}{\nu_A^2} = \nu_\zeta^2 - \frac{(\sqrt{m_\zeta})''}{\sqrt{m_\zeta}},$$
$$\nu_B^2 + \frac{1}{2}\frac{\nu_B''}{\nu_B} - \frac{3}{4}\frac{\nu_B'^2}{\nu_B^2} = \nu_S^2 - \frac{(\sqrt{m_S})''}{\sqrt{m_S}}.$$

The Hamiltonian in these variables is

$$\delta\mathcal{L}^{(2,s)} = P_A A' + P_B B' - \left[\frac{\nu_A P_A^2}{2\tilde{\lambda}_q} + \frac{\nu_B P_B^2}{2\tilde{\lambda}_q}\right.$$
$$\left. + \frac{\tilde{\lambda}_q \nu_A A^2}{2} + \frac{\tilde{\lambda}_q \nu_B B^2}{2} + y(P_A - L_A A)(P_B - L_B B)\right].$$

Integral Solution

Writing the fields in terms of

$$R_A^{\pm} = \sqrt{\tilde{\lambda}_q}A \pm \mathsf{i}\frac{P_A}{\sqrt{\tilde{\lambda}_q}} \quad \text{and} \quad R_B^{\pm} = \sqrt{\tilde{\lambda}_q}B \pm \mathsf{i}\frac{P_B}{\sqrt{\tilde{\lambda}_q}}.$$

The equations for these variables can be readily integrated, resulting in a set of integral equations,

$$R_A^{\pm} = e^{\mp \mathsf{i}\int \mathrm{d}\tau \nu_A}\left[R_{A0}^{\pm} + \int \mathrm{d}\tau y\left(\sqrt{\tilde{\lambda}_q} \pm \mathsf{i}\frac{L_A}{\sqrt{\tilde{\lambda}_q}}\right)(P_B - L_B B)\, e^{\pm \mathsf{i}\int \mathrm{d}\tau \nu_A}\right],$$

$$R_B^{\pm} = e^{\mp \mathsf{i}\int \mathrm{d}\tau \nu_B}\left[R_{B0}^{\pm} + \int \mathrm{d}\tau y\left(\sqrt{\tilde{\lambda}_q} \pm \mathsf{i}\frac{L_B}{\sqrt{\tilde{\lambda}_q}}\right)(P_A - L_A A)\, e^{\pm \mathsf{i}\int \mathrm{d}\tau \nu_B}\right].$$

Adiabatic Approximation

If ν_A and ν_B are large enough, we can approximate the integral solutions by

$$R_A^{\pm} = e^{\mp \mathrm{i} \int \mathrm{d}\tau \nu_A} R_{A0}^{\pm} \pm \frac{1}{\mathrm{i}\nu_A} y \left(\sqrt{\tilde{\lambda}_q} \pm \mathrm{i} \frac{L_A}{\sqrt{\tilde{\lambda}_q}} \right) (P_B - L_B B)\,,$$

$$R_B^{\pm} = e^{\mp \mathrm{i} \int \mathrm{d}\tau \nu_B} R_{B0}^{\pm} \pm \frac{1}{\mathrm{i}\nu_B} y \left(\sqrt{\tilde{\lambda}_q} \pm \mathrm{i} \frac{L_B}{\sqrt{\tilde{\lambda}_q}} \right) (P_A - L_A A)\,.$$

- This approximation assumes only that ν_A and ν_B are large.
- Therefore, it can be applied even when the coupling y is large.
- The equations above provide a linear system in terms of $R_A^{\pm}$ and $R_B^{\pm}$.

Conclusions

- We obtained the second order Hamiltonian for a multiple fluids system.
- This Hamiltonian was obtained without assuming a dynamics for the background.
- We suggested a well defined procedure to find a vacuum for a multiple components system.
- This in turn leads to a WKB approximation which can be easily applied to a multiple components system.