Institut d'Astrophysique de Paris
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Rencontres du Vietnam

Hot Topics in General Relativity & Gravitation

POST-NEWTONIAN THEORY VERSUS BLACK HOLE PERTURBATIONS

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Gravitation et Cosmologie ($\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$)
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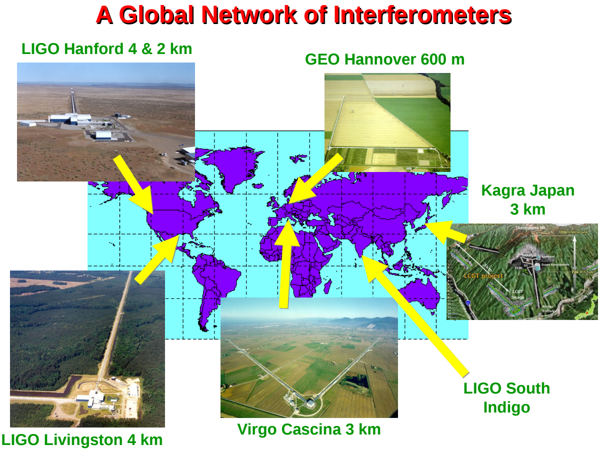
World-wide network of gravitational wave detectors

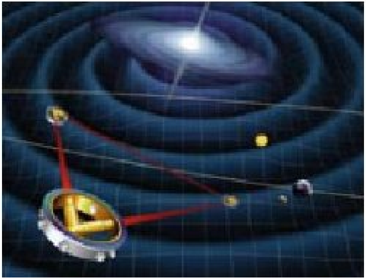


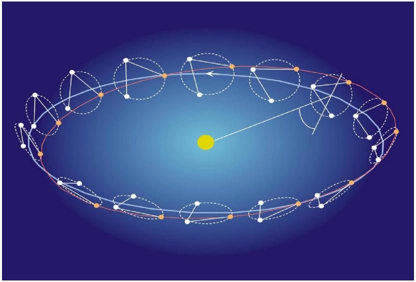
The network will observe the GWs in the high-frequency band

$$10\,\mathrm{Hz} \lesssim f \lesssim 10^3\,\mathrm{Hz}$$

Space-based laser interferometric detector

eLISA

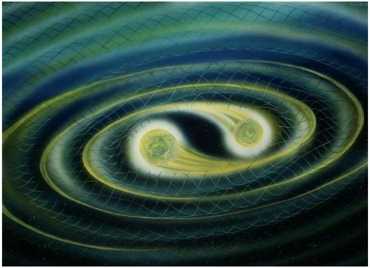




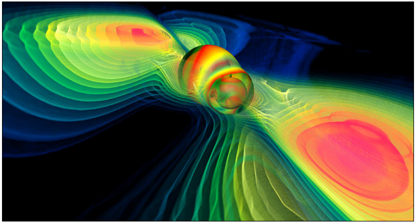
eLISA will observe the GWs in the low-frequency band

$$10^{-4}\,\mathrm{Hz} \lesssim f \lesssim 10^{-1}\,\mathrm{Hz}$$

The inspiral and merger of compact binaries



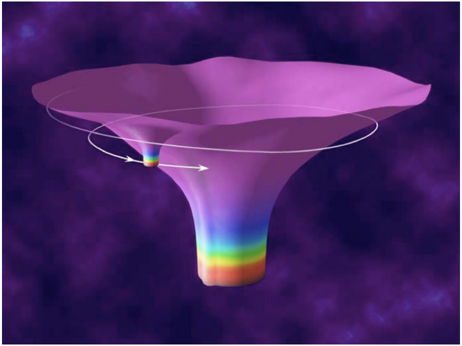
Neutron stars spiral and coalesce



Black holes spiral and coalesce

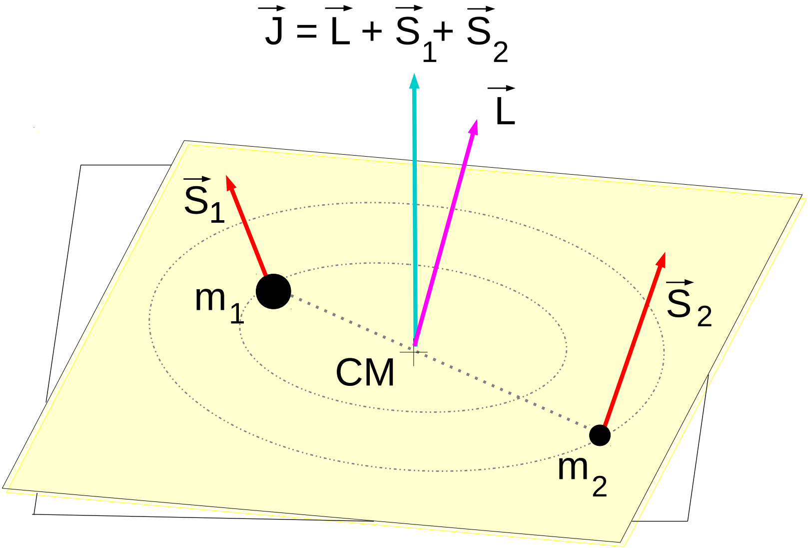
1. Neutron star ($M = 1.4\,M_\odot$) events will be detected by ground-based detectors LIGO/VIRGO
2. Stellar size black hole ($5\,M_\odot \lesssim M \lesssim 20\,M_\odot$) events will also be detected by ground-based detectors
3. Supermassive black hole ($10^5\,M_\odot \lesssim M \lesssim 10^8\,M_\odot$) events will be detected by the space-based detector eLISA

Extreme mass ratio inspirals (EMRI) for eLISA

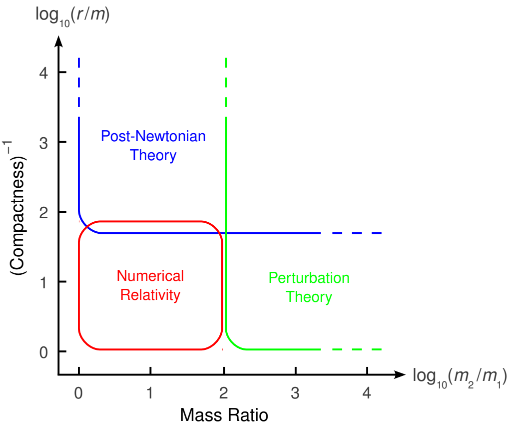


- A neutron star or a stellar black hole follows a highly relativistic orbit around a supermassive black hole. The gravitational waves generated by the orbital motion are computed using black hole perturbation theory
- Observations of EMRIs will permit to test the no-hair theorem for black holes, i.e. to verify that the central black hole is described by the Kerr geometry

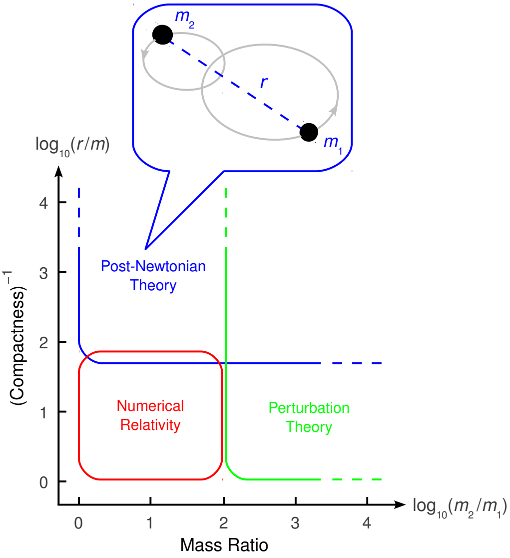
Modelling the compact binary inspiral



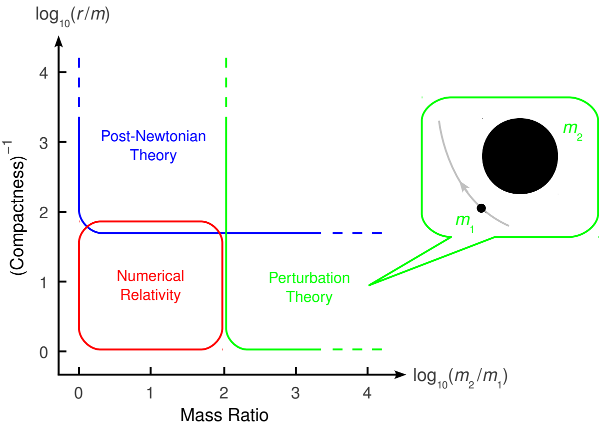
Methods to compute GW templates



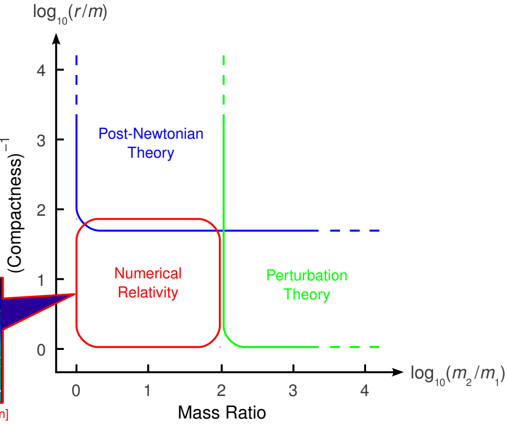
Methods to compute GW templates

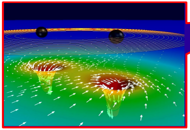


Methods to compute GW templates



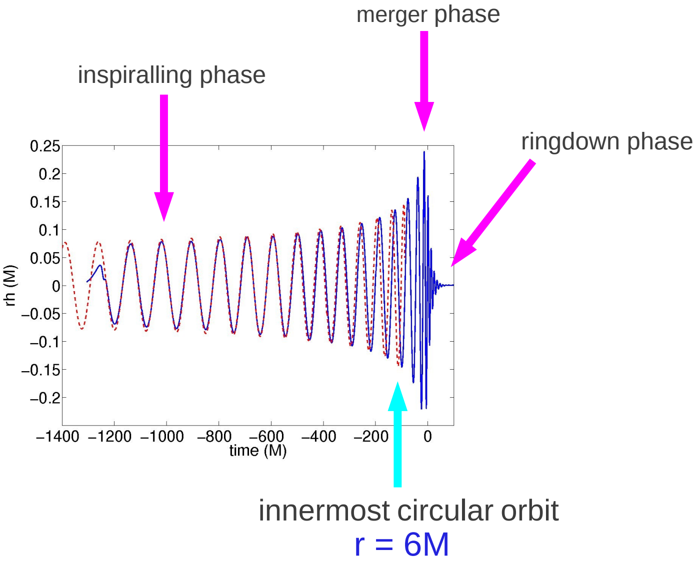
Methods to compute GW templates



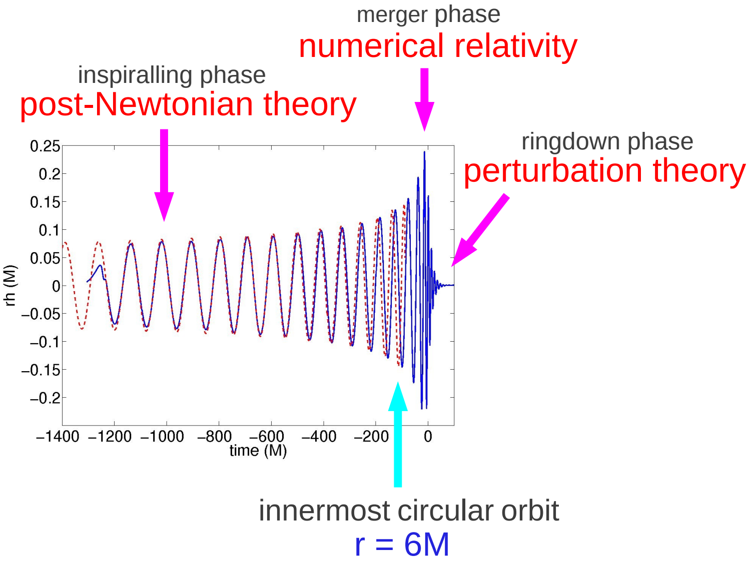


[Caltech/Cornell/CITA collaboration]

The gravitational chirp of compact binaries

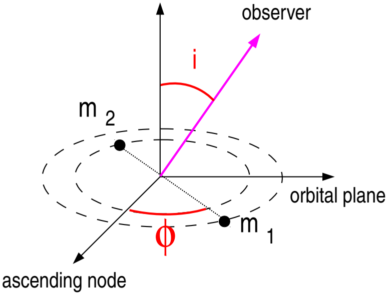


The gravitational chirp of compact binaries



Inspiralling binaries require high-order PN modelling

[Cutler, Flanagan, Poisson & Thorne 1992; Blanchet & Schäfer 1993]



$$\phi(t) = \underbrace{\phi_0 - \frac{M}{\mu}\left(\frac{GM\omega}{c^3}\right)^{-5/3}}_{\text{result of the quadrupole formalism (sufficient for the binary pulsar)}} \left\{ 1 + \underbrace{\frac{1\text{PN}}{c^2} + \frac{1.5\text{PN}}{c^3} + \cdots + \frac{3\text{PN}}{c^6} + \cdots}_{\text{needs to be computed with 3PN precision at least}} \right\}$$

Short History of the PN Approximation

EQUATIONS OF MOTION

- 1PN equations of motion [Lorentz & Droste 1917; Einstein, Infeld & Hoffmann 1938]
- Radiation-reaction controvercy [Ehlers *et al* 1979; Walker & Will 1982]
- 2.5PN equations of motion and GR prediction for the binary pulsar [Damour & Deruelle 1982; Damour 1983]
- The "3mn" Caltech paper [Cutler, Flanagan, Poisson & Thorne 1993]
- 3.5PN equations of motion [Jaranowski & Schäfer 1999; BF 2001; ABF 2002; BI 2003; Itoh & Futamase 2003; Foffa & Sturani 2011]
- Ambiguity parameters resolved [DJS 2001; BDE 2003]
- 4PN [DJS, BBBFM]

RADIATION FIELD

- 1918 Einstein quadrupole formula
- 1940 Landau-Lifchitz formula
- 1960 Peters-Mathews formula
- EW multipole moments [Thorne 1980]
- BD moments and wave generation formalism [BD 1989; B 1995, 1998]
- 1PN orbital phasing [Wagoner & Will 1976; BS 1989]
- 2PN waveform [BDIWW 1995]
- 3.5PN phasing and 3PN waveform [BFIJ 2003; BFIS 2007]
- Ambiguity parameters resolved [BI 2004; BDEI 2004, 2005]
- 4.5PN (?)

4PN equations of motion of compact binaries

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2} n_{12}^i + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3} + \frac{4G^2m_2^2}{r_{12}^3} + \cdots\right] n_{12}^i + \cdots\right\}}^{\text{1PN Lorentz-Droste-EIH term}} + \underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}[\cdots]}_{\substack{\text{2.5PN} \\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\substack{\text{3.5PN} \\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8}[\cdots]}_{\substack{\text{4PN} \\ \text{conservative \& radiation tail}}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

3PN	[Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001]	ADM Hamiltonian
	[Blanchet & Faye 2000; de Andrade, Blanchet & Faye 2001]	Harmonic equations of motion
	[Itoh, Futamase & Asada 2001; Itoh & Futamase 2003]	Surface integral method
	[Foffa & Sturani 2011]	Effective field theory
4PN	[Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014]	ADM Hamiltonian
	[See the talk of Laura Bernard in this meeting]	Harmonic Lagrangian

3.5PN energy flux of compact binaries (4.5PN?)

[Blanchet, Faye, Iyer & Joguet 2002]

$$\mathcal{F}^{\mathrm{GW}} = -\frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right) x + \overbrace{4\pi x^{3/2}}^{\text{1.5PN tail}}$$
$$+ \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right) x^2 + \underbrace{\left[\cdots\right] x^{5/2}}_{\text{2.5PN tail}}$$
$$+ \underbrace{\left[\cdots\right] x^3}_{\substack{\text{3PN} \\ \text{includes a tail-of-tail}}} + \underbrace{\left[\cdots\right] x^{7/2}}_{\text{3.5PN tail}} + \underbrace{\left[\cdots\right] x^4}_{\text{4PN (?)}} + \underbrace{\left[\cdots\right] x^{9/2}}_{\text{4.5PN (?)}} + \mathcal{O}\left(x^5\right) \Bigg\}$$

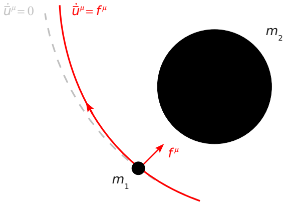
The orbital frequency and phase for quasi-circular orbits are deduced from an energy balance argument

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}^{\mathrm{GW}}$$

Spin contributions are also known to high order [Bohé, Marsat, Faye & Blanchet 2013]

General problem of the gravitational perturbation

- A particle is moving on a background space-time
- Its own stress-energy tensor modifies the background gravitational field
- Because of the "back-reaction" the motion of the particle deviates from a background geodesic hence the appearance of a gravitational self force (GSF)

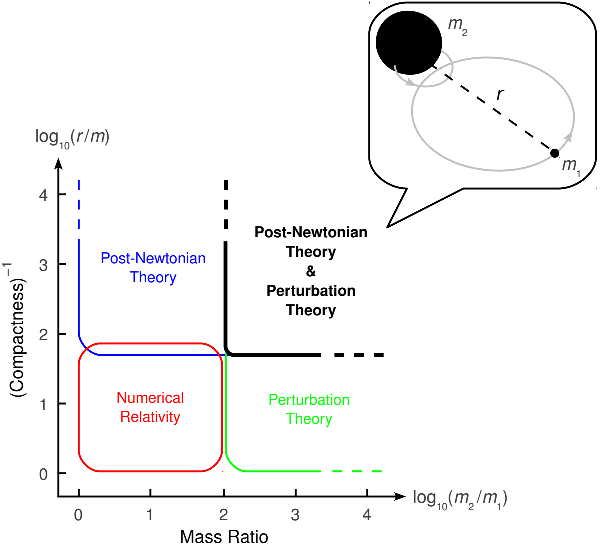


The self acceleration of the particle is proportional to its mass

$$\frac{\mathrm{D}\bar{u}^\mu}{\mathrm{d}\tau} = f^\mu = \mathcal{O}\left(\frac{m_1}{m_2}\right)$$

- The self force is computed by numerical methods [Sago, Barack & Detweiler 2008]

Common regime of validity of GSF and PN



Why and how comparing PN and GSF predictions?

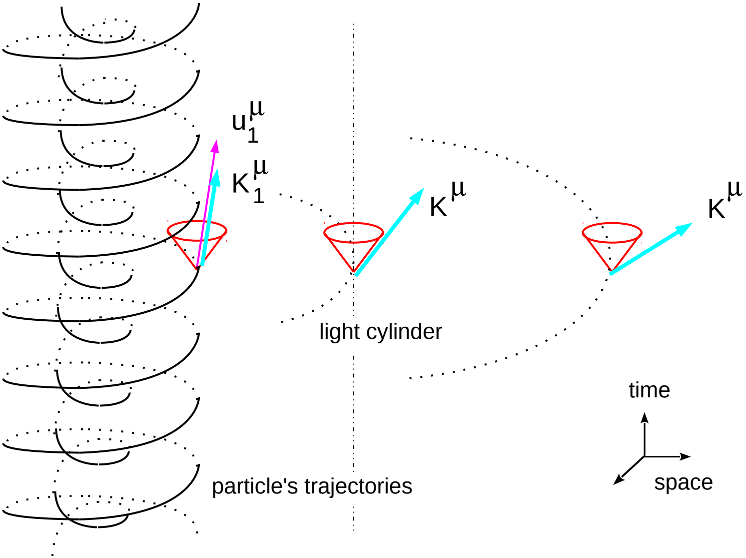
Both the PN and GSF approaches use a self-field regularization for point particles followed by a renormalization. However, the prescription are very different

1. GSF theory is based on a prescription for the Green function G_{R} that is at once regular and causal [Detweiler & Whiting 2003]
2. PN theory uses dimensional regularization and it was shown that subtle issues appear at the 3PN order due to the appearance of poles $\propto (d-3)^{-1}$

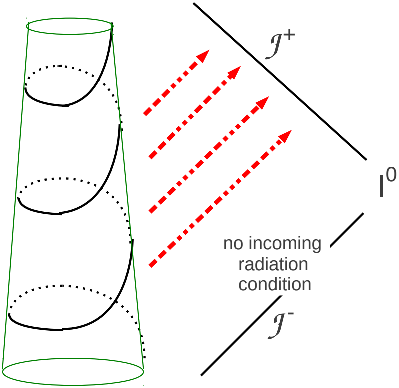
How can we make a meaningful comparison?

1. Restrict attention to the conservative part (circular orbits) of the dynamics
2. Find a gauge-invariant observable computable in both formalisms

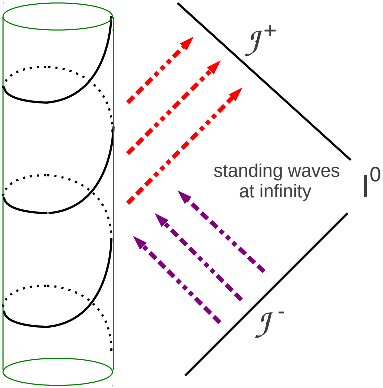
Circular orbit means Helical Killing symmetry



Looking at the conservative part of the dynamics



Physical situation



Situation with the HKV

Choice of a gauge-invariant observable [Detweiler 2008]

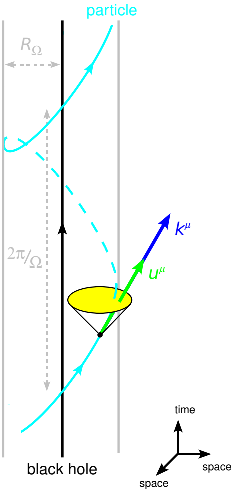
1. For exactly circular orbits the geometry admits a helical Killing vector with

$$K^\mu \partial_\mu = \partial_t + \Omega\, \partial_\varphi \quad \text{(asymptotically)}$$

2. The four-velocity of the particle is necessarily tangent to the Killing vector hence

$$K_1^\mu = z_1\, u_1^\mu$$

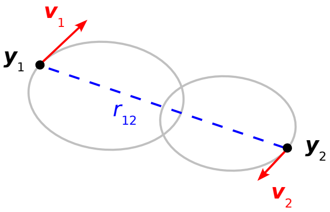
3. This z_1 is the Killing energy of the particle associated with the HKV and is also a redshift
4. The relation $z_1(\Omega)$ is well-defined in both PN and SF approaches and is gauge-invariant



Post-Newtonian calculation of the redshift factor

In a coordinate system such that $K^\mu \partial_\mu = \partial_t + \Omega\, \partial_\varphi$ everywhere this invariant quantity reduces to the zero-th component of the particle's four-velocity,

$$u_1^t = \frac{1}{z_1} = \Bigg(- \underbrace{(g_{\mu\nu})_1}_{\text{regularized metric}} \frac{v_1^\mu v_1^\nu}{c^2} \Bigg)^{-1/2}$$



One needs a self-field regularization

- Hadamard regularization will yield an ambiguity at 3PN order
- Dimensional regularization will be free of any ambiguity at 3PN order

High-order PN result for the redshift factor

[Blanchet, Detweiler, Le Tiec & Whiting 2010ab]

- The redshift factor of particle 1 through 3PN order and augmented by 4PN and 5PN logarithmic terms is

$$
\begin{aligned}
u_1^t =& 1 + \left(\frac{3}{4} - \frac{3}{4}\sqrt{1-4\nu} - \frac{\nu}{2}\right) x + \overbrace{[\cdots]\, x^2}^{\text{1PN}} + \overbrace{[\cdots]\, x^3}^{\text{2PN}} + \overbrace{[\cdots]\, x^4}^{\text{3PN}} \\
& + \Big(\cdots \underbrace{+ [\cdots]\, \nu \ln x}_{\text{4PN log}}\Big)\, x^5 + \Big(\cdots \underbrace{+ [\cdots]\, \nu \ln x}_{\text{5PN log}}\Big)\, x^6 + \mathcal{O}\left(x^7\right)
\end{aligned}
$$

where we pose $\nu = m_1 m_2 / m^2$ and $x = (G m \Omega / c^3)^{3/2}$

- The logarithms are due to the (conservative part of) radiation reaction tails

High-order PN result for the redshift factor

[Blanchet, Detweiler, Le Tiec & Whiting 2010ab]

- We re-expand in the small mass-ratio limit $q = m_1/m_2 \ll 1$ so that

$$u^t = u^t_{\text{Schw}} + \underbrace{q\, u^t_{\text{SF}}}_{\text{self-force}} + \underbrace{q^2\, u^t_{\text{PSF}}}_{\text{post-self-force}} + \mathcal{O}(q^3)$$

- Posing $y = \left(\frac{G m_2 \Omega}{c^3}\right)^{3/2}$ we find

$$u^t_{\text{SF}} = -y - 2y^2 - 5y^3 + \overbrace{\left(-\frac{121}{3} + \frac{41}{32}\pi^2\right) y^4}^{\text{3PN}} + \underbrace{\left(a_4 - \frac{64}{5}\ln y\right) y^5}_{\text{4PN}} + \underbrace{\left(a_5 - \frac{956}{105}\ln y\right) y^6}_{\text{5PN}} + \mathcal{O}(y^7)$$

High-order PN fit to the numerical self-force

[Blanchet, Detweiler, Le Tiec & Whiting 2010ab]

- The 3PN prediction agrees with the GSF value with 7 significant digits

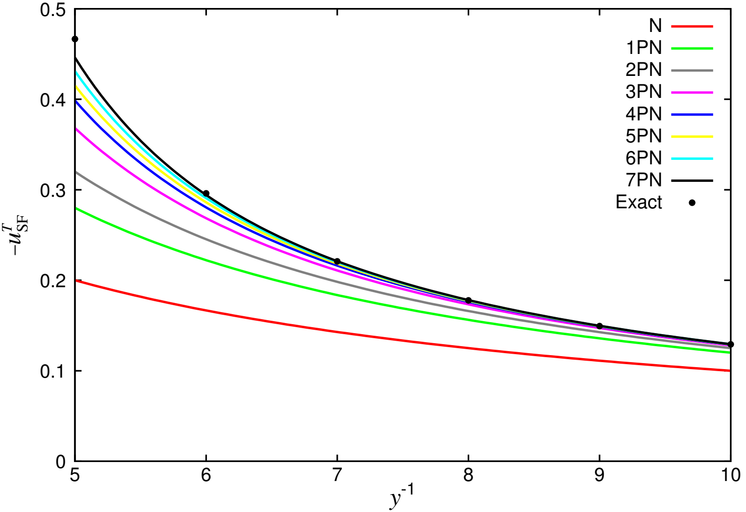
3PN value	GSF fit
$a_{3\mathrm{PN}} = -\frac{121}{3} + \frac{41}{32}\pi^2 = -27.6879026\cdots$	$-27.6879034 \pm 0.0000004$

- Post-Newtonian coefficients are fitted up to 7PN order

PN coefficient	GSF value
$a_{4\mathrm{PN}}$	$-114.34747(5)$
$a_{5\mathrm{PN}}$	$-245.53(1)$
$a_{6\mathrm{PN}}$	$-695(2)$
$b_{6\mathrm{PN}}$	$+339.3(5)$
$a_{7\mathrm{PN}}$	$-5837(16)$

High-order PN fit to the numerical self-force

[Blanchet, Detweiler, Le Tiec & Whiting 2010ab]



More recent developments

1. 4PN coefficient also known analytically [Bini & Damour 2013]

$$a_{4\mathrm{PN}} = -\frac{1157}{15} + \frac{677}{512}\pi^2 - \frac{256}{5}\ln 2 - \frac{128}{5}\gamma_\mathrm{E}$$

   and agrees with previous numerical value [Le Tiec, Blanchet & Whiting 2012]

2. Super-high precision analytical and numerical GSF calculations of the redshift factor up to 10PN order [Shah, Friedman & Whiting 2013]

3. Alternative approach to GSF calculations [Bini & Damour 2014] based on the post-Minkowskian expansion of the RWZ equation [Mano, Susuki & Takasuki 1996]

Analytically known GSF terms [Shah, Friedman & Whiting 2014]

In addition to super-high precision numerical high-order terms we have

$$\begin{aligned}
u^t_{\mathrm{SF}} = & - y - 2y^2 - 5y^3 + \left(-\frac{121}{3} + \frac{41}{32}\pi^2\right) y^4 \\
& + \left(-\frac{1157}{15} + \frac{677}{512}\pi^2 - \frac{128}{5}\gamma_{\mathsf{E}} - \frac{64}{5}\ln(16y)\right) y^5 \\
& - \frac{956}{105} y^6 \ln y - \frac{13696\pi}{525} y^{13/2} - \frac{51256}{567} y^7 \ln y + \frac{81077\pi}{3675} y^{15/2} \\
& + \frac{27392}{525} y^8 \ln^2 y + \frac{82561159\pi}{467775} y^{17/2} - \frac{27016}{2205} y^9 \ln^2 y \\
& - \frac{11723776\pi}{55125} y^{19/2} \ln y - \frac{4027582708}{9823275} y^{10} \ln^2 y \\
& + \frac{99186502\pi}{1157625} y^{21/2} \ln y + \frac{23447552}{165375} y^{11} \ln^3 y + \cdots
\end{aligned}$$

Notice the occurence of half-integral PN terms starting at 5.5PN order

Half-integral conservative PN terms

[Blanchet, Faye & Whiting 2014ab]

1. Half-integral conservative PN terms (of type $\frac{n}{2}$PN) that are instantaneous are in fact zero for circular orbits

$$(z_1)_{\text{inst}} \sim \sum_{j,k,p,q} \nu^j \left(\frac{Gm}{rc^2}\right)^k \left(\frac{\boldsymbol{v}^2}{c^2}\right)^p \left(\frac{\boldsymbol{n}\cdot\boldsymbol{v}}{c}\right)^q$$

2. They come from hereditary-type (non-local-in-time) integrals and their first occurence is due to tail-of-tail multipole interactions

$$M \times M \times M_{ij}$$

arising precisely at the 5.5PN order

Half-integral conservative PN terms

[Blanchet, Faye & Whiting 2014ab]

1. We have to solve many d'Alembertian equations of the type

$$\Box h \sim \sum \frac{G^3 M^2}{c^n r^k} \int_1^{\infty} \mathrm{d}x \, Q_m(x) \, M_L^{(a)}(t - rx/c)$$

2. The solution in the near-zone $r \to 0$ reads [Blanchet 1993]

$$h \quad \sim \quad \underbrace{\partial \left\{ \frac{G(t-r/c) - G(t+r/c)}{r} \right\}}_{\text{retarded-minus-advanced homogeneous solution}} \quad + \Box^{-1}_{\text{inst}} S$$

$$\text{where} \quad G(u) \quad \sim \quad \underbrace{\sum \frac{G^3 M^2}{c^n} \int_0^{\infty} \mathrm{d}\tau \, \ln \tau \, M_L^{(a)}(u - \tau)}_{\text{tail-of-tail integral}}$$

3. Only the homogeneous solution contribute to half-integral PN terms

Half-integral conservative PN terms

[Blanchet, Faye & Whiting 2014ab]

1. Split the dynamics into conservative and dissipative pieces and keep only the conservative part (neglecting readiation reaction dissipative effects)

$$G^{\text{cons}}(u) \sim \sum \frac{G^3 M^2}{c^n} \underbrace{\int_0^{\infty} \mathrm{d}\tau \, \frac{M_L^{(a)}(u-\tau) + M_L^{(a)}(u+\tau)}{2}}_{\text{symmetric-in-time integral}}$$

With that prescription one checks that the equations of motion are indeed conservative, i.e. that the acceleration is purely radial

2. The final result for the redshift factor is in full agreement with analytical and numerical GSF computations

$$\boxed{a_{5.5\text{PN}} = -\frac{13696}{525}\pi\,, \qquad a_{6.5\text{PN}} = \frac{81077}{3675}\pi\,, \qquad a_{7.5\text{PN}} = \frac{82561159}{467775}\pi}$$

Conclusions

1. Compact binary star systems are the most important source for gravitational wave detectors LIGO/VIRGO and eLISA
2. Post-Newtonian theory has proved to be the appropriate tool for describing the inspiral phase of compact binaries up to the ISCO
3. For massive BH binaries the PN templates should be matched to the results of numerical relativity for the merger and ringdown phases
4. The PN approximation is now tested against different approaches such as the perturbative GSF and performs extremely well