Mukhanov-Sasaki equations in Hybrid Loop Quantum Cosmology

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MOTIVATION

The theory of **cosmological perturbations**, combined with the **inflationary** paradigm, conciliates the theoretical models of the early universe with observations:

- ✓ Anisotropies of the cosmic microwave background (CMB)
- ✓ Formation of structures at large scales

Need for a quantum theory:

- Very quantum nature of the perturbations
- Singularity-free and consistent description of the evolution of the universe

HYBRID APPROACH

[Garay, Martín-Benito, Mena Marugán]

- Quantization of **inhomogeneous** cosmological systems.

- Strategy:

PHASE SPACE
- Homogeneous sector → Loop Quantum Cosmology
- Inhomogeneous sector → Fock quantization

- Assumption: main (loop) quantum effects on geometry are those concerning the homogeneous degrees of freedom of the background.

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Our purpose is to investigate whether it is possible to find information about the quantum nature of the geometry of spacetime in the quantum fluctuations in the early universe.

HYBRID APPROACH

QUANTUM GRAVITY

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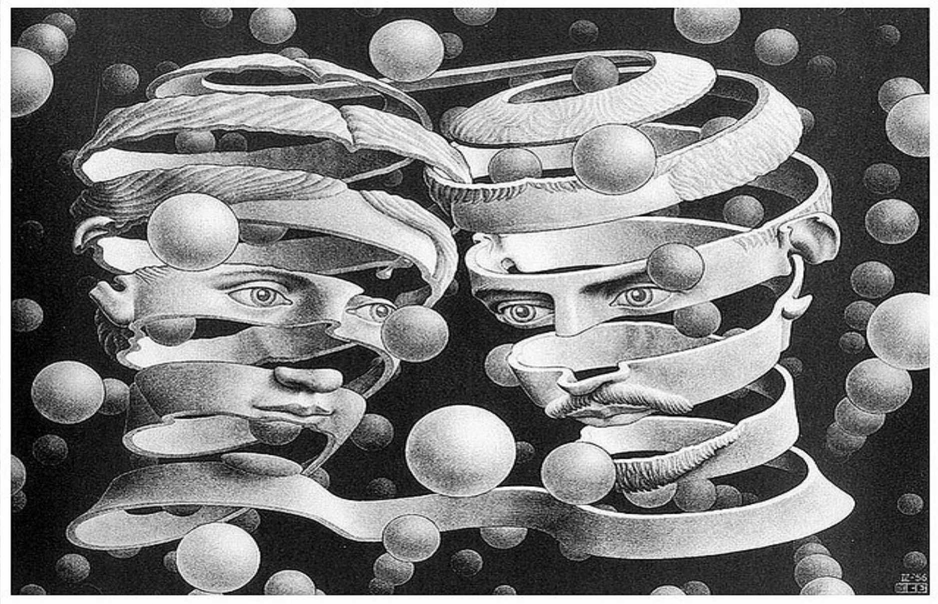

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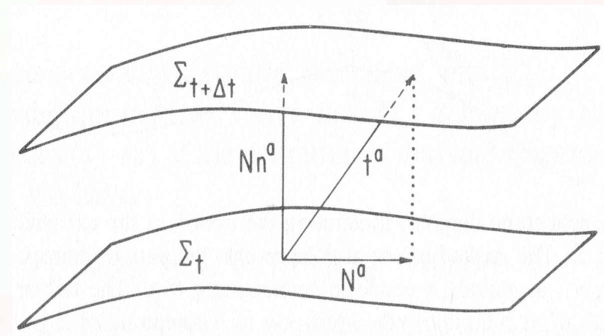
QUANTUM FIELD THEORY
on curved spacetime

LOOP QUANTUM COSMOLOGY

Adresses the quantization of cosmological systems following the ideas and techniques of Loop Quantum Gravity, a **canonical**, **non-perturvative** and **background independent** program for the quantization of general relativity (GR):

- Starting from a Hamiltonian formalism of GR.



- Classical phase space: **Ashtekar-Barbero variables**.

$$E^a_i = \sqrt{h}\, e^a_i\,; \qquad A^i_a = \Gamma^i_a + \gamma K^i_a \longrightarrow \{A^a_i(x), E^j_b(y)\} = 8\pi G \gamma \delta^a_b \delta^j_i \delta^3(x-h)$$

  - ☆ Triads allow the coupling of fermions.

- Dirac's canonical quantization programme for constrained systems.

Avoids the Big Bang

PERTURBED FLRW

Scalar perturbations in:

FLRW universe + massive coupled scalar field

Approximation:

Truncation at quadratic perturbative order in the action

- Compact flat spatial topology T^3.
- Scalar field is subject to a quadratic potential.

PERTURBED FLRW

- Mode expansion of the inhomogeneities: metric and field.

- Adopt the real modes of the Laplace-Beltrami operator compatible with the metric.

- We call $g_{\vec{n},\pm}$ and $k_{\vec{n},\pm}$ the (properly scaled) Fourier coefficients of the lapse function and shift vector.

- Truncated action at quadratic order:

$$H = N_0\left[H_{|0} + \sum H^{\vec{n},\pm}_{|2}\right] + \sum g_{\vec{n},\pm}\,\widetilde{H}^{\vec{n},\pm}_{|1} + \sum k_{\vec{n},\pm}\,\widetilde{H}^{\vec{n},\pm}_{\_1}$$

COVARIANT DESCRIPTION

- Canonical transformation for perturbations:

$$\{X_l^{\vec{n},\pm}\} \equiv \{a_{\vec{n},\pm}, b_{\vec{n},\pm}, f_{\vec{n},\pm};\ \pi_{a_{\vec{n},\pm}}, \pi_{b_{\vec{n},\pm}}, \pi_{f_{\vec{n},\pm}}\} \longrightarrow \{V_l^{\vec{n},\pm}\} \equiv \{\boldsymbol{v}_{\vec{n},\pm}, C_{|1}^{\vec{n},\pm}, C_{\_1}^{\vec{n},\pm};\ \pi_{v_{\vec{n},\pm}}, \breve{\boldsymbol{H}}_{|1}^{\vec{n},\pm}, \widetilde{\boldsymbol{H}}_{\_1}^{\vec{n},\pm}\}$$

- **Mukhanov-Sasaki variable** (gauge-invariant)

$$v_{\vec{n},\pm} = e^{\alpha}\left[f_{\vec{n},\pm} + \frac{\pi_\varphi}{\pi_\alpha}\left(a_{\vec{n},\pm} + b_{\vec{n},\pm}\right)\right]$$

- **Abelianize the algebra of constraints**

$$\breve{H}_{|1}^{\vec{n},\pm} = \widetilde{H}_{|1}^{\vec{n},\pm} - 3e^{3\alpha} H_{|0}\, a_{\vec{n},\pm}$$

- Canonical transformation for homogeneous sector:

$$\{q_A;\ \pi_{q_A}\} \equiv \{\alpha, \varphi;\ \pi_\alpha, \pi_\varphi\} \longrightarrow \{\tilde{q}_A;\ \tilde{\pi}_{q_A}\} \equiv \{\tilde{\alpha}, \tilde{\varphi};\ \tilde{\pi}_\alpha, \tilde{\pi}_\varphi\}$$

$$q_A = \tilde{q}_A - \frac{1}{2}\sum\left[X_{ql}^{\vec{n},\pm}\frac{\partial X_{pl}^{\vec{n},\pm}}{\partial \pi_{\tilde{q}_A}} - \frac{\partial X_{ql}^{\vec{n},\pm}}{\partial \pi_{\tilde{q}_A}} X_{pl}^{\vec{n},\pm}\right], \qquad \pi_{q_A} = \tilde{\pi}_{q_A} + \frac{1}{2}\sum\left[X_{ql}^{\vec{n},\pm}\frac{\partial X_{pl}^{\vec{n},\pm}}{\partial \tilde{q}_A} - \frac{\partial X_{ql}^{\vec{n},\pm}}{\partial \tilde{q}_A} X_{pl}^{\vec{n},\pm}\right].$$

Symplectic struture of the total system is preserved

The Hamiltonian constraint in the new formulation: $H_{|0}\left(\tilde{q}_A, \tilde{\pi}_{q_A}\right) + \sum \widetilde{H}^{\vec{n},\pm}_{|2}\left(\tilde{q}_A, \tilde{\pi}_{q_A}, V^{\vec{n},\pm}_{l}\right)$

where

$$\widetilde{H}^{\vec{n},\pm}_{|2} = H^{\vec{n},\pm}_{|2}\left(\tilde{q}_A, \tilde{\pi}_{q_A}, V^{\vec{n},\pm}_{l}\right) + \sum \left[\left(q_A - \tilde{q}_A\right)\frac{\partial H_{|0}}{\partial \tilde{q}_A} + \left(\pi_{q_A} - \tilde{\pi}_{q_A}\right)\frac{\partial H_{|0}}{\partial \tilde{\pi}_{q_A}}\right]$$

$$= \breve{H}^{\vec{n},\pm}_{|2} + F^{\vec{n},\pm}_{|2} H_{|0} + F^{\vec{n},\pm}_{|1} \breve{H}^{\vec{n},\pm}_{|1} + \left(F^{\vec{n},\pm}_{\_1} - 3\frac{e^{-3\tilde{\alpha}}}{\tilde{\pi}_{\alpha}} \breve{H}^{\vec{n},\pm}_{|1} + \frac{9}{2} e^{-3\tilde{\alpha}} \widetilde{H}^{\vec{n},\pm}_{\_1}\right) \widetilde{H}^{\vec{n},\pm}_{\_1}$$

Hamiltonian :

$$H = \overline{N}_0\left[H_{|0} + \sum \breve{H}^{\vec{n},\pm}_{|2}\right] + \sum G_{\vec{n},\pm} \breve{H}^{\vec{n},\pm}_{|1} + \sum K_{\vec{n},\pm} \widetilde{H}^{\vec{n},\pm}_{\_1}$$

HAMILTONIAN

The Hamiltonian constraint in the new formulation: $H_{|0}\left(\tilde{q}_A, \tilde{\pi}_{q_A}\right) + \sum \widetilde{H}^{\vec{n},\pm}_{|2}\left(\tilde{q}_A, \tilde{\pi}_{q_A}, V^{\vec{n},\pm}_{l}\right)$

where

$$\widetilde{H}^{\vec{n},\pm}_{|2} = H^{\vec{n},\pm}_{|2}\left(\tilde{q}_A, \tilde{\pi}_{q_A}, V^{\vec{n},\pm}_{l}\right) + \sum \left[\left(q_A - \tilde{q}_A\right)\frac{\partial H_{|0}}{\partial \tilde{q}_A} + \left(\pi_{q_A} - \tilde{\pi}_{q_A}\right)\frac{\partial H_{|0}}{\partial \tilde{\pi}_{q_A}}\right]$$

$$= \breve{H}^{\vec{n},\pm}_{|2} + F^{\vec{n},\pm}_{|2} H_{|0} + F^{\vec{n},\pm}_{|1} \breve{H}^{\vec{n},\pm}_{|1} + \left(F^{\vec{n},\pm}_{-1} - 3\frac{e^{-3\tilde{\alpha}}}{\tilde{\pi}_{\alpha}}\breve{H}^{\vec{n},\pm}_{|1} + \frac{9}{2}e^{-3\tilde{\alpha}}\widetilde{H}^{\vec{n},\pm}_{-1}\right)\widetilde{H}^{\vec{n},\pm}_{-1}$$

Hamiltonian :

$$H = \overline{N}_0\left[H_{|0} + \sum \breve{H}^{\vec{n},\pm}_{|2}\right] + \sum G_{\vec{n},\pm}\breve{H}^{\vec{n},\pm}_{|1} + \sum K_{\vec{n},\pm}\widetilde{H}^{\vec{n},\pm}_{-1}$$

$$\breve{H}^{\vec{n},\pm}_{|2} = \frac{e^{-\tilde{\alpha}}}{2}\left\{\left[\omega_n^2 + e^{-4\tilde{\alpha}}\pi^2_{\tilde{\alpha}} + \tilde{m}^2 e^{2\tilde{\alpha}}\left(1 + 15\tilde{\varphi}^2 - 12\tilde{\varphi}\frac{\pi_{\tilde{\varphi}}}{\pi_{\tilde{\alpha}}} - 18\tilde{m}^2 e^{6\tilde{\alpha}}\frac{\tilde{\varphi}^4}{\pi^2_{\tilde{\alpha}}}\right)\right]\left(v_{\vec{n},\pm}\right)^2 + \left(\pi_{v_{\vec{n},\pm}}\right)^2\right\}$$

MS Hamiltonian

QUANTIZATION

- Quantum representation of the constrains:

  Linear perturbative constraints are represented by momenta operators that act as derivatives.

  → Physical states only depend on the homogeneous variables and the MS gauge invariants

  Hilbert space: $H_{kin}^{FRW-LQC} \otimes H_{kin}^{matt} \otimes \mathscr{F}$.

- We must still impose the **scalar constraint** given by: $e^{-3\tilde{\alpha}}\left(H_{|0} + \breve{H}_{|2}\right) = 0$

- Hybrid quantization: Fock representation for the inhomogeneous sector

  Annihilation and **creation operators** for our (rescaled) MS gauge invariantes and naturally associated with the massless scalar field.

CLASSICAL SYSTEM IN LQC

Massive scalar field minimally coupled to a compact, flat FLRW universe:

Geometry: $A^i_a = c\,{}^0e^i_a(2\pi)^{-1};\quad E^a_i = p\sqrt{{}^0e}\,{}^0e^a_i(2\pi)^{-2}\qquad \{c, p\} = 8\pi G\gamma/3$

$$a^2 = e^{2\alpha} = |p|(2\pi\sigma)^{-2};\quad \pi_\alpha = -pc(\gamma 8\pi^3\sigma^2)^{-1}$$

Matter: $\varphi = (2\pi)^{3/2}\sigma\phi;\quad \pi_\varphi = (2\pi)^{-3/2}\sigma^{-1}\pi_\phi$

Hamiltonian:

$$C_0 = -\frac{6}{\gamma^2}\sqrt{|p|}c^2 + \frac{8\pi G}{V}(\pi_\phi^2 + m^2V^2\phi^2)$$

$$\sigma^2 = G(6\pi^2)^{-1}$$

$$V = |p|^{3/2}$$

QUANTIZATION: HOMOGENEOUS SECTOR

The kinematic **Hilbert space** is $H_{kin}^{FRW-LQC} \otimes H_{kin}^{matt}$.

For the FLRW-geometry sector:

- Adopt a basis of **volume eigenstates** $\{|v\rangle ; v \in \mathbb{R}\}$, with $\hat{v} \propto |\hat{p}|^{3/2}$.
- The inner product is **discrete**: $\forall v_1, v_2 \in \mathbb{R}$, $\langle v_1 | v_2 \rangle = \delta_{v_2}^{v_1}$.
- On straight edges, holonomy elements are linear in $N_{\bar{\mu}} := e^{i \bar{\mu} c/2}$.
- We adhere to the so-called **improved dynamics** prescription of LQC

→ Δ, minimum non-vanishing area

$$\hat{N}_{\bar{\mu}} |v\rangle := |v+1\rangle, \quad \hat{v}|v\rangle = v|v\rangle$$

QUANTIZATION: HOMOGENEOUS SECTOR

- The inverse volume is regularized as usual in LQC.

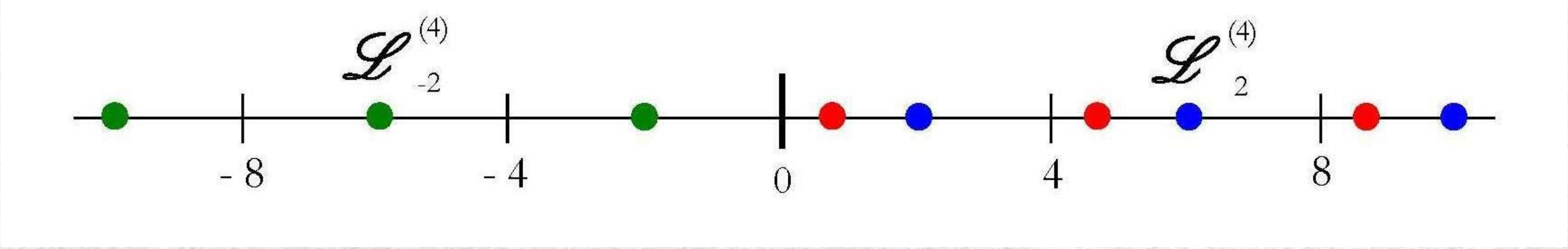
- After **decoupling the zero-volume** state, we change the constraint densitization

$$\hat{C}_0 = \left[\widehat{\frac{1}{V}}\right]^{1/2} \hat{\mathcal{C}}_0 \left[\widehat{\frac{1}{V}}\right]^{1/2} \qquad \hat{\mathcal{C}}_0 = -\frac{3}{4\pi G \gamma^2}\hat{\Omega}_0^2 + \hat{\pi}_\phi^2 + m^2 \hat{\phi}^2 \hat{V}^2$$

- With **our proposal**, the gravitational part is a difference operator:

$$\widehat{\Omega_0^2}|v\rangle = f_+(v)|v+4\rangle + f(v)|v\rangle + f_-(v)|v-4\rangle$$

that acts on the **superselection sectors** $\mathscr{L}_{\pm\epsilon}^{(4)} := \{\pm(\epsilon + 4n),\ \ n \in \mathbb{N}\},\ \epsilon \in (0,4].$



HYBRID QUANTIZATION

- To quantize the quadratic contribution of the perturbations to the Hamiltonian we adapt the **proposals of the homogeneous sector** and use a symmetric factor ordering:

  - **Symmetrized products** of the type $f(\hat{\tilde{\varphi}})\hat{\pi}_{\tilde{\varphi}}$.
  - **Symmetric geometric** factor ordering $V^k A \rightarrow \hat{V}^{k/2}\hat{A}\hat{V}^{k/2}$.
  - Adopting the LQC representation $(cp)^{2m} \rightarrow \left[\hat{\Omega}_0^2\right]^m$.
  - In order to preserve the FLRW **superselection sectors**, we adopt the prescription $(cp)^{2m+1} \rightarrow \left[\hat{\Omega}_0^2\right]^{m/2}\hat{\Lambda}_0\left[\hat{\Omega}_0^2\right]^{m/2}$, where $\hat{\Lambda}_0$ is defined like $\hat{\Omega}_0$ but with double steps.

- The Hamiltonian constraint reads then $\hat{C}_0 - \sum \hat{\Theta}_e^{\vec{n},\pm} - \sum (\hat{\Theta}_o^{\vec{n},\pm}\hat{\pi}_{\tilde{\varphi}})_{sym} = 0.$

BORN-OPPENHEIMER ANSATZ

- Consider states whose evolution in the inhomogeneities and the FLRW geometry presents different rates of variation:

$$\Psi=\chi_0(\tilde{\alpha},\tilde{\varphi})\psi(N,\tilde{\varphi}), \qquad \chi_0(\tilde{\alpha},\tilde{\varphi})=\boldsymbol{P}\left[\exp\left(i\int_{\tilde{\varphi}_0}^{\tilde{\varphi}} d\varphi\ \hat{H}_0(\varphi)\right)\right]\chi_0(\tilde{\alpha}).$$

  - The FLRW state is peaked and evolves unitarily.

- Disregard **nondiagonal** elements for the FLRW geometry sector in the constraint.

$$-\partial_{\tilde{\varphi}}^2\psi-i\left(2\langle\hat{H}_0\rangle_\chi-\langle\hat{\Theta}_o\rangle_\chi\right)\partial_{\tilde{\varphi}}\psi=\left[\langle\hat{\Theta}_e+(\hat{\Theta}_o\hat{H}_0)_{sym}\rangle_\chi+i\left\langle \boldsymbol{d}_{\tilde{\varphi}}\hat{H}_0-\frac{1}{2}\boldsymbol{d}_{\tilde{\varphi}}\hat{\Theta}_o\right\rangle_\chi\right]\psi$$

where $\boldsymbol{d}_{\tilde{\varphi}}\hat{O}=\partial_{\tilde{\varphi}}\hat{O}-i\ [\hat{H}_0,\hat{O}]$.

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- If we can **neglect** :
  a) The second derivative of ψ,
  b) The total $\tilde{\varphi}$ -derivative of $2\hat{H}_0-\hat{\Theta}_o$,

$$-i\,\partial_{\tilde{\varphi}}\psi=\frac{\langle\hat{\Theta}_e+(\hat{\Theta}_o\hat{H}_0)_{sym}\rangle_\chi}{2\langle\hat{H}_0\rangle_\chi}\psi$$

Schrödinger-like equation

BORN-OPPENHEIMER ANSATZ

- There are **restrictions** on the range of validiy.
- The extra terms are negligible if so are the $\tilde{\varphi}$ -derivatives of

$$\langle \hat{H}_0 \rangle_\chi, \ \langle \hat{\Theta}_e \rangle_\chi, \ \langle \hat{\Theta}_o \rangle_\chi, \ \langle (\hat{H}_0 \hat{\Theta}_o)_{sym} \rangle_\chi.$$

- These derivatives contain two types of terms. One comes from the explicit dependence, and is proportional to powers of the **mass**.
- The other comes from commutators with $\hat{H}_0$ in the FLRW geometry.
- Contributions arising from $[\hat{\Omega}_0^{2,} \ \hat{V}]$ can be relevant. In the effetive regime.

EFFECTIVE MUKHANOV-SASAKI EQUATIONS

- Starting from the **Born-Oppenheimer** form of the constraint and assuming a direct **effective** counterpart for the **inhomogeneities**:

$$d^2_{\eta_\chi} v_{\vec{n},\pm} = -v_{\vec{n},\pm}\left[\tilde{\omega}_n^2 + \langle \hat{\theta}_{e,(v)} + \hat{\theta}_{o,(v)} \rangle_\chi\right]$$

where

$$\hat{\theta}_o = \sqrt{\frac{12G}{\pi}}\, 2\gamma m^2 \hat{\varphi} \hat{V}^{2/3} |\hat{\Omega}_0|^{-1} \hat{\Lambda}_0 |\hat{\Omega}_0|^{-1} \hat{V}^{2/3},$$

$$\hat{\theta}_e = \frac{3}{2G} \hat{V}^{2/3},$$

$$\hat{\theta}_e^q = \frac{2G}{3} \left[\frac{\hat{1}}{V}\right]^{1/3} \hat{H}_0^{(2)} \left(19 - 32\pi^2 G^2 \gamma^2 \hat{\Omega}_0^{-2} \hat{H}_0^{(2)}\right) \left[\frac{\hat{1}}{V}\right]^{1/3}$$

$$+ \frac{3m^2}{8\pi^2 G} \hat{V}^{4/3} \left(1 - \frac{8\pi G}{3} \hat{\varphi}^2\right).$$

CONCLUSIONS

- We have considered the **hybrid loop quantization** of a FLRW universe with a massive scalar field perturbed at quadratic order in the action.

- The system is a constrained symplectic manifold. Backreaction is included at the considered truncation order.

- The model has been described in terms of the Mukhanov-Sasaki gauge-invariant.

- A Born-Oppenheimer ansatz leads to a Schrödinger equation for the inhomogeneities. We have discussed the range of validity.

- Finally, we have derived the effective **Mukhanov-Sasaki equations**. The ultraviolet regime is hyperbolic.