Black Holes — The First 99 Years

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ABSTRACT

It has been 99 years since Schwarzschild constructed his eponymous solution in Einstein's theory of gravity. It is suprising that exact solutions can exist in such a nonlinear theory. In this talk, we shall review the tremendous progress made in the past 99 years, focusing on exact local solutions. We also discuss the current problems and possible future directions.

8:30-9:15am, August 11, 2015

@ICISE, Quy Nhon, Viet-Nam

Motivation

2015 marks the centennial of Einstein's General Theory of Relativity.

In Chinese tradition, possibly in some other asian traditions as well, one celebrates the 9'th rather than the 10'th.

It is thus appropriate to celebrate black holes, whose first solution was published 99 years ago, by Karl Schwarzschild.

Karl Schwarzschild died in May 11, 1916. (Ref: Wikipedia)

Scope of the talk

- I shall focus only on exact solutions.

  – In all branches of physics, exact solutions play the most important role in understanding and developing the concepts.

  – Numerical progress is much harder to summarize in less than 40 minutes.

- There is recently a book called "Exact solutions of Einstein's field equations," but

- I shall talk about mainly black holes and black hole related topics.

- I shall also focus on pure gravity. When matter is involved, it is string or supergravity matter, or at least has a Lagrangian formulation.

- Also, since in general relativity, we typically do not know where we are and what time it is, so I shall only present the local solutions. (The global analysis deserves a special talk for each black hole.)

$\boxed{\text{outline}}$

- Review of past work (30mins)
- My recent work (15mins)
  - Black hole formation
  - Black holes in $D = 4$ higher-derivative gravity

$\boxed{\text{Einstein Theory of Gravity(1915)}}$

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$G_{\mu\nu}$ is called Einstein tensor. Vacuum solution $G_{\mu\nu} = 0$.

To be specific, consider $ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu , \quad x^\mu = \{t, x, y, z\}$

$$\begin{aligned}
g^{\mu\nu} : &\qquad g_{\mu\lambda} g^{\nu\lambda} = \delta^\nu_\mu \,, \\
\Gamma^\rho{}_{\mu\nu} &= \tfrac{1}{2} g^{\rho\lambda} (\partial_\nu g_{\lambda\mu} + \partial_\mu g_{\lambda\nu} - \partial_\lambda g_{\mu\nu}) \,, \\
R^\rho{}_{\sigma\mu\nu} &= \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda} \Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda} \Gamma^\lambda{}_{\mu\sigma} \,, \\
R_{\mu\nu} &= R^\rho{}_{\mu\rho\nu} \,, \qquad\qquad R = R^\mu{}_\mu \,.
\end{aligned}$$

Thus we see that Einstein's gravity is described by a set of very complicated non-linear differential equations, involving six independent functions ($\frac{1}{2}(4 \times 5) - 4 = 6$) and four variables (t, x, y, z).

At first sight, one does not expect any non-trivial solutions.

Trivial solution: Minkowski spacetime $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$. In other words, $g_{\mu\nu}$ is constant in the Cartesian coordinates, and hence all the connection $\Gamma^\rho{}_{\mu\nu}$, Riemann tensor $R^\mu{}_{\nu\rho\sigma}$, Ricci tensor $R_{\mu\nu}$ and Ricci scalar R all vanish. Einstein's vacuum equation of motion is then automatically satisfied.

Static and spherically-symmetric black hole

Schwarzschild (1916): (Spherically symmetric and static)

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2$$

$$f = 1 - \frac{2m}{r}, \qquad d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$$

This is a vacuum solution $G_{\mu\nu} = 0$.

How come vacuum has a black hole?

Also, what is this solution good for? Even if there exist black holes in our universe, there is no evidence of black holes on the earth or in our solar system.

Vacuum versus vacuum solution

These concepts are analogous to those in electromagnetism (EM). Vacuum solution is not the same as vacuum; the former is a local concept whilst the latter is a global one.

Vacuum solution involves localized matter.

Minkowski spacetime is a vacuum of Einstein gravity, whilst the Schwarzschild black hole is a vacuum solution.

As in EM, the electric potential outside a spherical ball of uniformly-distributed charge is the same as that created by a point charge of equal total charge.

The gravitational field outside the Sun is the same as that created by the black hole of the same mass (and angular momentum).

Thus the Schwarzschild black hole solution can be used to test General Relativity, even if there is no black hole in the solar system.

Back to the Schwarzschild metric

$$
\begin{aligned}
ds^2 &= -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2 \,, \\
f &= 1 - \frac{2m}{r} \,, \qquad d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2 \,.
\end{aligned}
$$

Since when $r \to +\infty$, $f \to 1$, the solution is asymptotic to flat Minkowski spacetime. The metric has two singularities, one is at $r = r_0 = 2m$, and the other is at $r = 0$. The former gives rise to the event horizon and it is a consequence of coordinate choice, whilst the latter corresponds to the true spacetime singularity.

An interesting property

$$
-f dt^2 + \frac{dr^2}{f} = -f(dt^2 - \frac{dr^2}{f^2}) = -f(dt + \frac{dr}{f})(dt - \frac{dr}{f})
$$

Define $du = dt + \frac{dr}{f}$, we have

$$
\begin{aligned}
ds^2 &= 2du dr - du^2 + r^2 d\Omega_2^2 + \frac{2m}{r} du^2 \\
&= -d\tilde{t}^2 + dr^2 + r^2 d\Omega_2^2 + \frac{2m}{r}(d\tilde{t} + dr)^2 \,,
\end{aligned}
$$

where $u \to \tilde{t} + r$. In other words, a black hole is a linear perturbation of the Minkowski spacetime.

Some important black hole properties

Definition: having an (enclosing) event horizon.

- Black hole has inevitable singularity, indicating that the general relativity is not classically complete. (Interestingly, Newtonian gravity has no such a problem, if mass is assumed to be proportional to volume.)
- The singularity is hidden within an event horizon.
- There is Hawking radiation due to the (semi-classical) quantum effect (Temperature $\sim \hbar$, entropy $\sim 1/\hbar$.)
- Black hole no-hair theorem: The properties outside the horizon is completely specified by the conserved quantities such as mass, angular momentum and charges. suggesting it is the purest thermal system.
- The first law of black hole thermodynamics : $dM = TdS + \Omega dJ + \cdots$.

Hawking radiation

Temperature:

$$T = \frac{\hbar c^3}{8\pi G M k_B} \approx \frac{1.227 \times 10^{23}\text{kg}}{M} K$$

Black hole with mass of the Sun: $T \approx 10^{-7} K$.
LHC black hole:
(energy released by nuclear bomb: $M \sim 10 g$, $r_s \sim 10^{-25} m$, $T \sim 10^{27} K$.)

Life time:

$$t = \frac{5120\pi G^2 M_0^3}{\hbar c^4} \approx 8.671 \frac{M_0^3}{M_p^3} \times 10^{-40} s$$

M_p Planck mass。
Black hole with mass of the Sun: $t \approx 10^{74} s$。
LHC mass: $t << 10^{-22} s$.

Even if Hawking is wrong and there is no Hawking radiation, microscopic black hole created by LHC is rather harmless and they are as much inert as dark matter. In fact, if it were not for Hawking radiation, microscopic black holes are very good candidates for dark matter.

Schwarzschild-(A)dS

Schwarzschild black hole is asymptotic flat, but our universe appears to have a cosmological constant.

Maximal symmetric spacetime in Einstein theory is Minkowski

Maximal symmetric spacetime in Einstein theory with a cosmological constant is (Anti-)de Sitter or (A)dS.

Schwarzschild-(A)dS:

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2$$

$$f = 1 - \tfrac{1}{3}\Lambda r^2 - \frac{2m}{r}$$

The cosmological constant in our universe is too small to be testable within our solar system.

But the Sun is Rotating

Effect of rotation on spacetime

In Newtonian gravity, the gravitational field created by the Sun (assuming its spherically symmetric) is independent of its rotation.

Einstein theory predicts that the rotation of the matter can drag the spacetime around it.

This of course provides a test to distinguish the two theories.

Rotating Black Hole: Kerr Solution

Roy Kerr metric (1963): $G_{\mu\nu} = 0$

$$
\begin{aligned}
ds^2 \;=\; & \rho^2\Big(\frac{dr^2}{\Delta_r} + d\theta^2\Big) + \frac{\sin^2\theta\,\Delta_\theta}{\rho^2}\Big(a dt - (r^2 + a^2) d\phi\Big)^2 \\
& -\frac{\Delta_r}{\rho^2}\Big(dt - a\sin^2\theta\, d\phi\Big)^2 ,
\end{aligned}
$$

$$
\rho^2 = r^2 + a^2\cos^2\theta\,, \qquad \Delta_r = r^2 + a^2 - 2mr
$$

- Mass: $M = m$
- Angular momentum: $J = ma$
- $J \le M^2$
- Extremal limit $|J| = M^2$

The metric is asymptotic flat. Note that when $M = 0$, the metric is Minkowski spacetime written in the rotating coordinates.

t: time; r: radial coordinate; θ: latitude $[0, \pi]$; ϕ: longitudinal $[0, 2\pi)$。

Black hole topology

The topology of a black hole is about the horizon shape. In four dimensional asymptotic Minkowski spacetime, a black hole horizon must have spherical topology. (Hawking, 1972.)

Schwarzschild black hole's horizon is a round sphere, whilst Kerr metric has elliptic shape.

The horizon ($r = r_0$) of the Kerr metric

$$ds_2^2 = (r_0^2 + a^2 \cos^2\theta) d\theta^2 + \frac{\sin^2\theta}{(r_0^2 + a^2\cos^2\theta)} (r_0^2 + a^2)^2 d\phi^2 \,.$$

Extremal case, $r_0 = a$. In general $r_0 \geq a$.

Rotating black hole?

Further on topology

Static black holes that are asymptotic to (A)dS spacetimes can have additional topologies

$$
\begin{aligned}
ds^2 &= -f^2dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2_{2,k}\,, \\
d\Omega^2_{2,k} &= \frac{du^2}{1-ku^2} + u^2 d\phi^2\,, \qquad\qquad k = 0, \pm 1\,,
\end{aligned}
$$

with

$$
f = -\frac{\Lambda}{3}r^2 + k - \frac{2m}{r}\,.
$$

$\boxed{\textbf{Kerr-(A)dS}}$

Carter (1968):

$$
\begin{aligned}
ds^2 \; &= \; \rho^2\Big(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\Big) + \frac{\sin^2\theta\,\Delta_\theta}{\rho^2}\Big(a dt - (r^2+a^2)\frac{d\phi}{\Xi}\Big)^2 \\
&\qquad\qquad\qquad\qquad - \frac{\Delta_r}{\rho^2}\Big(dt - a\sin^2\theta\frac{d\phi}{\Xi}\Big)^2 ,
\end{aligned}
$$

$$
\rho^2 = r^2 + a^2\cos^2\theta\,, \qquad\qquad \Xi = 1+\tfrac{1}{3}\Lambda a^2
$$

$$
\Delta_r = (r^2+a^2)(1-\tfrac{1}{3}\Lambda r^2) - 2mr
$$

$$
\Delta_\theta = 1 + \tfrac{1}{3}\Lambda a^2\cos^2\theta
$$

This is actually rather non-trivial to construct. If we let $m = 0$, the metric is (A)dS written in rotating coordinates.

$\boxed{\text{Generalize to higher dimensions}}$

- Asymptotic Minkowski: Meyer and Perry (1986)
- Asymptotic (A)dS: $D = 5$ Hawking, Hunter and Robinson (1998)
- Asymptotic (A)dS: Arbitrary D, Gibbons, L, Page and Pope (2004)

New topologies in higher dimensions

In higher dimensions, black objects can have new topologies other than spheres. For example, in five dimensions, in addition to S^3, the horizon topology can also be $S^2 \times S^1$. Such a solution is called black ring. The first example of Ricci-flat solution was obtained by Emparan and Reall. (Phys.Rev.Lett. 88 (2002) 101101)

How many such examples are there in higher dimensions? How to classify? What happens if one adds a cosmological constant? These are all interesting research projects, but after 13 years, all the progresses, albeit significant, are restricted to $D = 5$ asymptotic flat spacetime.

Generalization to Hawking's topology theorem: In higher dimensions the horizon is of the positive Yamabe type, i.e., admits metrics of positive scalar curvature. (Galloway, Shoen, gr-qc/0509107.)

Further topics

Kerr + NUT $\to$ Kerr-NUT or Plebanski metric $\to$ generalized to Kerr-AdS-NUT in general dimensions, Chen, L and Pope.

C-metric + Plebanski $\to$ Plebanski-Demianski metric (the most general type D metric) $\to$ generalized to $D = 5$ without a cosmological constant. L, Mei and Pope. Interestingly this metric contains the black ring limit.

What is the high-d generalization of Plebanski-Demianski metric?

Another related topic is that dS black hole + Euclideanization + BPS limit yield Einstein-Sasaki metrics of toric variety in odd dimensions. ($Y^{p,q}$, Gauntlett, Martelli, Sparks, Waldram; $L^{p,q,r}$ Cvetic, L, Page, Pope.)

Charged black holes

Einstein-Maxwell theory:

$$\mathcal{L} = \sqrt{-g}(R - F_{\mu\nu}F^{\mu\nu})\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,.$$

Reissner-Nordström (RN)black hole (1916-1918)

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2\,, \qquad f = 1 - \frac{2m}{r} + \frac{q^2}{r^2}\,,$$

with $A = \frac{q}{r}dt$.

With a cosmological constant

$$f \to -\tfrac{1}{3}\Lambda r^2 + 1 - \frac{2m}{r} + \frac{q^2}{r^2}\,.$$

This black hole has two parameters, mass m and charge q.

Charged and rotating: Kerr-Newman AdS

Charged AdS rotating black hole in four dimensions has long been known:

$$
\begin{aligned}
ds^2 &= \rho^2\Big(\frac{dr^2}{\Delta_r}+\frac{d\theta^2}{\Delta_\theta}\Big)+\frac{\Delta_\theta \sin^2\theta}{\rho^2}\Big(a\,dt-(r^2+a^2)\frac{d\phi}{\Xi}\Big)^2\\
&\quad -\frac{\Delta_r}{\rho^2}\Big(dt-a\sin^2\theta\frac{d\phi}{\Xi}\Big)^2\,,\\
A &= \frac{q\,r}{\rho^2}\Big(dt-a\sin^2\theta\frac{d\phi}{\Xi}\Big)+\frac{p\,\cos\theta}{\rho^2}\Big(a\,dt-(r^2+a^2)\frac{d\phi}{\Xi}\Big)\,,
\end{aligned}
$$

where,

$$
\rho^2=r^2+a^2\cos^2\theta\,,\qquad \Delta_\theta=1+\tfrac{1}{3}\Lambda a^2\cos^2\theta\,,\qquad \Xi=1+\tfrac{1}{3}\Lambda a^2\,,
$$
$$
\Delta_r=(r^2+a^2)(1-\tfrac{1}{3}\Lambda r^2)-2mr+p^2+q^2\,.
$$

Kerr-Newman (1965); Kerr-Newman-AdS is actually constructed by Carter in 1968.

Generalizing to Higher D

The RN black hole can be generalized easily to higher dimensions. However, within the framework of Einstein-Maxwell gravity, there is no known example of charged rotating black holes beyond four dimensions.

To be precise, black holes exist, but no known analytic solution.

Supergravity admits such a solution

Five dimensional Einstein-Maxwell gravity

$$\mathcal{L} = \sqrt{-g}(R - \tfrac{1}{4}F^2)\,.$$

Five-dimensional Einstein-Maxwell supergravity

$$\mathcal{L} = \sqrt{-g}(R - \tfrac{1}{4}F^2) + \tfrac{1}{12\sqrt{3}}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}F_{\rho\sigma}A_\lambda\,.$$

Five-dimensional Einstein-Maxwell gauged supergravity

$$\mathcal{L} = \sqrt{-g}(R - 2\Lambda - \tfrac{1}{4}F^2) + \tfrac{1}{12\sqrt{3}}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}F_{\rho\sigma}A_\lambda\,.$$

The most general charged rotating AdS black holes was constructed in 2005. Chong, Cvetic, L, Pope, hep-th/0506029.

Relevant references: Cvetic, Youm; Chong, Cvetic, L, Pope; Chow; Wu;...
Supersymmetric rotating solutions: Gutowski, Reall, Gauntlet, Martelli, etc.

Currently, the most able person in this type of construction is Professor Wu Shang-qing in China. Many new breakthrough was made by him recently.

Further interesting black holes

- Carrying Yang-Mills hair: Many BPS black holes in supergravities were constructed by Ortin et al.
- Carrying scalar hair: Many exact black holes that are asymptotic to flat or (A)dS spacetimes were recently constructed by many groups,e.g. Anabalon, Acena, Deruelle, Mann, Wen, Feng, L, etc.
- Black holes in cosmological backgrounds, Kastor and Trashen; Maiki and Shirashi; Maeda; Sabra; Gibbons, L, Pope; etc.
- AdS planar black holes...
- Lifshitz black holes...

Lifshitz spacetimes

$$ds^2 = \ell^2(-r^{2z}dt^2 + \frac{dr^2}{r^2} + r^2 dx^i dx^i)\,.$$

My recent work: Black hole formation

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2\,, \qquad f = 1 - \frac{2GM}{c^2 r}$$

Schwarzschild radius: $r_s = 2M$. (Using natural units $G = 1 = c$.)

Uniform mass distribution $M \propto r^3$

Thus matter can form a black hole even if it is in low density, as long as there are enough of them. (Example: Earth: $\rho \sim 5500 kg/m^3$. fixing ρ, $r_s \sim 1.7 \times 10^{11} m$, $M \sim 10^{38} kg$. Typical mass of a black hole $\geq 10 \times M_\odot$.)

Once an event horizon is formed, a singularity becomes inevitable inside the horizon, (at least in the framework of classical general relativity.)

Of course, the above discussion is static. Can the dynamical formation of a black hole described by an exact solution?

It is worth mentioning

Black holes that are formed due to gravitational collapse are necessarily big ones, with mass more than 10 times the mass of the Sun.

Microscopic black holes, if exist, were created at the early days of the universe when the temperature is extremely high.

Black hole formation can be described by Robinson-Trautman ansatz. (It is hard to imagine an exact solution from the general ansatz.)

Numerical analysis indicates that AdS spacetime is not stable and any finite perturbation in the asymptotic boundary can lead to formation on a black hole in the middle. (Bizon and Rostworowski; Buchel, Lehner and Liebling; Wu, etc.)

We would like to find an exact solution to demonstrate such instability.

$\boxed{\text{Vaidya metric}}$

Recall earlier-mentioned Kerr-Schild form

$$ds^2 = -f du(du \mp 2\frac{dr}{f}) + \cdots = \pm 2dudr - f du^2 + r^2 d\Omega_2^2 .$$

Vaidya metric

$$ds^2 = \pm 2dudr - \left(1 - \frac{2M(u)}{r}\right)du^2 + r^2 d\Omega_2^2 .$$

Energy-Momentum tensor $T_{uu} = \frac{2M'(u)}{r^2} \neq 0.$

$$T^{\mu\nu}T_{\mu\nu} = 0 \qquad \text{and} \qquad T^{\mu}{}_{\mu} = 0.$$

Pure radiative energy decaying or absorbtion.

Scalar-driven black hole formation

Einstein-Scalar theory: Fan and L, 1505.03557.

$$e^{-1}\mathcal{L}_n = \tfrac{n-2}{8(n-1)}(1-\phi^2)\,R - \tfrac{1}{2}(\partial\phi)^2 - V\,,$$
$$V = -\tfrac{1}{8}(n-2)^2\Big(g^2 + \alpha\phi^{\frac{2(n-1)}{n-2}}\big(\tfrac{1}{1-\phi^2} - {}_2F_1[1,\tfrac{n-1}{n-2};\tfrac{2n-3}{n-2};\phi^2]\big)\Big)\,.$$

The scalar has a fixed point $\phi = 0$ and hence the theory admits AdS vacuum

$$ds^2 = g^2r^2(-dt^2 + dx^i dx^i) + \frac{dr^2}{g^2r^2}\,,\qquad \phi = 0\,.$$

The scalar is conformally massless and satisfying the BF bound, and the AdS vacuum is therefore linearly stable.

We obtain the following dynamical solution:

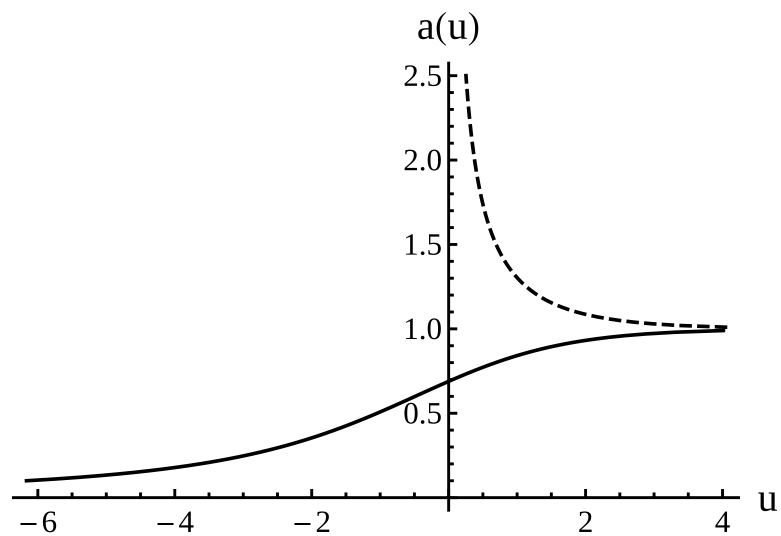
$$ds^2 = 2du dr - f du^2 + r^2 dx^i dx^i\,,\qquad \phi = \Big(\frac{a}{r}\Big)^{\frac{1}{2}(n-2)}\,,$$
$$f = g^2r^2 - \frac{\alpha a^{n-1}}{r^{n-3}}\,{}_2F_1[1,\tfrac{n-1}{n-2};\tfrac{2n-3}{n-2},\big(\tfrac{a}{r}\big)^{n-2}]\,,$$

where $a(u)$ satisfies

$$\dot{a} + \tilde{\alpha}\, a^2 \log\Big(\frac{a}{q}\Big) = 0\,, \qquad \tilde{\alpha} = \tfrac{1}{2}(n-1)\alpha\,.$$

There are two stationary points in this equation, one is $a = 0$, corresponding to the AdS vacuum, and the other is $a = q$, corresponding a static planar AdS black hole. There is a solution linking the two stationary points:

$$\mathrm{Ei}\Big(\log(\tfrac{q}{a})\Big) = -\tilde{\alpha} q\, u\,.$$



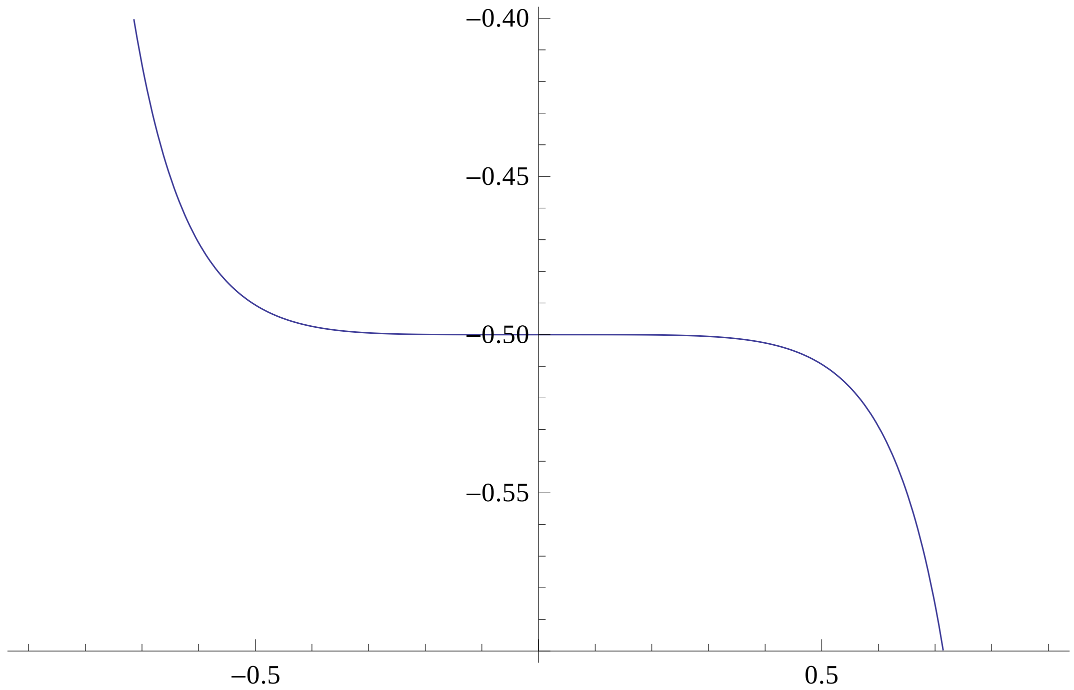
This solution provides an explicit demonstration of nonlinear instability of the planar AdS vacuum that is stable at the linear level.

See also Zhang and L, Phys.Lett. B736 (2014) 455-458.

Where does the energy come from

A black hole is formed out of the vacuum. Where does the energy come from? It comes from the scalar potential.

The scalar potential $V(\phi)$.



Another work: Higher derivative gravity

In quantum field theory, the nonrenormalizable problem can typically be solved by considering high derivatives. The same is true for gravity. However, high derivatives typically lead to ghost excitations. In gravity, the problem can be resolved by considering topological terms like Euler integrands. The lowest non-trivial order is the Gauss-Bonnet term. However, the benefit of avoiding ghost has a price that the theory is equally nonrenormalizable.

Black hole solutions of this type were constructed by Boulware, Deser; Cai.

There is no surprise here however in that the solution can be viewed as the Schwarzschild solution with higher-order corrections.

Four dimensional quadratic gravity

We are concerned with the four dimensional theory

$$\mathcal{L}_4 = \sqrt{-g}\Big(\kappa R + \alpha R^2 + \beta R^{\mu\nu} R_{\mu\nu} + \gamma R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}\Big) .$$

In four dimensions, the Gauss-Bonnet combination

$$\mathcal{L}_{GB} = \sqrt{-g}(R^2 - 4R^{\mu\nu} R_{\mu\nu} + R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma})$$

is a total derivative, and hence we can set $\gamma = 0$, or equally write

$$\mathcal{L} = \sqrt{-g}\left(\kappa R - \alpha C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \beta R^2\right) .$$

where C is the Weyl tensor.

- The theory is renomalizable for general parameters. (Stelle, 1976)
- The theory has ghost for non-vanishing β.
- The theory can be treated as a perturbative truncation of string theory.
- The higher derivative terms contribute no effective cosmological constant.

The effect of higher derivatives on black holes

How do the higher derivative terms affect the black hole solution? It turns out that in four dimensions, Schwarzschild and Kerr black holes receive no correction.

Is there a new black hole besides the Schwarzschild solution?

$$\mathcal{L} = \sqrt{-g}\left(\kappa R - \alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \beta R^2\right) .$$

If $\kappa = 0 = \beta$, the theory has called conformal gravity and new solutions do exist.

- Static: Riegert, Phys.Rev.Lett. 53 (1984) 315-318
- Rotating: Liu and L, JHEP 1302 (2013) 139

We now consider $\kappa \neq 0 \neq \beta$,

It can be proven, for asymptotic flat spacetime, if the solution has an event horizon, we must have $R = 0$. Nelson (Phys.Rev.D **82** (2010) 104026,) This means in particular there is no new black hole in Starobinsky $R + R^2$ gravity.

The effective theory

Thus the effective theory becomes

$$\mathcal{L} = \sqrt{-g}\,(\kappa R - \alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}) \ .$$

We find that in addition to the Schwarzschild black hole, there exist a new disconnect one. This was demonstrated numerically in Phys.Rev.Lett. 114 (2015) 17, 171601, L, Perkins, Pope and Stelle.

Asymptotic behavior

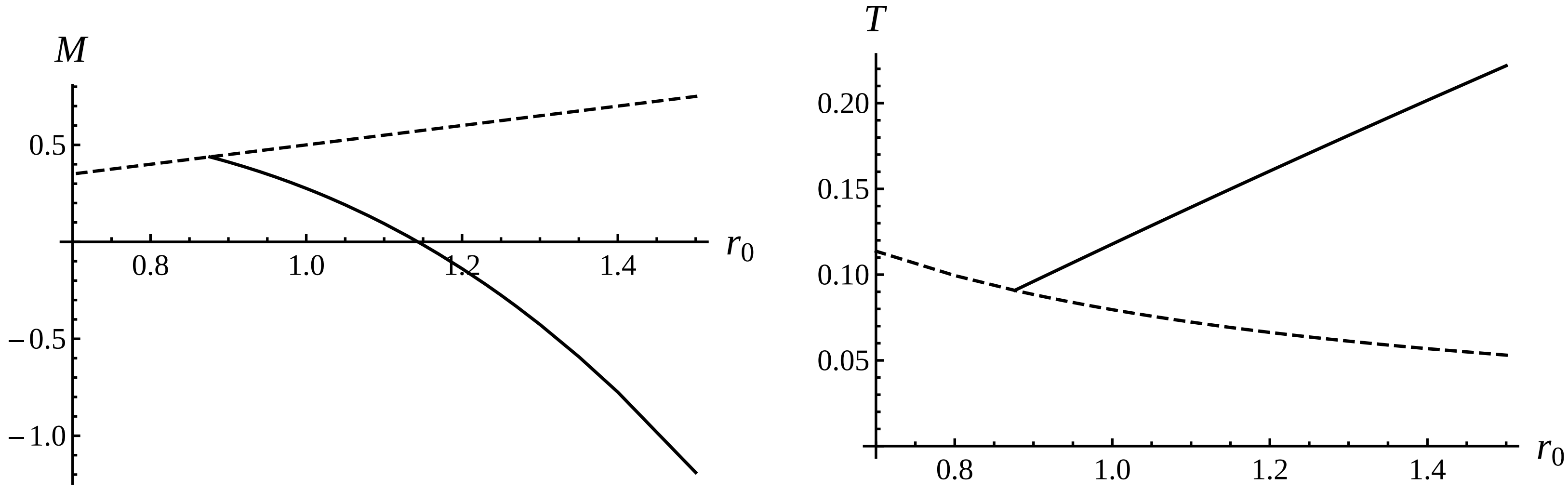
The metric: $ds^2 = -hdt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2$. Large r behavior

$$
\begin{aligned}
h &= 1 - \frac{2M}{r} - \frac{c_1 e^{-\mu_2 r}}{r} - \frac{c_2 e^{\mu_2 r}}{r}\,, \\
f &= 1 - \frac{2M}{r} - \tfrac{1}{2} c_1 (\mu_2 + \tfrac{1}{r}) e^{-\mu_2 r} + \tfrac{1}{2} c_2 (\mu_2 - \tfrac{1}{r}) e^{\mu_2 r}\,,
\end{aligned}
$$

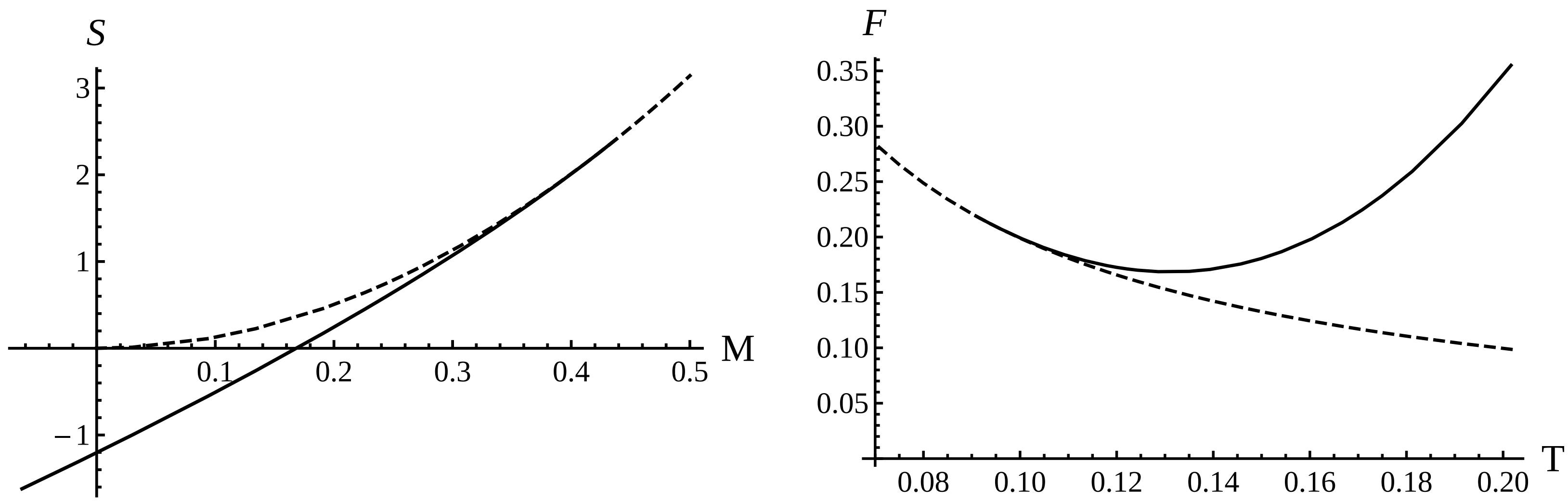
where $\mu_2^2 = 1/(2\alpha)$. If $c_1 = c_2 = 0$ gives rise to Schwarzschild black hole.

We must have $c_2 = 0$, but c_1 can be non-vanishing, which would give a new black hole. We demonstrate numerically that when mass $M \le M^* \equiv 0.434\sqrt{2\kappa\alpha}$, new disconnect black hole arises. In other words, given mass less than M^*, there are two black holes, one is the Schwarzschild and the other is this new one.

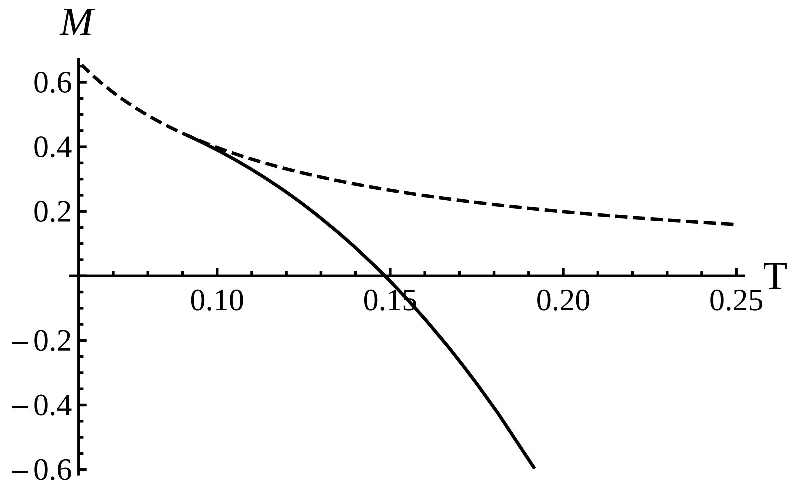
Black hole properties



The masses (left plot) and temperatures (right plot) of the Schwarzschild (dashed line) and non-Schwarzschild (solid line) black holes as a function of the horizon radius r_0.



The first plot shows the entropy as a function of mass, and the second shows the free energy $F = M - TS$ as a function of T, for the Schwarzschild (dashed line) and non-Schwarzschild (solid line) black holes.



The mass M as the function of temperature T.

It is clear that we have

$$C_{\text{new}} < C_{\text{Schw}} < 0\,.$$

Conclusions

There are two classes of physicists. One establishes equations; the other solves them. Einstein equations fortunately provide the highest enjoyment for people to solve them.

In the first 99 years, the results can be roughly summarised as follows

- More or less full understanding of static or stationary black holes with spherical topologies; few or possibly no surprise there.
- Few things are known on black holes with new topologies, except for black ring in $D = 5$, without a cosmological constant.
- Black hole formation and black holes in higher-derivative gravities remain largely untouched, (in the exact solution sense.)

Although tremendous progress has been made, but the subject is far from over, and calling the efforts as merely “the first 99 years” is not at all pretentious.

We are still at the Beginning rather than the End.

$\boxed{\text{The End}}$