名古屋大学

Wave optics in the Kerr spacetime and Black Hole Shadows

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Hot Topics in General Relativity and Gravitation @ Quy Nhon, Vietnam (2015)

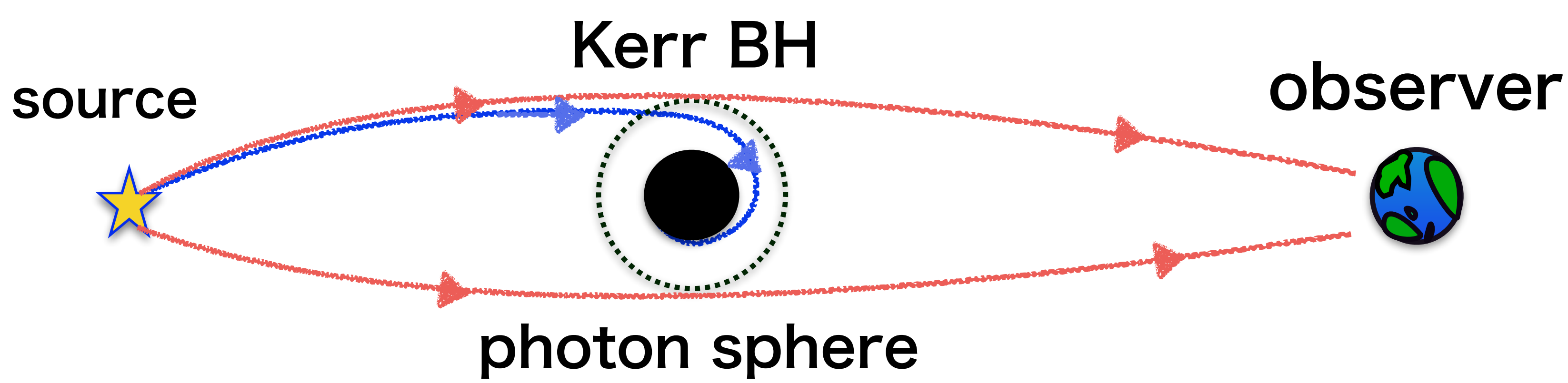
1. Introduction

2.Wave optics in the Kerr spacetime

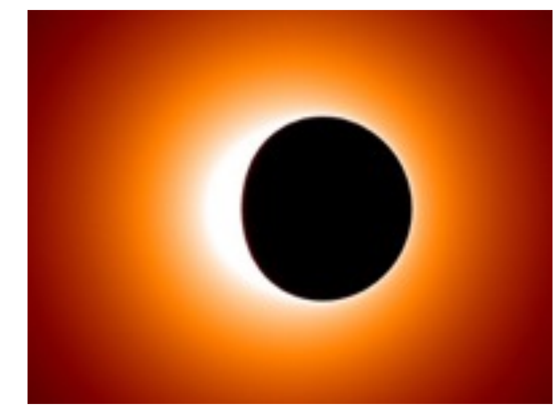
3. Result and discussion

Black Hole Shadow

○ Geometrical Optics (null geodesics)



Black Hole Shadow



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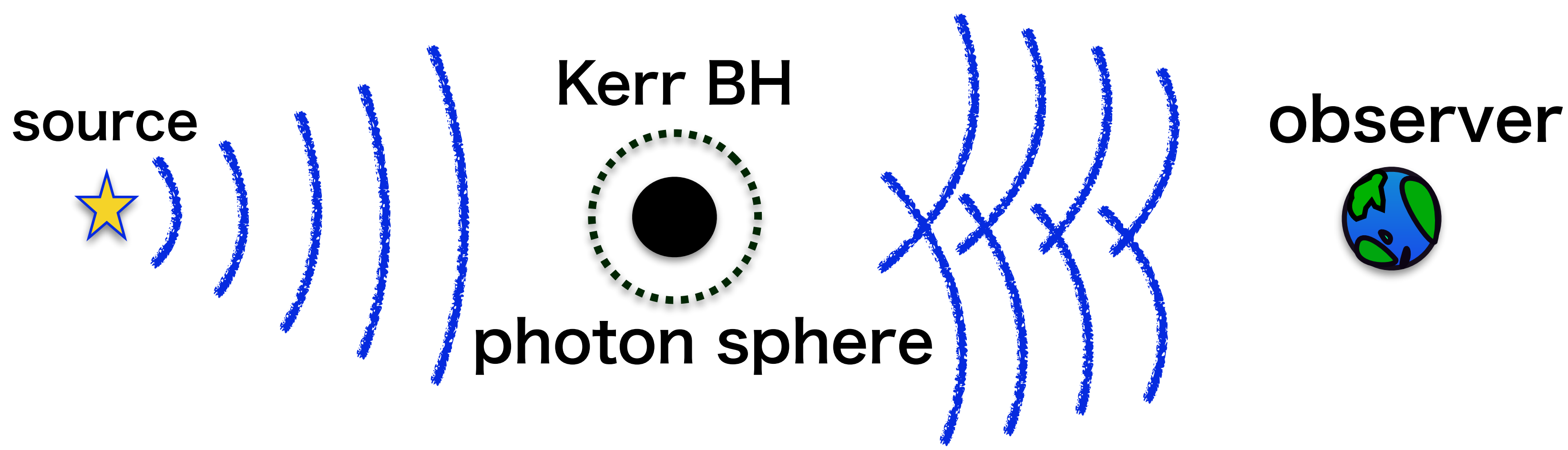
photon sphere ⟷ Rim of the shadow

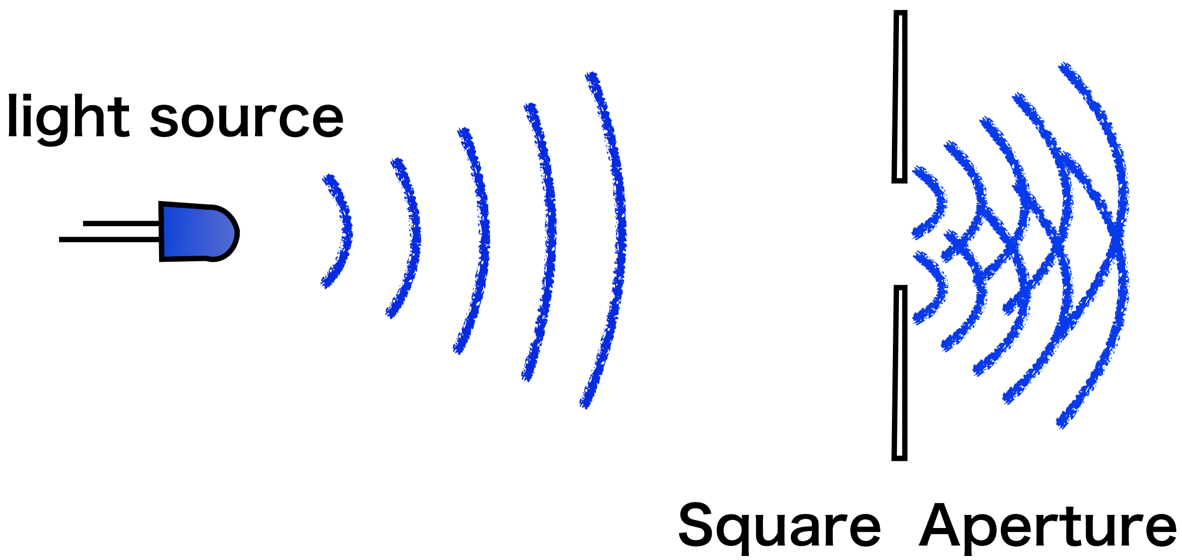
○ Wave Optics (Radio Wave or Gravitational Wave)

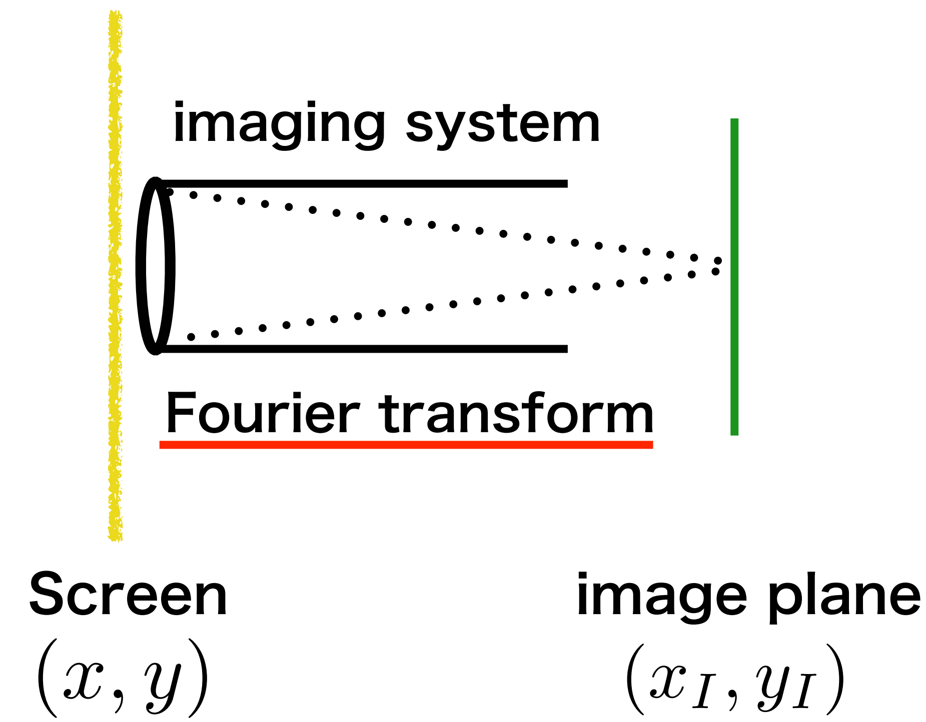


wave optical effects
more information

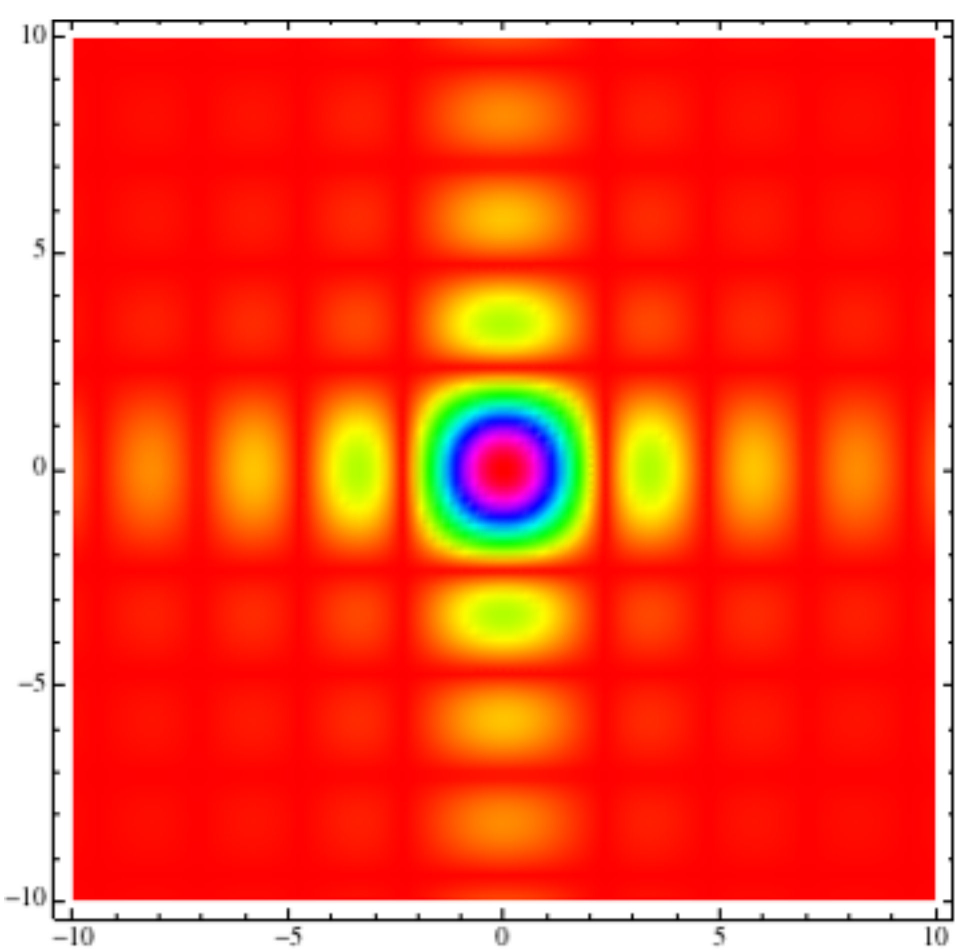
photon sphere ⟷ Quasi Normal Modes

Imaging in Wave Optics

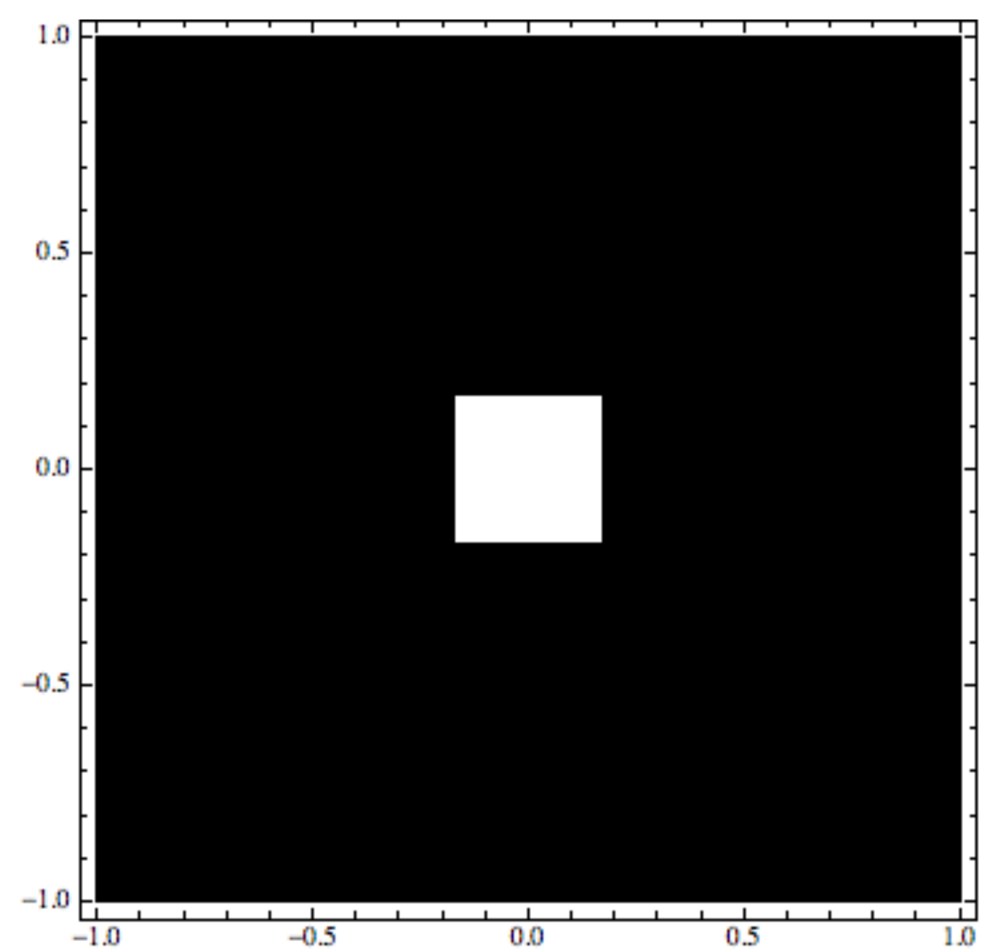




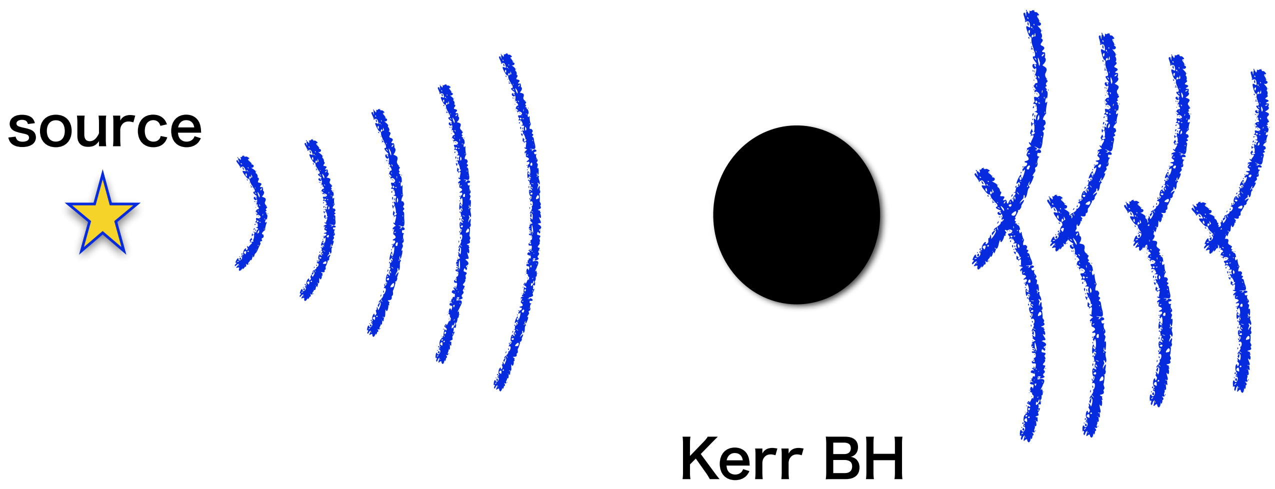
Interference pattern

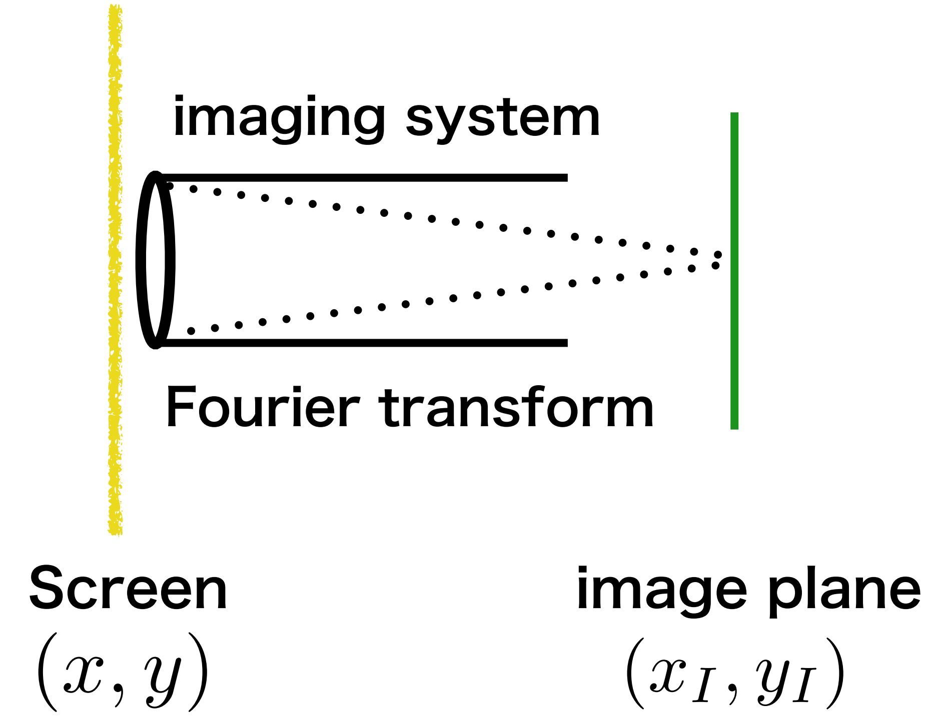


Image

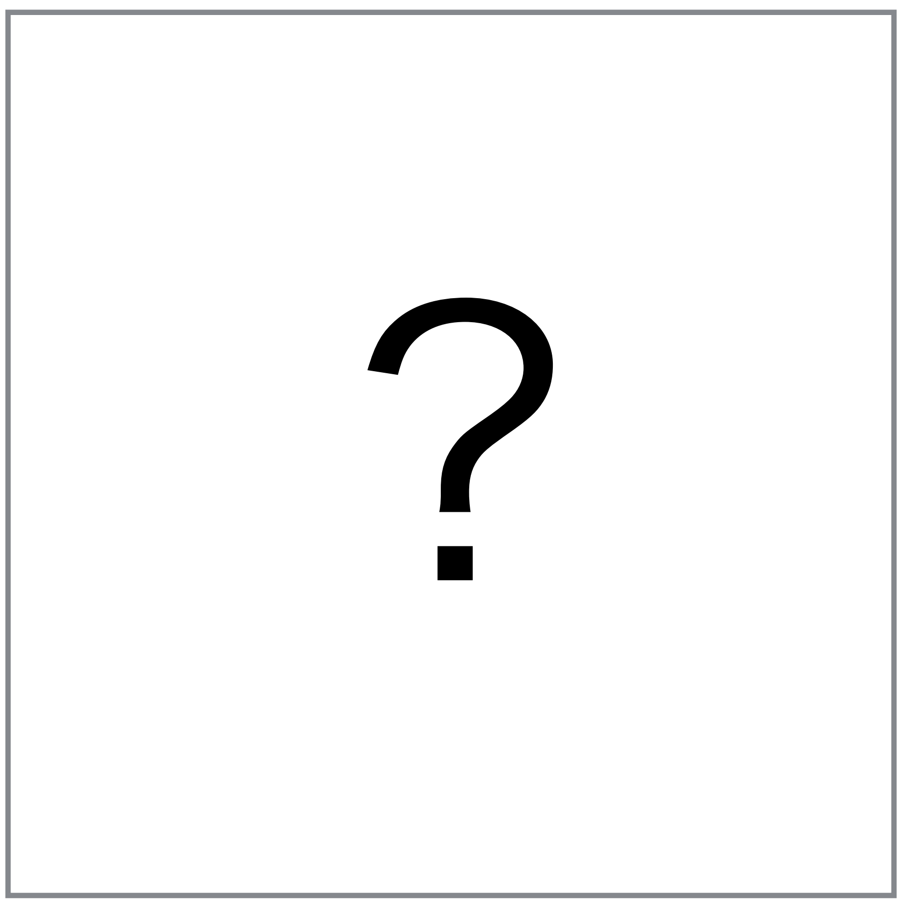


Wave Optics in the Kerr spacetime

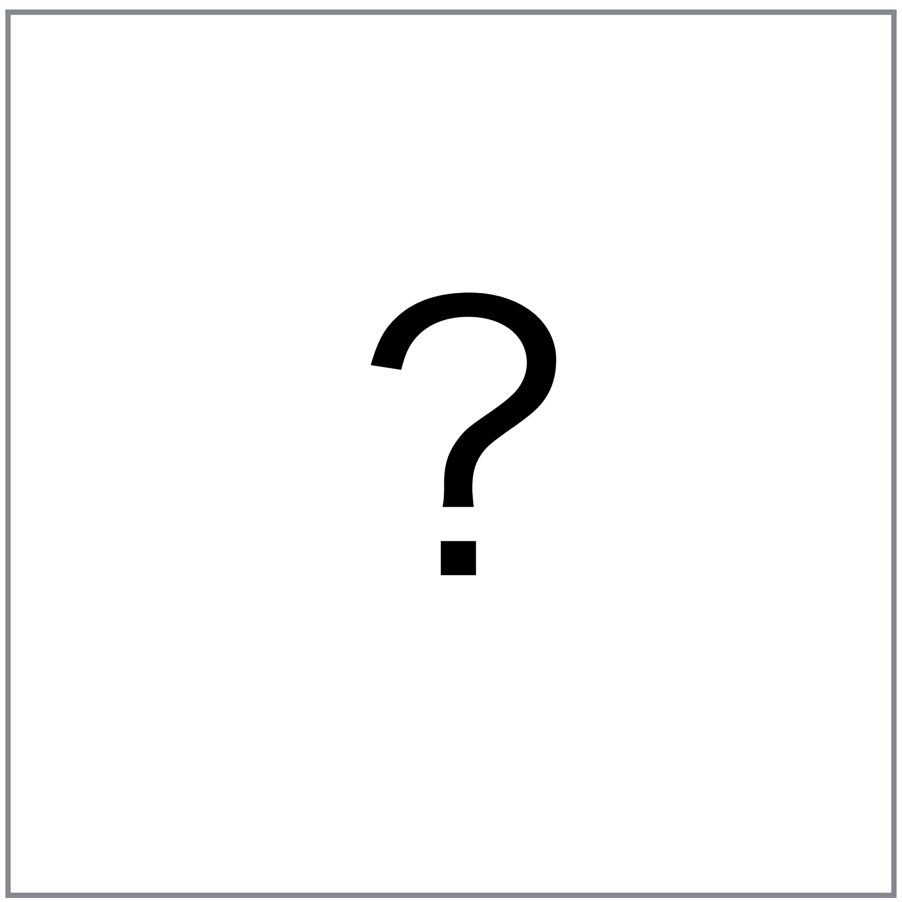




Interference pattern



Wave optical Image of a Black Hole



1. Introduction

2.Wave optics in the Kerr spacetime

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The Kerr spacetime

- Boyer Lindquist coordinate $\left(\frac{a}{M} = 0 \sim 1\right)$

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{A\sin^2\theta}{\Sigma} d\phi^2$$

$$\Delta \equiv r^2 - 2Mr + a^2 \qquad \Sigma \equiv r^2 + a^2\cos^2\theta \qquad A \equiv (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta$$

- Killing vectors

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$

$$\xi^\mu_{(t)} \qquad \xi^\mu_{(\phi)}$$

- Killing tensor

$$\nabla_{(\mu} K_{\nu\rho)} = 0$$

- Conserved quantities p_μ : 4-momentum

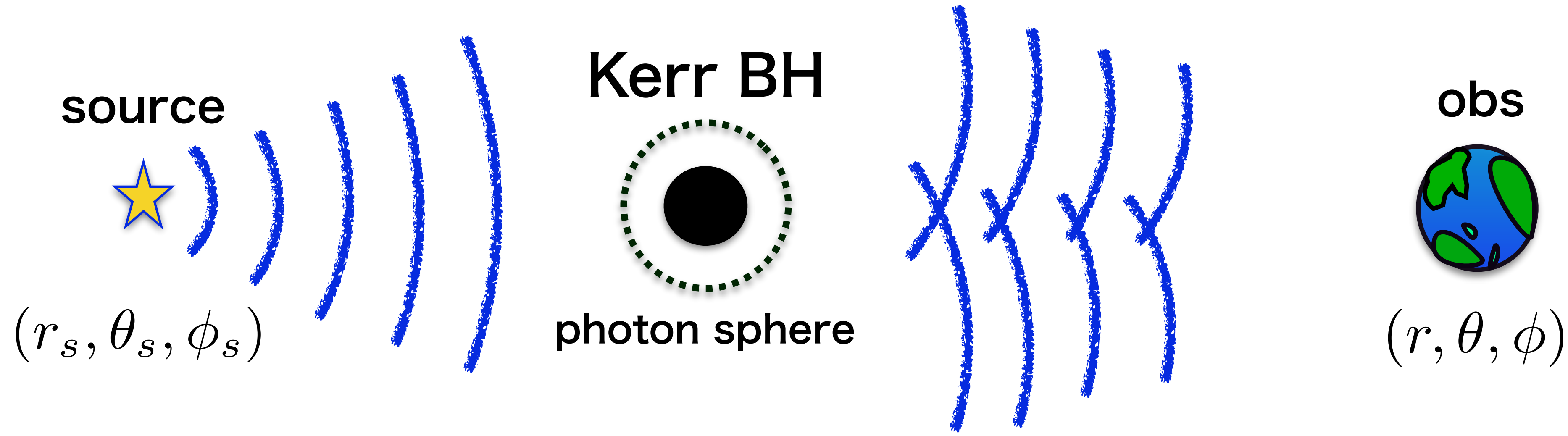
$$E = -p_\mu \xi^\mu_{(t)}$$ …Energy

$$L_z = p_\mu \xi^\mu_{(\phi)}$$ …Angular Momentum

$$\mathcal{K} = p_\mu p_\nu K^{\mu\nu}$$ …Carter constant

$$\mathcal{Q} = \mathcal{K} - (aE - L_z)^2$$

Wave scattering problem



- Source ・・・ point source , monochromatic , scalar wave

short wavelength
$\omega M \gg 1$

Klein Gordon eq.

$$\Box \Phi = S \qquad \Phi \sim e^{-i\omega t} \; , \; S \sim \delta^{(3)}(\vec{x} - \vec{x}_s)$$

- Green function

$$\nabla^2 G(\vec{x}, \vec{x_s}) = -\delta^{(3)}(\vec{x} - \vec{x_s})$$

partial wave expansion

$$G(x, x_s) = \sum_l \sum_m \frac{\tilde{G}_{lm}(r, r_s)}{\sqrt{r^2 + a^2}\sqrt{r_s^2 + a^2}} S_{lm}(\theta) S^*_{lm}(\theta_s) e^{im(\phi - \phi_s)}$$

spheroidal harmonics

The radial part short wavelength case $\omega M \gg 1$

- The radial part

$$\tilde{G}_{lm} = -\frac{u_{in}(r_s)u_{up}(r)}{w(u_{in}, u_{up})}$$

$w(u_{in}, u_{up})$: Wronskian

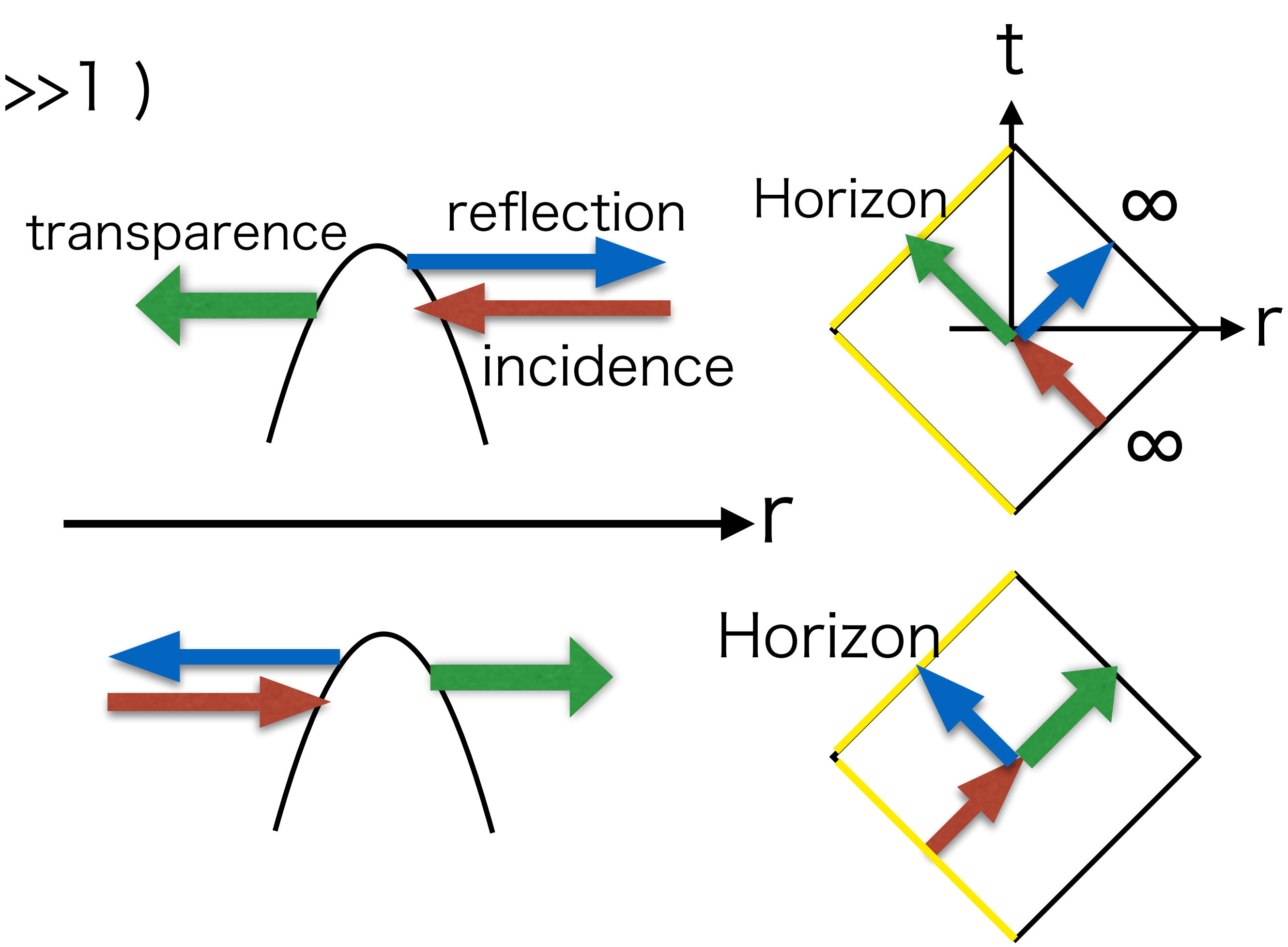
- The radial equation (homogeneous)

$$\frac{d^2u(r_*)}{dr_*^2} + Qu(r_*) = 0 \qquad Q = \frac{[\omega(r^2+a^2) - ma]^2 - \Delta(A_{lm} + a^2\omega^2 - 2am\omega)}{(r^2+a^2)^2}$$

- Independent linear **WKB** solutions (r >>1)

$$u_{in} \sim \sin\left(\omega r_* - \frac{\pi l}{2} + \delta_{lm} + \underline{\frac{A_{lm} + a^2\omega^2}{2\omega r}}\right)$$

$$u_{up} = \exp\left(i\left\{\omega r_* - \frac{\pi l}{2} + \delta_{lm} + \underline{\frac{A_{lm} + a^2\omega^2}{2\omega r}}\right\}\right)$$



Decomposition of the Green function

- Poisson's sum formula

$$\sum_{l=0}^{\infty} \to \sum_{W=-\infty}^{\infty} \int_C dL \; e^{i2\pi W(L-1/2)}$$

$L = l + 1/2$

W : integer

- Full Green function

$$G(x, x_s) = \sum_l \sum_m \frac{\tilde{G}_{lm}(r, r_s)}{\sqrt{r^2+a^2}\sqrt{r_s^2+a^2}} S_{lm}(\theta) S^*_{lm}(\theta_s) e^{im(\phi-\phi_s)}$$

WKB solution

$$S_{lm} \sim \frac{1}{\Theta^{1/4}}[e^{iS_\theta} + (-)^{l+m} e^{-iS_\theta}]$$

$$= G^{W=0} + G^{W\neq 0}$$

Direct part Winding part

- Winding part

$$G^{W\neq 0} = \frac{e^{i\omega(r^*+r_s^*)}}{2i\omega r r_s} \sum_{W\neq 0} \int_C dL \sum_{m=-l}^{l} e^{i2\pi W(L-1/2)} S_{lm}(\theta) S^*_{lm}(\theta_s) e^{2i\delta_{lm}} e^{im(\phi-\phi_s)} e^{i\frac{A_{lm}+a^2\omega^2}{2\omega}\left(\frac{1}{r}+\frac{1}{r_s}\right)}$$

Lens equation and Winding number

$S_{lm}e^{im\phi} \rightarrow Y_{lm}$

Schwarzschild case

$$G(x,x_s) = \frac{e^{i\omega(r^*+r_s^*)}}{2i\omega r r_s}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}(\theta,\phi)Y_{lm}^*(\theta_s,\phi_s)e^{i2\delta_l}e^{i\frac{(l+\frac{1}{2})^2}{2\omega}\left(\frac{1}{r}+\frac{1}{r_s}\right)}$$

addition theorem

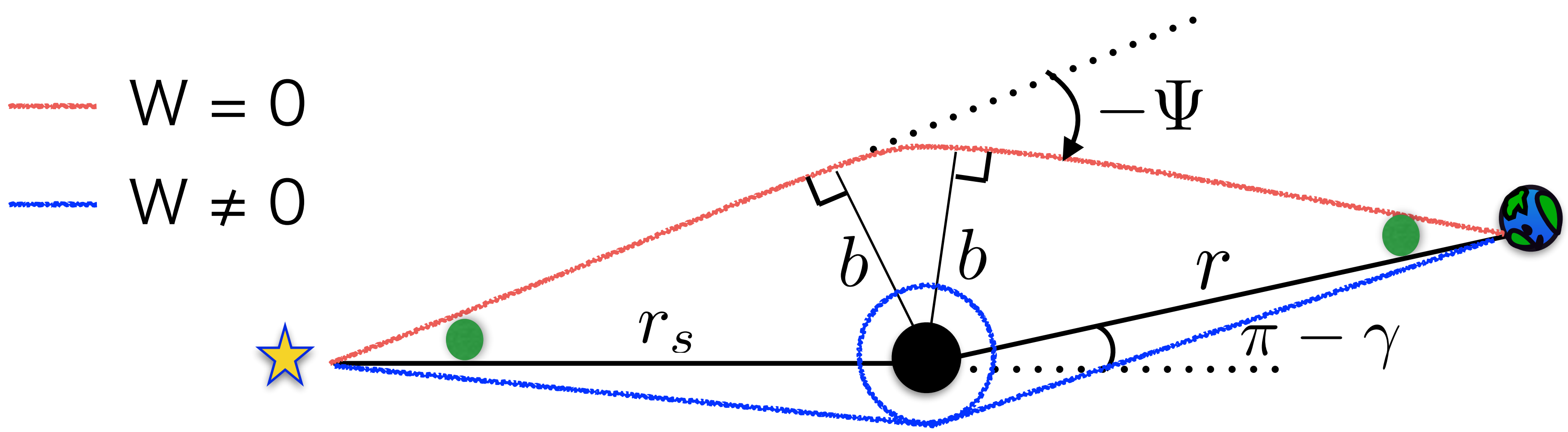
$$= \frac{e^{i\omega(r^*+r_s^*)}}{4\pi i\omega r r_s}\sum_{W=-\infty}^{\infty}\int_0^{\infty} e^{i2\pi W(L-1/2)} L P_L(\cos\gamma) e^{i2\delta_L} e^{i\frac{L^2}{2\omega r}\left(\frac{1}{r}+\frac{1}{r_s}\right)} dL$$

Lens equation (Stationary condition)

$$b\left(\frac{1}{r}+\frac{1}{r_s}\right) = -\Psi - 2\pi W \pm (\pi-\gamma)$$

$r, r_s \gg 1$

$\Psi \equiv 2\frac{d\delta_L}{dL} \quad b = \frac{L}{\omega}$



W : winding number

Sum over l

- Winding part (Kerr)

$$G^{W\neq 0} = \frac{e^{i\omega(r^*+r_s^*)}}{2i\omega r r_s} \sum_{W\neq 0} \int_C dL \sum_{m=-l}^{l} e^{i2\pi W(L-1/2)} S_{lm}(\theta) S^*_{lm}(\theta_s) \underline{e^{2i\delta_{lm}}} e^{im(\phi-\phi_s)} e^{i\frac{A_{lm}+a^2\omega^2}{2\omega}\left(\frac{1}{r}+\frac{1}{r_s}\right)}$$

S matrix

$$= \frac{e^{i\omega(r^*+r_s^*)}}{2i\omega r r_s} \sum_{W\neq 0} \sum_{m} 2\pi i \, \underline{\gamma^{(W)}(m)} = \frac{e^{i\omega(r^*+r_s^*)}\pi}{\omega r r_s} \sum_{m} \gamma(m)$$

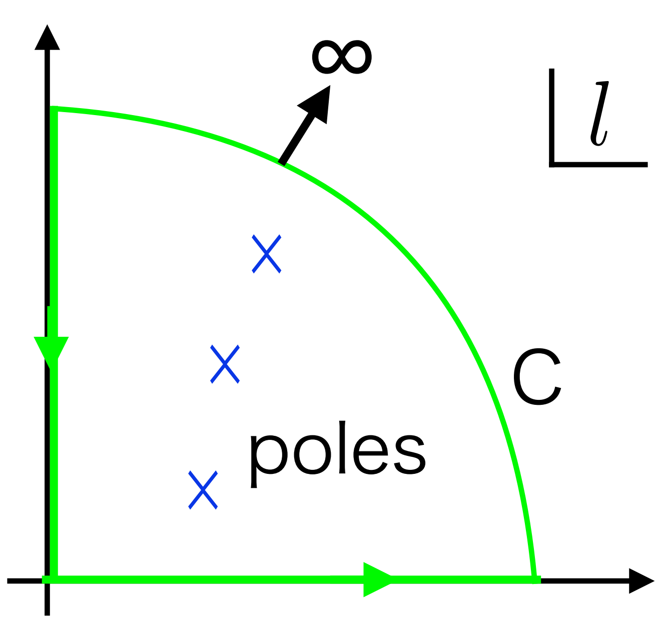
residue

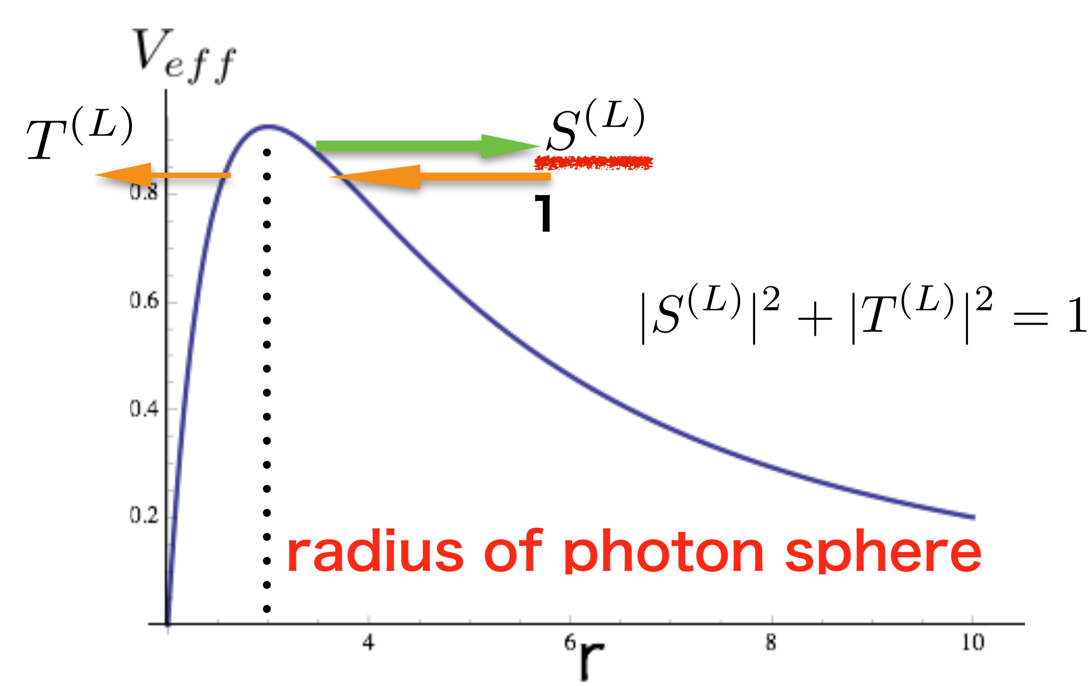
- S matrix

$$S(l,m) = e^{2i\delta_{lm}} = -(-)^l \frac{e^{i\pi\nu}}{\sqrt{2\pi}} \left(\nu + \frac{1}{2}\right)^{\nu+1/2} e^{-(\nu+1/2)} \underline{\Gamma(-\nu)}$$

ν =n (0,1,2, ・・・) → ∞

QNMs





Sum over m

$$G^{W\neq 0} = \frac{e^{i\omega(r^*+r_s^*)\pi}}{\omega r r_s} \sum_m \gamma(m)$$

Geometrical optics	Wave optics (WKB)
$H\left(\frac{\partial S}{\partial q}, q\right) = E$	$\Box\Phi = 0 \quad \Phi \sim e^{iS}$
E	ω
L_z	m
$\mathcal{Q}$	$A_{lm} - m^2$

- Radii of the photon sphere

$$\mathcal{Q} = \frac{r_c^3(4Ma^2 - r_c(r_c-3M)^2)}{a^2(M-r_c)^2}\,\omega^2 \qquad m = \frac{r_c^2(r_c-3M)+a^2(r_c+M)}{a(M-r_c)}\,\omega$$

$$\sum_m \to \int_{m_1}^{m_2} dm$$

$$G^{W\neq 0} = \frac{e^{i\omega(r^*+r_s^*)\pi}}{\omega r r_s}\int_{m_1}^{m_2} dm\,\gamma(m) = \frac{e^{i\omega(r^*+r_s^*)\pi}}{\omega r r_s}\int_{r_1}^{r_2} dr_c \frac{dm}{dr_c}\,\gamma(m(r_c))$$

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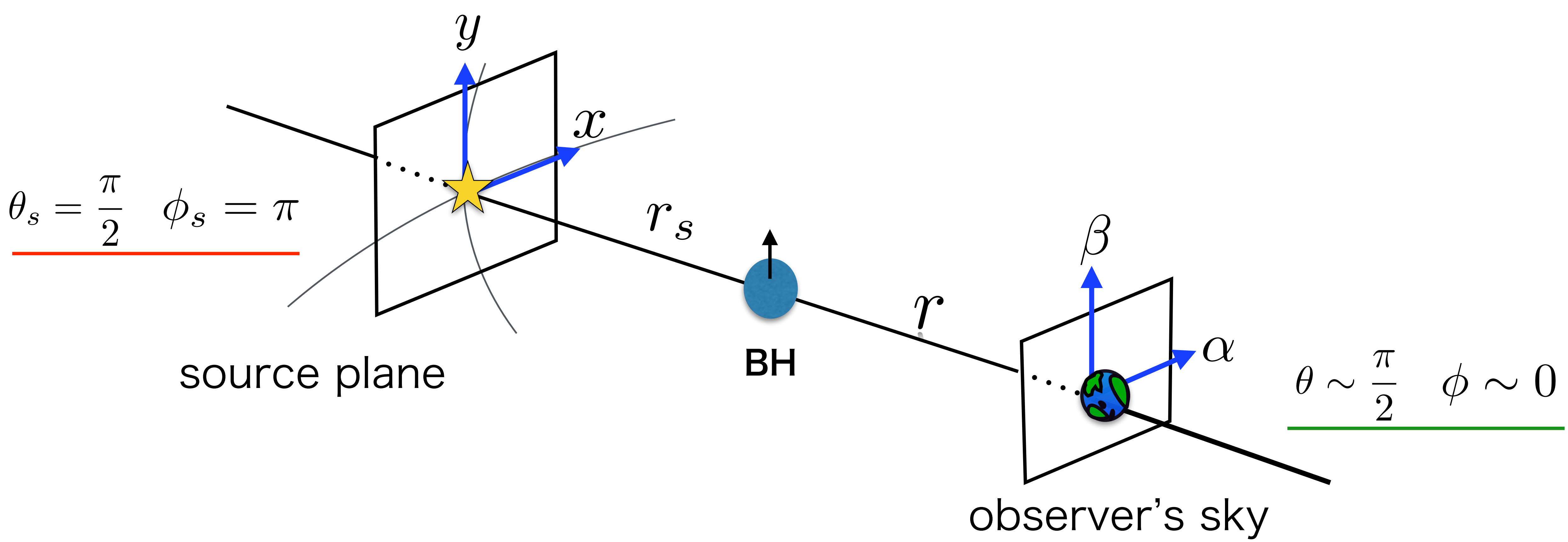
Configuration of the scattering problem

- Winding part of the Green function

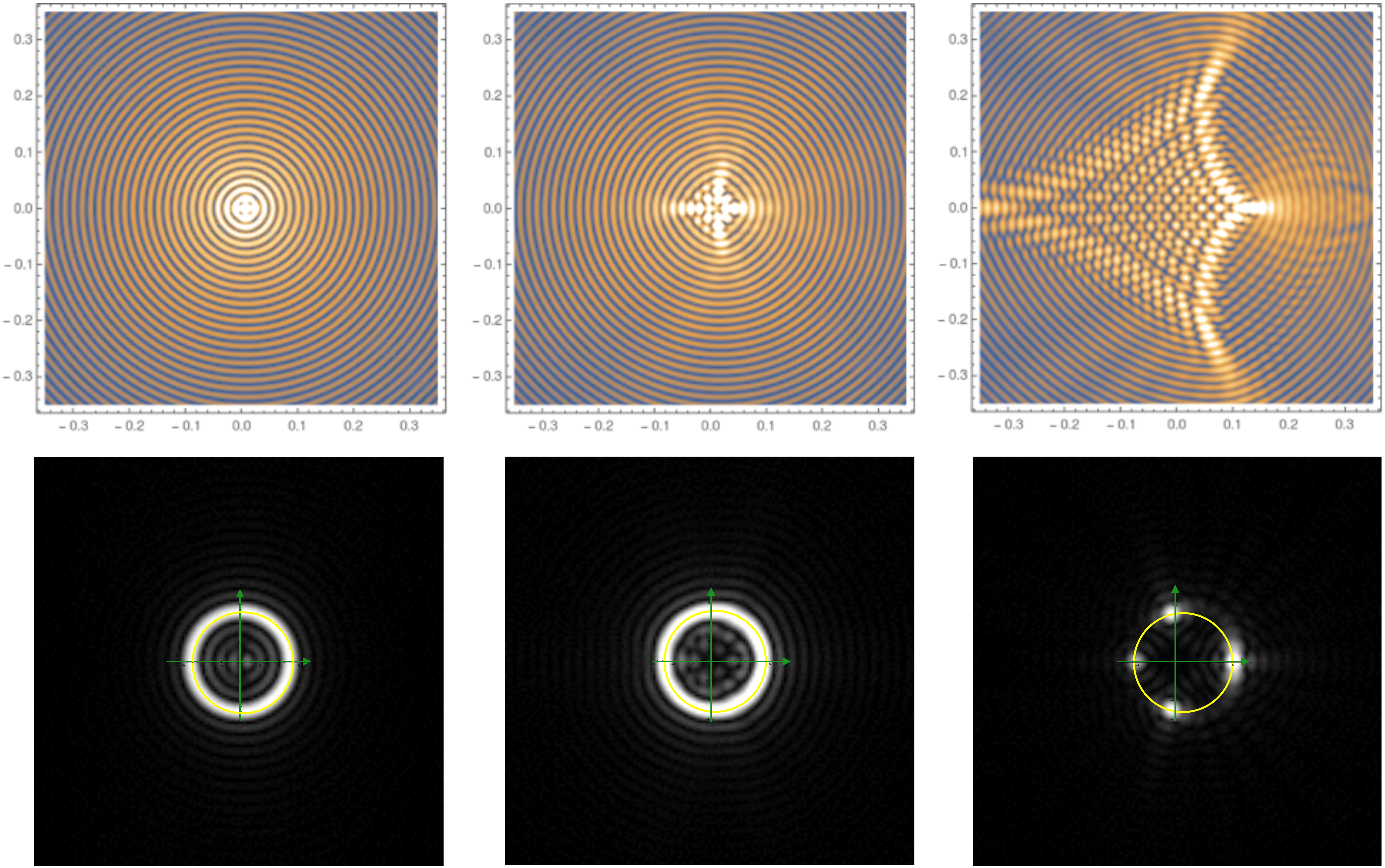
$$G^{W\neq 0}(\underbrace{r_s, \theta_s, \phi_s}_{\text{source}} \; ; \; \underbrace{r, \theta, \phi}_{\text{observer}})$$

→ interference pattern on the observer's sky



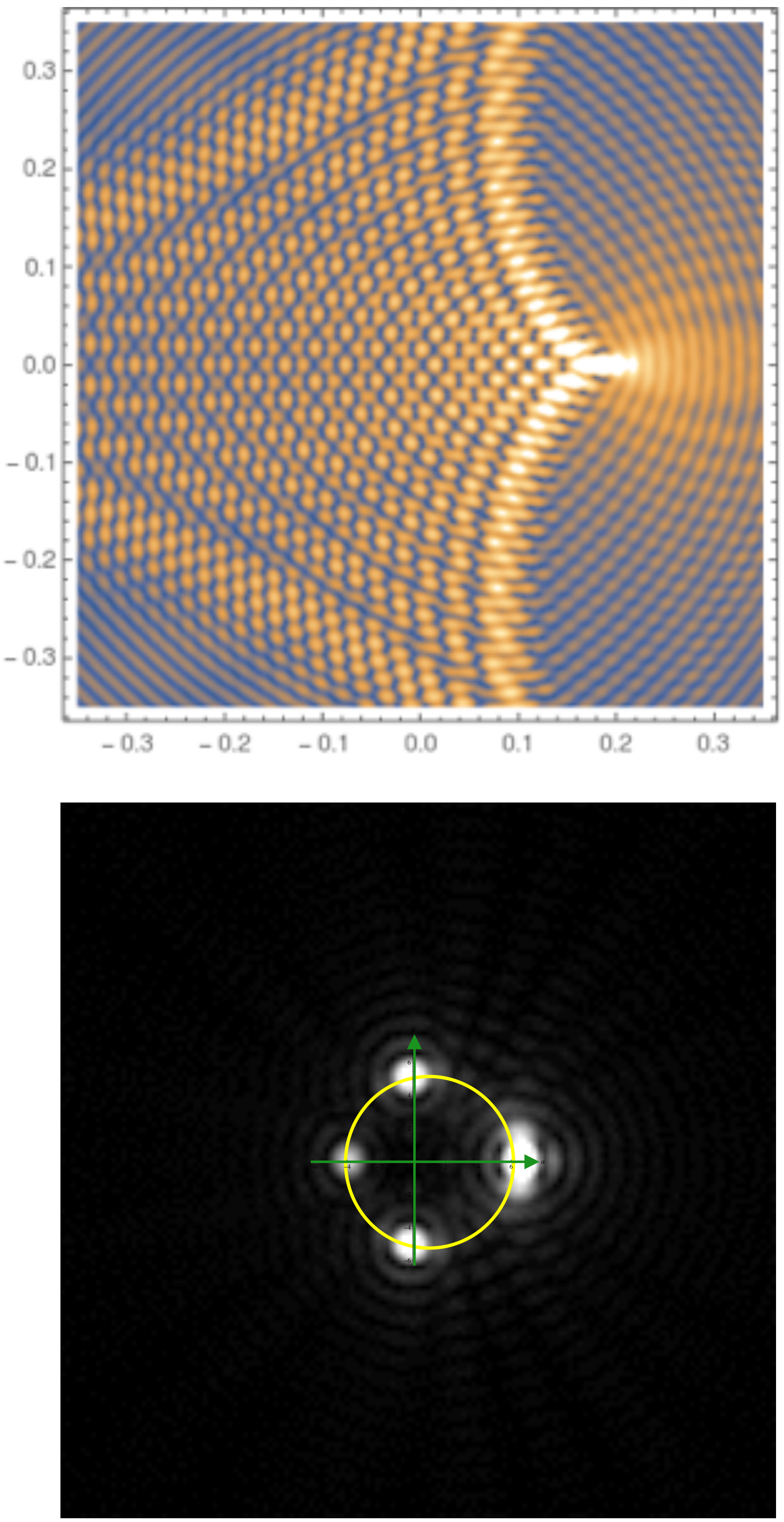
Interference patterns & images

a=0.1 | a=0.2 | a=0.4

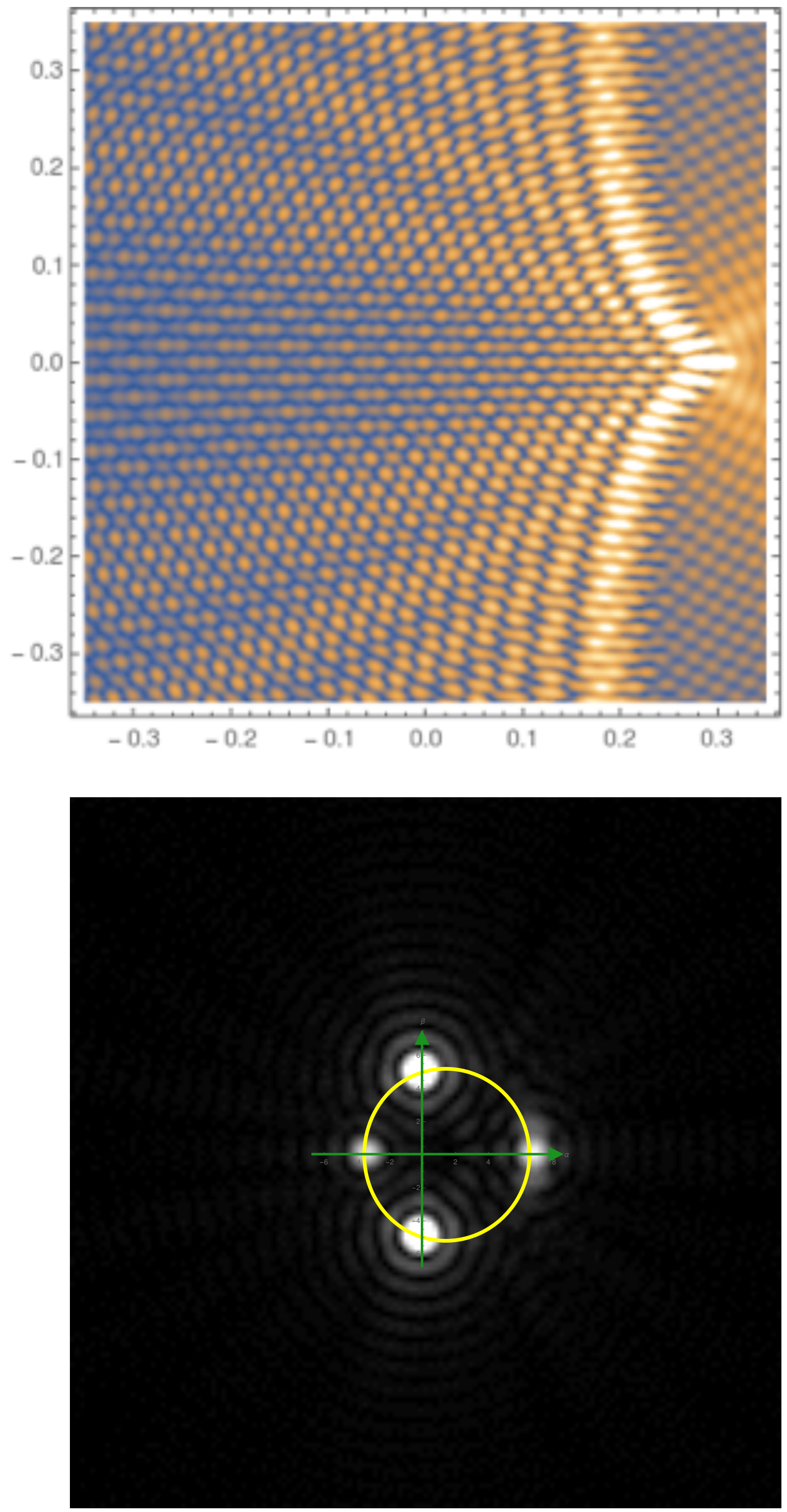


Interference patterns & images

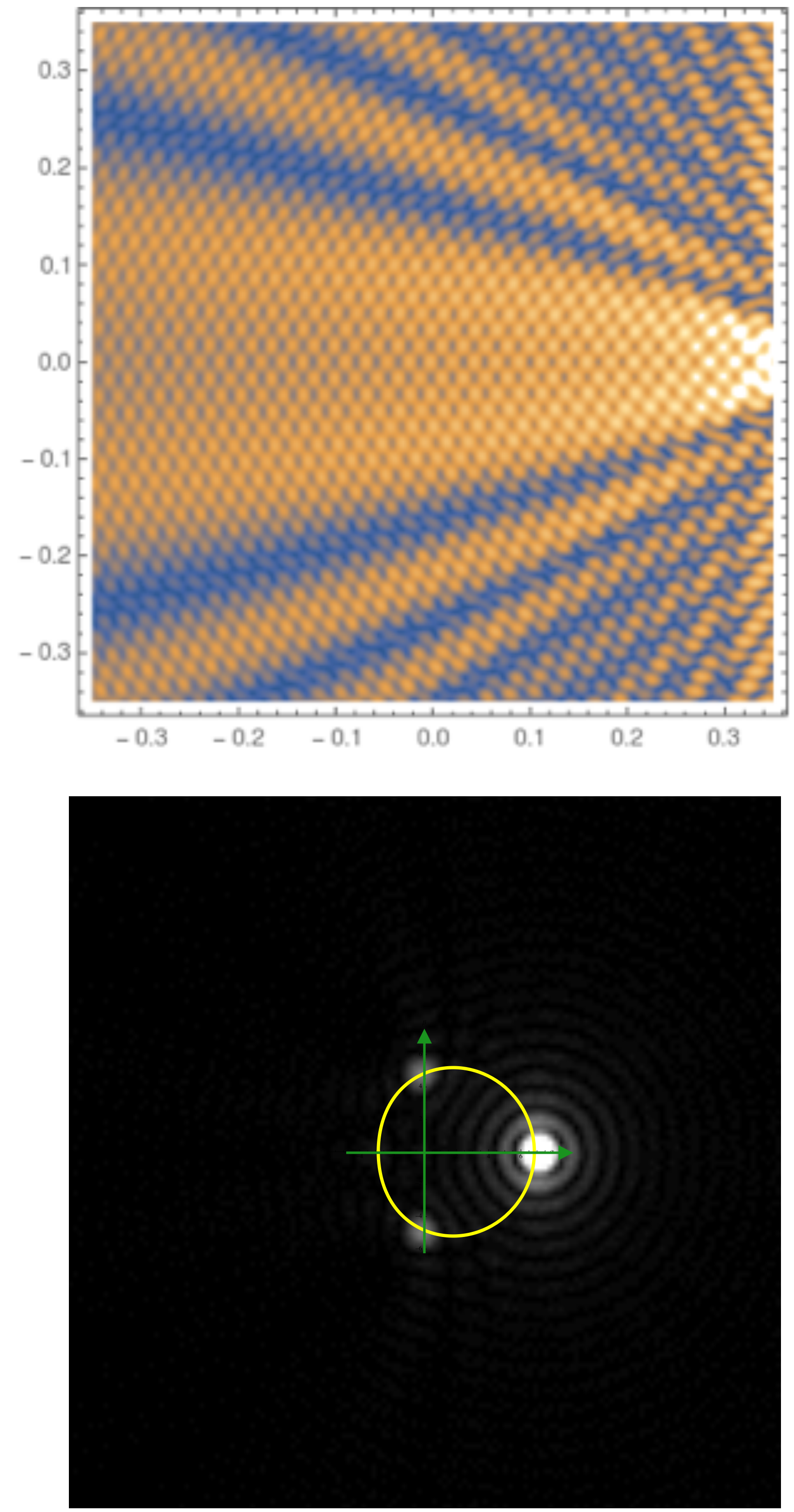
a=0.5



a=0.7

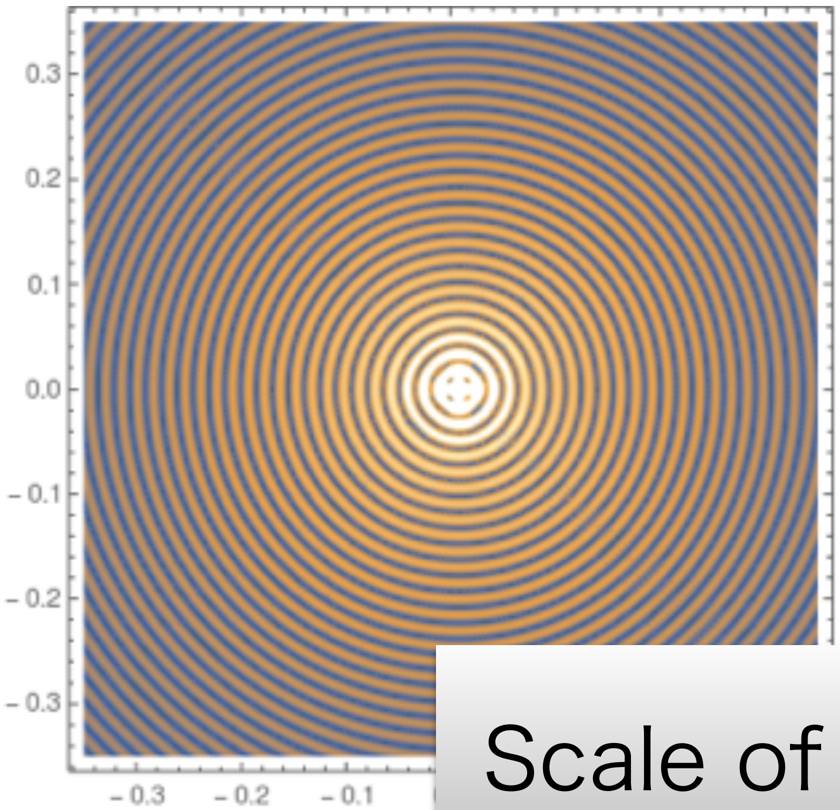


a=0.9

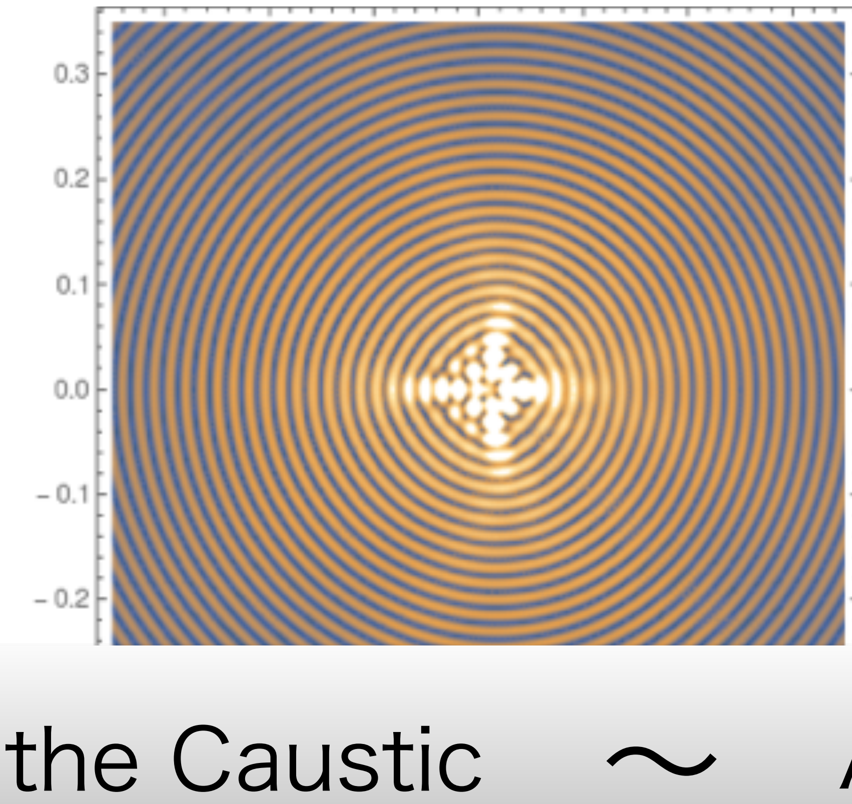


Difference in images

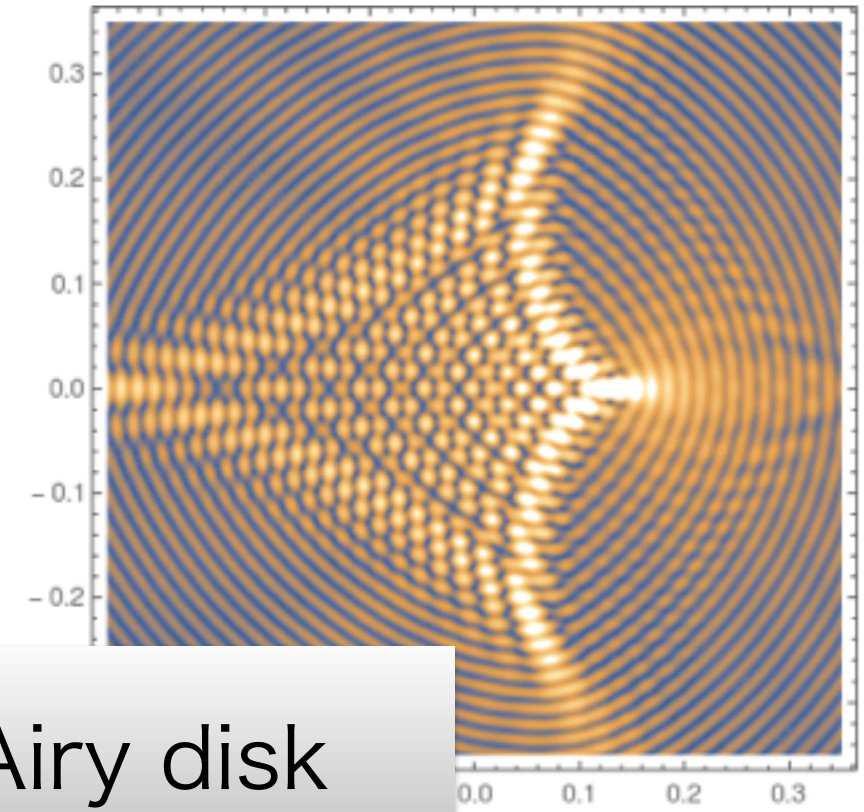
a=0.1



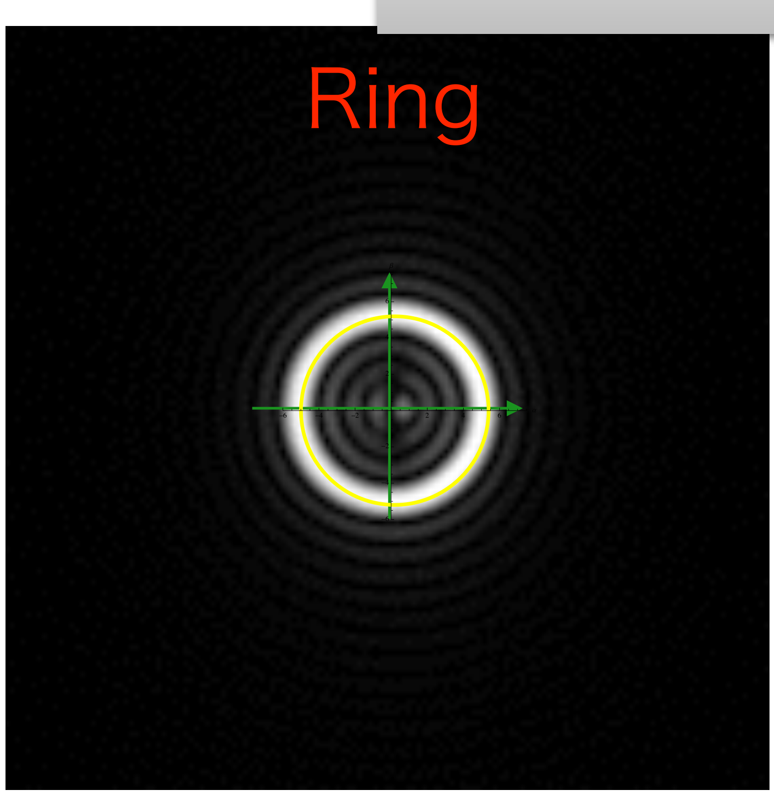
a=0.2

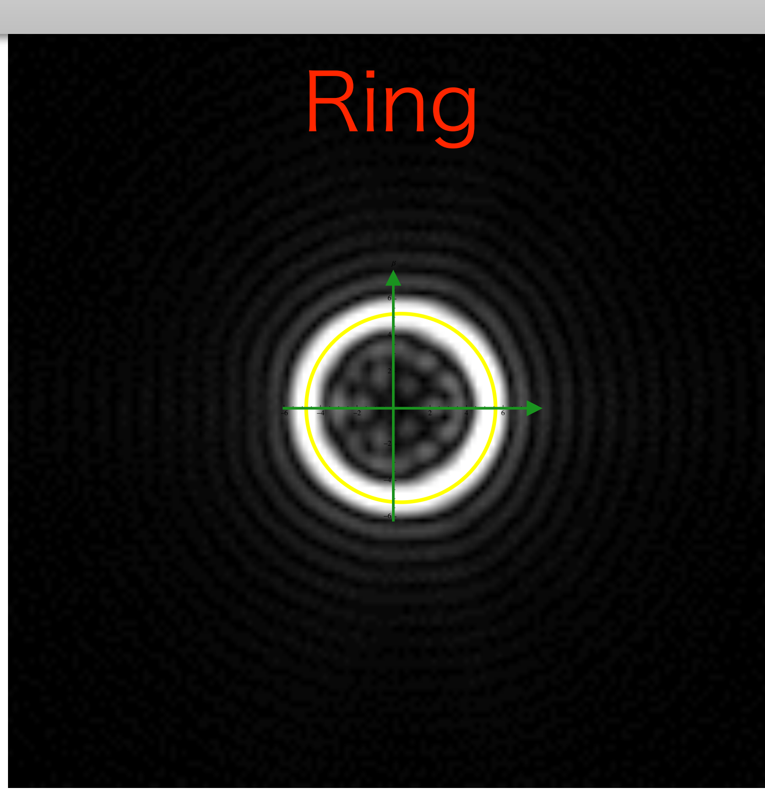


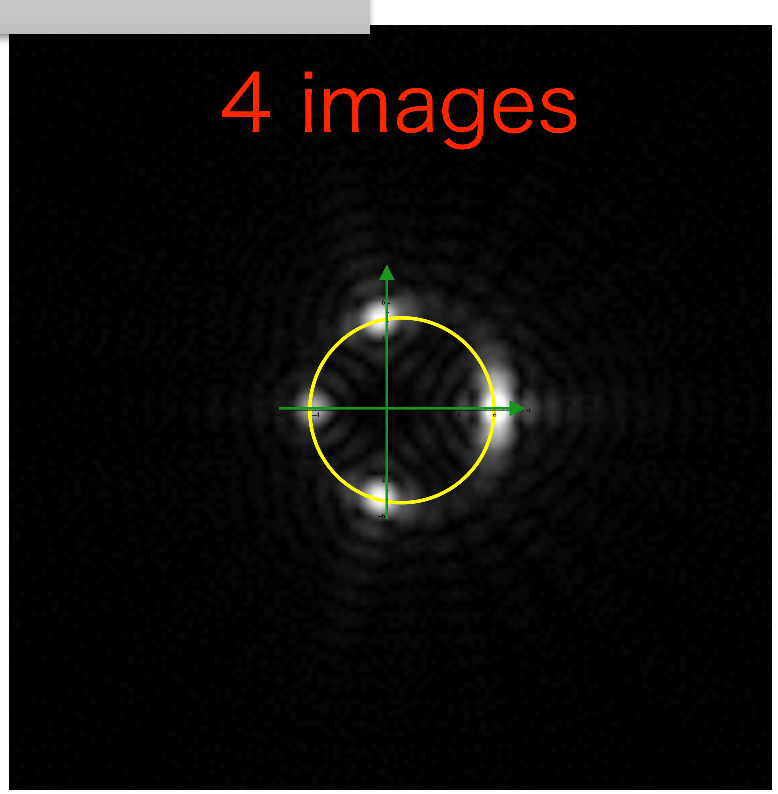
a=0.4



Scale of the Caustic ∼ Airy disk







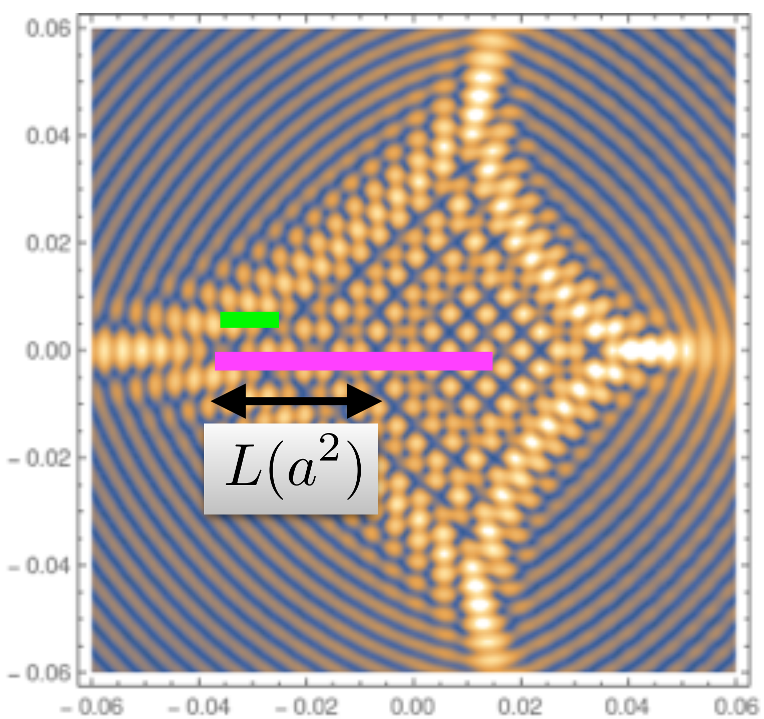
Estimation of a

○ Rayleigh criterion

smaller structure than the size of Airy disk → ~~resolution~~

The size of Airy disk $\sim \dfrac{1}{\omega D}$ ω : frequency D : aperture

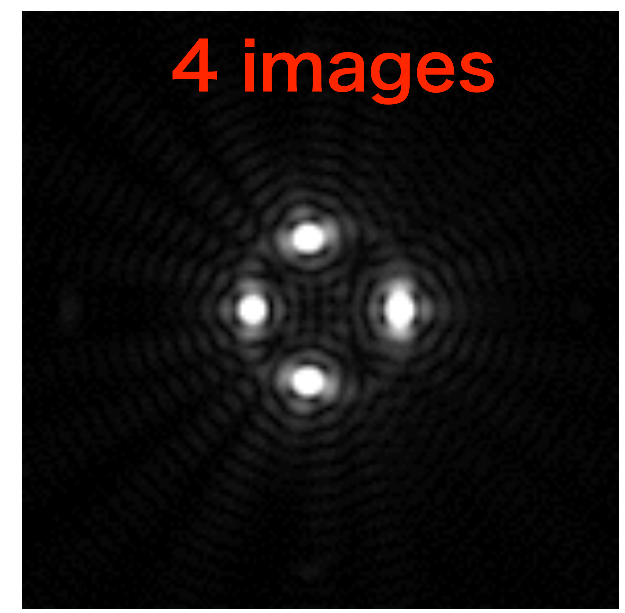
a=0.2



$L(a^2) > \dfrac{1}{\omega D}$

large aperture

$L(a^2) < \dfrac{1}{\omega D}$



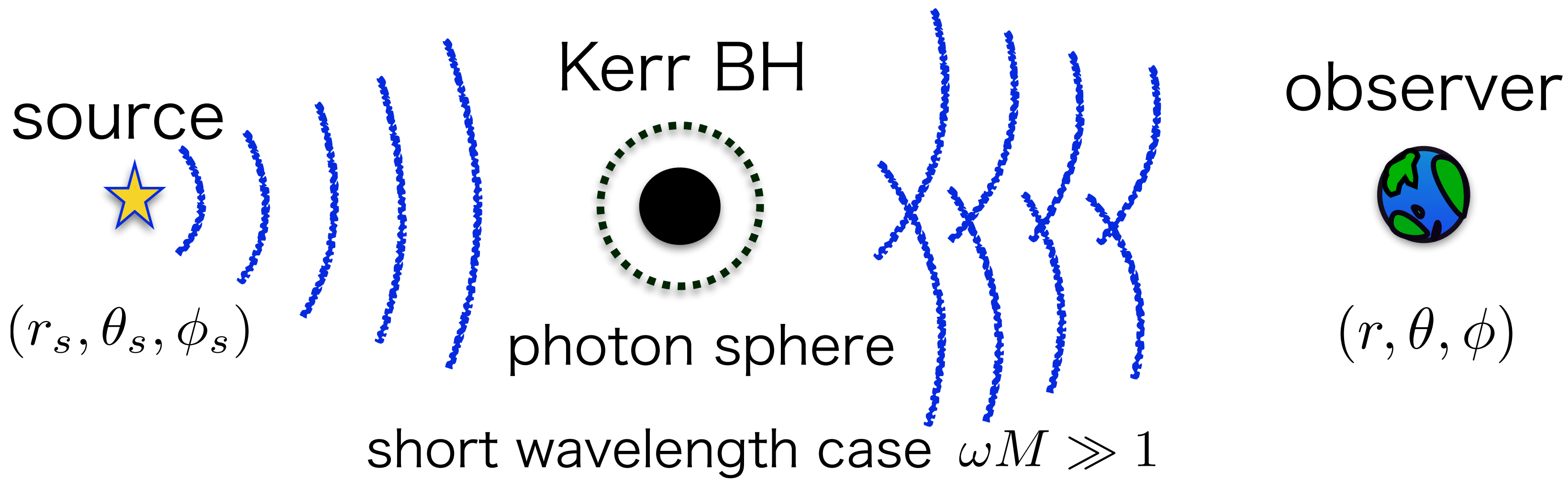
small aperture

For critical ω and D,

$$L(a^2) \sim \frac{1}{\omega_c D_c}$$

→ We can estimate a.

Summary



Green function

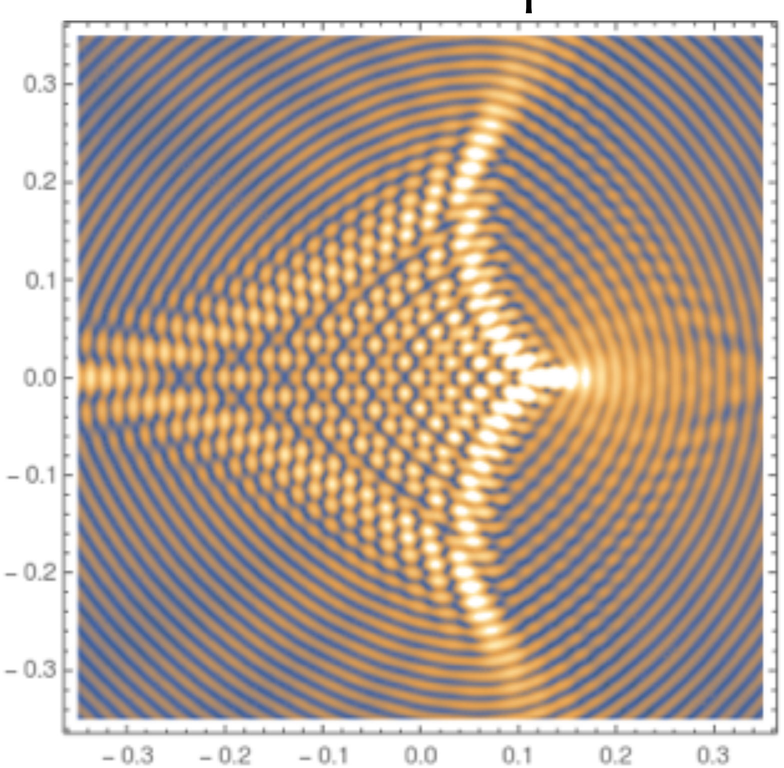
$$G(x_s, x) = G^{W=0} + G^{W\neq 0}$$

winding part

sum over l , m →

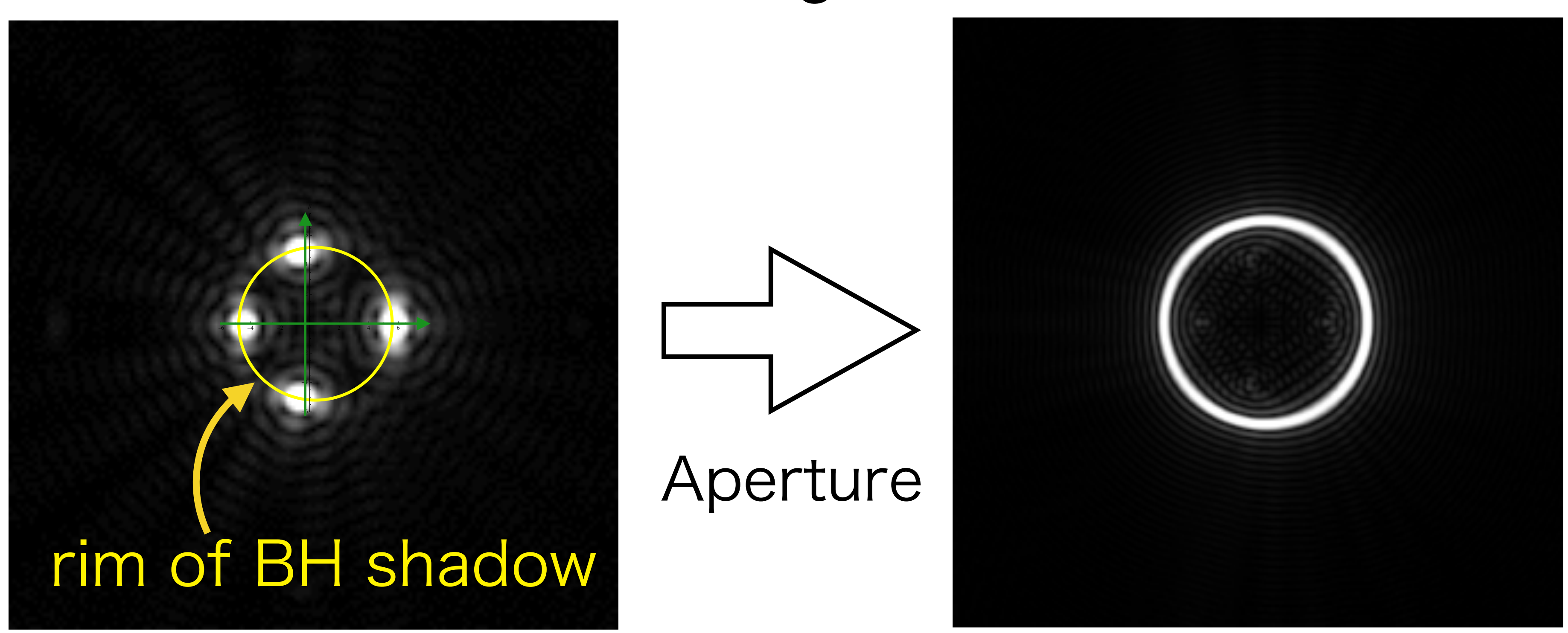
- QNMs $e^{2i\delta_{lm}} \to \infty$
- Photon sphere

Interference pattern



caustic

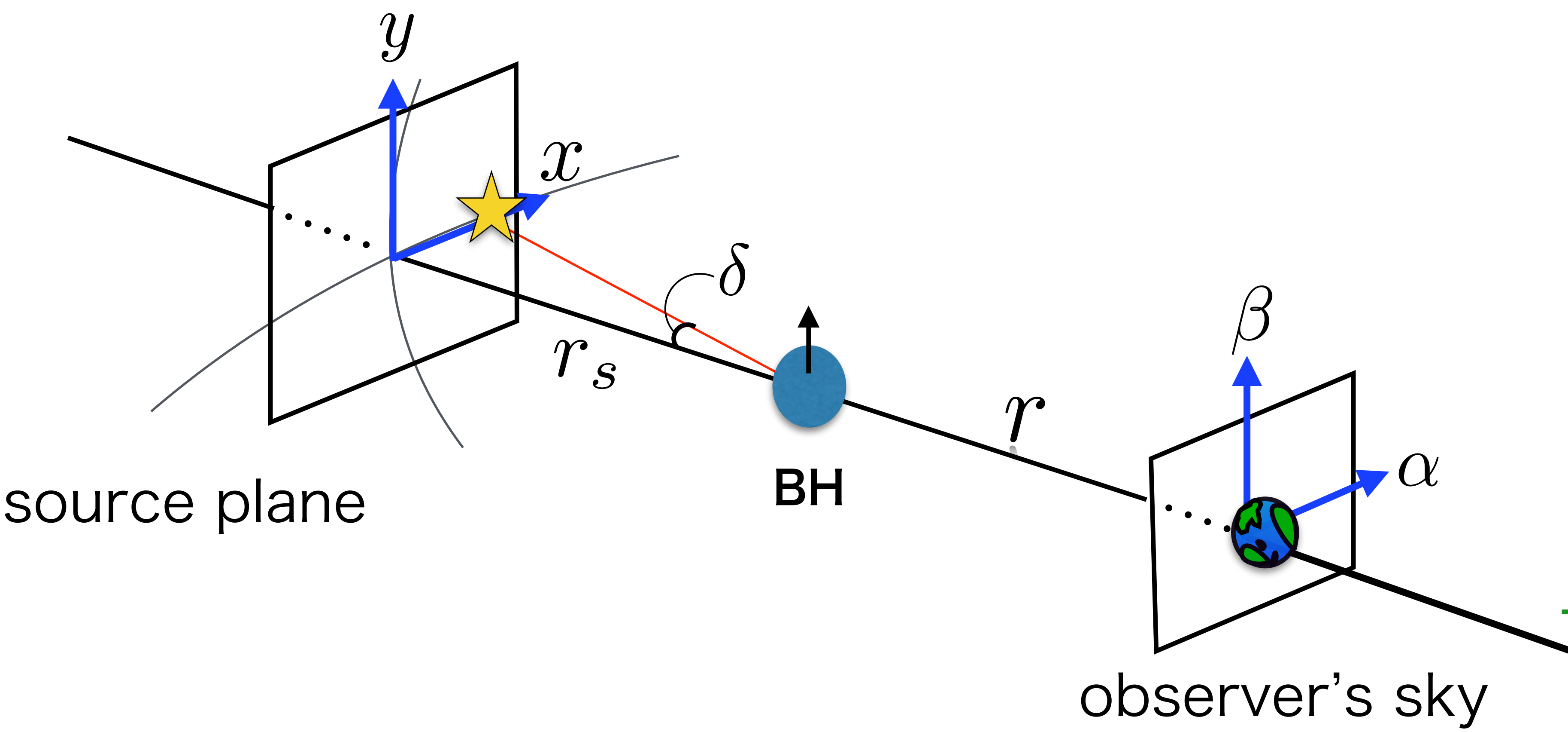
Images



4 images

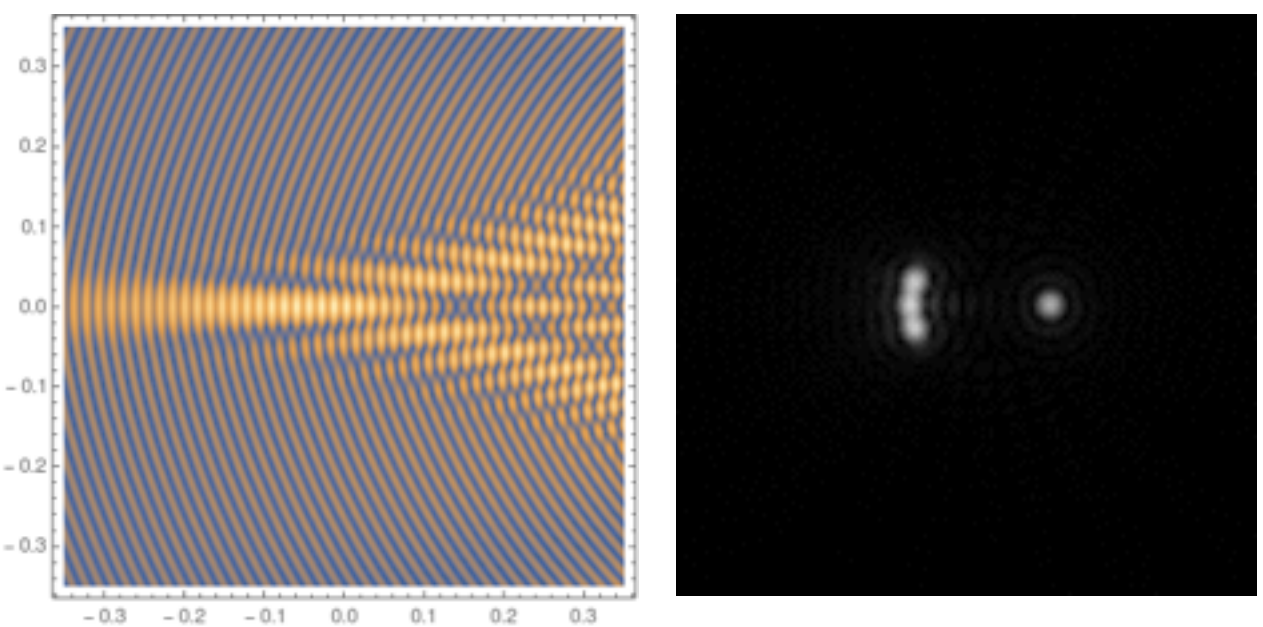
Ring

Other cases

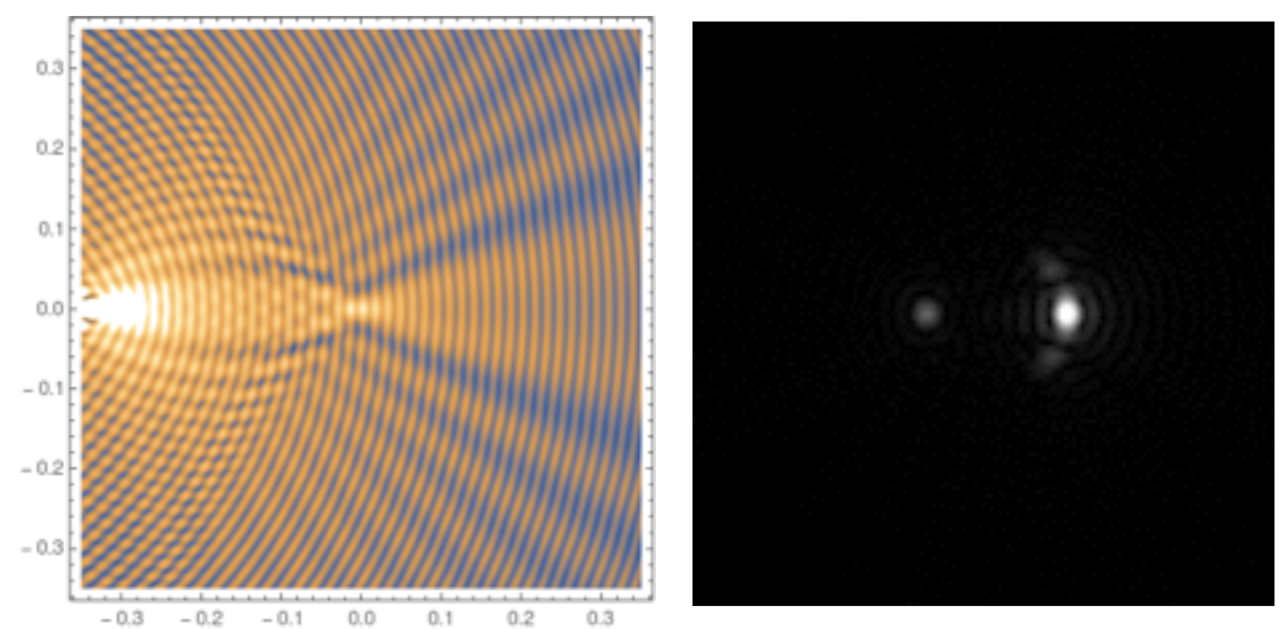


$\theta \sim \frac{\pi}{2} \quad \phi \sim 0$

$\delta = -45°$



$\delta = 30°$



Caustics