INSTITUT D'ASTROPHYSIQUE DE PARIS
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Hot Topics in General Relativity and Gravitation

POST-NEWTONIAN MODELLING of INSPIRALLING COMPACT BINARIES

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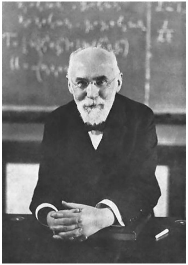
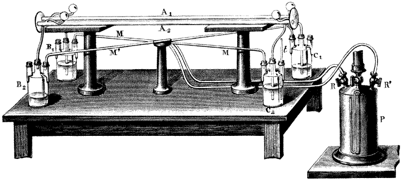
Gravitation et Cosmologie ($\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$)
Institut d'Astrophysique de Paris

31 juillet 2017

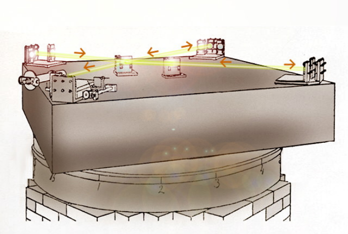
Special Relativity and “ondes gravifiques”

[Lorentz 1904; Poincaré 1905; Einstein 1905]





[Fizeau 1851]



[Michelson & Morley 1887]

100 years of gravitational waves [Einstein 1916]

688 Sitzung der physikalisch-mathematischen Klasse vom 22. Juni 1916

Näherungsweise Integration der Feldgleichungen der Gravitation.

Von A. Einstein.

Bei der Behandlung der meisten speziellen (nicht prinzipiellen) Probleme auf dem Gebiete der Gravitationstheorie kann man sich damit begnügen, die $g_{\mu\nu}$ in erster Näherung zu berechnen. Dabei bedient man sich mit Vorteil der imaginären Zeitvariable $x_4 = it$ aus denselben Gründen wie in der speziellen Relativitätstheorie. Unter »erster Näherung« ist dabei verstanden, daß die durch die Gleichung

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu} \qquad (1)$$

⟸ small perturbation of Minkowski's metric

definierten Größen $\gamma_{\mu\nu}$, welche linearen orthogonalen Transformationen gegenüber Tensorcharakter besitzen, gegen 1 als kleine Größen behandelt werden können, deren Quadrate und Produkte gegen die ersten Potenzen vernachlässigt werden dürfen. Dabei ist $\delta_{\mu\nu} = 1$ bzw. $\delta_{\mu\nu} = 0$, je nachdem $\mu = \nu$ oder $\mu \neq \nu$.

Wir werden zeigen, daß diese $\gamma_{\mu\nu}$ in analoger Weise berechnet werden können wie die retardierten Potentiale der Elektrodynamik.

Quadrupole moment formalism [Einstein 1918; Landau & Lifchitz 1947]

$$4\pi R^2 \bar{\mathfrak{J}} = \frac{\varkappa}{40\pi}\left[\sum_{\mu\nu} \dddot{J}_{\mu\nu}^2 - \frac{1}{3}\left(\sum_\mu \dddot{J}_{\mu\mu}\right)^2\right].$$

1. Einstein quadrupole formula

$$\boxed{\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\mathrm{GW}} = \frac{G}{5c^5}\left\{\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{v}{c}\right)^2\right\}}$$

2. Amplitude quadrupole formula

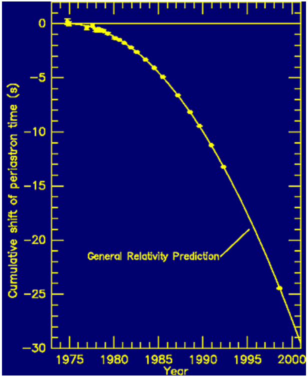
$$\boxed{h_{ij}^{\mathrm{TT}} = \frac{2G}{c^4 D}\left\{\frac{\mathrm{d}^2 Q_{ij}}{\mathrm{d}t^2}\left(t - \frac{D}{c}\right) + \mathcal{O}\left(\frac{v}{c}\right)\right\}^{\mathrm{TT}} + \mathcal{O}\left(\frac{1}{D^2}\right)}$$

3. Radiation reaction formula [Chandrasekhar & Esposito 1970; Burke & Thorne 1970]

$$\boxed{F_i^{\mathrm{reac}} = -\frac{2G}{5c^5}\rho\, x^j \frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5} + \mathcal{O}\left(\frac{v}{c}\right)^7}$$

The quadrupole formula works for the binary pulsar

[Taylor & Weisberg 1982]



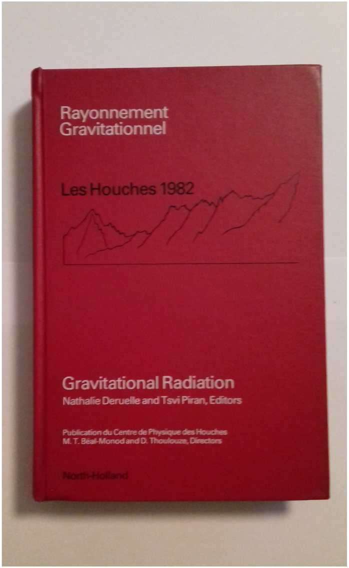
$$\dot{P} = -\frac{192\pi}{5c^5}\nu\left(\frac{2\pi G\,M}{P}\right)^{5/3}\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}} \approx -2.4\times 10^{-12}$$

[Peters & Mathews 1963, Esposito & Harrison 1975, Wagoner 1975, Damour & Deruelle 1983]

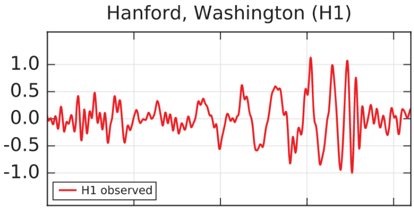
The radiation reaction controversy

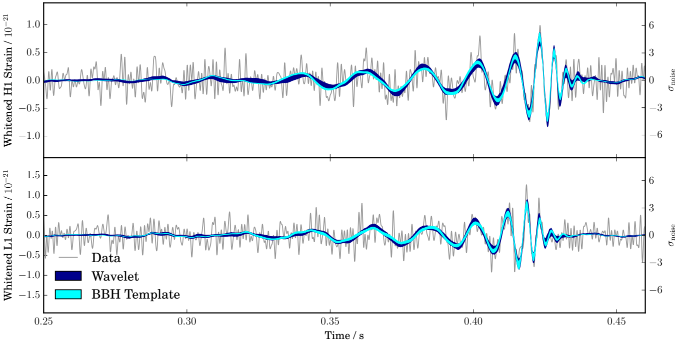
[Ehlers, Rosenblum, Goldberg & Havas 1976; Walker & Will 1980]

- Gravitational radiation: a review
  [Kip Thorne]
- GWs and motion of compact bodies
  [Thibault Damour]
- Methods of numerical relativity
  [Tsvi Piran]
- Interferometric detectors of GWs
  [Ron Drever]
- Optical detectors of GWs
  [Alain Brillet & Philippe Tourrenc]
- EOMs: a round table discussion
  [moderator: Abhay Ashtekar]



Binary black-hole event GW150914 [LIGO/VIRGO collaboration 2016]





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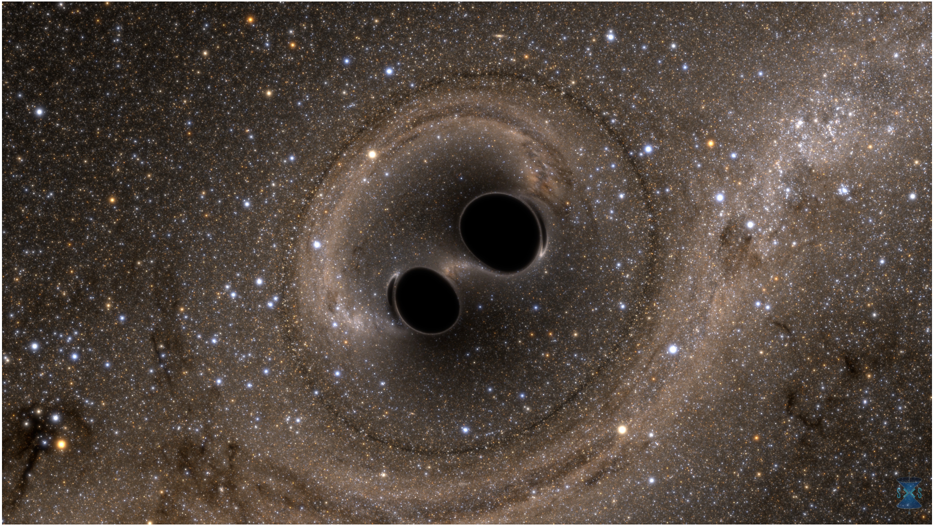
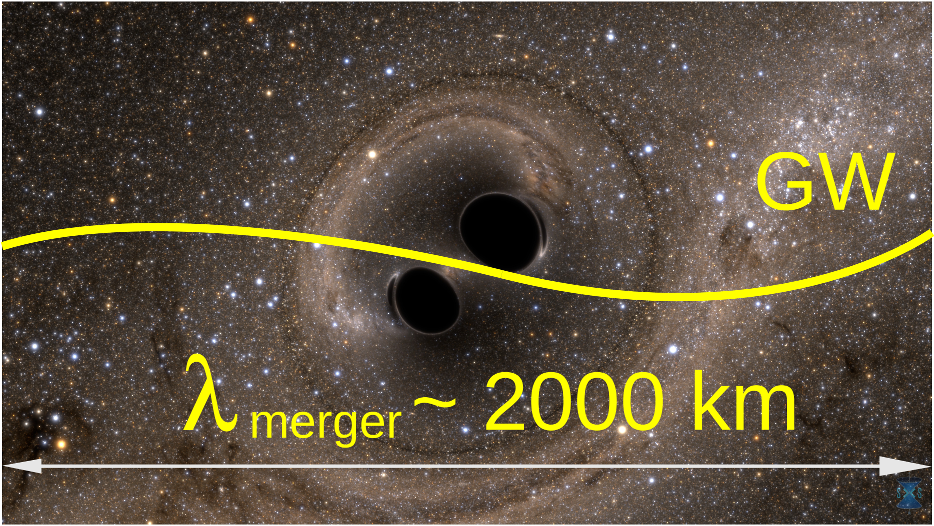


Binary black-hole event GW150914 [LIGO/VIRGO collaboration 2016]



The quadrupole formula works also for GW150914 !

1. The GW frequency is given in terms of the chirp mass $\mathcal{M} = \mu^{3/5} M^{2/5}$ by

$$f = \frac{1}{\pi}\left[\frac{256}{5}\frac{G\mathcal{M}^{5/3}}{c^5}(t_\mathsf{f} - t)\right]^{-3/8}$$

2. Therefore the chirp mass is directly measured as

$$\boxed{\mathcal{M} = \left[\frac{5}{96}\frac{c^5}{G\pi^{8/3}} f^{-11/3}\dot{f}\right]^{3/5}}$$

which gives $\mathcal{M} = 30 M_\odot$ thus $M \geqslant 70 M_\odot$

3. The GW amplitude is predicted to be

$$h_\text{eff} \sim 4.1 \times 10^{-22} \left(\frac{\mathcal{M}}{M_\odot}\right)^{5/6} \left(\frac{100\,\text{Mpc}}{D}\right) \left(\frac{100\,\text{Hz}}{f_\text{merger}}\right)^{-1/6} \sim 1.6 \times 10^{-21}$$

4. The distance $D = 400\,\text{Mpc}$ is measured from the signal itself

Total energy radiated by GW150914

1. The ADM energy of space-time is constant and reads (at any time t)

$$E_{\text{ADM}} = (m_1 + m_2)c^2 - \frac{G m_1 m_2}{2r} + \frac{G}{5c^5} \int_{-\infty}^{t} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2 (t')$$

2. Initially $E_{\text{ADM}} = (m_1 + m_2)c^2$ while finally (at time t_{f})

$$E_{\text{ADM}} = M_{\text{f}} c^2 + \frac{G}{5c^5} \int_{-\infty}^{t_{\text{f}}} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2 (t')$$

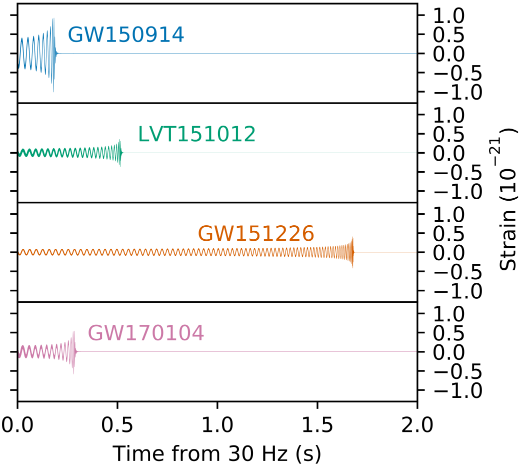
3. The total energy radiated in GW is

$$\boxed{\Delta E^{\text{GW}} = (m_1 + m_2 - M_{\text{f}})c^2 = \frac{G}{5c^5} \int_{-\infty}^{t_{\text{f}}} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2 (t') = \frac{G m_1 m_2}{2 r_{\text{f}}}}$$

4. The total power released is

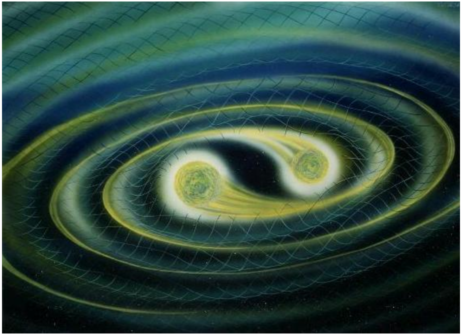
$$\boxed{\mathcal{P}^{\text{GW}} \sim \frac{3 M_\odot c^2}{0.2\,\text{s}} \sim 10^{49}\,\text{W} \sim 10^{-3}\,\frac{c^5}{G}}$$

Gravitational wave BBH events [LIGO/VIRGO collaboration 2016, 2017]



For BH binaries the detectors are mostly sensitive to the merger phase and a few cycles are observed before coalescence

The inspiral and merger of neutron star binaries

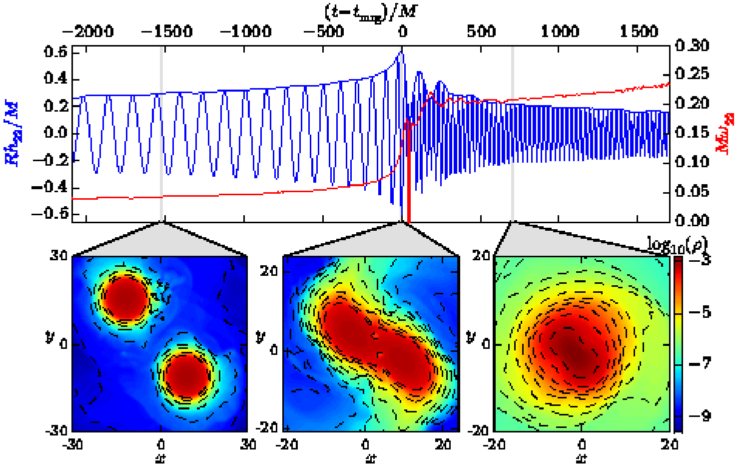


Many physical results are expected

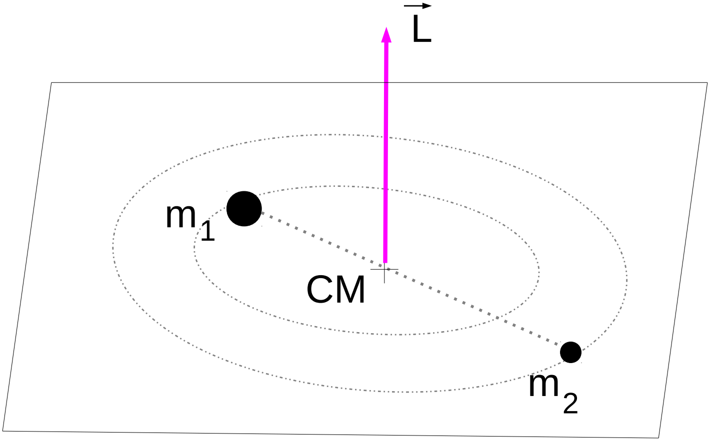
- Measurement of the equation of state of nuclear matter, formation and properties of hypermassive neutron stars, etc.
- Constraints on the existence of boson stars (gravitationally bound conglomerates of scalar particles [Ruffini & Bonazzola 1969])

Post-merger waveform of neutron star binaries

[Shibata *et al.*, Rezzolla *et al.* 1990-2010s]



Modelling the compact binary dynamics



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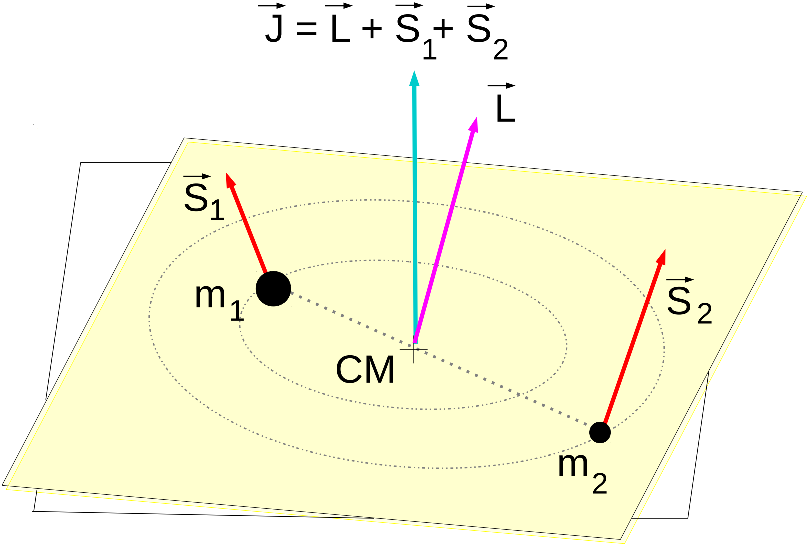
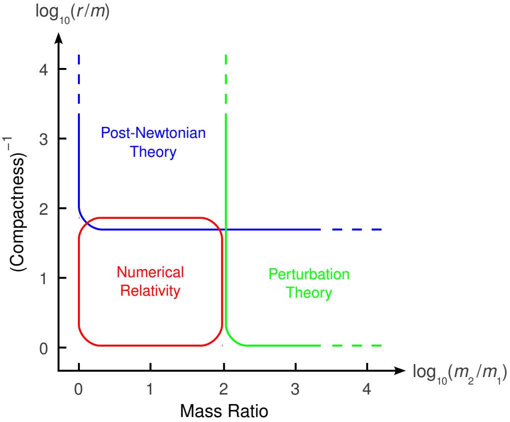


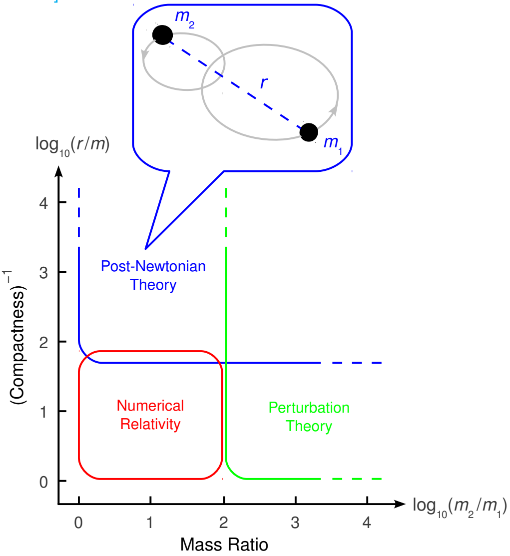
Methods to compute GW templates



[courtesy Alexandre Le Tiec]

Methods to compute GW templates

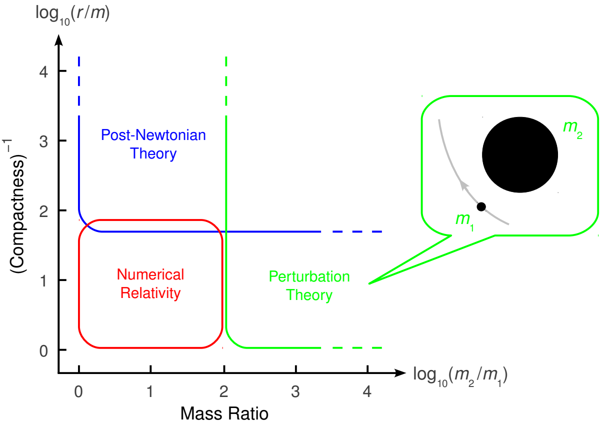
[see Blanchet 2014 for a review]



[courtesy Alexandre Le Tiec]

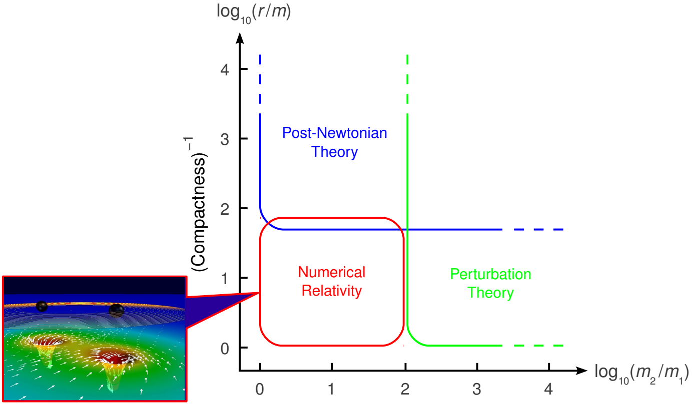
Methods to compute GW templates

[Detweiler 2008; Barack 2009]



[courtesy Alexandre Le Tiec]

Methods to compute GW templates

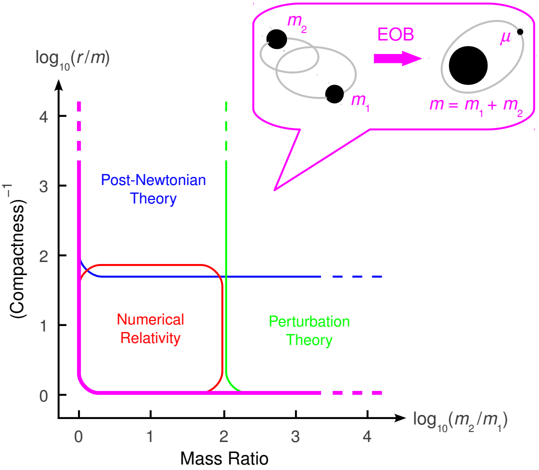


[Caltech/Cornell/CITA collaboration]

[courtesy Alexandre Le Tiec]

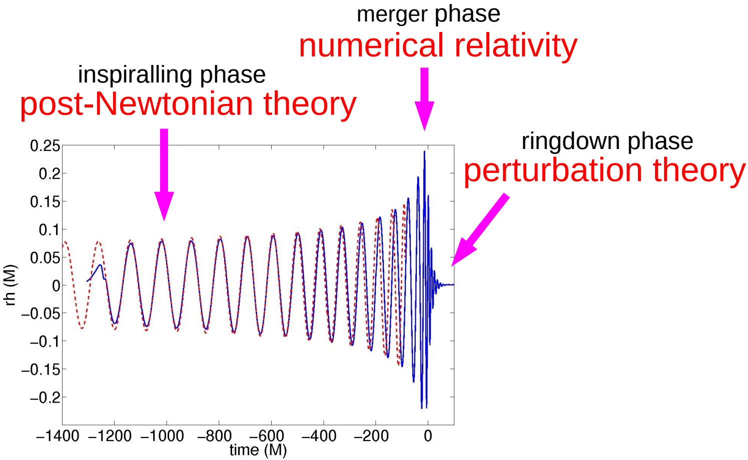
Methods to compute GW templates

[Buonanno & Damour 1998]



[courtesy Alexandre Le Tiec]

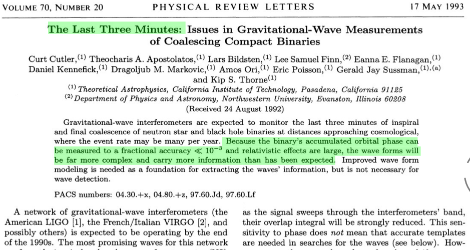
The gravitational chirp of compact binaries



Effective methods such as EOB that interpolate between the PN and NR are also very important notably for the data analysis

Why inspiralling binaries require high PN modelling

[Caltech “3mn paper” 1992; Blanchet & Schäfer 1993]



VOLUME 70, NUMBER 20 PHYSICAL REVIEW LETTERS 17 MAY 1993

The Last Three Minutes: Issues in Gravitational-Wave Measurements of Coalescing Compact Binaries

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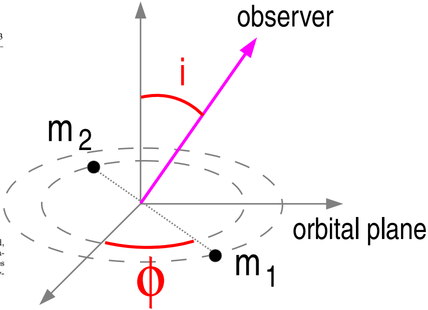
(Received 24 August 1992)

Gravitational-wave interferometers are expected to monitor the last three minutes of inspiral and final coalescence of neutron star and black hole binaries at distances approaching cosmological, where the event rate may be many per year. Because the binary's accumulated orbital phase can be measured to a fractional accuracy $\ll 10^{-3}$ and relativistic effects are large, the wave forms will be far more complex and carry more information than has been expected. Improved wave form modeling is needed as a foundation for extracting the waves' information, but is not necessary for wave detection.

PACS numbers: 04.30.+x, 04.80.+z, 97.60.Jd, 97.60.Lf

A network of gravitational-wave interferometers (the American LIGO [1], the French/Italian VIRGO [2], and possibly others) is expected to be operating by the end of the 1990s. The most promising waves for this network ... as the signal sweeps through the interferometers' band, their overlap integral will be strongly reduced. This sensitivity to phase does *not* mean that accurate templates are needed in searches for the waves (see below). How-



$$\phi(t) = \phi_0 - \underbrace{\frac{M}{\mu}\left(\frac{GM\omega}{c^3}\right)^{-5/3}}_{\text{quadrupole formalism}} \left\{1 + \underbrace{\frac{1\text{PN}}{c^2} + \frac{1.5\text{PN}}{c^3} + \cdots + \frac{3\text{PN}}{c^6} + \cdots}_{\text{needs to be computed with 3PN precision at least}}\right\}$$

Here 3PN means 5.5PN as a radiation reaction effect !

Methods to compute PN equations of motion

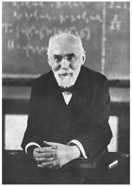
1. ADM Hamiltonian canonical formalism [Ohta *et al.* 1973; Schäfer 1985]
2. EOM in harmonic coordinates [Damour & Deruelle 1985; Blanchet & Faye 1998, 2000]
3. Extended fluid balls [Grishchuk & Kopeikin 1986]
4. Surface-integral approach [Itoh, Futamase & Asada 2000]
5. Effective-field theory (EFT) [Goldberger & Rothstein 2006; Foffa & Sturani 2011]

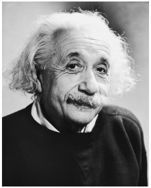
- EOM derived in a general frame for arbitrary orbits
- Dimensional regularization is applied for UV divergences[1]
- Radiation-reaction dissipative effects added separately by matching
- Spin effects can be computed within a pole-dipole approximation
- Tidal effects incorporated at leading 5PN and sub-leading 6PN orders

[1] Except in the surface-integral approach

The 1PN equations of motion

[Lorentz & Droste 1917; Einstein, Infeld & Hoffmann 1938]





$$
\begin{aligned}
\frac{\mathrm{d}^2\boldsymbol{r}_A}{\mathrm{d}t^2} = -\sum_{B\neq A}\frac{Gm_B}{r_{AB}^2}\boldsymbol{n}_{AB}\bigg[1 - 4\sum_{C\neq A}\frac{Gm_C}{c^2 r_{AC}} - \sum_{D\neq B}\frac{Gm_D}{c^2 r_{BD}}\left(1-\frac{\boldsymbol{r}_{AB}\cdot\boldsymbol{r}_{BD}}{r_{BD}^2}\right) \\
+\frac{1}{c^2}\left(\boldsymbol{v}_A^2 + 2\boldsymbol{v}_B^2 - 4\boldsymbol{v}_A\cdot\boldsymbol{v}_B - \frac{3}{2}(\boldsymbol{v}_B\cdot\boldsymbol{n}_{AB})^2\right)\bigg] \\
+\sum_{B\neq A}\frac{Gm_B}{c^2 r_{AB}^2}\boldsymbol{v}_{AB}[\boldsymbol{n}_{AB}\cdot(3\boldsymbol{v}_B - 4\boldsymbol{v}_A)] - \frac{7}{2}\sum_{B\neq A}\sum_{D\neq B}\frac{G^2 m_B m_D}{c^2 r_{AB} r_{BD}^3}\boldsymbol{n}_{BD}
\end{aligned}
$$

4PN: state-of-the-art on equations of motion

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2} n_{12}^i + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3} + \frac{4G^2m_2^2}{r_{12}^3} + \cdots\right] n_{12}^i + \cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}} + \underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}[\cdots]}_{\substack{\text{2.5PN}\\\text{radiation reaction}}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\substack{\text{3.5PN}\\\text{radiation reaction}}} + \underbrace{\frac{1}{c^8}[\cdots]}_{\substack{\text{4PN}\\\text{conservative \& radiation tail}}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

2PN:

[Otha, Okamura, Kimura & Hiida 1973, 1974; Damour & Schäfer 1985]	ADM Hamiltonian
[Damour & Deruelle 1981; Damour 1983]	Harmonic coordinates
[Kopeikin 1985; Grishchuk & Kopeikin 1986]	Extended fluid balls
[Blanchet, Faye & Ponsot 1998]	Direct PN iteration
[Itoh, Futamase & Asada 2001]	Surface integral method

4PN: state-of-the-art on equations of motion

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2}n_{12}^i + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3}+\frac{4G^2m_2^2}{r_{12}^3}+\cdots\right]n_{12}^i+\cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}}$$
$$+ \underbrace{\frac{1}{c^4}\left[\cdots\right]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}\left[\cdots\right]}_{\substack{\text{2.5PN}\\\text{radiation reaction}}} + \underbrace{\frac{1}{c^6}\left[\cdots\right]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}\left[\cdots\right]}_{\substack{\text{3.5PN}\\\text{radiation reaction}}} + \underbrace{\frac{1}{c^8}\left[\cdots\right]}_{\substack{\text{4PN}\\\text{conservative \& radiation tail}}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

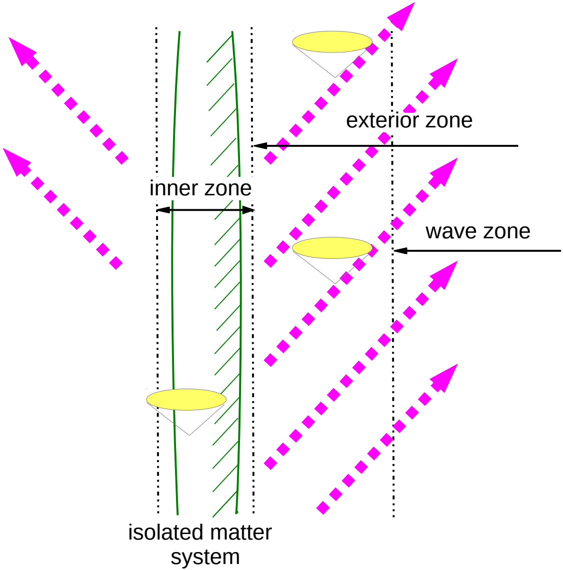
3PN	[Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001ab]	ADM Hamiltonian
	[Blanchet-Faye-de Andrade 2000, 2001; Blanchet & Iyer 2002]	Harmonic EOM
	[Itoh & Futamase 2003; Itoh 2004]	Surface integral method
	[Foffa & Sturani 2011]	Effective field theory
4PN	[Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014]	ADM Hamiltonian
	[Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2015, 2016, 2017ab]	Fokker Lagrangian
	[Foffa & Sturani 2012, 2013] (partial results)	Effective field theory

Methods to compute PN radiation field

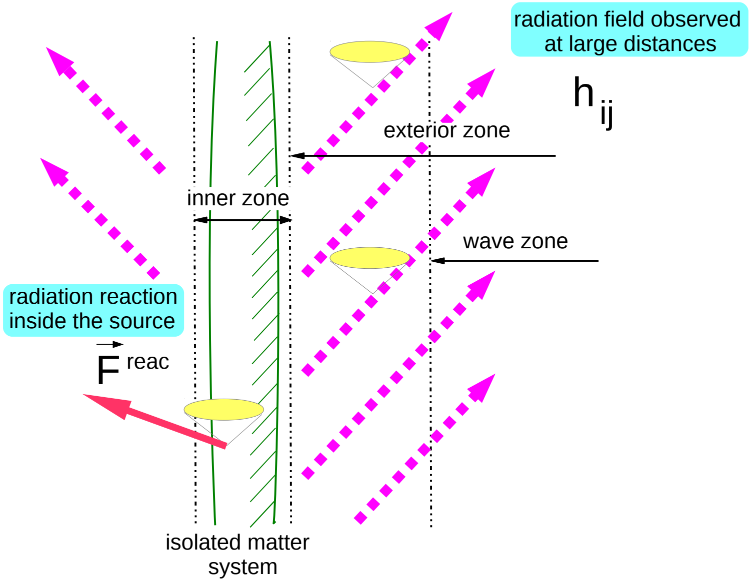
1. Multipolar-post-Minkowskian (MPM) & PN [Blanchet-Damour-Iyer 1986, . . . , 1998]
2. Direct iteration of the relaxed field equations (DIRE) [Will-Wiseman-Pati 1996, . . .]
3. Effective-field theory (EFT) [Hari Dass & Soni 1982; Goldberger & Ross 2010]

- Involves a machinery of tails and related non-linear effects
- Uses dimensional regularization to treat point-particle singularities
- Phase evolution relies on balance equations valid in adiabatic approximation
- Spin effects are incorporated within a pole-dipole approximation
- Provides polarization waveforms for DA & spin-weighted spherical harmonics decomposition for NR

Isolated matter system in general relativity

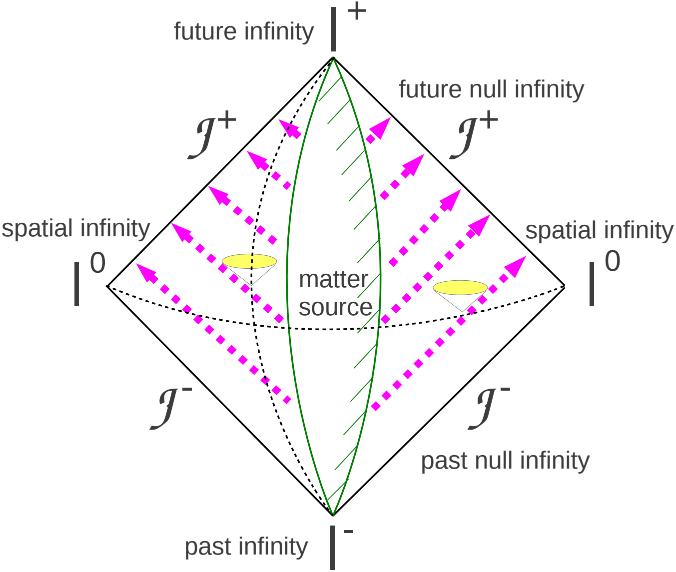


Isolated matter system in general relativity



Asymptotic structure of radiating space-time

[Bondi-Sachs-Penrose formalism 1960s]



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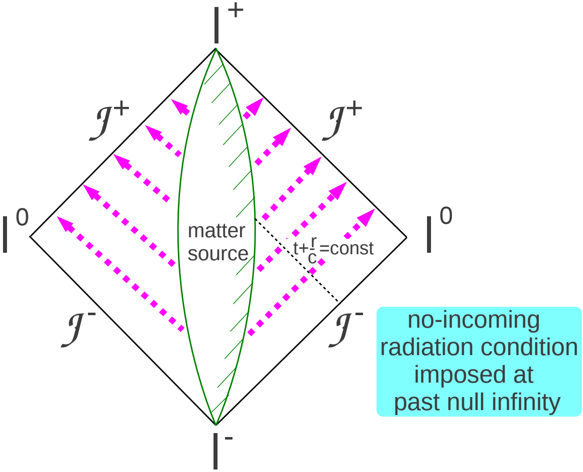


$$\lim_{\substack{r\to+\infty \\ t+\frac{r}{c}=\text{const}}} \left(\frac{\partial}{\partial r}+\frac{\partial}{c\partial t}\right)\left(r h^{\alpha\beta}\right)=0$$

Linearized multipolar vacuum solution [Thorne 1980]

General solution of linearized vacuum field equations in harmonic coordinates

$$\boxed{\Box h_1^{\alpha\beta} = \partial_\mu h_1^{\alpha\mu} = 0}$$

$$h_1^{00} = -\frac{4}{c^2}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left(\frac{1}{r}I_L(u)\right)$$

$$h_1^{0i} = \frac{4}{c^3}\sum_{\ell=1}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left(\frac{1}{r}I^{(1)}_{iL-1}(u)\right) + \frac{\ell}{\ell+1}\epsilon_{iab}\partial_{aL-1}\left(\frac{1}{r}J_{bL-1}(u)\right)\right\}$$

$$h_1^{ij} = -\frac{4}{c^4}\sum_{\ell=2}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-2}\left(\frac{1}{r}I^{(2)}_{ijL-2}(u)\right) + \frac{2\ell}{\ell+1}\partial_{aL-2}\left(\frac{1}{r}\epsilon_{ab(i}J^{(1)}_{j)bL-2}(u)\right)\right\}$$

- multipole moments $I_L(u)$ and $J_L(u)$ arbitrary functions of $u = t - r/c$
- mass $M = I = \mathrm{const}$, center-of-mass position $X_i \equiv I_i/M = \mathrm{const}$, linear momentum $P_i \equiv I_i^{(1)} = 0$, angular momentum $J_i = \mathrm{const}$

Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988, 1992; Blanchet 1987, 1993, 1998]

- The linearized solution is the starting point of an explicit MPM algorithm

$$h^{\alpha\beta}_{\text{MPM}} = \sum_{n=1}^{+\infty} G^n \, h^{\alpha\beta}_n$$

- Hierarchy of perturbation equations is solved by induction over n

$$\Box h^{\alpha\beta}_n = \Lambda^{\alpha\beta}_n[h_1, h_2, \ldots, h_{n-1}]$$
$$\partial_\mu h^{\alpha\mu}_n = 0$$

- A regularization is required in order to cope with the divergency of the multipolar expansion when $r \to 0$

Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988, 1992; Blanchet 1987, 1993, 1998]

1. Multiply source term by r^B where $B \in \mathbb{C}$ and integrate

$$u_n^{\alpha\beta}(B) = \Box_{\text{ret}}^{-1}\left[r^B \Lambda_n^{\alpha\beta}\right]$$

2. Consider Laurent expansion when $B \to 0$

$$u_n^{\alpha\beta}(B) = \sum_{j=j_{\text{min}}}^{+\infty} u_{jn}^{\alpha\beta} B^j \quad \text{then} \quad \begin{cases} j < 0 & \Longrightarrow \quad \Box u_{jn}^{\alpha\beta} = 0 \\ j \geqslant 0 & \Longrightarrow \quad \Box u_{jn}^{\alpha\beta} = \frac{(\ln r)^j}{j!}\Lambda_n^{\alpha\beta} \end{cases}$$

3. Define the finite part (FP) when $B \to 0$ to be the zeroth coefficient $u_{0n}^{\alpha\beta}$

$$u_n^{\alpha\beta} = \mathrm{FP}\Box_{\text{ret}}^{-1}\left[r^B \Lambda_n^{\alpha\beta}\right] \quad \text{then} \quad \Box u_n^{\alpha\beta} = \Lambda_n^{\alpha\beta}$$

Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988, 1992; Blanchet 1987, 1993, 1998]

1. Harmonic gauge condition is not yet satisfied

$$w_n^\alpha = \partial_\mu u_n^{\alpha\mu} = \mathsf{FP}\,\Box_{\text{ret}}^{-1}\left[B\,r^{B-1} n_i \Lambda_n^{\alpha i}\right]$$

2. But $\Box w_n^\alpha = 0$ hence we can compute $v_n^{\alpha\beta}$ such that at once

$$\Box v_n^{\alpha\beta} = 0 \quad \text{and} \quad \partial_\mu v_n^{\alpha\mu} = -w_n^\alpha$$

3. Thus we define

$$\boxed{h_n^{\alpha\beta} = u_n^{\alpha\beta} + v_n^{\alpha\beta}}$$

Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988, 1992; Blanchet 1987, 1993, 1998]

Theorem 1:
The MPM solution is the most general solution of Einstein's vacuum equations outside an isolated matter system

Theorem 2:
The general structure of the PN expansion is

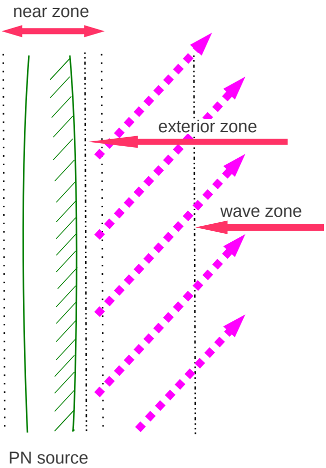
$$h^{\alpha\beta}_{\mathsf{PN}}(\mathbf{x}, t, c) = \sum_{\substack{p \geqslant 2 \\ q \geqslant 0}} \frac{(\ln c)^q}{c^p} h^{\alpha\beta}_{p,q}(\mathbf{x}, t)$$

Theorem 3:
The MPM solution is asymptotically simple at future null infinity in the sense of Penrose [1963, 1965] and agrees with the Bondi-Sachs [1962] formalism

$$\underbrace{M_{\mathsf{B}}(u)}_{\text{Bondi mass}} = \underbrace{M}_{\text{ADM mass}} - \frac{G}{5c^5}\int_{-\infty}^{u} \mathrm{d}\tau \, I^{(3)}_{ij}(\tau) I^{(3)}_{ij}(\tau)$$
$$+ \quad \text{higher multipoles and higher PM computable to any order}$$

The MPM-PN formalism [Blanchet-Damour-Iyer formalism 1980-90s]

A multipolar post-Minkowskian (MPM) expansion in the exterior zone is matched to a general post-Newtonian (PN) expansion in the near zone



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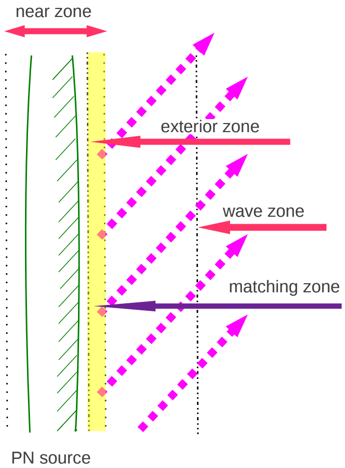


The matching equation [Kates 1980; Anderson *et al.* 1982; Blanchet 1998]

This is a variant of the theory of matched asymptotic expansions

$$\text{match} \quad \begin{cases} \text{the multipole expansion} \quad \mathcal{M}(h^{\alpha\beta}) \equiv h^{\alpha\beta}_{\mathsf{MPM}} \\ \qquad\qquad \text{with} \\ \text{the PN expansion} \quad \bar{h}^{\alpha\beta} \equiv h^{\alpha\beta}_{\mathsf{PN}} \end{cases}$$

$$\boxed{\overline{\mathcal{M}(h^{\alpha\beta})} = \mathcal{M}(\bar{h}^{\alpha\beta})}$$

- Left side is the NZ expansion $(r \to 0)$ of the exterior MPM field
- Right side is the FZ expansion $(r \to \infty)$ of the inner PN field

1. The matching equation has been implemented at any post-Minkowskian order in the exterior field and any PN order in the inner field
2. It gives a unique (formal) multipolar-post-Newtonian solution valid everywhere inside and outside the source
3. The solution recovers the [Bondi-Sachs-Penrose] formalism at $\mathcal{J}^+$

General solution for the multipolar field [Blanchet 1995, 1998]

$$\mathcal{M}(h^{\mu\nu}) = \text{FP}\Box^{-1}_{\text{ret}}\mathcal{M}(\Lambda^{\mu\nu}) + \underbrace{\sum_{\ell=0}^{+\infty}\partial_L\left\{\frac{M_L^{\mu\nu}(t-r/c)}{r}\right\}}_{\text{homogeneous retarded solution}}$$

where
$$M_L^{\mu\nu}(t) = \text{FP}\int \mathrm{d}^3\mathbf{x}\,\hat{x}_L\int_{-1}^{1}\mathrm{d}z\,\delta_\ell(z)\;\underbrace{\bar{\tau}^{\mu\nu}(\mathbf{x},t-zr/c)}_{\text{PN expansion of the pseudo-tensor}}$$

- The FP procedure plays the role of an UV regularization in the non-linearity term but an IR regularization in the multipole moments
- From this one obtains the multipole moments of the source at any PN order solving the wave generation problem
- This is a formal PN solution *i.e.* a set of rules for generating the PN series regardless of the exact mathematical nature of this series
- The formalism is equivalent to the DIRE formalism [Will-Wiseman-Pati]

General solution for the inner PN field

[Poujade & Blanchet 2002; Blanchet, Faye & Nissanke 2004]

$$\bar{h}^{\mu\nu} = \mathrm{FP}\Box^{-1}_{\text{ret}}\bar{\tau}^{\mu\nu} + \underbrace{\sum_{\ell=0}^{+\infty}\partial_L\left\{\frac{R_L^{\mu\nu}(t-r/c) - R_L^{\mu\nu}(t+r/c)}{r}\right\}}_{\text{homogeneous antisymmetric solution}}$$

where $$R_L^{\mu\nu}(t) = \mathrm{FP}\int \mathrm{d}^3\mathbf{x}\,\hat{x}_L\int_1^\infty \mathrm{d}z\,\gamma_\ell(z)\;\underbrace{\mathcal{M}(\tau^{\mu\nu})(\mathbf{x}, t - zr/c)}_{\text{multipole expansion of the pseudo-tensor}}$$

- The radiation reaction effects starting at 2.5PN order appropriate to an isolated system are determined to any order
- In particular nonlinear radiation reaction effects associated with tails are contained in the second term and start at 4PN order

Radiation reaction potentials to 4PN order

[Burke & Thorne 1972; Blanchet 1996]

$$V^{\text{reac}}(\mathbf{x},t) = -\frac{G}{5c^5}x^{ij}I^{(5)}_{ij}(t) + \frac{G}{c^7}\left[\frac{1}{189}x^{ijk}\,I^{(7)}_{ijk}(t) - \frac{1}{70}\mathbf{x}^2 x^{ij}\,I^{(7)}_{ij}(t)\right] \underbrace{- \frac{4G^2M}{5c^8}\,x^{ij}\int_0^{+\infty}\mathrm{d}\tau\,I^{(7)}_{ij}(t-\tau)\left[\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{11}{12}\right]}_{\text{4PN radiation reaction tail}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

$$V_i^{\text{reac}}(\mathbf{x},t) = \frac{G}{c^5}\left[\frac{1}{21}\hat{x}^{ijk}\,I^{(6)}_{jk}(t) - \frac{4}{45}\,\epsilon_{ijk}\,x^{jl}\,J^{(5)}_{kl}(t)\right] + \mathcal{O}\left(\frac{1}{c^7}\right)$$

Radiative moments at future null infinity

1. Correct for the logarithmic deviation of retarded time in harmonic coordinates with respect to the actual null coordinate

$$\underbrace{T - \frac{R}{c}}_{\text{radiative coordinates}} = \underbrace{t - \frac{r}{c}}_{\text{harmonic coordinates}} - \frac{2GM}{c^3} \ln\left(\frac{r}{c\tau_0}\right) + \mathcal{O}\left(\frac{1}{r}\right)$$

2. In radiative coordinates the field admits an expansion in powers of $1/R$ without any powers of $\ln R$ at $\mathcal{I}^+$ [Bondi *et al.* 1962]

3. The asymptotic waveform is then parametrized by radiative multipole moments U_L and V_L [Thorne 1980]

$$h_{ij}^{\mathrm{TT}} = \frac{1}{R}\sum_{\ell=2}^{\infty} N_{L-2}\,\underbrace{U_{ijL-2}(T-R/c)}_{\text{mass-type moment}} + \epsilon_{ab(i} N_{aL-1}\,\underbrace{V_{j)bL-2}(T-R/c)}_{\text{current-type moment}} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

The 4.5PN radiative quadrupole moment

[Marchand, Blanchet & Faye 2016]

$$U_{ij}(t) = I^{(2)}_{ij}(t) + \underbrace{\frac{GM}{c^3}\int_0^{+\infty} d\tau\, I^{(4)}_{ij}(t-\tau)\left[2\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{11}{6}\right]}_{\text{1.5PN tail integral}}$$

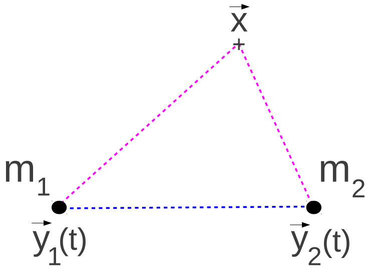
$$+\frac{G}{c^5}\left\{\underbrace{-\frac{2}{7}\int_0^{+\infty} d\tau\, I^{(3)}_{a<i}I^{(3)}_{j>a}(t-\tau)}_{\text{2.5PN memory integral}} + \text{instantaneous terms}\right\}$$

$$+\underbrace{\frac{G^2M^2}{c^6}\int_0^{+\infty} d\tau\, I^{(5)}_{ij}(t-\tau)\left[2\ln^2\left(\frac{\tau}{2\tau_0}\right)+\frac{57}{35}\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{124627}{22050}\right]}_{\text{3PN tail-of-tail integral}}$$

$$+\underbrace{\frac{G^3M^3}{c^9}\int_0^{+\infty} d\tau\, I^{(6)}_{ij}(t-\tau)\left[\frac{4}{3}\ln^3\left(\frac{\tau}{2\tau_0}\right)+\cdots+\frac{129268}{33075}+\frac{428}{315}\pi^2\right]}_{\text{4.5PN tail-of-tail-of-tail integral}}$$

$$+\mathcal{O}\left(\frac{1}{c^{10}}\right)$$

Problem of point particles and UV divergences



$$U(\mathbf{x},t) = \frac{Gm_1}{|\mathbf{x}-\mathbf{y}_1(t)|} + \frac{Gm_2}{|\mathbf{x}-\mathbf{y}_2(t)|}$$

$$\frac{\mathrm{d}^2\mathbf{y}_1}{\mathrm{d}t^2} = (\boldsymbol{\nabla} U)\,(\mathbf{y}_1(t),t) \overset{?}{=} -Gm_2\frac{\mathbf{y}_1-\mathbf{y}_2}{|\mathbf{y}_1-\mathbf{y}_2|^3}$$

- For extended bodies the self-acceleration of the body cancels out by Newton's action-reaction law
- For point particles one needs a self-field regularization to remove the infinite self-field of the particle

Dimensional regularization for UV divergences

[t'Hooft & Veltman 1972; Bollini & Giambiagi 1972; Breitenlohner & Maison 1977]

1. Einstein's field equations are solved in d spatial dimensions (with $d \in \mathbb{C}$) with distributional sources. In Newtonian approximation

$$\Delta U = -4\pi \frac{2(d-2)}{d-1} G\rho$$

2. For two point-particles $\rho = m_1 \delta_{(d)}(\mathbf{x} - \mathbf{y}_1) + m_2 \delta_{(d)}(\mathbf{x} - \mathbf{y}_2)$ we get

$$U(\mathbf{x}, t) = \frac{2(d-2)k}{d-1} \left(\frac{G m_1}{|\mathbf{x} - \mathbf{y}_1|^{d-2}} + \frac{G m_2}{|\mathbf{x} - \mathbf{y}_2|^{d-2}} \right) \quad \text{with} \quad k = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}}$$

3. Computations are performed when $\Re(d)$ is a large negative number, and the result is analytically continued for any $d \in \mathbb{C}$ except for isolated poles
4. Dimensional regularization is then followed by a renormalization of the worldline of the particles so as to absorb the poles $\propto (d-3)^{-1}$

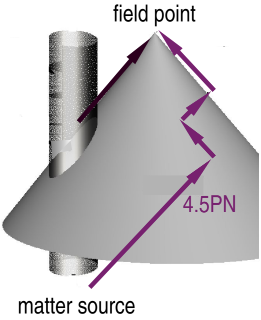
3.5PN energy flux of compact binaries

[BDIWW 1995; B 1996, 1998; BFIJ 2002; BDEI 2006]

$$
\begin{aligned}
\mathcal{F} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ 1 &+ \overbrace{\left(-\frac{1247}{336} - \frac{35}{12}\nu\right) x}^{\text{1PN}} + \overbrace{4\pi x^{3/2}}^{\text{1.5PN tail}} \\
&+ \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right) x^2 + \overbrace{\left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2}}^{\text{2.5PN tail}} \\
&+ \Bigg[\frac{6643739519}{69854400} + \overbrace{\frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_\mathsf{E} - \frac{856}{105}\ln(16\,x)}^{\text{3PN tail-of-tail}} \\
&\qquad + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\Bigg] x^3 \\
&+ \underbrace{\left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2}}_{\text{3.5PN tail}} + \mathcal{O}\left(\frac{1}{c^8}\right) \Bigg\}
\end{aligned}
$$

4.5PN tail interactions between moments

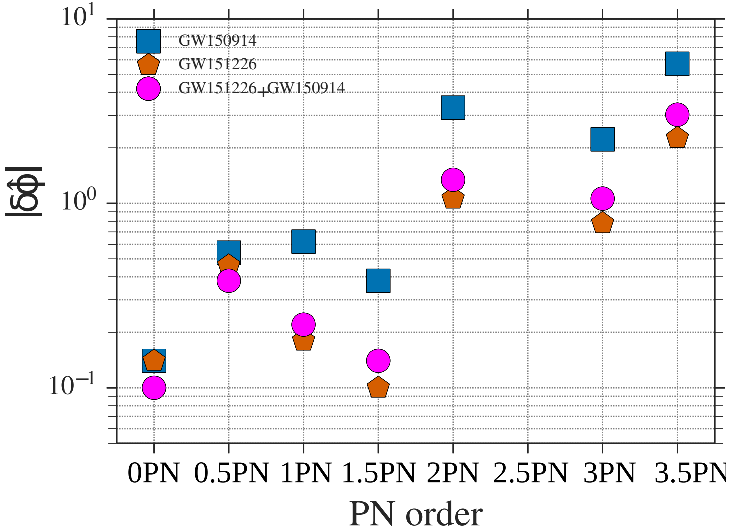
[Marchand, Blanchet & Faye 2016]



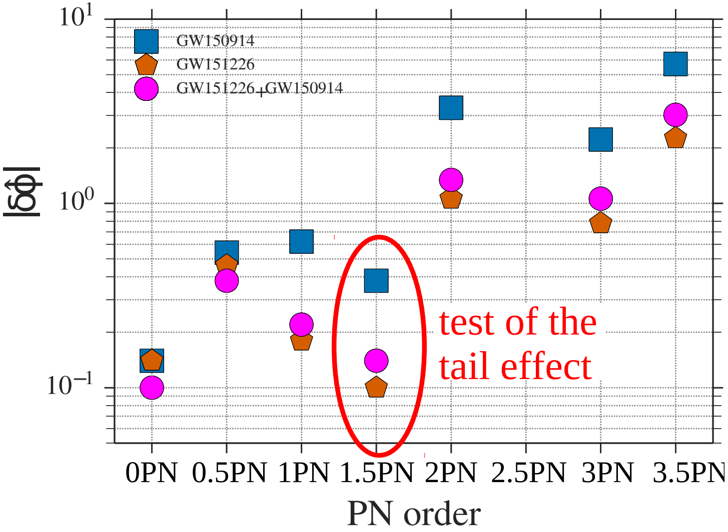
$$\mathcal{F}^{\text{4.5PN}} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ \Bigg(\frac{265978667519}{745113600} - \frac{6848}{105}\gamma_\text{E} - \frac{3424}{105}\ln(16x) + \left[\frac{2062241}{22176} + \frac{41}{12}\pi^2\right]\nu - \frac{133112905}{290304}\nu^2 - \frac{3719141}{38016}\nu^3 \Bigg) \pi x^{9/2} \Bigg\}$$

- Perfect agreement with results from BH perturbation theory in the small mass ratio limit $\nu \to 0$ [Tanaka, Tagoshi & Sasaki 1996]
- However the 4PN term in the flux is still in progress [Marchand *et al.* 2017]

Measurement of PN parameters [LIGO/VIRGO collaboration 2016, 2017]



Measuring GW tails [Blanchet & Sathyaprakash 1994, 1995]



Summary of known PN orders

Method	Equations of motion	Energy flux	Waveform
Multipolar-post-Minkowskian & post-Newtonian (MPM-PN)	4PN non-spin 3.5PN (NNL) SO 3PN (NL) SS 3.5PN (NL) SSS	3.5PN non-spin[2] 4PN (NNL) SO 3PN (NL) SS 3.5PN (NL) SSS	3.5PN non-spin 1.5PN (L) SO 2PN (L) SS
Canonical ADM Hamiltonian	4PN non-spin 3.5PN (NNL) SO 4PN (NNL) SS 3.5PN (NL) SSS		
Effective Field Theory (EFT)	3PN non-spin 2.5PN (NL) SO 4PN (NNL) SS	2PN non-spin 3PN (NL) SS	
Direct Integration of Relaxed Equations (DIRE)	2.5PN non-spin 1.5PN (L) SO 2PN (L) SS	2PN non-spin 1.5PN (L) SO 2PN (L) SS	2PN non-spin 1.5PN (L) SO 2PN (L) SS
Surface Integral	3PN non-spin		

Many works devoted to spins:

- Spin effects (SO, SS, SSS) are known in EOM up to 4PN order
- SO effects are known in radiation field up to 4PN
- SS in radiation field known to 3PN

[2] The 4.5PN coefficient is also known