Hot Topics in General Relativity and Gravitation

FIRST LAW OF COMPACT BINARY MECHANICS AT 4PN ORDER

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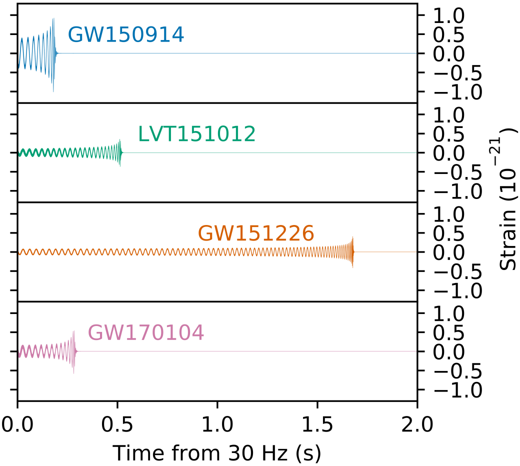
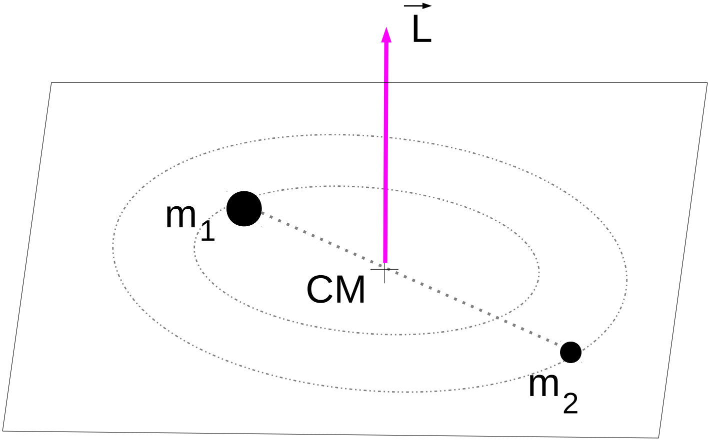
Gravitational wave BBH events [LIGO/VIRGO collaboration 2016, 2017]



For BH binaries the detectors are mostly sensitive to the merger phase and a few cycles are observed before coalescence

Modelling the compact binary dynamics



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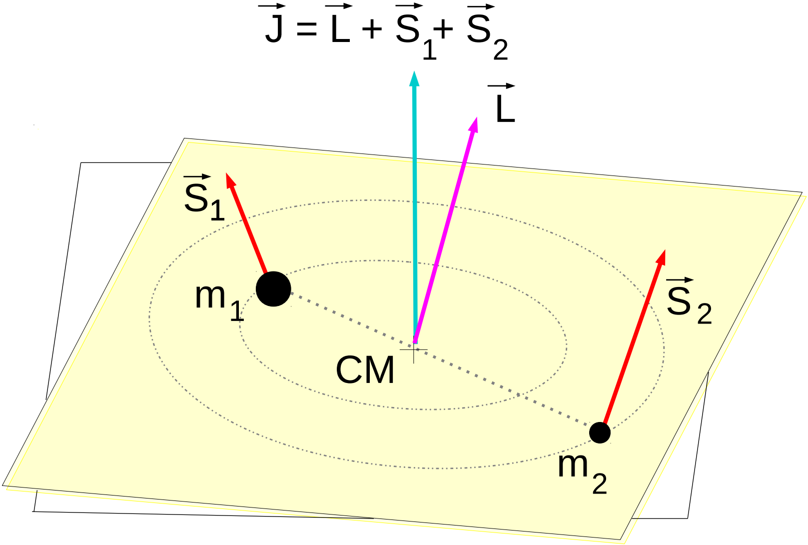
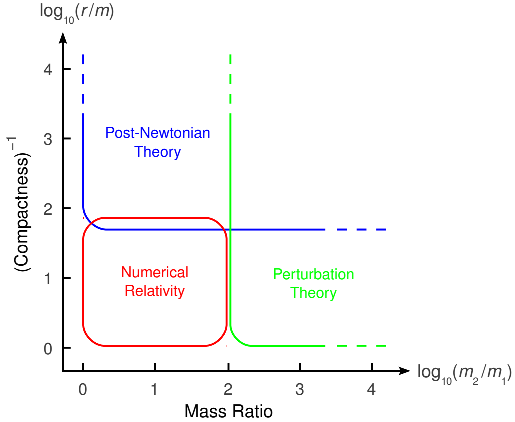


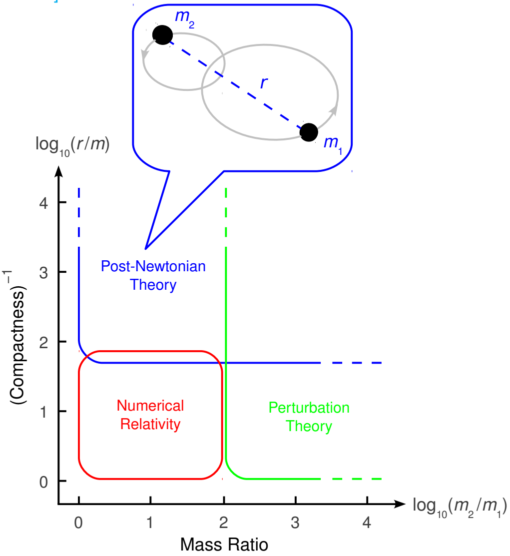
Methods to compute GW templates



[courtesy Alexandre Le Tiec]

Methods to compute GW templates

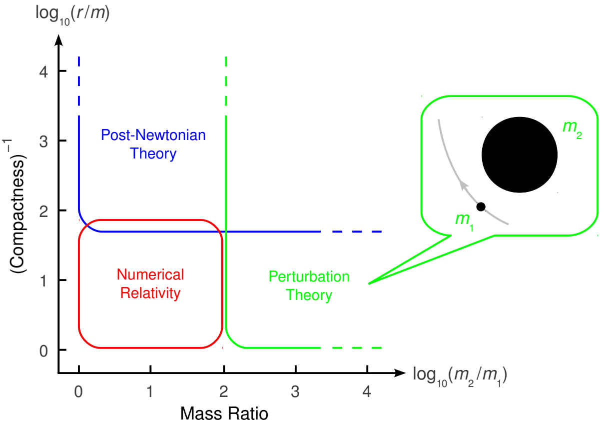
[see Blanchet 2014 for a review]



[courtesy Alexandre Le Tiec]

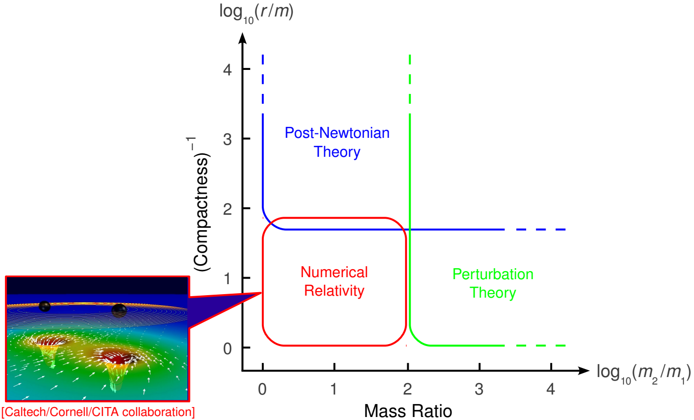
Methods to compute GW templates

[Detweiler 2008; Barack 2009]



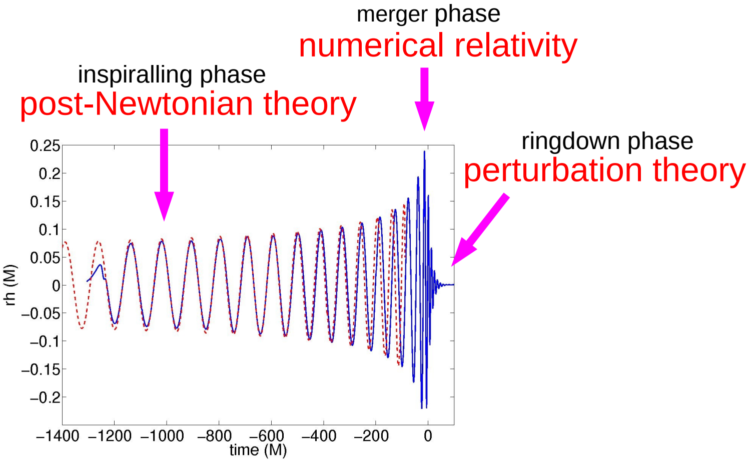
[courtesy Alexandre Le Tiec]

Methods to compute GW templates



[courtesy Alexandre Le Tiec]

The gravitational chirp of compact binaries



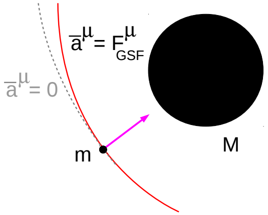
Effective methods such as EOB that interpolate between the PN and NR are also very important notably for the data analysis

COMPARISONS BETWEEN THE PN AND GRAVITATIONAL SELF-FORCES

Problem of the gravitational self-force (GSF)

[Mino, Sasaki & Tanaka 1997; Quinn & Wald 1997; Detweiler & Whiting 2003]

- A particle is moving on a background space-time of a massive black hole
- Its stress-energy tensor modifies the background gravitational field
- Because of the back-reaction the motion of the particle deviates from a background geodesic hence the gravitational self force

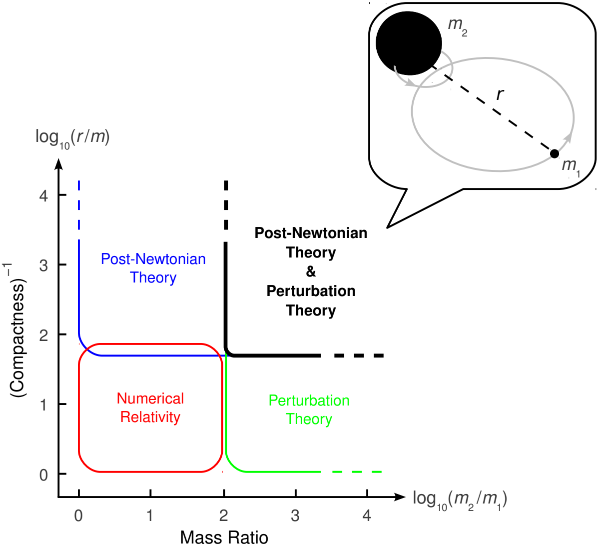


$$\boxed{\bar{a}^{\mu} = F^{\mu}_{\mathsf{GSF}} = \mathcal{O}\left(\frac{m}{M}\right)}$$

The GSF is computed to high accuracy by

- numerical methods [Sago, Barack & Detweiler 2008; Shah, Friedmann & Whiting 2014]
- analytical ones [Mano, Susuki & Takasugi 1996; Bini & Damour 2013, 2014]

Common regime of validity of GSF and PN



Why and how comparing PN and GSF predictions?

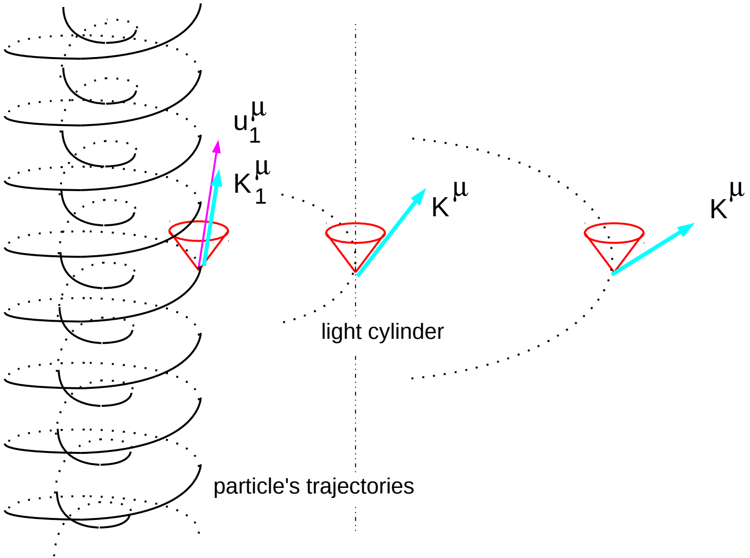
Both the PN and SF approaches use a self-field regularization for point particles followed by a renormalization. However, the prescription are very different

1. SF theory is based on a prescription for the Green's function G_R based on Hadamard's elementary solution [Detweiler & Whiting 2003]
2. PN theory uses dimensional regularization and it was shown that subtle issues appear at the 3PN order due to the appearance of poles $\propto (d-3)^{-1}$

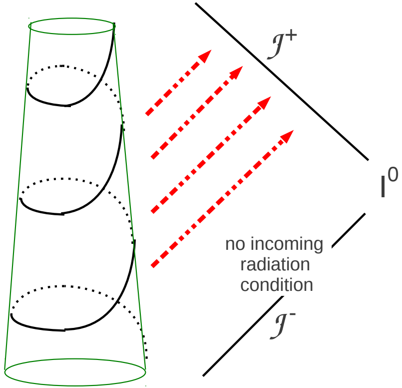
How can we make a meaningful comparison?

1. Restrict attention to the conservative part (circular orbits) of the dynamics
2. Find a gauge-invariant observable computable in both formalisms

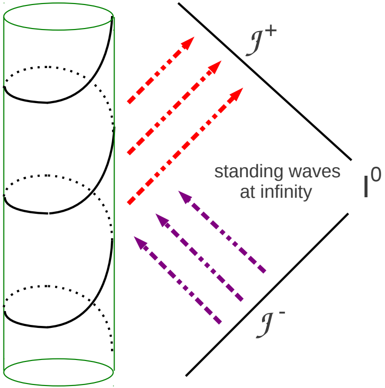
Circular orbit means Helical Killing symmetry



Looking at the conservative part of the dynamics



Physical situation



Situation with the HKV

The redshift observable [Detweiler 2008]

1. For exactly circular orbits the geometry admits a helical Killing vector with

$$K^\mu \partial_\mu = \partial_t + \Omega\, \partial_\varphi$$

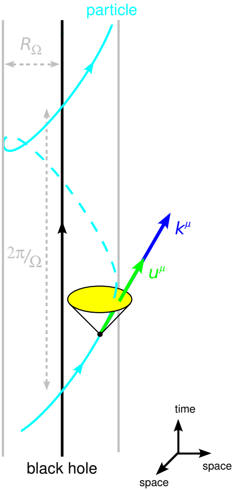
2. The four-velocity of the particle is tangent to the Killing vector hence

$$\boxed{K_1^\mu = z_1\, u_1^\mu}$$

3. This z_1 is the Killing energy of the particle associated with the HKV and can also be viewed as a redshift factor

4. For eccentric orbits one considers the averaged redshift [Barack & Sago 2011]

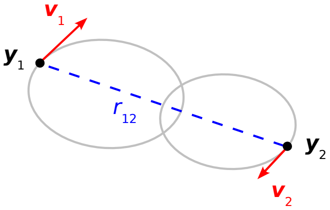
$$\boxed{\langle z_1 \rangle = \frac{1}{P} \int_0^P \mathrm{d}t\, z_1(t)}$$



Post-Newtonian calculation of the redshift factor

In a coordinate system such that $K^\mu \partial_\mu = \partial_t + \omega\, \partial_\varphi$ we have

$$\boxed{z_1 = \frac{1}{u_1^t} = \left(- \underbrace{(g_{\mu\nu})_1}_{\text{regularized metric}} \frac{v_1^\mu v_1^\nu}{c^2} \right)^{1/2}}$$



One needs a self-field regularization

- Hadamard's partie finie regularization is extremely useful in practical calculations but yields (UV and IR) ambiguity parameters at high PN orders
- Dimensional regularization is an extremely powerful regularization which seems to be free of ambiguities at any PN order

High-order PN result for the redshift factor

[Blanchet, Detweiler, Le Tiec & Whiting 2010, 2011]

The redshift factor of particle 1 through 3PN order and augmented by 4PN and 5PN logarithmic terms is

$$\begin{aligned} u_1^t \quad &= \quad 1 + \left(\frac{3}{4} - \frac{3}{4}\sqrt{1-4\nu} - \frac{\nu}{2}\right) x + \overbrace{[\cdots]\; x^2}^{\text{1PN}} + \overbrace{[\cdots]\; x^3}^{\text{2PN}} + \overbrace{[\cdots]\; x^4}^{\text{3PN}} \\ &\qquad + \Bigg(\cdots \underbrace{+ [\cdots]\, \nu \ln x}_{\text{4PN log}}\Bigg) x^5 + \Bigg(\cdots \underbrace{+ [\cdots]\, \nu \ln x}_{\text{5PN log}}\Bigg) x^6 + \mathcal{O}\left(x^7\right) \end{aligned}$$

where we pose $\nu = \frac{m_1 m_2}{m^2}$ and $x = \left(\frac{G m \Omega}{c^3}\right)^{3/2}$

High-order PN result for the redshift factor

[Blanchet, Detweiler, Le Tiec & Whiting 2010, 2011]

- We re-expand in the small mass-ratio limit $q = m_1/m_2 \ll 1$ so that

$$u^T = u^T_{\text{Schw}} + \underbrace{q\, u^T_{\text{SF}}}_{\text{self-force}} + \underbrace{q^2\, u^T_{\text{PSF}}}_{\text{post-self-force}} + \mathcal{O}(q^3)$$

- Posing $y = \left(\frac{G m_2 \Omega}{c^3}\right)^{3/2}$ we find

$$\begin{aligned} u^T_{\text{SF}} &= -y - 2y^2 - 5y^3 + \overbrace{\left(-\frac{121}{3} + \frac{41}{32}\pi^2\right) y^4}^{\text{3PN}} \\ &+ \underbrace{\left(a_4 + \frac{64}{5}\ln y\right) y^5}_{\text{4PN}} + \underbrace{\left(a_5 - \frac{956}{105}\ln y\right) y^6}_{\text{5PN}} + o(y^6) \end{aligned}$$

High-order PN fit to the numerical self-force

- Numerical SF data is fitted with a PN series in $y = \left(\frac{G m_2 \Omega}{c^3}\right)^{2/3}$

$$z_1 = \sum_a \left[a_{n\mathrm{PN}} + b_{n\mathrm{PN}} \ln y + \cdots \right] y^{n+1}$$

- The 3PN prediction agrees with the SF value with 7 significant digits

3PN value	SF fit
$a_{3\mathrm{PN}} = -\frac{121}{3} + \frac{41}{32}\pi^2 = -27.6879026\cdots$	$-27.6879034 \pm 0.0000004$

- Logarithmic coefficients $b_{4\mathrm{PN}}$ and $b_{5\mathrm{PN}}$ also perfectly agree
- Post-Newtonian coefficients are measured up to 7PN order

$a_{4\mathrm{PN}}$	$-114.34747(5)$
$a_{5\mathrm{PN}}$	$-245.53(1)$
$a_{6\mathrm{PN}}$	$-695(2)$
$b_{6\mathrm{PN}}$	$+339.3(5)$
$a_{7\mathrm{PN}}$	$-5837(16)$

Further developments

1. 4PN coefficient known analytically by GSF calculation [Bini & Damour 2013]

$$\boxed{a_{\mathsf{4PN}} = -\frac{1157}{15} + \frac{677}{512}\pi^2 - \frac{256}{5}\ln 2 - \frac{128}{5}\gamma_{\mathsf{E}}}$$

and agrees with numerical value [Blanchet, Detweiler, Le Tiec & Whiting 2011]

2. Super-high precision analytical and numerical GSF calculations of the redshift factor up to 10PN order, including a previously unexpected existence of half-integral PN terms starting at 5.5PN order [Shah, Friedman & Whiting 2013]

3. Half-integral conservative PN terms [Blanchet, Faye & Whiting 2013, 2014]

$$\boxed{a_{\mathsf{5.5PN}} = -\frac{13696}{525}\pi\,, \qquad a_{\mathsf{6.5PN}} = \frac{81077}{3675}\pi\,, \qquad a_{\mathsf{7.5PN}} = \frac{82561159}{467775}\pi}$$

Standard PN theory agrees with GSF calculations

$$
\begin{aligned}
u^t_{\mathrm{SF}} &= -y - 2y^2 - 5y^3 + \left(-\frac{121}{3} + \frac{41}{32}\pi^2\right) y^4 \\
&+ \left(-\frac{1157}{15} + \frac{677}{512}\pi^2 - \frac{128}{5}\gamma_{\mathsf{E}} - \frac{64}{5}\ln(16y)\right) y^5 \\
&- \frac{956}{105}y^6\ln y - \frac{13696\pi}{525}y^{13/2} - \frac{51256}{567}y^7\ln y + \frac{81077\pi}{3675}y^{15/2} \\
&+ \frac{27392}{525}y^8\ln^2 y + \frac{82561159\pi}{467775}y^{17/2} - \frac{27016}{2205}y^9\ln^2 y \\
&- \frac{11723776\pi}{55125}y^{19/2}\ln y - \frac{4027582708}{9823275}y^{10}\ln^2 y \\
&+ \frac{99186502\pi}{1157625}y^{21/2}\ln y + \frac{23447552}{165375}y^{11}\ln^3 y + \cdots
\end{aligned}
$$

1. Integral PN terms such as 3PN permit checking dimensional regularization
2. Half-integral PN terms starting at 5.5PN order permit checking the non-linear tails (and tail-of-tails)

Standard PN theory agrees with GSF calculations

$$\boxed{\begin{aligned} u^t_{\mathrm{SF}} = &-y - 2y^2 - 5y^3 + \left(-\frac{121}{3} + \frac{41}{32}\pi^2\right) y^4 \\ &+ \left(-\frac{1157}{15} + \frac{677}{512}\pi^2 - \frac{128}{5}\gamma_{\mathrm{E}} - \frac{64}{5}\ln(16y)\right) y^5 \\ &- \frac{956}{105} y^6 \ln y - \frac{13696\pi}{525} y^{13/2} - \frac{51256}{567} y^7 \ln y + \frac{81077\pi}{3675} y^{15/2} \\ &+ \frac{27392}{525} y^8 \ln^2 y + \frac{82561159\pi}{467775} y^{17/2} - \frac{27016}{2205} y^9 \ln^2 y \\ &- \frac{11723776\pi}{55125} y^{19/2} \ln y - \frac{4027582708}{9823275} y^{10} \ln^2 y \\ &+ \frac{99186502\pi}{1157625} y^{21/2} \ln y + \frac{23447552}{165375} y^{11} \ln^3 y + \cdots \end{aligned}}$$

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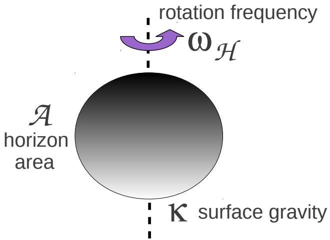
FIRST LAW OF COMPACT BINARY MECHANICS

Four laws of black hole dynamics



ZEROTH LAW

Surface gravity κ is constant over the horizon of a stationary black hole

FIRST LAW

Mass M and angular momentum J of BH change according to [Bardeen, Carter & Hawking 1973]

$$\delta M - \omega_{\mathcal{H}}\, \delta J = \frac{\kappa}{8\pi} \delta \mathcal{A}$$

SECOND LAW

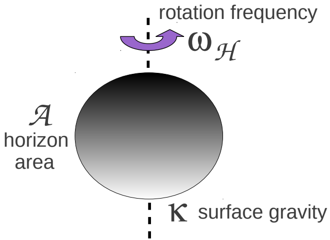
In any physical process involving one or several BHs with or without an environment [Hawking 1971]

$$\delta A \geqslant 0$$

THIRD LAW

It is impossible to achieve $\kappa = 0$ in any process

Four laws of black hole dynamics



ZEROTH LAW

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FIRST LAW

Mass M and angular momentum J of BH change according to [Christodoulou 1970, Smarr 1973]

$$M - 2\omega_{\mathcal{H}}\, J = \frac{\kappa}{4\pi}\mathcal{A}$$

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In any physical process involving one or several BHs with or without an environment [Hawking 1971]

$$\delta A \geqslant 0$$

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It is impossible to achieve $\kappa = 0$ in any process

Black hole thermodynamics [Bekenstein 1972, Hawking 1976]

- Using arguments involving a piece of matter with entropy thrown into a BH, Bekenstein derived the BH entropy

$$S_{\mathrm{BH}} = \alpha\,\mathcal{A}$$

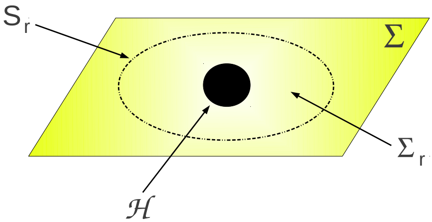
This would require $T_{\mathrm{BH}} = \frac{\kappa}{8\pi\alpha}$ but the thermodynamic temperature of a *classical* BH is absolute zero since a BH is a perfect absorber

- However Hawking proved that *quantum* particle creation effects near a BH result in a black body temperature $T_{\mathrm{BH}} = \frac{\kappa}{2\pi}$
- This yields the famous Bekenstein-Hawking entropy of a stationary black hole

$$\boxed{S_{\mathrm{BH}} = \frac{c^3 k}{\hbar G}\frac{\mathcal{A}}{4}}$$

- The analogy between BH dynamics and the laws of thermodynamics is complete although still mysterious today

Toward a generalized first law for a system of BHs



The mass and angular momentum of the BH are given by Komar surface integrals at spatial infinity

$$
\begin{aligned}
M &= -\frac{1}{8\pi}\lim_{r\to\infty}\oint_{S_r}\nabla^\mu t^\nu \,\mathrm{d}S_{\mu\nu} \\
J &= \frac{1}{16\pi}\lim_{r\to\infty}\oint_{S_r}\nabla^\mu \phi^\nu \,\mathrm{d}S_{\mu\nu}
\end{aligned}
$$

where t^μ and ϕ^μ are the two stationary and axi-symmetric Killing vectors

Toward a generalized first law for a system of BHs

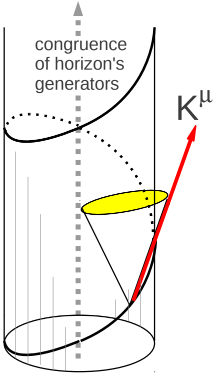
The first law of BH dynamics expresses the change

$$\delta Q = \delta M - \omega_{\mathcal{H}}\, \delta J$$

in the Noether charge Q between two nearby BH configurations, where Q is associated with the Killing vector

$$K^\mu = t^\mu + \omega_{\mathcal{H}}\, \phi^\mu$$

which is the null generator of the BH horizon



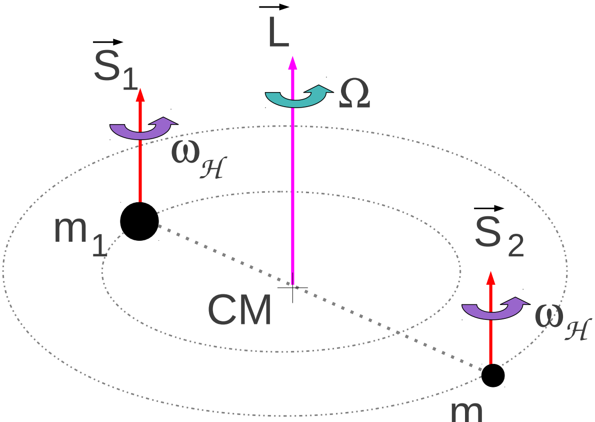
Toward a generalized first law for a system of BHs

- A generalized First Law valid for systems of BHs can be obtained when the geometry admits a Helical Killing Vector (HKV)

$$K^\mu \partial_\mu = \partial_t + \Omega\, \partial_\varphi$$

  where ∂_t is time-like and ∂_φ is space-like (with closed orbits), even when ∂_t and ∂_φ are not separately Killing vectors
- This applies to the case of two Kerr BHs moving on exactly circular orbits with orbital frequency Ω
- The two BHs should be in corotation, so that $\omega_{\mathcal{H}}$ should approximately be equal to Ω
- In particular the spins should be aligned with the orbital angular momentum

Toward a generalized first law for a system of BHs



$$\omega_{\mathcal{H}} = \Omega$$

Mass and angular momentum of compact binaries

The mass M and angular momentum J are checked to satisfy for all the terms up to 3PN order, and also for the 4PN and 5PN log terms, the thermodynamic relation valid for circular orbits

$$\boxed{\frac{\partial M}{\partial \Omega} = \Omega \, \frac{\partial J}{\partial \Omega}}$$

which constitutes the first ingredient in the First Law of binary black holes

- The thermodynamic relation states that the flux of energy emitted in the form of gravitational waves is proportional to the flux of angular momentum
- It is used in numerical computations of the binary evolution based on a sequence of quasi-equilibrium configurations [Gourgoulhon *et al* 2002]

First law of binary point particle mechanics

[Le Tiec, Blanchet & Whiting 2011]

1. We find by direct computation that the redshift factors z_1 and z_2 are related to the ADM mass and angular momentum by

$$\frac{\partial M}{\partial m_1} - \Omega \frac{\partial J}{\partial m_1} = z_1 \quad \text{and} \quad (1 \leftrightarrow 2)$$

2. Finally those relations can be summarized into the First law of binary point-particles mechanics

$$\boxed{\delta M - \Omega\, \delta J = z_1\, \delta m_1 + z_2\, \delta m_2}$$

The first law tells how the ADM quantities change when the individual masses m_1 and m_2 of the particles vary

The generalized first law [Friedman, Uryū & Shibata 2002]

- Space-time generated by black holes and perfect fluid matter distributions
- Globally defined HKV field
- Asymptotic flatness

Generalized law of perfect fluid and black hole mechanics

$$\boxed{\delta M - \Omega \delta J = \int_\Sigma \left[\bar{\mu}\, \Delta(\mathrm{d}m) + \bar{T}\, \Delta(\mathrm{d}S) + w^\mu \Delta(\mathrm{d}C_\mu)\right] + \sum_a \frac{\kappa_a}{8\pi}\, \delta A_a}$$

where Δ denotes the Lagrangian variation of the matter fluid, where $\mathrm{d}m$ is the conserved baryonic mass element, and where $\overline{T} = zT$ and $\overline{\mu} = z(h - Ts)$ are the redshifted temperature and chemical potential

First law for binary point particles with spins

[Blanchet, Buonanno & Le Tiec 2012]

$$\boxed{\delta M - \Omega\,\delta J = \sum_{a=1}^{2}\Big[z_a\,\delta m_a + (\Omega_a - \Omega)\,\delta S_a\Big]}$$

1. The precession frequency Ω_a of the spins obeys

$$\frac{\mathrm{d}\boldsymbol{S}_a}{\mathrm{d}t} = \boldsymbol{\Omega}_a \times \boldsymbol{S}_a$$

2. The total angular momentum is related to the orbital angular momentum by

$$J = L + S_1 + S_2$$

3. For point particles which have no finite extension the notion of rotation frequency of the body is meaningless and the law is valid for arbitrary spins

The first law for binary corotating black holes

1. To describe extended bodies such as black holes one must suplement the point particles with some internal constitutive relation of the type

$$m_a = m_a\left(m_a^{\text{irr}}, S_a\right)$$

where S_a is the spin and m_a^{irr} is some irreducible constant mass

2. We define the response coefficients associated with the internal structure

$$c_a = \left(\frac{\partial m_a}{\partial m_a^{\text{irr}}}\right)_{S_a}, \qquad \omega_a = \left(\frac{\partial m_a}{\partial S_a}\right)_{m_a^{\text{irr}}}$$

where in particular ω_a is the rotation frequency of the body

3. The First Law becomes

$$\delta M - \Omega\,\delta J = \sum_{a=1}^{2}\Big[z_a\,c_a\,\delta m_a^{\text{irr}} + \left(z_a\,\omega_a + \Omega_a - \Omega\right)\delta S_a\Big]$$

The first law for binary corotating black holes

This yields the corotation condition for extended particles

$$\boxed{z_a\,\omega_a = \Omega - \Omega_a}$$

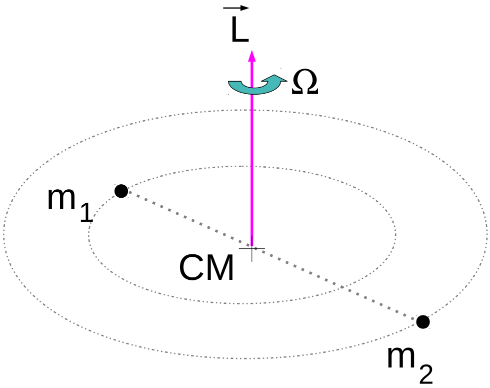
The First Law is then in agreement with the first law for two corotating black holes [Friedman, Uryū & Shibata 2002]

$$\delta M - \Omega\,\delta J = \sum_{a=1}^{2} \frac{\kappa_a}{8\pi} \delta\mathcal{A}_a$$

provided that we make the identifications

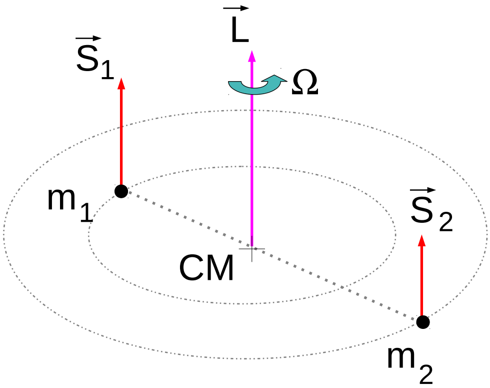
$$m_a^{\text{irr}} \quad \longleftrightarrow \quad \sqrt{\frac{\mathcal{A}_a}{16\pi}}$$
$$z_a\,c_a \quad \longleftrightarrow \quad 4m_a^{\text{irr}}\kappa_a$$

First law of mechanics for binary point particles



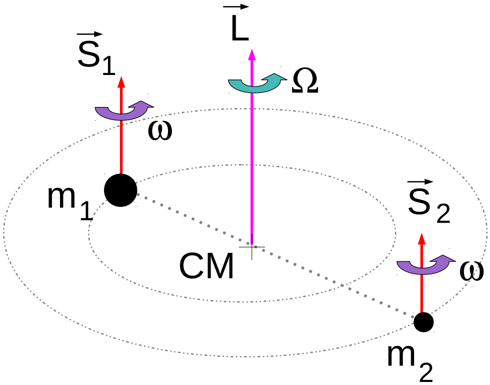
$$\boxed{\delta M - \Omega\,\delta L = \sum_{a=1}^{2} \underbrace{z_a}_{\substack{\text{helical}\\ \text{Killing energy}}} \delta m_a}$$

First law for binary point particles with spins



$$\boxed{\delta M - \Omega\,\delta J = \sum_{a=1}^{2}\Big[z_a\,\delta m_a + \big(\underbrace{\Omega_a}_{\text{precession frequency}} - \Omega\big)\delta S_a\Big]}$$

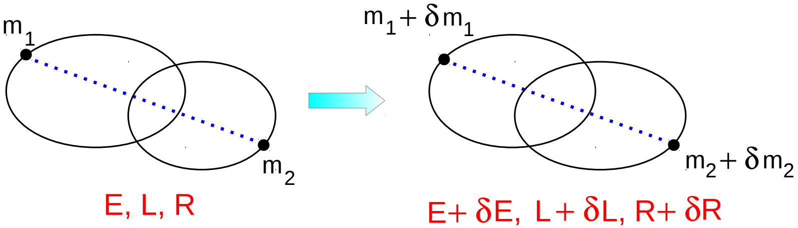
First law of mechanics for corotating binary BH



$$\boxed{\delta M - \Omega\,\delta J = \sum_{a=1}^{2} \underbrace{\kappa_a}_{\text{surface gravity}} \frac{\delta A_a}{8\pi}}$$

FIRST LAW OF MECHANICS AT THE 4PN ORDER

First law for eccentric orbits [Le Tiec 2015]



$$\boxed{\delta E = \omega\, \delta L + n\, \delta R + \langle z_1 \rangle\, \delta m_1 + \langle z_2 \rangle\, \delta m_2}$$

- E, L : ADM energy and angular momentum
- $\boxed{R = \dfrac{1}{2\pi} \oint p_r \mathrm{d}r}$: radial action integral
- n, ω : radial and azimuthal frequencies

First law versus non-local dynamics

1. The basic variable computed by GSF techniques is the averaged redshift $\langle z_a \rangle$ in the test-mass limit $m_1/m_2 \to 0$
2. The first law permits to derive from $\langle z_a \rangle$ the binary's conserved energy E and periastron advance K for circular orbits

$$K = \frac{\omega}{n}$$

3. These results are then used to fix the ambiguity parameters in the 4PN equations of motion in
   - Hamiltonian formalism [Damour, Jaranowski & Schäfer 2014, 2016]
   - Lagrangian formalism [Bernard, Blanchet, Bohé, Faye & Marsat 2015, 2016, 2017]
4. However the first law has been derived from a local Hamiltonian but at 4PN order the dynamics becomes non-local due to the tail term

Are we still allowed to use the first law in standard form for the non-local dynamics at the 4PN order ?

The 4PN non-local-in-time dynamics

1. At 4PN order the dynamics becomes non-local due to the tail term

$$H = H_0(r, p_r, p_\varphi; m_a) + H_{\mathsf{tail}}[r, \varphi, p_r, p_\varphi; m_a]$$

with $$\boxed{H_{\mathsf{tail}} = -\frac{m}{5} I^{(3)}_{ij}(t) \int_{-\infty}^{+\infty} \frac{\mathrm{d}t'}{|t-t'|} I^{(3)}_{ij}(t')}$$

2. Hamilton's equations involve functional derivatives

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\delta H}{\delta p_i} \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\delta H}{\delta x^i}$$

3. For the non-local dynamics H and p_φ are no longer conserved but instead

$$\begin{aligned} E &= H + \Delta H^{\mathsf{DC}} + \Delta H^{\mathsf{AC}} \\ L &= p_\varphi + \Delta p_\varphi^{\mathsf{DC}} + \Delta p_\varphi^{\mathsf{AC}} \end{aligned}$$

where H^{AC} and p_φ^{AC} (given by Fourier series) average to zero and

$$\boxed{\Delta H^{\mathsf{DC}} = -2m\,\mathcal{F}^{\mathsf{GW}} \qquad \Delta p_\varphi^{\mathsf{DC}} = -2m\,\mathcal{G}^{\mathsf{GW}}}$$

Conserved energy for circular orbits at 4PN order

- The 4PN energy for circular orbits in the small mass ratio limit is known from GSF of the redshift variable [Le Tiec, Blanchet & Whiting 2012; Bini & Damour 2013]
- This permits to fix the ambiguity parameter α and to complete the 4PN equations of motion

$$\boxed{\begin{aligned} E^{\text{4PN}} = -\frac{\mu c^2 x}{2}\Bigg\{ & 1 + \left(-\frac{3}{4} - \frac{\nu}{12}\right) x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24}\right) x^2 \\ & + \left(-\frac{675}{64} + \left[\frac{34445}{576} - \frac{205}{96}\pi^2\right]\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3\right) x^3 \\ & + \left(-\frac{3969}{128} + \left[-\frac{123671}{5760} + \frac{9037}{1536}\pi^2 + \frac{896}{15}\gamma_E + \frac{448}{15}\ln(16x)\right]\nu \right. \\ & \left. + \left[-\frac{498449}{3456} + \frac{3157}{576}\pi^2\right]\nu^2 + \frac{301}{1728}\nu^3 + \frac{77}{31104}\nu^4\right) x^4 \Bigg\} \end{aligned}}$$

Periastron advance for circular orbits at 4PN order

The periastron advanced (or relativistic precession) constitutes a second invariant which is also known in the limit of circular orbits from GSF calculations

$$\boxed{\begin{aligned} K^{\mathsf{4PN}} = 1 + 3x + \left(\frac{27}{2} - 7\nu\right) x^2 \\ + \left(\frac{135}{2} + \left[-\frac{649}{4} + \frac{123}{32}\pi^2\right]\nu + 7\nu^2\right) x^3 \\ + \left(\frac{2835}{8} + \left[-\frac{275941}{360} + \frac{48007}{3072}\pi^2 - \frac{1256}{15}\ln x\right.\right. \\ \left. - \frac{592}{15}\ln 2 - \frac{1458}{5}\ln 3 - \frac{2512}{15}\gamma_\mathsf{E}\right]\nu \\ \left. + \left[\frac{5861}{12} - \frac{451}{32}\pi^2\right]\nu^2 - \frac{98}{27}\nu^3\right) x^4 \end{aligned}}$$

Derivation of the first law at 4PN order

[Blanchet & Le Tiec 2017]

1. We perform an unconstrained variation of the Hamiltonian

$$\begin{aligned}\delta H &= \dot{\varphi}\delta p_\varphi - \dot{p}_\varphi\delta\varphi + \dot{r}\delta p_r - \dot{p}_r\delta r + \frac{2m}{5}\left(I_{ij}^{(3)}\right)^2\frac{\delta n}{n}\\ &+ \sum_a z_a\delta m_a + \Delta\end{aligned}$$

   where Δ is a complicated double Fourier series but such that $\boxed{\langle\Delta\rangle = 0}$

2. By averaging we obtain

$$\begin{aligned}\langle\dot{r}\,\delta p_r - \dot{p}_r\,\delta r\rangle &= n\,\delta R\\ \langle\dot{\varphi}\,\delta p_\varphi - \dot{p}_\varphi\,\delta\varphi\rangle &= \omega\,\delta L + \omega\,\delta\big(2m\,\mathcal{G}^{\mathsf{GW}}\big) - n\,\delta\left(\frac{1}{2\pi}\oint\Delta p_\varphi^{\mathsf{AC}}\,\mathrm{d}\varphi\right)\end{aligned}$$

3. Here the radial action integral is

$$\boxed{R = \frac{1}{2\pi}\oint p_r\mathrm{d}r}$$

Derivation of the first law at 4PN order

[Blanchet & Le Tiec 2017]

1. Combining all the terms we obtain a first law in standard form

$$\boxed{\delta E = \omega\,\delta L + n\,\delta\mathcal{R} + \sum_a \langle z_a\rangle\,\delta m_a}$$

but where the radial action integral gets corrected at 4PN order

$$\boxed{\mathcal{R} = R + 2m\left(\mathcal{G}^{\mathsf{GW}} - \frac{\mathcal{F}^{\mathsf{GW}}}{\omega}\right) - \frac{1}{2\pi}\oint \Delta p_\varphi^{\mathsf{AC}}\,\mathrm{d}\varphi}$$

2. The first law admits the first integral relationship

$$\boxed{E = 2\omega L + 2n\mathcal{R} + \sum_a m_a\,\langle z_a\rangle}$$

3. We have proved that z_a is the redshift in the sense that

$$z_a = \frac{\delta H}{\delta m_a} = \sqrt{-g_{\mu\nu}(y_a)v_a^\mu v_a^\nu}$$

Derivation of the first law at 4PN order

[Blanchet & Le Tiec 2017]

1. By performing a non-local-in-time shift of canonical variables

$$(r, \varphi, p_r, p_\varphi) \longrightarrow (r^{\mathsf{loc}}, \varphi^{\mathsf{loc}}, p_r^{\mathsf{loc}}, p_\varphi^{\mathsf{loc}})$$

   the non-local Hamiltonian can be transformed into an ordinary local Hamiltonian [Damour, Jaranowski & Schäfer 2016]

2. Once this is done one can perform an ordinary derivation of the first law

$$\delta E = \omega\, \delta L + n\, \delta R^{\mathsf{loc}} + \sum_a \langle z_a \rangle\, \delta m_a$$

3. The modified action integral in non-local coordinates is identical to the local one when expressed in terms of E, L and the masses

$$\mathcal{R}(E, L, m_a) = R^{\mathsf{loc}}(E, L, m_a) = \frac{1}{2\pi} \oint \mathrm{d}r^{\mathsf{loc}}\, p_r^{\mathsf{loc}}(r^{\mathsf{loc}}, E, L, m_a)$$

4. With the present derivation of the first law at 4PN order we have fully confirmed the expressions of E^{4PN} and K^{4PN} in the test-mass limit