Higher dimensional massive (bi-)gravity: Constructions and solutions

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Contents

I. Motivations

- The massive gravity [gravitons have tiny but non-zero mass] has had a long and rich history since the seminal paper by Fierz & Pauli [PRSA173(1939)211].
- van Dam & Veltman [NPB22(1970)397] and Zakharov [PZETF12(1970)447] showed that in the massless limit, it cannot recover GR.
- Vainshtein pointed out that the nonlinear extensions of FP theory can solve the vDVZ discontinuity problem [PLB39(1972)393].
- Boulware & Deser claimed that there exists a ghost associated with the sixth mode in graviton coming from nonlinear levels [PRD6(1972)3368].
- Building a ghost-free nonlinear massive gravity, in which a massive graviton carries only five "physical" degrees of freedom, has been a great challenge for physicists.
- de Rham, Gabadadze & Tolley (dRGT) have successfully constructed a ghost-free nonlinear massive gravity [1011.1232, 1007.0443].
- The dRGT theory has been proved to be ghost-free for general fiducial metric by some different approaches, e.g., Hassan & Rosen [1106.3344, 1109.3230].
- The dRGT theory might be a solution to the cosmological constant problem.

I. Motivations

- An interesting extension of dRGT theory is the massive bi-metric gravity (bi-gravity) proposed by Hassan & Rosen, in which the reference metric is introduced to be full dynamical as the physical metric [1109.3515].
- For interesting review papers, see de Rham [1401.4173]; K. Hinterbichler [1105.3735]; Schmidt-May & von Strauss [1512.00021].
- It is noted that most of previous papers have focused only on four-dimensional frameworks, which involve only the first three massive graviton terms, $\mathcal{L}_2$, $\mathcal{L}_3$, and $\mathcal{L}_4$.
- There have been a few papers discussing higher dimensional scenarios of massive (bi)gravity theories, e.g., Hinterbichler & Rosen [1203.5783]; Hassan, Schmidt-May & von Strauss [1212.4525]; Huang, Zhang & Zhou [1306.4740]. However, these papers have not studied the well-known metrics in higher dimensions, e.g., the Friedmann-Lemaitre-Robertson-Walker (FLRW), Bianchi type I, and Schwarzschild-Tangherlini metrics.
- We would like to investigate whether the five-dimensional (bi)gravity theories admit the above metrics as their solutions.

II. Cayley-Hamilton theorem and ghost-free graviton terms

- Recall the four-dimensional action of the dRGT massive gravity [1011.1232, 1007.0443]:

$$S_{4\mathrm{d}} = \frac{M_p^2}{2}\int d^4x\sqrt{-g}\Big\{R + m_g^2\left(\mathcal{L}_2 + \alpha_3\mathcal{L}_3 + \alpha_4\mathcal{L}_4\right)\Big\},$$

where M_p the Planck mass, m_g the graviton mass, $\alpha_{3,4}$ free parameters, and the massive graviton terms $\mathcal{L}_i$ defined as

$$\mathcal{L}_2 = [\mathcal{K}]^2 - [\mathcal{K}^2];\ \mathcal{L}_3 = \frac{1}{3}[\mathcal{K}]^3 - [\mathcal{K}][\mathcal{K}^2] + \frac{2}{3}[\mathcal{K}^3],$$

$$\mathcal{L}_4 = \frac{1}{12}[\mathcal{K}]^4 - \frac{1}{2}[\mathcal{K}]^2[\mathcal{K}^2] + \frac{1}{4}[\mathcal{K}^2]^2 + \frac{2}{3}[\mathcal{K}][\mathcal{K}^3] - \frac{1}{2}[\mathcal{K}^4].$$

- Square brackets:

$$[\mathcal{K}] \equiv \mathrm{tr}\mathcal{K}^\mu{}_\nu;\ [\mathcal{K}]^2 \equiv \left(\mathrm{tr}\mathcal{K}^\mu{}_\nu\right)^2;\ [\mathcal{K}^2] \equiv \mathrm{tr}\mathcal{K}^\mu{}_\alpha\mathcal{K}^\alpha{}_\nu;\ \text{and so on.}$$

- The square matrix $\mathcal{K}^\mu{}_\nu$ is defined as

$$\mathcal{K}^\mu{}_\nu \equiv \delta^\mu{}_\nu - \sqrt{f_{ab}\partial_\mu\phi^a\partial_\alpha\phi^b g^{\alpha\nu}},$$

$\phi^a \sim$ Stückelberg fields; $g_{\mu\nu} \sim$ (dynamical) physical metric,

$f_{ab} \sim$ non-dynamical reference (fiducial) metric of massive gravity.

II. Cayley-Hamilton theorem and ghost-free graviton terms

- Recall the four-dimensional action of the massive bi-gravity [1109.3515]:

$$S_{4d} = M_g^2 \int d^4x \sqrt{g} R(g) + M_f^2 \int d^4x \sqrt{f} R(f) + 2m^2 M_{\text{eff}}^2 \int d^4x \sqrt{g} \Big(\mathcal{U}_2 + \alpha_3 \mathcal{U}_3 + \alpha_4 \mathcal{U}_4 \Big),$$

where

$$\mathcal{U}_i = \frac{1}{2} \mathcal{L}_i; \;\; M_{\text{eff}}^2 \equiv \left(\frac{1}{M_g^2} + \frac{1}{M_f^2} \right)^{-1}.$$

- The square matrix $\mathcal{K}^\mu{}_\nu$ is defined as

$$\mathcal{K}^\mu{}_\nu \equiv \delta^\mu{}_\nu - \sqrt{f_{\mu\alpha} g^{\alpha\nu}},$$

$g_{\mu\nu} \sim$ (dynamical) physical metric,

$f_{\mu\nu} \sim$ full dynamical reference (fiducial) metric.

II. Cayley-Hamilton theorem and ghost-free graviton terms

- We will construct higher dimensional terms $\mathcal{L}_{n>4}$ by applying the well-known Cayley-Hamilton theorem for the square matrix $\mathcal{K}^{\mu}{}_{\nu}$.
- In algebra, there exists the well-known Cayley-Hamilton theorem: **any square matrix must obey its characteristic equation**. In particular, given a $n \times n$ matrix K with its characteristic equation, $\mathcal{P}(\lambda) \equiv \det(\lambda I_n - K) = 0$, then

$$\mathcal{P}(K) \equiv K^n - \mathcal{D}_{n-1}K^{n-1} + \mathcal{D}_{n-2}K^{n-2} - ... \\ +(-1)^{n-1}\mathcal{D}_1 K + (-1)^n \det(K) I_n = 0,$$

  where $\mathcal{D}_{n-1} = \mathrm{tr}K \equiv [K]$ and $\mathcal{D}_{n-j}$ $(2 \leq j \leq n-1)$ are coefficients of the characteristic polynomial.
- For $n = 2$, the following characteristic equation:

$$K^2 - [K]K + \det K_{2\times 2} I_2 = 0,$$

  which implies after taking the trace

$$\det K_{2\times 2} = \frac{1}{2}\Big\{[K]^2 - [K^2]\Big\} \sim \frac{\mathcal{L}_2}{2}.$$

II. Cayley-Hamilton theorem and ghost-free graviton terms

- For $n = 3$, the corresponding characteristic equation:

$$K^3 - [K]K^2 + \frac{1}{2}\left\{[K]^2 - [K^2]\right\} K - \det K_{3\times 3} I_3 = 0,$$

which leads to

$$\det K_{3\times 3} = \frac{1}{6}\left\{[K]^3 - 3[K^2][K] + 2[K^3]\right\} \sim \frac{\mathcal{L}_3}{2}.$$

- For $n = 4$, the corresponding characteristic equation:

$$K^4 - [K]K^3 + \frac{1}{2}\left\{[K]^2 - [K^2]\right\} K^2$$
$$-\frac{1}{6}\left\{[K]^3 - 3[K^2][K] + 2[K^3]\right\} K + \det K_{4\times 4} I_4 = 0,$$

which gives

$$\det K_{4\times 4} = \frac{1}{24}\left\{[K]^4 - 6[K]^2[K^2] + 3[K^2]^2 + 8[K][K^3] - 6[K^4]\right\} \sim \frac{\mathcal{L}_4}{2}.$$

- The higher dimensional graviton terms $\mathcal{L}_{n>4}$ must vanish in all four-dimensional spacetimes.

II. Cayley-Hamilton theorem and ghost-free graviton terms

- The higher dimensional terms $\mathcal{L}_{n>4} = \det \mathcal{K}_{n\times n}/2$ can be constructed from the Cayley-Hamilton theorem to be

$$\frac{\mathcal{L}_5}{2} = \frac{1}{120}\Big\{[\mathcal{K}]^5 - 10[\mathcal{K}]^3[\mathcal{K}^2] + 20[\mathcal{K}]^2[\mathcal{K}^3] - 20[\mathcal{K}^2][\mathcal{K}^3] + 15[\mathcal{K}][\mathcal{K}^2]^2 - 30[\mathcal{K}][\mathcal{K}^4] + 24[\mathcal{K}^5]\Big\},$$

$$\frac{\mathcal{L}_6}{2} = \frac{1}{720}\Big\{[\mathcal{K}]^6 - 15[\mathcal{K}]^4[\mathcal{K}^2] + 40[\mathcal{K}]^3[\mathcal{K}^3] - 90[\mathcal{K}]^2[\mathcal{K}^4] + 45[\mathcal{K}]^2[\mathcal{K}^2]^2 - 15[\mathcal{K}^2]^3 + 40[\mathcal{K}^3]^2 - 120[\mathcal{K}^3][\mathcal{K}^2][\mathcal{K}] + 90[\mathcal{K}^4][\mathcal{K}^2] + 144[\mathcal{K}^5][\mathcal{K}] - 120[\mathcal{K}^6]\Big\},$$

$$\frac{\mathcal{L}_7}{2} = \frac{1}{5040}\Big\{[\mathcal{K}]^7 - 21[\mathcal{K}]^5[\mathcal{K}^2] + 70[\mathcal{K}]^4[\mathcal{K}^3] - 210[\mathcal{K}]^3[\mathcal{K}^4] + 105[\mathcal{K}]^3[\mathcal{K}^2]^2 - 420[\mathcal{K}]^2[\mathcal{K}^2][\mathcal{K}^3] + 504[\mathcal{K}]^2[\mathcal{K}^5] - 105[\mathcal{K}^2]^3[\mathcal{K}] + 210[\mathcal{K}^2]^2[\mathcal{K}^3] - 504[\mathcal{K}^2][\mathcal{K}^5] + 280[\mathcal{K}^3]^2[\mathcal{K}] - 420[\mathcal{K}^3][\mathcal{K}^4] + 630[\mathcal{K}^4][\mathcal{K}^2][\mathcal{K}] - 840[\mathcal{K}^6][\mathcal{K}] + 720[\mathcal{K}^7]\Big\}.$$

II. Cayley-Hamilton theorem and ghost-free graviton terms

- A five-dimensional scenario of massive gravity [1602.05672]:

$$S = \frac{M_p^2}{2} \int d^5x \sqrt{-g} \Big\{ R + m_g^2 \left(\mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \alpha_4 \mathcal{L}_4 + \alpha_5 \mathcal{L}_5 \right) \Big\},$$

- The corresponding five-dimensional Einstein field equations:

$$\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + m_g^2 \left(X_{\mu\nu} + \sigma Y_{\mu\nu} + \alpha_5 W_{\mu\nu} \right) = 0,$$

$$X_{\mu\nu} = -\frac{1}{2} \left(\alpha \mathcal{L}_2 + \beta \mathcal{L}_3 \right) g_{\mu\nu} + \tilde{X}_{\mu\nu},$$

$$\tilde{X}_{\mu\nu} = \mathcal{K}_{\mu\nu} - [\mathcal{K}] g_{\mu\nu} - \alpha \left\{ \mathcal{K}^2_{\mu\nu} - [\mathcal{K}] \mathcal{K}_{\mu\nu} \right\} + \beta \left\{ \mathcal{K}^3_{\mu\nu} - [\mathcal{K}] \mathcal{K}^2_{\mu\nu} + \frac{\mathcal{L}_2}{2} \mathcal{K}_{\mu\nu} \right\},$$

$$Y_{\mu\nu} = -\frac{\mathcal{L}_4}{2} g_{\mu\nu} + \tilde{Y}_{\mu\nu}; \quad \tilde{Y}_{\mu\nu} = \frac{\mathcal{L}_3}{2} \mathcal{K}_{\mu\nu} - \frac{\mathcal{L}_2}{2} \mathcal{K}^2_{\mu\nu} + [\mathcal{K}] \mathcal{K}^3_{\mu\nu} - \mathcal{K}^4_{\mu\nu},$$

$$W_{\mu\nu} = -\frac{\mathcal{L}_5}{2} g_{\mu\nu} + \tilde{W}_{\mu\nu},$$

$$\tilde{W}_{\mu\nu} = \frac{\mathcal{L}_4}{2} \mathcal{K}_{\mu\nu} - \frac{\mathcal{L}_3}{2} \mathcal{K}^2_{\mu\nu} + \frac{\mathcal{L}_2}{2} \mathcal{K}^3_{\mu\nu} - [\mathcal{K}] \mathcal{K}^4_{\mu\nu} + \mathcal{K}^5_{\mu\nu},$$

II. Cayley-Hamilton theorem and ghost-free graviton terms

- Here $\alpha = \alpha_3 + 1$, $\beta = \alpha_3 + \alpha_4$, and $\sigma = \alpha_4 + \alpha_5$.
- Note that $Y_{\mu\nu} = 0$ in four dimensional spacetimes [Do & Kao, PRD88(2013)063006] but $\neq 0$ in higher-than-four dimensional ones .
- Similarly, $W_{\mu\nu} = 0$ in five dimensional spacetimes but $\neq 0$ in higher-than-five dimensional ones.
- The constraint equations associated with the existence of fiducial metric:

$$t_{\mu\nu} \equiv \tilde{X}_{\mu\nu} + \sigma \tilde{Y}_{\mu\nu} + \alpha_5 \tilde{W}_{\mu\nu} - \frac{1}{2}\left(\alpha_3 \mathcal{L}_2 + \alpha_4 \mathcal{L}_3 + \alpha_5 \mathcal{L}_4\right) g_{\mu\nu} = 0.$$

- Due to these constraint equations the Einstein field equations for $g_{\mu\nu}$ become

$$(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) - \frac{m_g^2}{2} \mathcal{L}_M g_{\mu\nu} = 0; \;\; \mathcal{L}_M \equiv \mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \alpha_4 \mathcal{L}_4 + \alpha_5 \mathcal{L}_5,$$

$$\Rightarrow (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) + \Lambda_M g_{\mu\nu} = 0 \;(\text{Bianchi constraint}, \partial^\nu \mathcal{L}_M = 0),$$

with $\Lambda_M \equiv -m_g^2 \mathcal{L}_M / 2$ as an effective cosmological constant.

II. Cayley-Hamilton theorem and ghost-free graviton terms

Ghost free issue

- Follow the analysis of dRGT papers [1011.1232, 1007.0443] by considering the tensor $X^{(n)}_{\mu\nu}$ and its the recursive relation:

$$X^{(n)}_{\mu\nu}(g_{\mu\nu},\mathcal{K}) = \sum_{m=0}^{n}(-1)^m \frac{n!}{2(n-m)!}\mathcal{K}^m_{\mu\nu}\mathcal{L}^{(n-m)}_{\text{der}}(\mathcal{K})$$

$$X^{(n)}_{\mu\nu} = -\, n\mathcal{K}^{\alpha}_{\mu}X^{(n-1)}_{\alpha\nu} + \mathcal{K}^{\alpha\beta}X^{(n-1)}_{\alpha\beta}g_{\mu\nu}.$$

- For the 4D case $X^{(4)}_{\mu\nu}(g_{\mu\nu},\mathcal{K}) \sim Y_{\mu\nu} = 0 \rightarrow X^{(n>4)}_{\mu\nu}(g_{\mu\nu},\mathcal{K}) = 0 \rightarrow$ no ghostlike pathology arises at the quartic or higher order levels with arbitrary physical and fiducial metrics.
- Similarly, for the 5D case $X^{(5)}_{\mu\nu}(g_{\mu\nu},\mathcal{K}) \sim W_{\mu\nu} = 0 \rightarrow X^{(n>5)}_{\mu\nu}(g_{\mu\nu},\mathcal{K}) = 0 \rightarrow$ any ghostlike pathology arising at the quintic or higher order levels must disappear, no matter the form of physical and fiducial metrics.
- The similar conclusion is also valid for higher-than-five massive gravity theories.

III. Simple solutions for a five-dimensional massive gravity

- Solve the constraint Euler-Lagrange equations of fiducial metric's scale factors, which are indeed equivalent with $t_{\mu\nu} = 0$, in order to obtain the value of Λ_M.
- These constraint equations are not differential but algebraic.
- Solve the corresponding Einstein field equations to obtain the value of physical metric's scale factors.
- The fiducial metrics will be chosen to be compatible with the physical ones, i.e., they have the similar forms.
- FLRW (isotropic):

$$
\begin{aligned}
ds^2_{5d}(g_{\mu\nu}) &= -N_1^2(t)dt^2 + a_1^2(t)\left(d\vec{x}^2 + du^2\right), \\
ds^2_{5d}(f_{ab}) &= -N_2^2(t)dt^2 + a_2^2(t)\left(d\vec{x}^2 + du^2\right).
\end{aligned}
$$

III. Simple solutions for a five-dimensional massive gravity

- Bianchi type I (anisotropic):

$$
\begin{aligned}
ds^2_{5d}(g_{\mu\nu}) =& -N_1^2(t)dt^2 + \exp\left[2\alpha_1(t) - 4\sigma_1(t)\right]dx^2 \\
& + \exp\left[2\alpha_1(t) + 2\sigma_1(t)\right]\left(dy^2 + dz^2\right) + \exp\left[2\beta_1(t)\right]du^2, \\
ds^2_{5d}(f_{ab}) =& -N_2^2(t)dt^2 + \exp\left[2\alpha_2(t) - 4\sigma_2(t)\right]dx^2 \\
& + \exp\left[2\alpha_2(t) + 2\sigma_2(t)\right]\left(dy^2 + dz^2\right) + \exp\left[2\beta_2(t)\right]du^2,
\end{aligned}
$$

- Schwarzschild-Tangherlini black holes:

$$
\begin{aligned}
ds^2_{5d}(g_{\mu\nu}) =& -N_1^2(t,r)\,dt^2 + \frac{dr^2}{F_1^2(t,r)} + 2D_1(t,r)\,dtdr + \frac{r^2 d\Omega_3^2}{H_1^2(t,r)}, \\
ds^2_{5d}(f_{ab}) =& -N_2^2(t,r)\,dt^2 + \frac{dr^2}{F_2^2(t,r)} + 2D_2(t,r)\,dtdr + \frac{r^2 d\Omega_3^2}{H_2^2(t,r)},
\end{aligned}
$$

with $d\Omega_3^2 = d\theta^2 + \sin^2\theta d\varphi^2 + \sin^2\theta\sin^2\varphi d\psi^2$.

III. Simple solutions for a five-dimensional massive gravity

- FLRW: $\frac{\partial \mathcal{L}_M}{\partial N_2} = \frac{\partial \mathcal{L}_M}{\partial a_2} = 0$.
- Bianchi type I: $\frac{\partial \mathcal{L}_M}{\partial N_2} = \frac{\partial \mathcal{L}_M}{\partial \alpha_2} = \frac{\partial \mathcal{L}_M}{\partial \sigma_2} = 0$.
- Schwarzschild-Tangherlini: $\frac{\partial \mathcal{L}_M}{\partial N_2} = \frac{\partial \mathcal{L}_M}{\partial F_2} = \frac{\partial \mathcal{L}_M}{\partial H_2} = 0$.
- Note again that the Euler-Lagrange equations $\Leftrightarrow$ $t_{\mu\nu} = 0$.
- The constraint equations are non-linear algebraic equations $\rightarrow$ we obtain several values of Λ_M [see the paper 1602.05672 for more details].
- Recall the Einstein field equations for physical metric:

$$\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right) + \Lambda_M g_{\mu\nu} = 0.$$

- The corresponding solution for FLRW physical metric:

$$a_1(t) = \exp\left[\sqrt{\frac{\Lambda_M}{6}}\, t\right].$$

III. Simple solutions for a five-dimensional massive gravity

- The corresponding solutions for the Bianchi type I physical metric:

$$V_1 \equiv \exp[3\alpha_1] = \exp[3\alpha_0]\left[\cosh\left(3\tilde{H}_1 t\right) + \frac{\dot{\alpha}_0}{\tilde{H}_1}\sinh\left(3\tilde{H}_1 t\right)\right],$$

$$V_2 \equiv \exp[\beta_1] = \exp[\beta_0]\left[\cosh\left(3\bar{H}_1 t\right) + \frac{\dot{\beta}_0}{3\bar{H}_1}\sinh\left(3\bar{H}_1 t\right)\right],$$

$$\sigma_1 = \sigma_0 + \sqrt{\dot{\alpha}_0^2 + \dot{\alpha}_0\dot{\beta}_0 - H_1^2} \times \int\left\{\left[\cosh\left(3\tilde{H}_1 t\right) + \frac{\dot{\alpha}_0}{\tilde{H}_1}\sinh\left(3\tilde{H}_1 t\right)\right]\right.$$

$$\left.\times\left[\cosh\left(3\bar{H}_1 t\right) + \frac{\dot{\beta}_0}{3\bar{H}_1}\sinh\left(3\bar{H}_1 t\right)\right]\right\}^{-1} dt,$$

with with $\tilde{H}_1^2 = 4H_1^2/9(1-V_0)$, $\bar{H}_1^2 = V_0\tilde{H}_1^2$, and $H_1^2 \equiv \frac{\Lambda_M}{3}$. Additionally, $\alpha_0,\ \dot{\alpha}_0,\ \beta_0,\ \dot{\beta}_0,\ \sigma_0 \sim$ initial values.

III. Simple solutions for a five-dimensional massive gravity

- The Schwarzschild-Tangherlini solution [Tangherlini, Nuovo Cimento 27(1963)636] to the 5D massive gravity:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2,$$
$$N_1^2(t,r) = F_1^2(t,r) = f(r) = 1 - \frac{\mu}{r^2} - \frac{\Lambda_M}{6} r^2,$$
$$H_1^2(t,r) = 1,\ D_1^2(t,r) = 0.$$

  where $\mu = \frac{8G_5 M}{3\pi} \sim$ mass parameter, $M \sim$ the mass of source, and $G_5 \sim$ 5D Newton constant.

- $\Lambda_M > 0$ (< 0) $\sim$ Schwarzschild-Tangherlini-(Anti-) de Sitter metric.

IV. Simple solutions for a five-dimensional massive bi-gravity

- The action of five-dimensional massive bi-gravity [1604.07568]:

$$S_{5\mathrm{d}} = M_g^2 \int d^5x\sqrt{g}R(g) + M_f^2 \int d^5x\sqrt{f}R(f) + 2m^2 M_{\mathrm{eff}}^2 \int d^5x\sqrt{g}\Big(\mathcal{U}_2 + \alpha_3\mathcal{U}_3 + \alpha_4\mathcal{U}_4 + \alpha_5\mathcal{U}_5\Big),$$

where

$$\mathcal{U}_2 = \frac{1}{2}\Big\{[\mathcal{K}]^2 - [\mathcal{K}^2]\Big\},\ \mathcal{U}_3 = \frac{1}{6}\Big\{[\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]\Big\},$$

$$\mathcal{U}_4 = \frac{1}{24}\Big\{[\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}][\mathcal{K}^3] - 6[\mathcal{K}^4]\Big\},$$

$$\mathcal{U}_5 = \frac{\mathcal{L}_5}{2} = \frac{1}{120}\Big\{[\mathcal{K}]^5 - 10[\mathcal{K}]^3[\mathcal{K}^2] + 20[\mathcal{K}]^2[\mathcal{K}^3] - 20[\mathcal{K}^2][\mathcal{K}^3] + 15[\mathcal{K}][\mathcal{K}^2]^2 - 30[\mathcal{K}][\mathcal{K}^4] + 24[\mathcal{K}^5]\Big\}.$$

IV. Simple solutions for a five-dimensional massive bi-gravity

- The Einstein field equations for physical metric (identical to ones for massive gravity):

$$M_g^2\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)+m^2M_{\rm eff}^2\mathcal{H}^{(5)}_{\mu\nu}(g)=0,$$

$$\mathcal{H}^{(5)}_{\mu\nu}(g)=X^{(5)}_{\mu\nu}+\sigma Y^{(5)}_{\mu\nu}+\alpha_5 W_{\mu\nu},$$

$$X^{(5)}_{\mu\nu}=-\left(\alpha\mathcal{U}_2+\beta\mathcal{U}_3\right)g_{\mu\nu}+\tilde{X}^{(5)}_{\mu\nu},$$

$$\tilde{X}^{(5)}_{\mu\nu}=\mathcal{K}_{\mu\nu}-[\mathcal{K}]g_{\mu\nu}-\alpha\left\{\mathcal{K}^2_{\mu\nu}-[\mathcal{K}]\mathcal{K}_{\mu\nu}\right\}+\beta\left\{\mathcal{K}^3_{\mu\nu}-[\mathcal{K}]\mathcal{K}^2_{\mu\nu}+\mathcal{U}_2\mathcal{K}_{\mu\nu}\right\},$$

$$Y^{(5)}_{\mu\nu}=-\mathcal{U}_4 g_{\mu\nu}+\tilde{Y}^{(5)}_{\mu\nu},$$

$$\tilde{Y}^{(5)}_{\mu\nu}=\mathcal{U}_3\mathcal{K}_{\mu\nu}-\mathcal{U}_2\mathcal{K}^2_{\mu\nu}+[\mathcal{K}]\mathcal{K}^3_{\mu\nu}-\mathcal{K}^4_{\mu\nu},$$

$$W_{\mu\nu}=-\mathcal{U}_5 g_{\mu\nu}+\tilde{W}_{\mu\nu},$$

$$\tilde{W}_{\mu\nu}=\mathcal{U}_4\mathcal{K}_{\mu\nu}-\mathcal{U}_3\mathcal{K}^2_{\mu\nu}+\mathcal{U}_2\mathcal{K}^3_{\mu\nu}-[\mathcal{K}]\mathcal{K}^4_{\mu\nu}+\mathcal{K}^5_{\mu\nu}.$$

IV. Simple solutions for a five-dimensional massive bi-gravity

- The Einstein-like field equations for reference metric:

$$\sqrt{f} M_f^2 \left(R_{\mu\nu}(f) - \frac{1}{2} f_{\mu\nu} R(f) \right) + \sqrt{g} m^2 M_{\text{eff}}^2 s_{\mu\nu}^{(5)}(f) = 0,$$

$$s_{\mu\nu}^{(5)}(f) \equiv -\hat{\mathcal{K}}_{\mu\nu} + \left\{ [\mathcal{K}] + \alpha_3 \mathcal{U}_2 + \alpha_4 \mathcal{U}_3 + \alpha_5 \mathcal{U}_4 \right\} f_{\mu\nu} + \alpha \left\{ \hat{\mathcal{K}}_{\mu\nu}^2 - [\mathcal{K}] \hat{\mathcal{K}}_{\mu\nu} \right\}$$
$$- \beta \left\{ \hat{\mathcal{K}}_{\mu\nu}^3 - [\mathcal{K}] \hat{\mathcal{K}}_{\mu\nu}^2 + \mathcal{U}_2 \hat{\mathcal{K}}_{\mu\nu} \right\} - \sigma \left\{ \mathcal{U}_3 \hat{\mathcal{K}}_{\mu\nu} - \mathcal{U}_2 \hat{\mathcal{K}}_{\mu\nu}^2 + [\mathcal{K}] \hat{\mathcal{K}}_{\mu\nu}^3 - \hat{\mathcal{K}}_{\mu\nu}^4 \right\}$$
$$- \alpha_5 \left\{ \mathcal{U}_4 \hat{\mathcal{K}}_{\mu\nu} - \mathcal{U}_3 \hat{\mathcal{K}}_{\mu\nu}^2 + \mathcal{U}_2 \hat{\mathcal{K}}_{\mu\nu}^3 - [\mathcal{K}] \hat{\mathcal{K}}_{\mu\nu}^4 + \hat{\mathcal{K}}_{\mu\nu}^5 \right\}.$$

- $\hat{\mathcal{K}}$'s are defined as $\hat{\mathcal{K}}_{\mu\nu} = \mathcal{K}_\mu^\sigma f_{\sigma\nu}$, $\hat{\mathcal{K}}_{\mu\nu}^2 = \mathcal{K}_\mu^\rho \mathcal{K}_\rho^\sigma f_{\sigma\nu}$, and so on.
- These equations are differential, not algebraic as ones for the reference metric in the massive gravity $\rightarrow$ the massive graviton terms $\mathcal{U}_i$'s will not easily turn out to be effective constants $\rightarrow$ need the help of the Bianchi identities for both physical and reference metrics:

$$D_g^\mu G_{\mu\nu}(g) = 0 \;\; \rightarrow \;\; D_g^\mu \mathcal{H}_{\mu\nu}^{(5)}(g) = 0 \text{ (physical metric)},$$

$$D_f^\mu G_{\mu\nu}(f) = 0 \;\; \rightarrow \;\; D_f^\mu \left[\frac{\sqrt{g}}{\sqrt{f}} s_{\mu\nu}^{(5)}(f) \right] = 0 \text{ (reference metric)}.$$

IV. Simple solutions for a five-dimensional massive bi-gravity

- Solving these Bianchi constraint equations for the FLRW, Bianchi type I, and Schwarzschild-Tangherlini metrics will yield a solution:

$$f_{\mu\nu} = (1-\mathcal{C})^2 g_{\mu\nu} \;\; (\text{proportional to } g_{\mu\nu}).$$

- $\mathcal{C}$ is a constant obeying the following algebraic equation:

$$\sigma\mathcal{C}^5 - 2(\sigma - 2\beta)\mathcal{C}^4 + \left(\sigma - 8\beta + 6\alpha + \alpha_5\tilde{M}^2\right)\mathcal{C}^3$$
$$+4\left(\beta - 3\alpha + \alpha_4\tilde{M}^2 + 1\right)\mathcal{C}^2 + 2\left(3\alpha + 3\alpha_3\tilde{M}^2 - 4\right)\mathcal{C} + 4\left(\tilde{M}^2 + 1\right) = 0,$$
$$\tilde{M}^2 = \tilde{M}_g^2/\tilde{M}_f^2;\;\; \tilde{M}_g^2 = M_g^2/(m^2 M_{\rm eff}^2);\;\; \tilde{M}_f^2 = M_f^2/(m^2 M_{\rm eff}^2).$$

- Once $\mathcal{C}$ is solved, the corresponding value of effective cosmological constant $\Lambda_M \equiv -m^2 M_{\rm eff}^2 \mathcal{U}_M$ will be defined as:

$$\Lambda_M = -\,m^2 M_{\rm eff}^2 \mathcal{C}\left[\left(\sigma\mathcal{C}^3 + 4\beta\mathcal{C}^2 + 6\alpha\mathcal{C} + 4\right) + (\mathcal{C}-1)\left(\alpha_5\mathcal{C}^3 + 4\alpha_4\mathcal{C}^2 + 6\alpha_3\mathcal{C} + 4\right)\right].$$
$$= \Lambda_0^g\left[M_g^2 + M_f^2(1-\mathcal{C})^3\right] \neq \Lambda_0^g M_g^2.$$

- Note that in massive gravity, $\Lambda_M = M_g^2\Lambda_0^g$ due to $M_f = 0$.

IV. Simple solutions for a five-dimensional massive bi-gravity

- For the FLRW metric:

$$a_1(t) = \exp\left[\sqrt{\Lambda_0^g/6t}\right];\ a_2(t) = (1-\mathcal{C})a_1(t).$$

- For the Bianchi type I metric:

$$\exp[3\alpha_1] = \exp[3\alpha_{01}]\left[\cosh\left(3\tilde{H}_1 t\right) + \frac{\dot{\alpha}_{01}}{\tilde{H}_1}\sinh\left(3\tilde{H}_1 t\right)\right],$$

$$\exp[\beta_1] = \exp[\beta_{01}]\left[\cosh\left(3\bar{H}_1 t\right) + \frac{\dot{\beta}_{01}}{3\bar{H}_1}\sinh\left(3\bar{H}_1 t\right)\right],$$

$$\sigma_1 = \sigma_{01} + \sqrt{\dot{\alpha}_{01}^2 + \dot{\alpha}_{01}\dot{\beta}_{01} - \frac{\Lambda_0^g}{3}}\int\left\{\left[\cosh\left(3\tilde{H}_1 t\right) + \frac{\dot{\alpha}_{01}}{\tilde{H}_1}\sinh\left(3\tilde{H}_1 t\right)\right]\right.$$

$$\left.\times\left[\cosh\left(3\bar{H}_1 t\right) + \frac{\dot{\beta}_{01}}{3\bar{H}_1}\sinh\left(3\bar{H}_1 t\right)\right]\right\}^{-1} dt.$$

$$\exp[\alpha_2] = (1-\mathcal{C})\exp[\alpha_1];\ \exp[\beta_2] = (1-\mathcal{C})\exp[\beta_1];\ \sigma_2 = \sigma_1,$$

$$\tilde{H}_1^2 = 4\Lambda_0^g/27(1-V_0^g);\ \bar{H}_1^2 = V_0^g\tilde{H}_1^2.$$

IV. Simple solutions for a five-dimensional massive bi-gravity

- For the Schwarzschild-Tangherlini black hole:

$$N_1^2(r) = F_1^2(r) = f(r) = 1 - \frac{\mu}{r^2} - \frac{\Lambda_0^g}{6} r^2, \ H_1^2(r) = 1,$$

$$g_{\mu\nu}^{\text{5d}} dx^\mu dx^\nu = - f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2,$$

$$f_{\mu\nu}^{\text{5d}} dx^\mu dx^\nu = (1 - \mathcal{C})^2 \left[-f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2 \right].$$

V. Conclusions

- An effective method based on the Cayley-Hamilton theorem to construct arbitrary dimensional graviton potential terms has been proposed.
- We have shown that the five-dimensional massive (bi)gravity theories with additional massive graviton term $\mathcal{L}_5$ are indeed physically non-trivial.
- The nature of cosmological constant Λ_M can be realized in the context of massive (bi)gravity. In particular, all complicated massive terms in $\mathcal{L}_M$ are behind in a simple constant Λ_M.
- We have found that some well-known metrics such as the FLRW, Bianchi type I, and Schwarzschild-Tangherlini spacetimes are indeed solutions of the five-dimensional massive (bi)gravity under assumptions that the physical metrics are compatible/proportional with/to the fiducial ones.

(Possible) further investigations

- A full ghost-free proof for higher dimensional massive (bi-)gravity [this task might be straightforward as claimed in Hassan, Schmidt-May & von Strauss, 1212.4525] ?
- The stability of Schwarzschild-Tangherlini-(A)dS black holes in the context of 5D massive (bi-)gravity ?
- The fate of cosmic no-hair conjecture in massive (bi-)gravity ?
- Higher-than-five dimensional scenarios of massive (bi-)gravity ?
- Gravitational waves in higher dimensional massive (bi-)gravity ?
- Bound of graviton mass in the massive (bi-)gravity [de Rham, Deskins, Tolley & Zhou, 1606.08462] ?
  Note that LIGO has announced a bound: $m_g < 1.2 \times 10^{-22}\ eV/c^2$ from data of GW150914 [1602.03837].
  For a comparison: Mass of electron is given by $m_e \simeq 0.51 \times 10^6\ eV/c^2$, Mass of neutrino is determined as $m_\nu = 0.320 \pm 0.081\ eV/c^2$.

Thank you for your attention!