This Talk's Three Key "Takeaways"

• Relativistic time dilation *is incompatible with Newton's Second Law*. However *fixing to unity* the g_{00} metric component for a dust ball *extinguishes relativistic time dilation and produces Friedmann's Newtonian solution*.

• Extended *Oppenheimer-Snyder transformation* of the $g_{00} = 1$ Friedmann dust-ball solution to "standard" metric form, which has *unconstrained* g_{00}*, enables relativistic time dilation, producing non-Newtonian dust motion*.

• Gravitational time dilation *causes all expanding dust balls to undergo early-epoch "accelerative inflation"*. For favorable initial conditions that acceleration of expansion *can persist throughout "standard" time* (although it decays toward zero), which apparently renders the "dark-energy" cosmological-constant hypothesis *unnecessary*.

Acceleration of Dust-Ball Expansion due to GR Gravitational Time Dilation

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A. Acceleration versus force in the presence of relativistic time dilation

A1. Assuming purely radial motion, conservation of Newtonian kinetic plus gravitational potential energy, namely,

$$\tfrac{1}{2}m\dot{r}^2 - GmM/r = T_0 + V_0,$$

implies the Newtonian Second Law *proportionality of acceleration to the gravitational force*, namely $\ddot{r} = -GM/r^2$.

A2. However, conservation of *special relativistic* kinetic plus gravitational potential energy, namely,

$$mc^2[(1 - \dot{r}^2/c^2)^{-\frac{1}{2}} - 1] - GmM/r = T_0 + V_0,$$

disrupts Newtonian Second Law proportionality of acceleration to gravitational force $-GmM/r^2$, implying *instead*,

$$\ddot{r} = \frac{-GM/r^2}{[1 + (T_0 + V_0)/(mc^2) + (GM)/(c^2 r)]^3}.$$

We note in particular that as $r \to 0$ *and the gravitational force becomes infinite*, the acceleration $\ddot{r} \to 0$! That occurs because special relativistic *time dilation* stops $|\dot{r}|$ from reaching c by driving $\ddot{r}$ *toward zero*.

A3. In GR, as the surface radius $\bar{r}_a$ of a simple FLRW dust ball *contracts toward its Schwarzschild value* r_S, the gravitational time-dilation factor at $\bar{r}_a$ *increases toward infinity*, which drives the contracting dust ball's *negative surface radial velocity* $d\bar{r}_a/d\bar{t}_a$ toward *zero*! Thus in GR a *contracting* dust ball's surface radius $\bar{r}_a$ will *ultimately* undergo *positive* (i.e., *outward*) acceleration *when its value gets close enough to the dust ball's Schwarzschild radius value* r_S*—in spite of the strong negative (i.e., inward) gravitational force on the dust at that radius*!

A4. Unfortunately this fascinating consequence of gravitational time dilation *cannot* be reflected *by the only known analytic solution of the Einstein equation for the simple FLRW dust ball*: that solution's metric-tensor component g_{00} *is fixed to unity*, which *extinguishes* gravitational time dilation *because* $(g_{00})^{-\frac{1}{2}}$ *is the gravitational time-dilation factor*! *Fixing* g_{00} *to unity*, however, is well-known *to require the clock readings of an infinite number of different observers* [Steven Weinberg, *Gravitation and Cosmology*, Section 11.8, page 338], a requirement which *neither* can be physically fulfilled *nor* is compatible with Einstein's observer-to-coordinate-system paradigm!

A5. That $g_{00} = 1$ *GR-unphysical* Einstein-equation solution for the FLRW dust ball was, for a *particular* initial condition, *transcended* in 1939 by Oppenheimer and Snyder, who, in an analytical tour-de-force, *worked out the transformation* of the (r, t) "comoving coordinates" of that solution's *GR-unphysical* spherically-symmetric metric,

$$ds^2 = (cdt)^2 - U(r,t)(dr)^2 - V(r,t)\left((d\theta)^2 + (\sin\theta d\phi)^2\right),$$

to the $(\bar{r}, \bar{t})$ "standard coordinates" of the *GR-physical* spherically-symmetric metric,

$$ds^2 = B(\bar{r},\bar{t})(cd\bar{t})^2 - A(\bar{r},\bar{t})(d\bar{r})^2 - \bar{r}^2\left((d\theta)^2 + (\sin\theta d\phi)^2\right),$$

which doesn't extinguish of gravitational time dilation since its g_{00} metric component $B(\bar{r},\bar{t})$ *isn't constrained*!

A6. The $g_{00} = 1$ GR-unphysical Einstein-equation solution's metric functions $U(r, t)$ and $V(r, t)$ are given by,

$$U(r, t) = (R(t))^2/(1 + \gamma(\omega r/c)^2) \text{ and } V(r, t) = r^2(R(t))^2,$$

where the dimensionless function $R(t)$ is given in terms of the *always-uniform* energy density $\rho(t)$ of the dust by,

$$R(t) = (\rho(t_0)/\rho(t))^{\frac{1}{3}},$$

defined so that $R(t_0) = 1$. In the $g_{00} = 1$ GR-unphysical "comoving coordinates" (r, t), the dust, as well as having *always-uniform* time-dependent energy density $\rho(t)$, also has *zero particle velocity*, so the dust ball's radius a *doesn't change in "comoving coordinates"*. The Einstein equation *within the dust ball in those "comoving coordinates"* (r, t), i.e., *within* the "comoving region" $0 \leq r \leq a$, implies that $R(t)$ satisfies *the Friedmann equation*, which is,

$$(\dot{R}(t))^2 = \omega^2((1/R(t)) + \gamma),$$

where, of course, $R(t_0) = 1$, and the constants ω and γ are related to $\rho(t_0)$ and $\dot{\rho}(t_0)$ as follows,

$$\omega^2 = (8\pi/3)G\rho(t_0)/c^2 \text{ and } \gamma = (\dot{R}(t_0)/\omega)^2 - 1 = (\dot{\rho}(t_0)/(3\omega\rho(t_0)))^2 - 1 \geq -1.$$

A7. The *entirely Newtonian character of the Friedmann equation* is *explicitly revealed* upon making the definitions,

$$r(t) \stackrel{\text{def}}{=} aR(t) \text{ and } M \stackrel{\text{def}}{=} (4\pi/3)a^3\rho(t_0)/c^2,$$

which imply that $\omega^2 = 2GM/a^3$, $\gamma = [a(\dot{r}(t_0))^2/(2GM)] - 1$ and $a = r(t_0)$ when they are combined with the relations given in A6. These three formulas plus $R(t) = (r(t)/a)$ change the Friedmann equation of A6 *to a simple Newtonian gravitational equation of motion of the type displayed in* A1, specifically to,

$$\tfrac{1}{2}(\dot{r}(t))^2 - GM/r(t) = \tfrac{1}{2}(\dot{r}(t_0))^2 - GM/r(t_0).$$

A8. Oppenheimer and Snyder solved for the transformation $(\bar{r}(r,t), \bar{t}(r,t))$ from the $g_{00} = 1$ GR-unphysical metric's (r,t) "comoving coordinates" to the GR-physical metric's $(\bar{r},\bar{t})$ "standard coordinates" *only when the particular initial condition* $\dot{\rho}(t_0) = 0$ *holds*, i.e., *only when* $\dot{R}(t_0) = 0$ *and* $\gamma = -1$. But the arduously intricate techniques they developed to obtain their limited transformation *are adequate to extend it to all values of* $\gamma \geq -1$.

A *key subtlety* resolved by Oppenheimer and Snyder is that the A5- and A6-described $g_{00} = 1$ metric specified *only* for $0 \leq r \leq a$ transforms into an A5 "standard" metric form *which is only determined up to an arbitrary function of one variable*. The Birkhoff theorem, however, states that outside of *and on* the dust ball's surface an A5 "standard" metric form is equal to the Schwarzschild metric, which fact *uniquely pins down* that arbitrary function of one variable.

B. The extended Oppenheimer-Snyder transformation

B1. The presentation of the extended Oppenheimer-Snyder transformation is more compact when all occurrences of the two constants ω and γ are systematically replaced by occurrences of the Schwarzschild radius r_S in "standard" coordinates and the dimensionless constant α: the constants r_S and α are defined as follows (M is discussed in A7),

$$r_S \stackrel{\text{def}}{=} \omega^2 a^3/c^2 = 2GM/c^2 \;\;\&\;\; \alpha \stackrel{\text{def}}{=} \gamma(r_S/a).$$

With r_S and α superseding ω and γ, the extended Oppenheimer-Snyder transformation is presented as,

$$\bar{r}(r,t) = rR(t) \;\;\&\;\; \bar{t}(r,t) = \bar{t}(a,t_0) \pm (a/c)(1+\alpha)^{\frac{1}{2}} \int_1^{S(r,t)} \frac{ds}{((r_S/(as))+\alpha)^{\frac{1}{2}}(1-(r_S/(as)))},$$

where the $\pm$ is the *sign* of $\dot{R}(t)$ if $\dot{R}(t) \neq 0$, but $\pm = -1$ if $\dot{R}(t) = 0$, and $S(r,t)$ is the following expression,

$$S(r,t) = R(t)\left(\frac{1+(r/a)^2\alpha}{1+\alpha}\right)^{\frac{1}{2}} - \left(\frac{r_S}{a\alpha}\right)\left[1 - \left(\frac{1+(r/a)^2\alpha}{1+\alpha}\right)^{\frac{1}{2}}\right],$$

where $0 \leq r \leq a$. Because $R(t)$ obeys the Friedmann equation, $d\bar{t}(r,t)/dt \to 1$ as $c \to \infty$.

B2. Note that the integral in the expression for $\bar{t}(r,t)$ *diverges to infinity unless* $a > r_S$ (and therefore $\alpha > -1$, since $\alpha = \gamma(r_S/a)$ and $\gamma \geq -1$) and $S(r,t) > (r_S/a)$. Thus a subset of "comoving space-time" is transformed to *infinite* "standard" time, *which makes that subset inaccessible in GR-physical "standard" space-time*. (This space-time subset inaccessibility arises from *the singular character of the transformation from GR-unphysical "comoving coordinates" to GR-physical "standard" coordinates*—it patently *wouldn't occur for the nonsingular transformation that necessarily obtains between two GR-physical metrics*.)

B3. The inaccessibility of a subset of GR-unphysical "comoving space-time" in GR-physical "standard" space-time *furthermore renders inaccessible in GR-physical "standard" coordinates those dust-ball configurations which occur in that inaccessible subset of GR-unphysical "comoving space-time"*. In the next section we shall see that *gravitational time dilation effects* which drive dust-particle *speeds* toward zero in GR-physical "standard" coordinates *enforce this inaccessibility in GR-physical "standard" coordinates of such disallowed dust-ball configurations*—just as *particle-speed time dilation effects* which drive particle *acceleration* toward zero in special relativity *enforce the inaccessibility in special relativity of disallowed particle speeds of* c *or greater*.

C. Equations of motion of the dust-ball interior-shell radii in "standard" coordinates

C1. In GR-unphysical "comoving coordinates" (r, t), the dust ball's *interior-shell radii* ϵa, $0 < \epsilon \leq 1$, *have zero velocity*, so *their world lines are just* $(\epsilon a, t)$. The B1 extended Oppenheimer-Snyder transformation maps these "comoving" world lines $(\epsilon a, t)$ to "standard" world lines $(\bar{r}_{\epsilon a}(t), \bar{t}_{\epsilon a}(t)) \stackrel{\text{def}}{=} (\bar{r}(\epsilon a, t), \bar{t}(\epsilon a, t))$. Therefore B1 yields,

$$\bar{r}_{\epsilon a}(t) = \epsilon a R(t) \;\;\&\;\; \bar{t}_{\epsilon a}(t) = \bar{t}(a, t_0) \pm (a/c)(1+\alpha)^{\frac{1}{2}} \int_1^{S(\epsilon a, t)} \frac{ds}{((r_S/(as)) + \alpha)^{\frac{1}{2}}(1 - (r_S/(as)))}$$
$$= \bar{t}(a, t_0) \pm (c\epsilon)^{-1}(1+\alpha)^{\frac{1}{2}} \int_{\epsilon a}^{\rho_{\epsilon a}(\bar{r}_{\epsilon a}(t))} \frac{d\rho}{((\epsilon r_S/\rho) + \alpha)^{\frac{1}{2}}(1 - (\epsilon r_S/\rho))} = \bar{t}_{\epsilon a}(\rho_{\epsilon a}(\bar{r}_{\epsilon a}(t))),$$

where the *initial* integration variable s is *changed to* $\rho = (\epsilon a)s$, and consequently,

$$\rho_{\epsilon a}(\bar{r}_{\epsilon a}(t)) = (\epsilon a)S(\epsilon a, t) = \bar{r}_{\epsilon a}(t)\left(\frac{1+\epsilon^2\alpha}{1+\alpha}\right)^{\frac{1}{2}} - \left(\frac{\epsilon r_S}{\alpha}\right)\left[1 - \left(\frac{1+\epsilon^2\alpha}{1+\alpha}\right)^{\frac{1}{2}}\right].$$

The above integral expression for $\bar{t}_{\epsilon a}(\rho_{\epsilon a}(\bar{r}_{\epsilon a}))$ *diverges to infinity* unless $\rho_{\epsilon a}(\bar{r}_{\epsilon a}) > \epsilon r_S$. Taking care to respect that *caveat*, the $d\rho$ integration can be carried out in closed form to produce an intricate *analytic result for* $\bar{t}_{\epsilon a}(\rho_{\epsilon a}(\bar{r}_{\epsilon a}))$. That analytic result is useful for creating *numerical plots of* $\bar{r}_{\epsilon a}(\bar{t}_{\epsilon a})$, but its intricacy *resists direct interpretation*.

C2. However the C1 integral result for $\bar{t}_{\epsilon a}(\rho_{\epsilon a}(\bar{r}_{\epsilon a}))$ *also* readily yields *a first-order differential equation for* $\bar{r}_{\epsilon a}(\bar{t}_{\epsilon a})$,

$$d\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a} = (d\bar{t}_{\epsilon a}(\rho_{\epsilon a}(\bar{r}_{\epsilon a}))/d\bar{r}_{\epsilon a})^{-1} = \pm c\epsilon \left(\frac{(\epsilon r_S/\rho_{\epsilon a}(\bar{r}_{\epsilon a})) + \alpha}{1 + \epsilon^2\alpha}\right)^{\frac{1}{2}} (1 - (\epsilon r_S/\rho_{\epsilon a}(\bar{r}_{\epsilon a}))),$$

which of course comes with the C1 *caveat* that $\rho_{\epsilon a}(\bar{r}_{\epsilon a}) > \epsilon r_S$. The details of the form of $\rho_{\epsilon a}(\bar{r}_{\epsilon a})$ are given in C1.

C3. The C2 equation and its *caveat* $\rho_{\epsilon a}(\bar{r}_{\epsilon a}) > \epsilon r_S$ imply an *upper bound* for the shell-radius *speed* $|d\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a}|$,

$$|d\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a}| < c \left(\frac{\epsilon^2 + \epsilon^2\alpha}{1 + \epsilon^2\alpha}\right)^{\frac{1}{2}} (1 - (\epsilon r_S/\rho_{\epsilon a}(\bar{r}_{\epsilon a}))).$$

Since the relationship of $\rho_{\epsilon a}(\bar{r}_{\epsilon a})$ to $\bar{r}_{\epsilon a}$ *is a linear one* (see C1), this upper bound drives $|d\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a}|$ *linearly toward zero* as $\rho_{\epsilon a}(\bar{r}_{\epsilon a}) \to \epsilon r_S+$, so it isn't possible for $\rho_{\epsilon a}(\bar{r}_{\epsilon a})$ to become equal to or smaller than ϵr_S *in any finite interval of "standard" local shell time* $\Delta\bar{t}_{\epsilon a}$. This *linear zeroing of the approach speed to the GR-unphysical configuration forbidden by the caveat reinforces that caveat*, which illustrates *the crucial role of gravitational time dilation*. (For the dust ball's *surface shell*, i.e., for $\epsilon = 1$, the caveat *simplifies* to $\bar{r}_a > r_S$ [see C1], so a dust ball's radius *always exceeds its Schwarzschild radius*, which implies that a dust ball *can't produce an event horizon*.)

C4. Since $r_S = 2GM/c^2$ and $\alpha = \gamma(r_S/a)$, in the $c \to \infty$ *nonrelativistic limit*, $\rho_{\epsilon a}(\bar{r}_{\epsilon a}) \to \bar{r}_{\epsilon a}$, and the C2 dust-ball shell-radii equations of motion reduce to,

$$d\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a} = \pm\epsilon^{\frac{3}{2}}(2GM)^{\frac{1}{2}}\left((1/\bar{r}_{\epsilon a}) + (\gamma/(\epsilon a))\right)^{\frac{1}{2}}.$$

These nonrelativistic-limit shell-radii equations of motion are *completely devoid* of both the *speed* and the *configuration* constraints which featured so prominently in C3. Squaring *both sides* of *each* above nonrelativistic-limit shell-radius equation of motion reveals that it *corresponds* to the *Newtonian Friedmann equation* $(\dot{R}(\bar{t}_{\epsilon a}))^2 = \omega^2((1/R(\bar{t}_{\epsilon a})) + \gamma)$, where $\omega^2 = (2GM/a^3)$, *via the simple scaling relationship* $\bar{r}_{\epsilon a}(\bar{t}_{\epsilon a}) = \epsilon a R(\bar{t}_{\epsilon a})$, $0 < \epsilon \leq 1$. The *second-order form* of each above nonrelativistic-limit shell-radius equation of motion is readily worked out to be,

$$d^2\bar{r}_{\epsilon a}/d\bar{t}^2_{\epsilon a} = -\epsilon^3 GM/\bar{r}^2_{\epsilon a},$$

which reflects *the nonrelativistic-limit* fact that the shell radius' acceleration arises from *the spherical effective mass* $\epsilon^3 M$, $0 < \epsilon \leq 1$, *which is the source of net Newtonian gravitational force on that shell*.

C5. The *full* C2 shell-radius *first-order* equation of motion in "standard" coordinates *also* has a *second-order form*, which illuminates *the modification of shell-radius acceleration caused by gravitational time dilation*. One differentiates both sides of the C2 equation with respect to $\bar{t}_{\epsilon a}$, and then replaces the overall factor of $d\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a}$ on that result's right-hand side by the right-hand side of the C2 equation to obtain,

$$d^2\bar{r}_{\epsilon a}/d\bar{t}^2_{\epsilon a} = (\epsilon/2)(c^2/r_S)\left[(-1+2\alpha)(\epsilon r_S/\rho_{\epsilon a}(\bar{r}_{\epsilon a}))^2 + 3(\epsilon r_S/\rho_{\epsilon a}(\bar{r}_{\epsilon a}))^3\right]\left[\frac{1-(\epsilon r_S/\rho_{\epsilon a}(\bar{r}_{\epsilon a}))}{(1+\epsilon^2\alpha)^{\frac{1}{2}}(1+\alpha)^{\frac{1}{2}}}\right],$$

which has *the same caveat* $\rho_{\epsilon a}(\bar{r}_{\epsilon a}) > \epsilon r_S$ as the C2 equation. Since $3 > (1-2\alpha)$ (because $\alpha > -1$), there *always* exists a *range of* $\rho_{\epsilon a}(\bar{r}_{\epsilon a})$ *values* which *both* satisfy the caveat $\rho_{\epsilon a}(\bar{r}_{\epsilon a}) > \epsilon r_S$ *and* produce *positive* (i.e., *outward*) shell-radius acceleration $d^2\bar{r}_{\epsilon a}/d\bar{t}^2_{\epsilon a}$—that is so *despite* the fact that the $c \to \infty$ *nonrelativistic limit* of the second-order equation is,

$$d^2\bar{r}_{\epsilon a}/d\bar{t}^2_{\epsilon a} = -\epsilon^3 GM/\bar{r}^2_{\epsilon a},$$

which implies *always negative* (i.e., *always inward*) acceleration. Furthermore, for all initial conditions such that $\alpha \geq \frac{1}{2}$, every shell-radius acceleration $d^2\bar{r}_{\epsilon a}/d\bar{t}^2_{\epsilon a}$ is *positive* (i.e., *outward*) at *all finite "standard" local times* $\bar{t}_{\epsilon a}$ (at any *finite* "standard" local time $\bar{t}_{\epsilon a}(\rho_{\epsilon a}(\bar{r}_{\epsilon a}))$, C1 implies that the caveat $\rho_{\epsilon a}(\bar{r}_{\epsilon a}) > \epsilon r_S$ *is automatically satisfied*). That it is *possible* for *every* shell-radius acceleration to be *positive* (i.e., *outward*) at *all* finite "standard" local times apparently *eliminates any need to postulate a nonzero "dark energy" cosmological constant*.