On the stability of a superspinar

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In collaboration with

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Based on **arXiv:1707.07242**

§ 1 Introduction

Kerr spacetime: vacuum, stationary, axi-symmetric, asymptotically flat

In Boyer-Lindquist coordinates

$$ds^2 = -\frac{\Sigma\Delta}{A}dt^2 + \frac{A}{\Sigma}\sin^2\theta\left(d\varphi - \frac{2aMr}{A}dt\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$

where $\Delta(r) = r^2 - 2Mr + a^2$

$$\Sigma(r,\theta) = r^2 + a^2\cos^2\theta \qquad A(r,\theta) = \left(r^2 + a^2\right)^2 - \Delta(r)a^2\sin^2\theta$$

$$-\infty < t < +\infty, \quad -\infty < r < +\infty, \quad 0 \le \theta \le \pi, \quad 0 \le \varphi < 2\pi$$

M : ADM mass $\qquad$ a : Kerr parameter (angular momentum $L=aM$)

$a^2 \le M^2$: larger root of $\Delta(r) = 0$ $\left(r = M + \sqrt{M^2 - a^2} \ge M\right)$ is the event horizon

$a^2 > M^2$: always $\Delta(r) > 0$ holds. No event horizon.

Ring singularity at $r = 0$, $\theta = \pi/2$ is naked.

Reissner-Nordström spacetime

Maxwell field, static, spherically symmetric, asymptotically flat

Metric
$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

where
$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$$

M : ADM mass $\quad$ Q : Charge parameter

Gauge one-form
$$A_\mu = \left(-\frac{Q}{r}, 0,0,0\right)$$

$Q^2 \leq M^2$: larger root of $f(r) = 0$ $\left(r = M + \sqrt{M^2 - Q^2} \geq M\right)$ is the event horizon

$Q^2 > M^2$: always $f(r) > 0$ holds. No event horizon.

Singularity at $r = 0$ is naked.

From a point of view of SUSY,

if the system is supersymmetric, BPS bound $Q^2 \leq M^2$ holds.

Kerr bound $a^2 \leq M^2$ does not have a special meaning.

In the framework of superstring theory,

over-spinning very compact entity named the superspinar may exist.

Stringy effects will make any singularities harmless.

Gimon and Horava (2007)

The over-spinning Kerr geometry around the naked singularity
is very interesting.

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$\frac{\partial}{\partial t}$: time coordinate basis=Killing vector field

Ergo-region (ergo-sphere): $\frac{\partial}{\partial t}$ is spacelike $\quad g\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right) = -\left(1 - \frac{2Mr}{\Sigma}\right) > 0$

$$M - \sqrt{M^2 - a^2\cos^2\theta} < r < M + \sqrt{M^2 - a^2\cos^2\theta}$$

Killing "energy" of a particle: $E = -\boldsymbol{u}\cdot\frac{\partial}{\partial t}$ can be negative in the ergo-region,

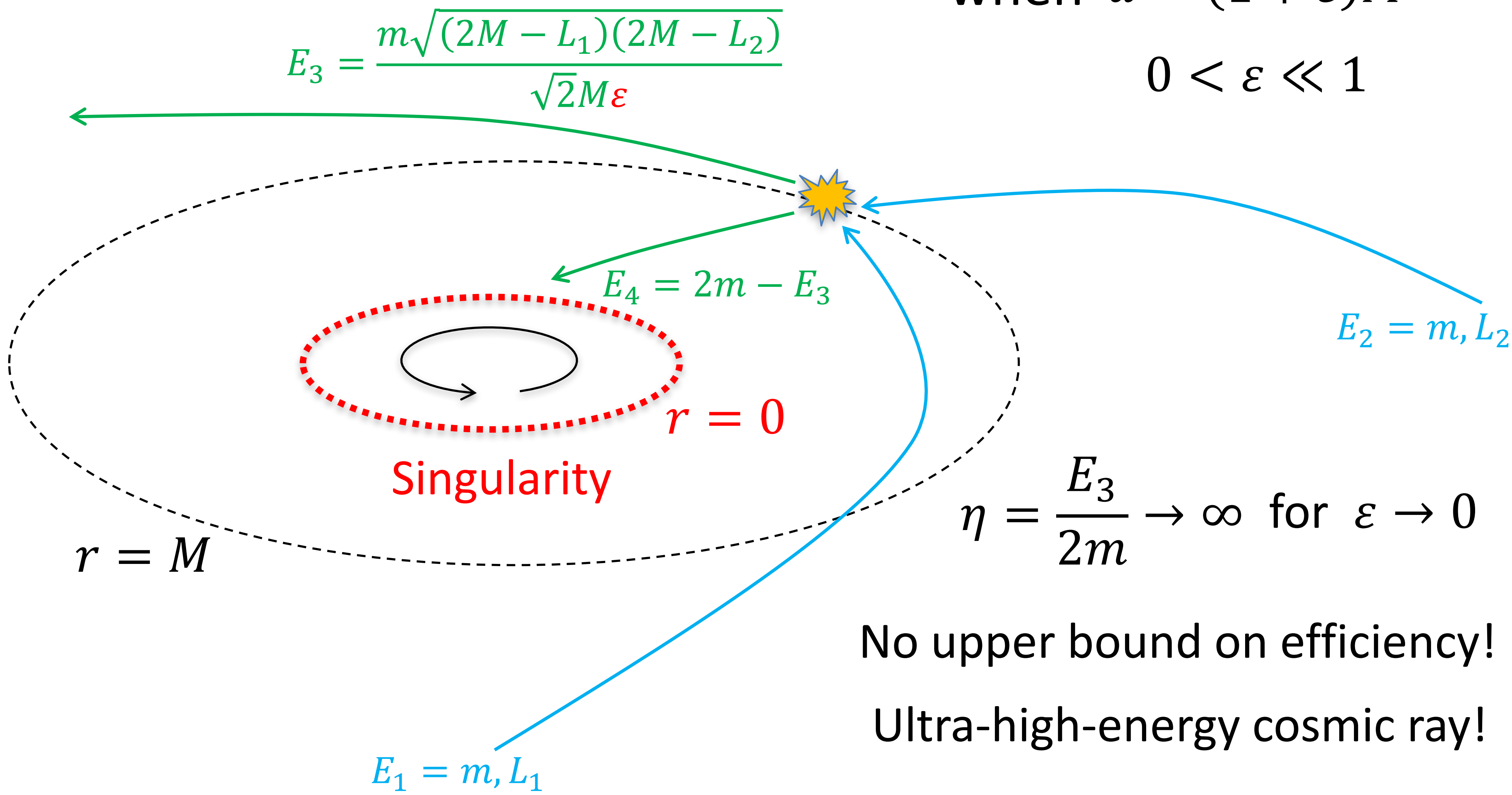
even if $\boldsymbol{u}$ is future directed timelike vector.

Collisional Penrose process

M. Patil, T. Harada, KN, P.S. Joshi and M. Kimura (2015)

when $a = (1+\varepsilon)M$

$0 < \varepsilon \ll 1$



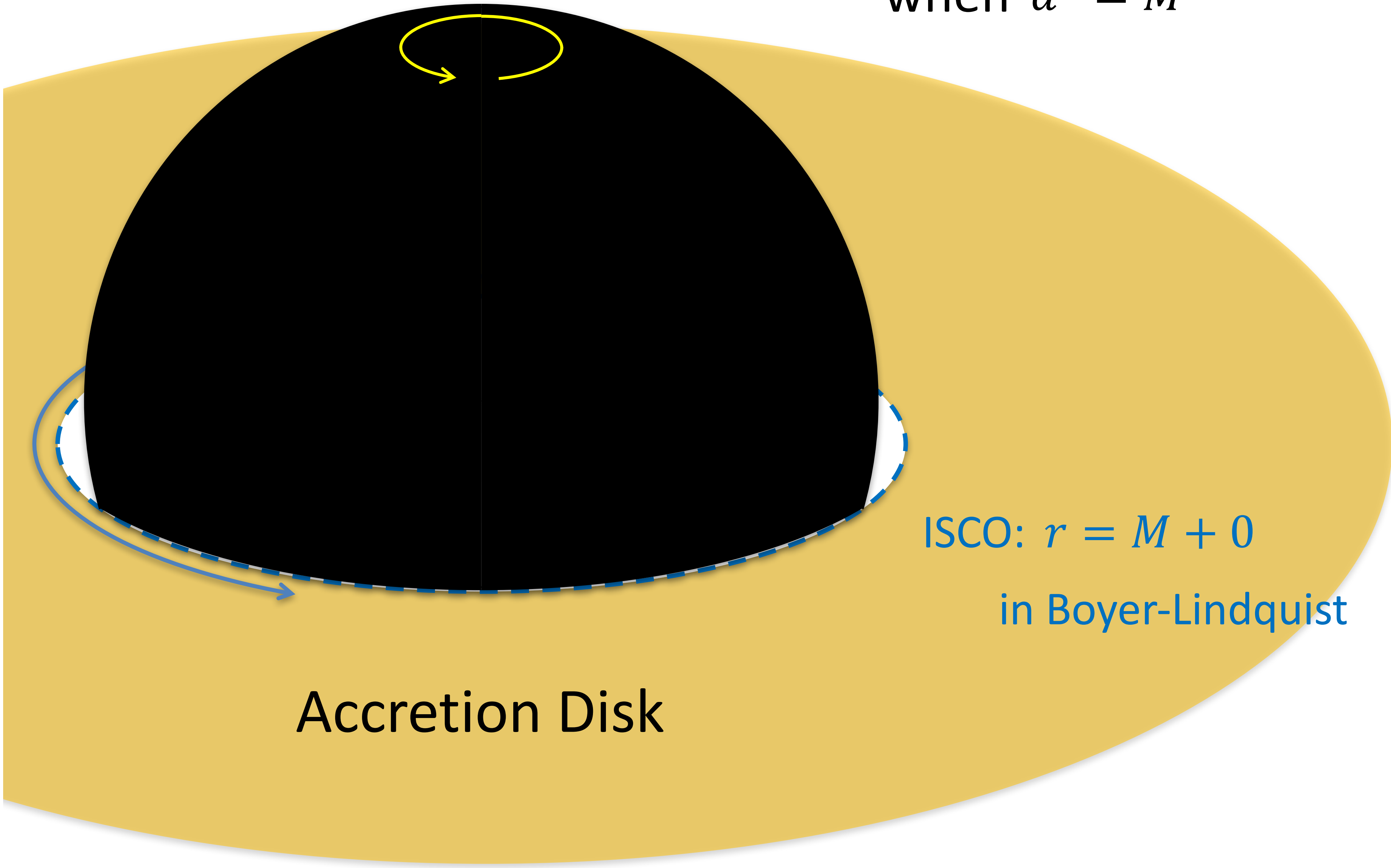
$$\eta = \frac{E_3}{2m} \to \infty \quad \text{for} \quad \varepsilon \to 0$$

No upper bound on efficiency!

Ultra-high-energy cosmic ray!

Efficiency of energy extraction from accreting matter≈42%,

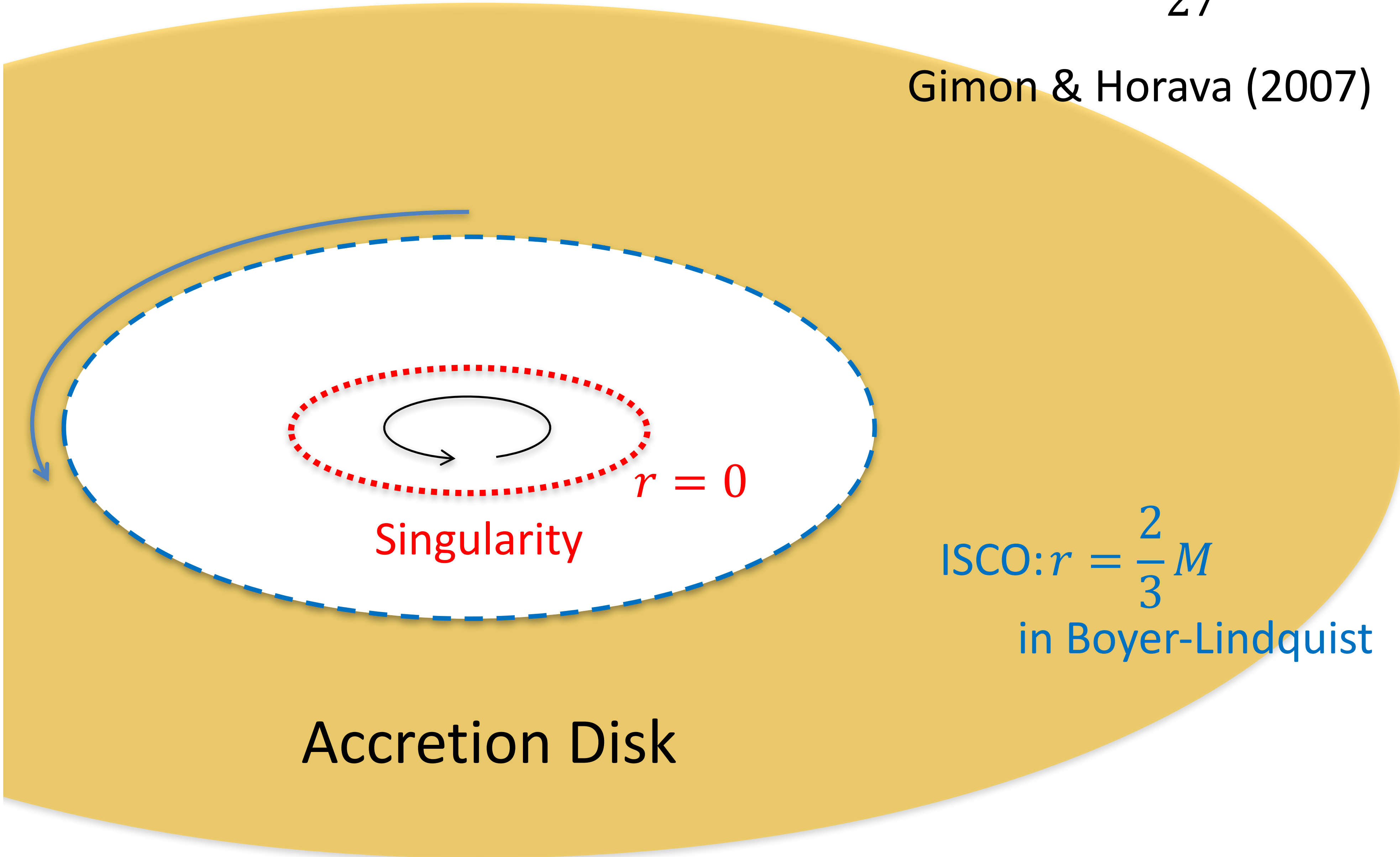
when $a^2 = M^2$



Efficiency of energy extraction from accreting matter=100%,

when $a^2 = \frac{32}{27}M^2$

Gimon & Horava (2007)



Is a superspinar stable?

How does a superspinar form?

$\boxed{\text{Is a superspinar stable?}}$

How does a superspinar form?

§ 2 Stability of Superspinar

Teukolsky equations determine perturbations in Kerr spacetime

Master variable of the perturbations: $\psi = e^{-i\omega t + im\varphi} R(r) S(\theta)$

$$\Delta^{-s} \frac{d}{dr}\left(\Delta^{s+1} \frac{dR}{dr}\right) + \left(\frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - \lambda\right) R = 0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dS}{d\theta}\right) + \left[(a\omega\cos\theta + s)^2 - \left(\frac{m + s\cos\theta}{\sin\theta}\right)^2 - s(s-1) + A\right] S = 0$$

where

$$K \coloneqq (r^2 + a^2)\omega - am$$

$$\lambda \coloneqq A + a^2\omega^2 - 2am\omega$$

$A = A_{\omega lms}$: Eigen value of the equation for S $\qquad l \geq max[|m|, |s|]$

$|s| = 0$: scalar $\qquad |s| = 1$: EM $\qquad |s| = 2$: GW

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Outgoing GW $\Psi_4 = (r - ia\cos\theta)^{-4}\psi$ with $s = -2$

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$R \to \frac{C}{r} e^{i\omega r}$ for $r \to \infty$ for Quasi-normal mode (QNM)

In black hole case

Horizon $\Delta(r) = 0$: singular point of the radial Teukolsky equation

Regularity at horizon $\longrightarrow$ Unique boundary condition

$R \to 0$ for $r \to r_+$

Master variable of the perturbations: $\psi = e^{-i\omega t + im\varphi} R(r) S(\theta)$

$$\Delta^{-s} \frac{d}{dr}\left(\Delta^{s+1} \frac{dR}{dr}\right) + \left(\frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - \lambda\right) R = 0$$

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In superspinar (over-spinning Kerr) case

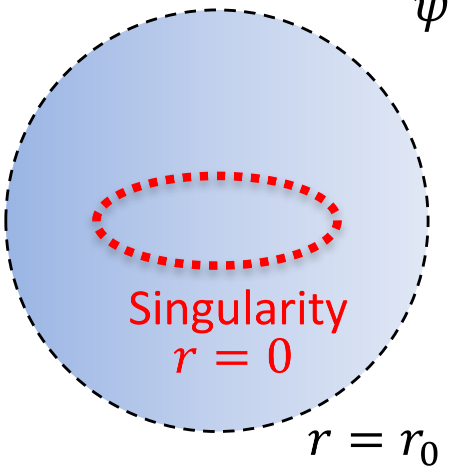
No horizon $\Delta(r) > 0$: no singular point of the radial Teukolsky equation on the real axis of r.

No unique boundary condition unless we know what is a superspinar.

§ 3 Stability of Superspinar (1)

Cardoso, Pani, Cadoni, Cavaglia (2008); Pani, Barausse, Berti and Cardoso (2010)

Quasi-normal modes of perturbations (no incoming wave at infinity)



$$\psi = e^{-i\omega t + im\varphi} R(r) S(\theta)$$

Boundary condition at $r = r_0$ =constant

- Reflecting BC: $R = 0$
- Absorbing BC: $Y' = -ikY$

where $Y = \Delta^{s/2}(r^2 + a^2)^{1/2} R$

$$Y'' + V(r)Y = 0$$

$$k = \sqrt{V(r_0)}$$

Imaginary part of ω is positive →unstable

TABLE I. Unstable gravitational ($s = 2$) frequencies with $l = m = 2$ for a superspinar with a perfect reflecting surface ($\mathcal{R} = 1$) and with a "stringy event horizon" ($\mathcal{R} = 0$) at $r = r_0$. All modes in this table have been computed using numerical values of ${}_sA_{lm}$ obtained via the continued fraction method [23].

	Reflecting BC $(\omega_R M, \omega_I M)$, $\mathcal{R} = 1$			Absorbing BC $(\omega_R M, \omega_I M)$, $\mathcal{R} = 0$		
r_0/M	$a = 1.1M$	$a = 1.01M$	$a = 1.001M$	$a = 1.1M$	$a = 1.01M$	$a = 1.001M$
0.01	(0.5690, 0.1085)	(0.9744, 0.0431)	(0.9810, 0.0097)	(0.5002, 0.0173)	(0.9498, 0.0062)	(1.0286, 0.0033)
0.1	(0.5548, 0.1237)	(0.9673, 0.0475)	(0.9794, 0.0110)	(0.4878, 0.0260)	(0.9435, 0.0093)	(1.0252, 0.0048)
0.5	(0.4571, 0.1941)	(0.9256, 0.0631)	(0.9688, 0.0155)	(0.3959, 0.0719)	(0.9016, 0.0237)	(1.0052., 0.0091)
0.8	(0.3081, 0.2617)	(0.8598, 0.0878)	(0.9507, 0.0202)	(0.2537, 0.1053)	(0.8298, 0.0376)	(0.9793, 0.0095)
1	(0.1364, 0.3095)	(0.6910, 0.1742)	(0.9003, 0.0640)	(0.0916, 0.1219)	(0.6530, 0.0821)	(0.8853, 0.0313)
1.1	(0.0286, 0.3248)	(0.4831, 0.2655)	(0.6071, 0.2207)	(−0.0078, 0.1233)	(0.4377, 0.1230)	(0.5696, 0.1064)

Reflection Boundary Condition

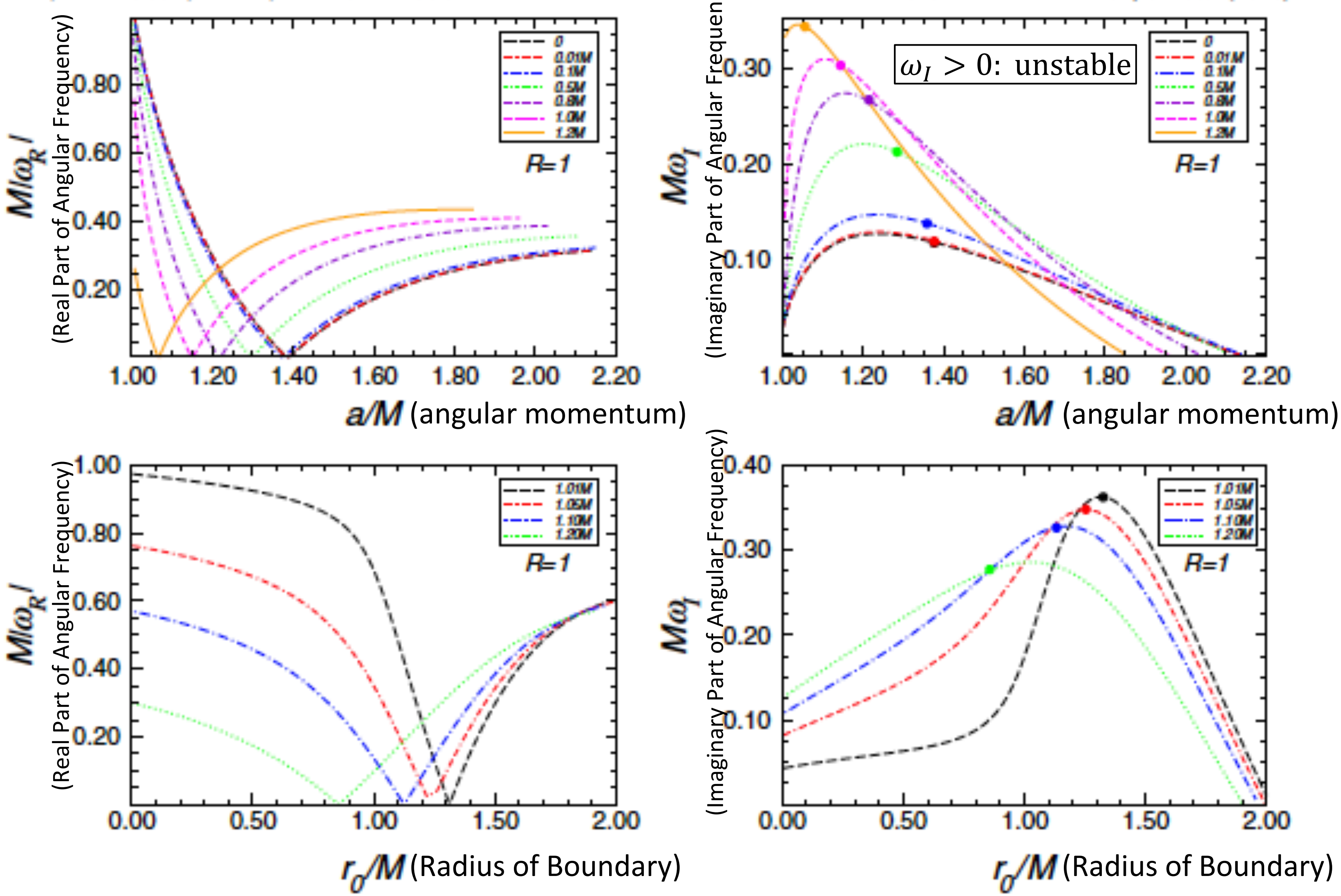


FIG. 1 (color online). Top: Real (left) and imaginary part (right) of unstable gravitational modes of a superspinar as a function of the spin parameter, a/M, for $l = m = 2$ and several fixed values of r_0. Bottom: Real (left) and imaginary part (right) of unstable gravitational modes of a superspinar as a function of the mirror location, r_0/M, for $l = m = 2$ and different fixed values of the spin parameter. Large dots indicate purely imaginary modes.

Absorbing Boundary Condition

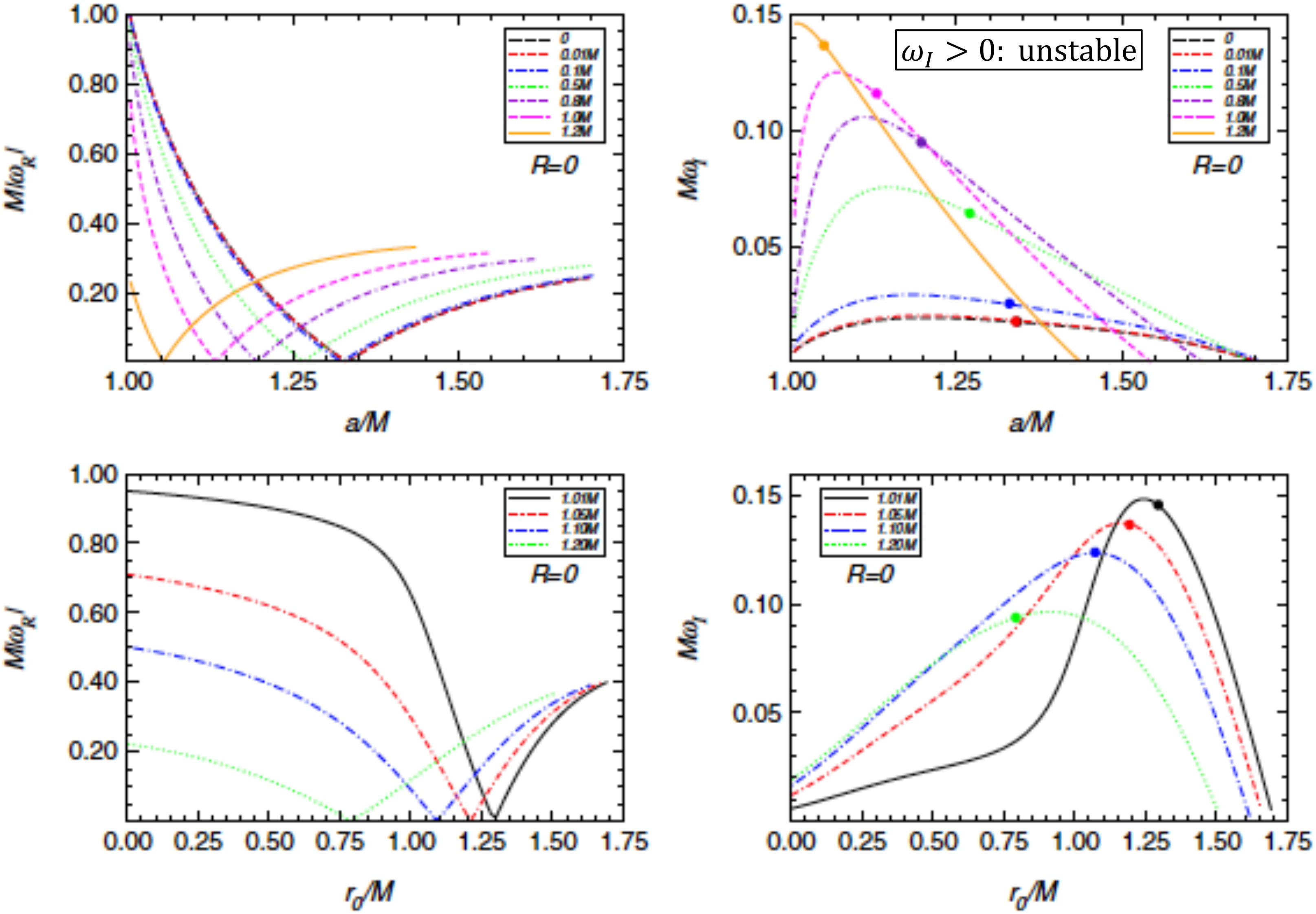


FIG. 5 (color online). Top: Real (left) and imaginary part (right) of unstable gravitational modes of a superspinar as a function of the spin parameter, a/M, for $l = m = 2$ and several fixed values of the horizon location r_0/M. Bottom: Real (left) and imaginary part (right) of unstable gravitational modes of a superspinar as a function of the horizon location, for $l = m = 2$ and fixed values of the spin parameter. Large dots indicate purely imaginary modes.

Reflecting Boundary Condition

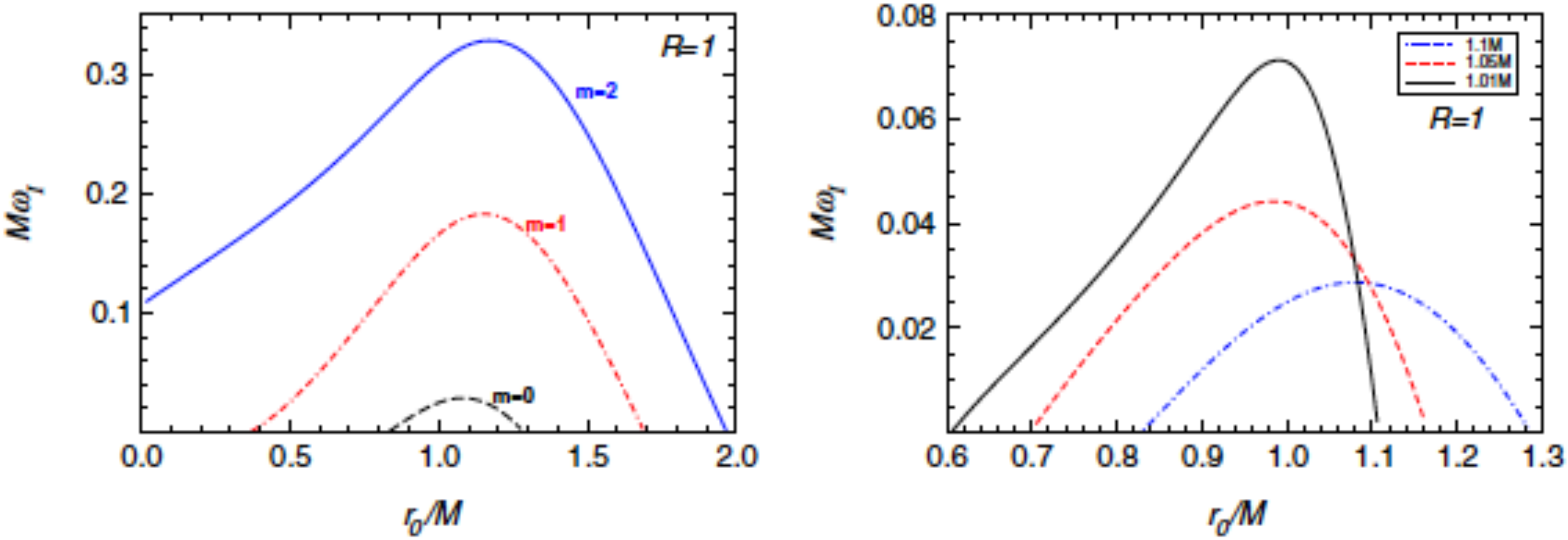


FIG. 3 (color online). Left: Imaginary part of unstable gravitational modes of a superspinar as a function of the mirror location, r_0/M, for $a = 1.1M$, $l = 2$ and $m = 0, 1, 2$. Right: Imaginary part of unstable gravitational modes of a superspinar as a function of the mirror location, r_0/M, for $l = 2$, $m = 0$ and several values of the spin parameter, a.

Pani et al state that the superspinar is, in general, unstable,
since perturbations grow exponentially
under both reflecting and absorbing boundary conditions.

However, there are boundary conditions
under which the superspinar is stable.

Is their conclusion right?

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Let’s consider!

In superspinar case, replace the the stability problem with

“Does there exist the boundary condition under which the superspinar is stable?”

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Answer: Yes!

We assume stable angular frequency, i.e., $\underbrace{\omega = \omega_R + i\omega_I}_{\text{Input Parameter}}$ with negative ω_I

Solve the angular Teukolsky equation by imposing regularities at $\theta = 0, \pi$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + \left[(a\omega\cos\theta + s)^2 - \left(\frac{m + s\cos\theta}{\sin\theta}\right)^2 - s(s-1) + A\right]S = 0$$

A is determined.

$$\lambda \coloneqq A + a^2\omega^2 - 2am\omega$$

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$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{dR}{dr}\right) + \left(\frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - \lambda\right)R = 0$$

No singular point on real axis of r $\Rightarrow$ obtained $R(r)$ is regular everywhere.

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No singular point on real axis of r → obtained $R(r)$ is regular everywhere.

→ The boundary condition at, for example, $r = 0$ is found.

Infinite number of boundary conditions under which the superspinar is stable.

§ 4 Summary and conclusion

The boundary condition := Physical nature of the superspinar

Unless we know the nature of the superspinar,
we cannot say anything on its stablity.

We should conclude

no one knows the stability of the superspinar at present.

Reflecting Boundary Condition

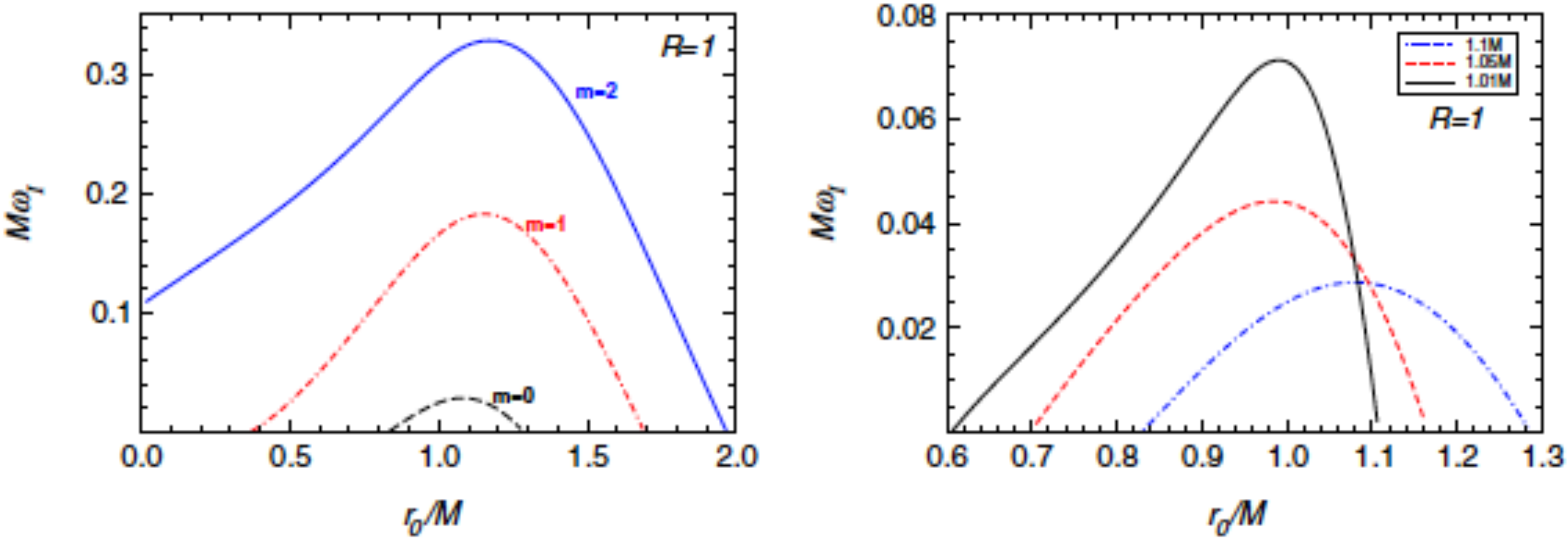

FIG. 3 (color online). Left: Imaginary part of unstable gravitational modes of a superspinar as a function of the mirror location, r_0/M, for $a = 1.1M$, $l = 2$ and $m = 0, 1, 2$. Right: Imaginary part of unstable gravitational modes of a superspinar as a function of the mirror location, r_0/M, for $l = 2$, $m = 0$ and several values of the spin parameter, a.

The reflecting boundary condition at *r=2M*
is equivalent to some other boundary condition at *r=M,*
since there is no singular point in the superspinar case.