Wave optics in the Kerr BH

Sousuke Noda (Nagoya Univ.)

Wave optics in black hole spacetimes: Schwarzschild case   Y. Nambu and S.N

Wave optics in black hole spacetimes: Kerr case (in prep.)   S.N and Y. Nambu

HTGRG3 2017 1 August @ Quy Nhon, Vietnam

On August 4 (Fri)

Analog rotating black holes in a MHD inflow

Sousuke Noda (Nagoya Univ. Japan)

Collaborators

Yasusada Nambu (Nagoya Univ.)

Masaaki Takahashi (Aichi Edu. Univ.)

Based on
Physical Rev. D .95, 104055 (2017)
S.N., Y.Nambu, and M.Takahashi

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1. Introduction & Motivation

2. Wave scattering by a Kerr BH

3. Interference patterns & Images

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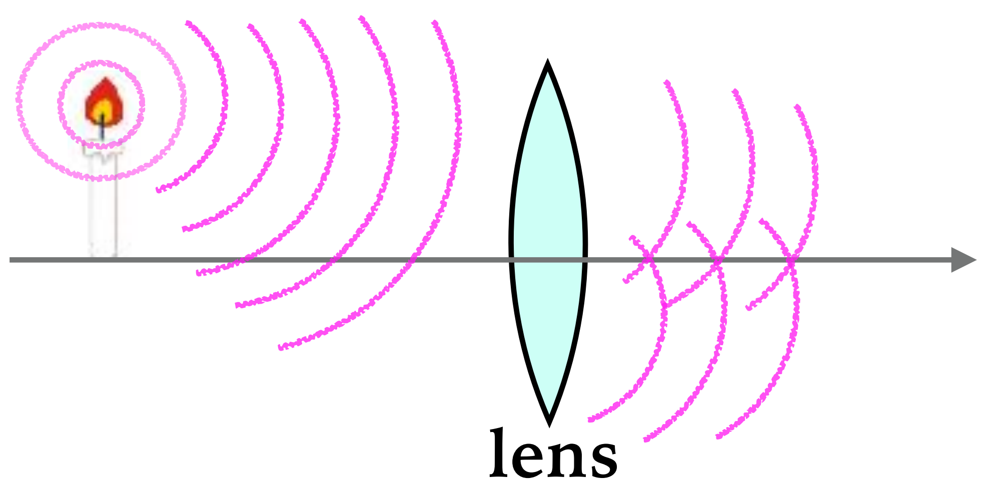
Why wave optics ?

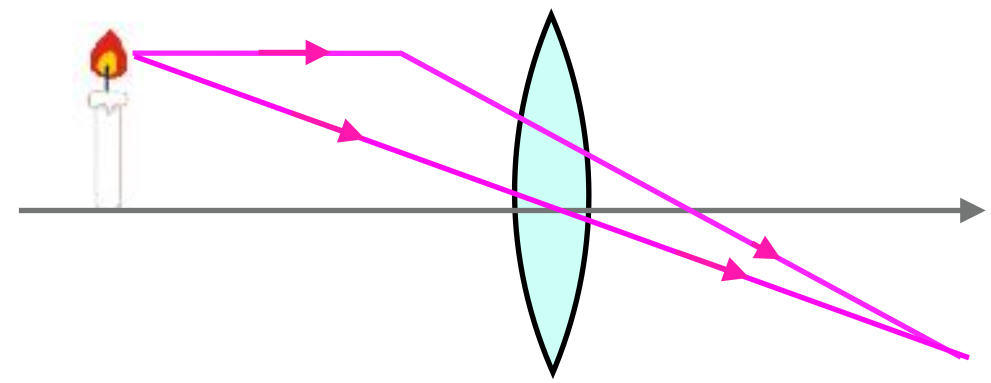
Wave optics



eikonal approx.

short wavelength

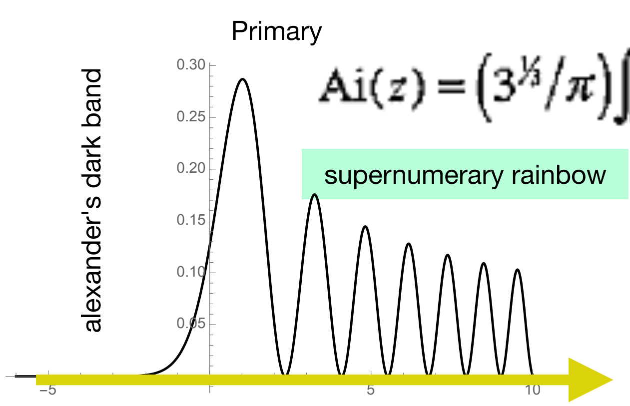
Geometrical optics



Violation of geometrical optics approximation

Supernumerary rainbow

Airy (1836)

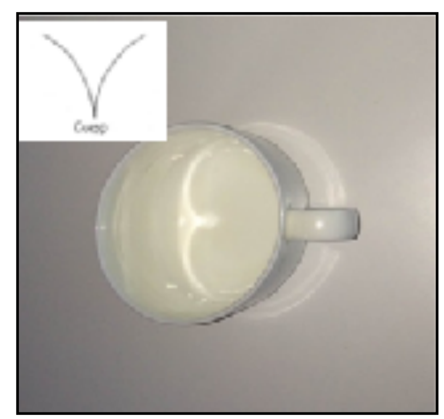


$$\mathrm{Ai}(z) = \left(3^{1/3}/\pi\right)\int_0^{\infty} \cos\left(t^3 + 3^{1/3} z t\right) dt$$

Brocken spectra

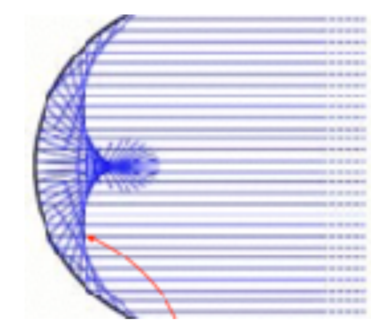
Mie scattering

Caustics



Brightness = ∞

in the geometrical optics

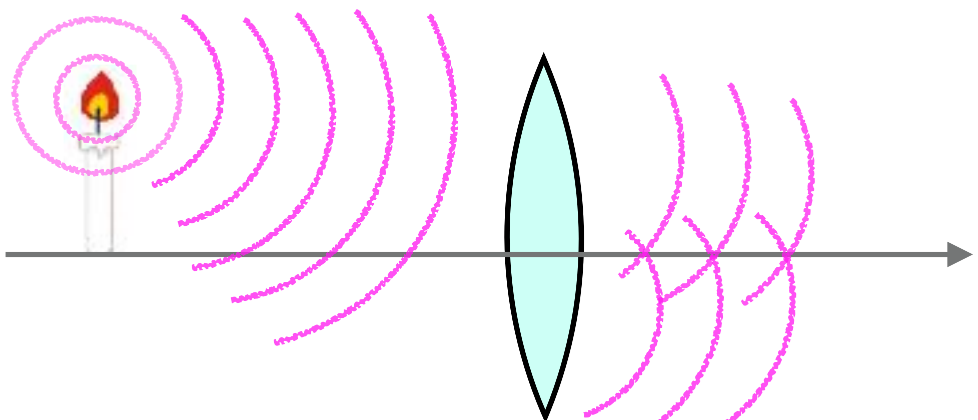


Envelope of rays

These phenomena cannot be understood in geometrical optics.

Why wave optics ?

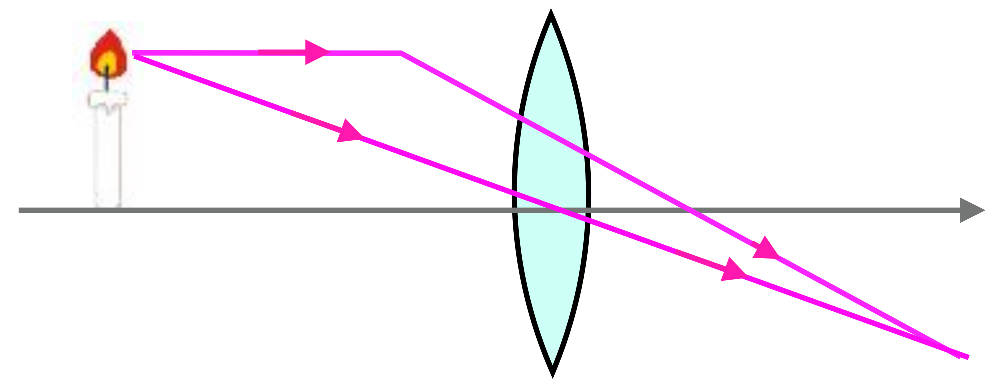
Wave optics



eikonal approx.

short wavelength

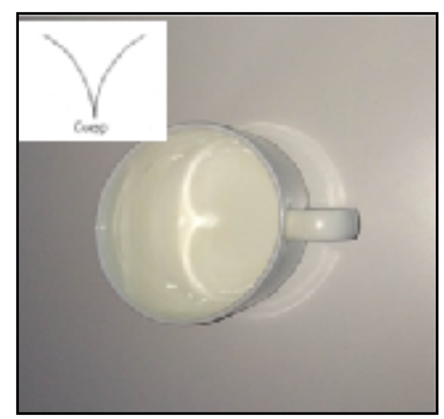
Geometrical optics



lens

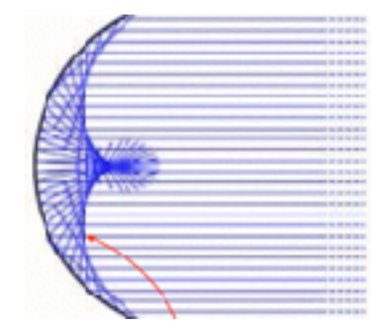
optics approximation

Caustics

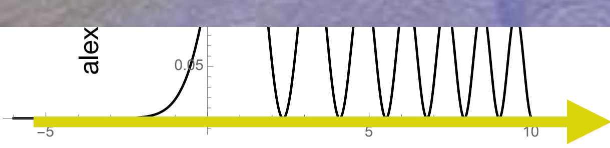


Brightness = ∞

in the geometrical optics



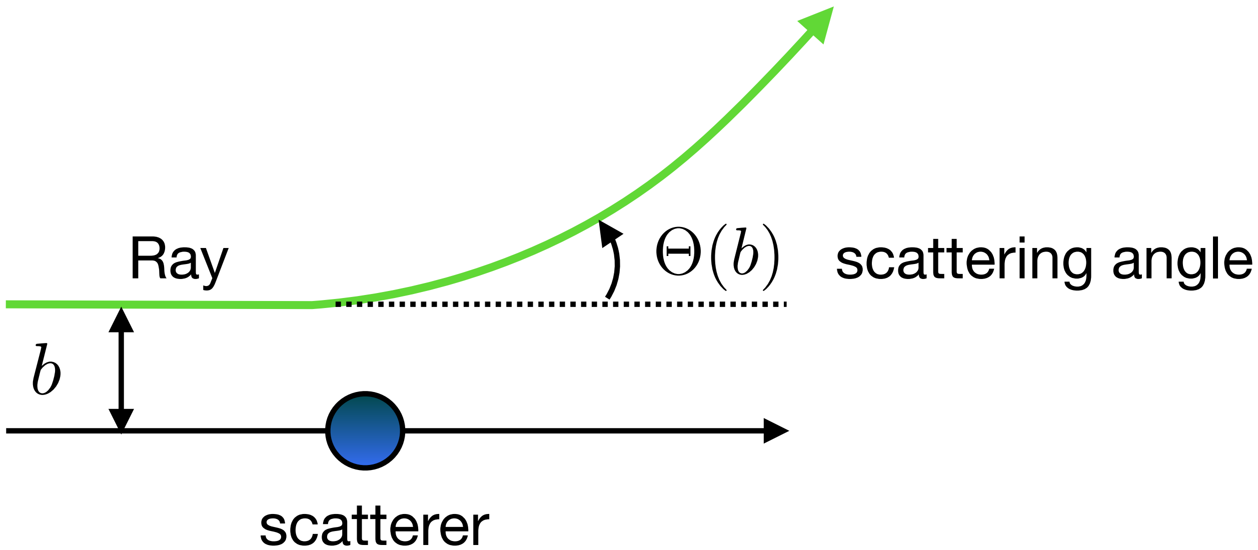
Envelope of rays



These phenomena cannot be understood in geometrical optics.

Why wave optics ?

Scattering in geometrical optics



Differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\Theta}\left(\frac{d\Theta}{db}\right)^{-1}$$

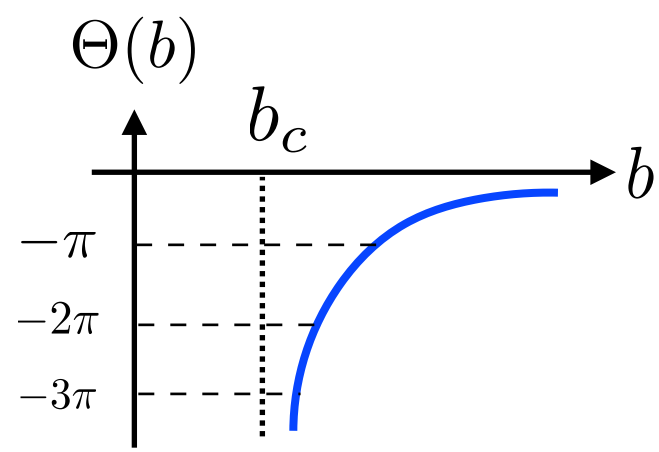
Violation of geometrical approximation

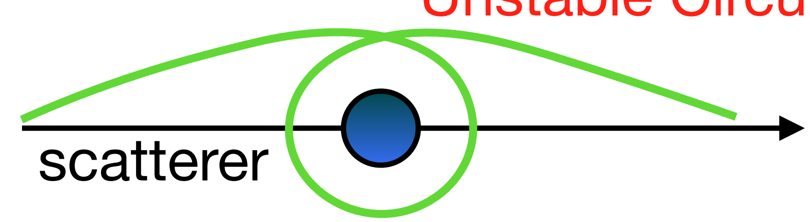
BH case

1. Orbiting (glory)

$$\sin\Theta = 0$$

$$\Theta = -\pi, -2\pi, \ldots$$

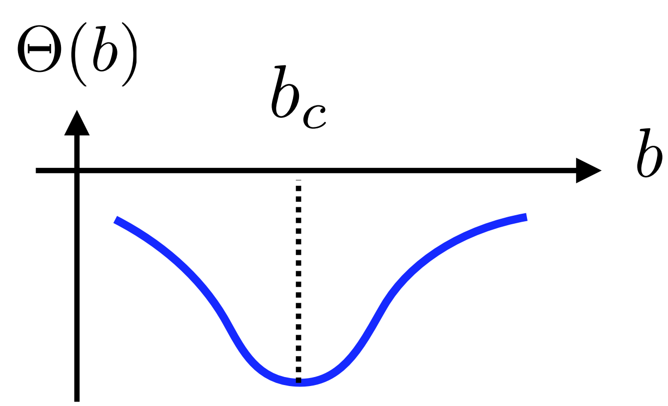


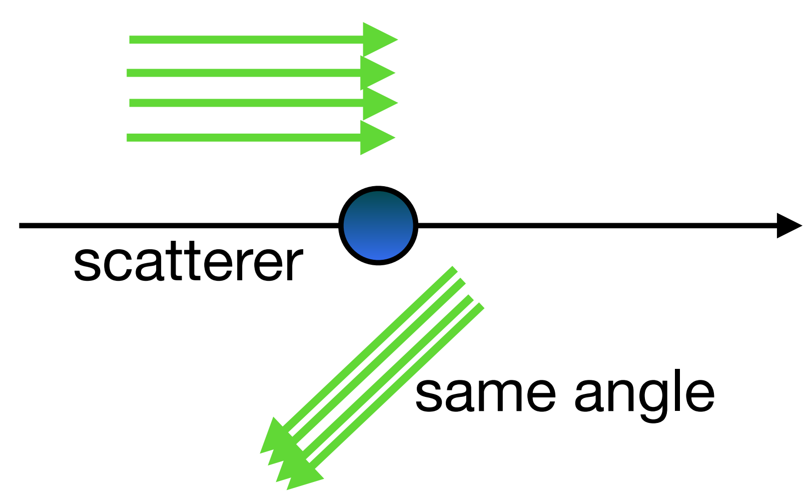


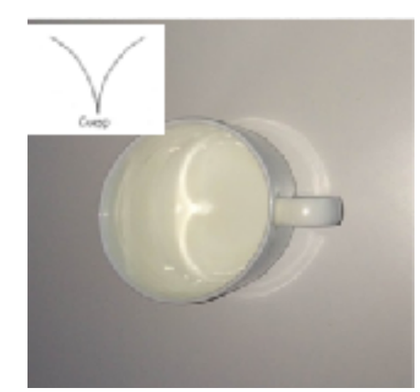
Caustics

2. Rainbow scattering

$$\left.\frac{d\Theta}{db}\right|_{b=b_c} = 0$$



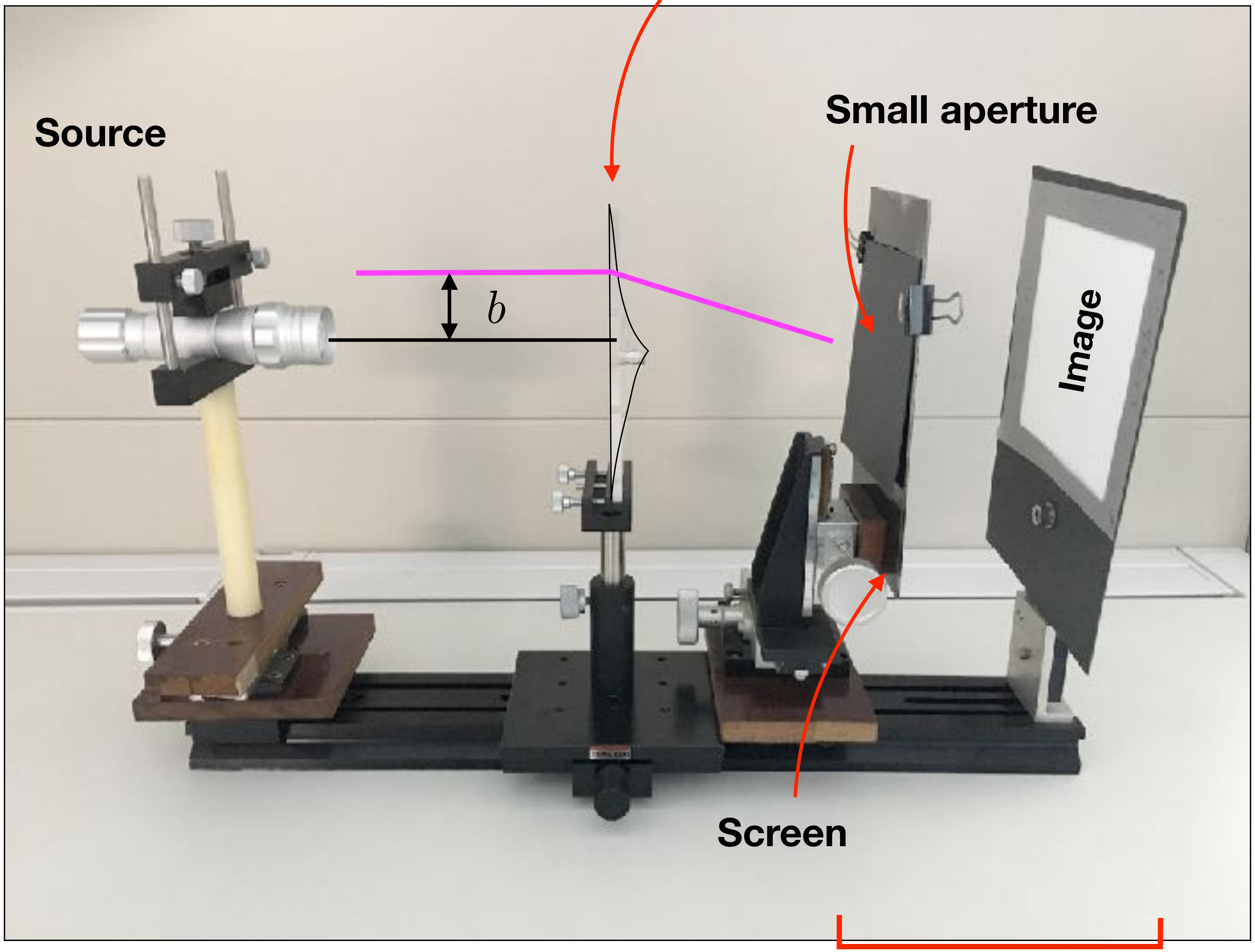




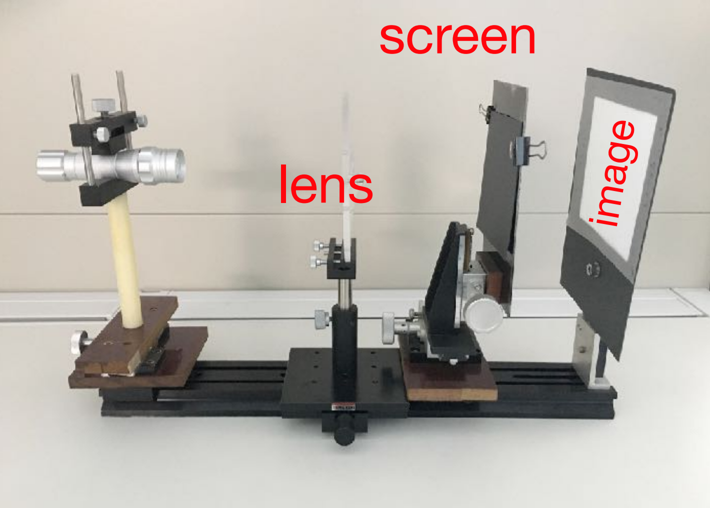
MODEL of a gravitational lensing

Lens **Deflection angle** $\propto 1/b$

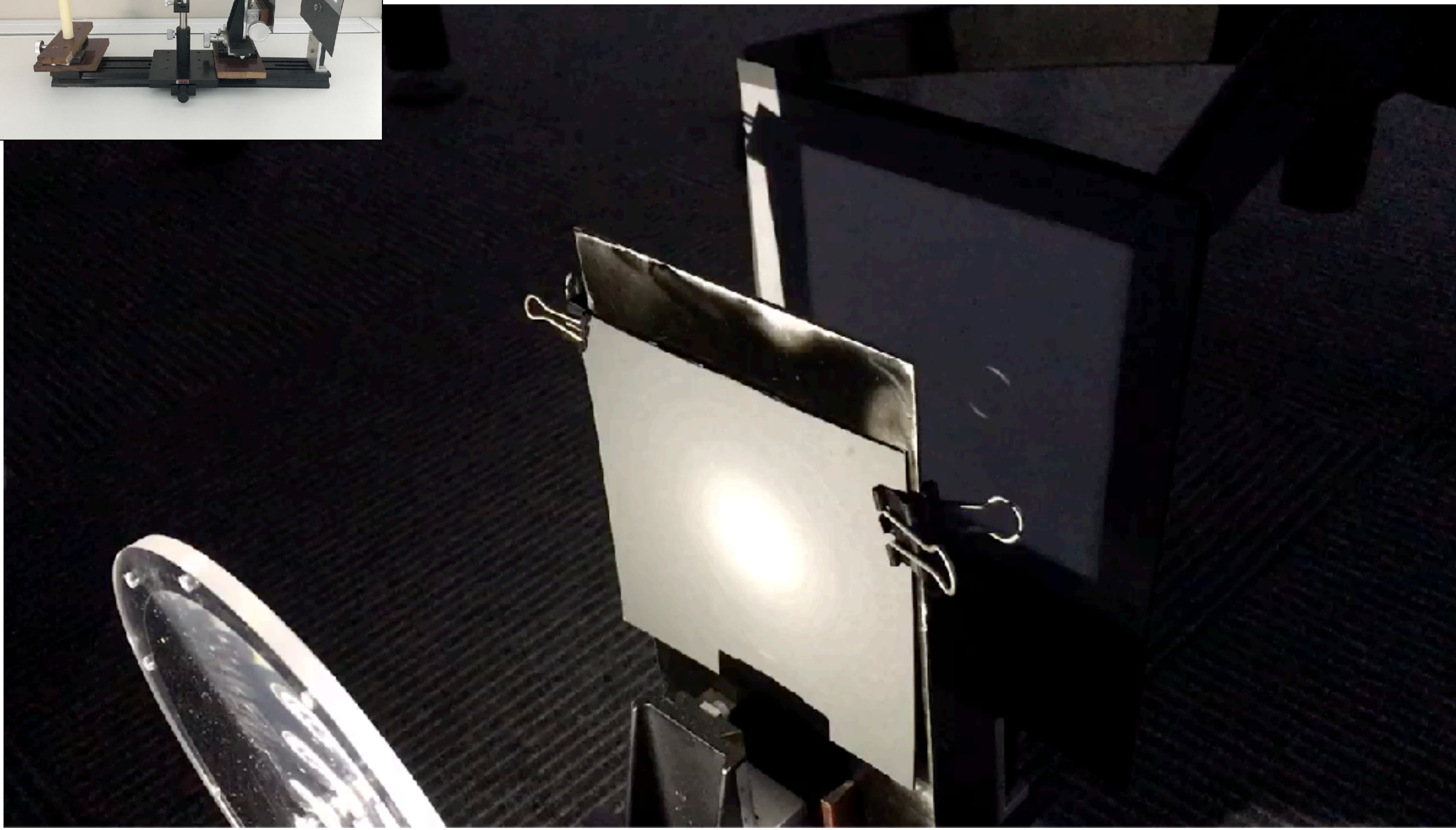
b : impact parameter



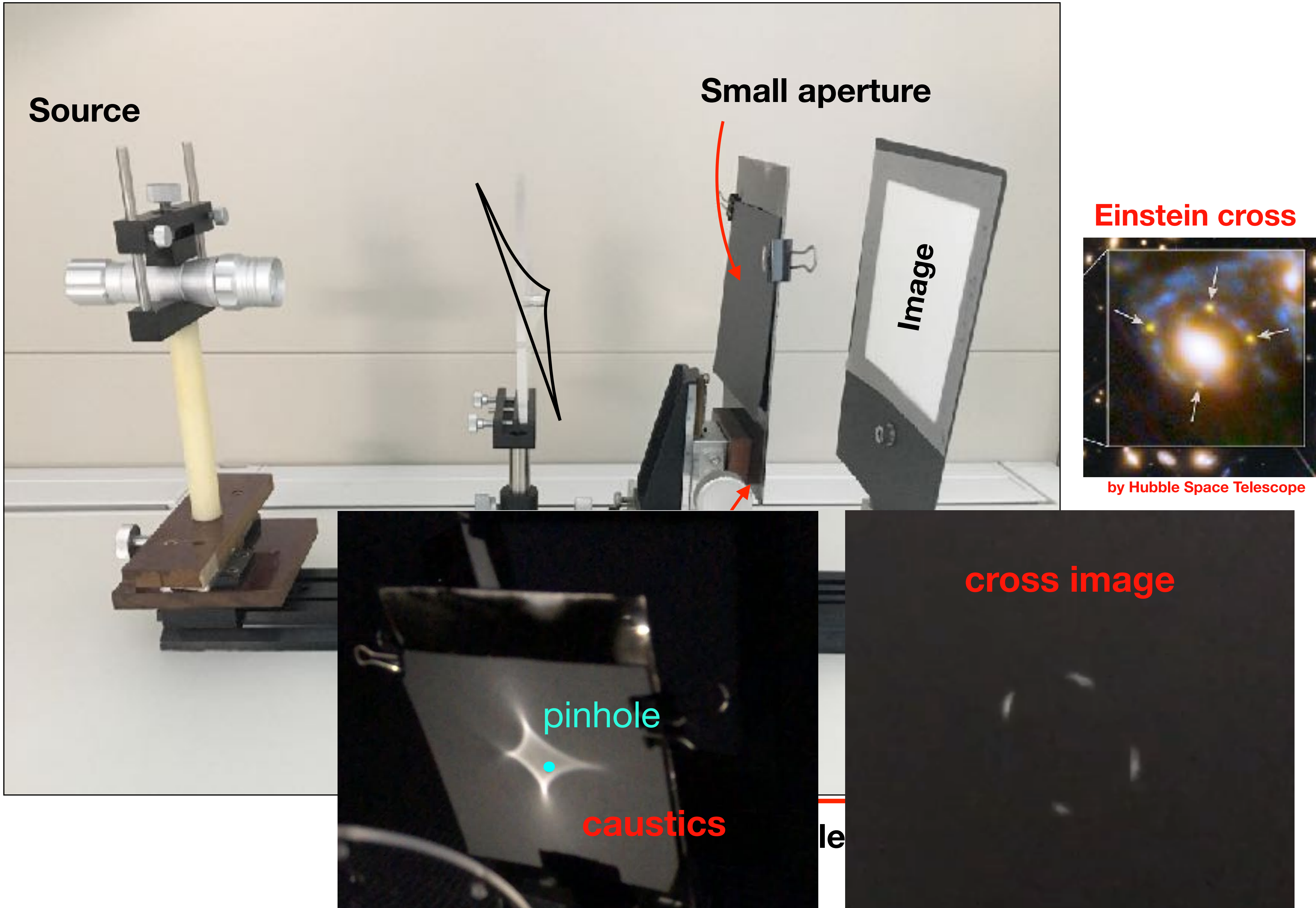
Pinhole camera



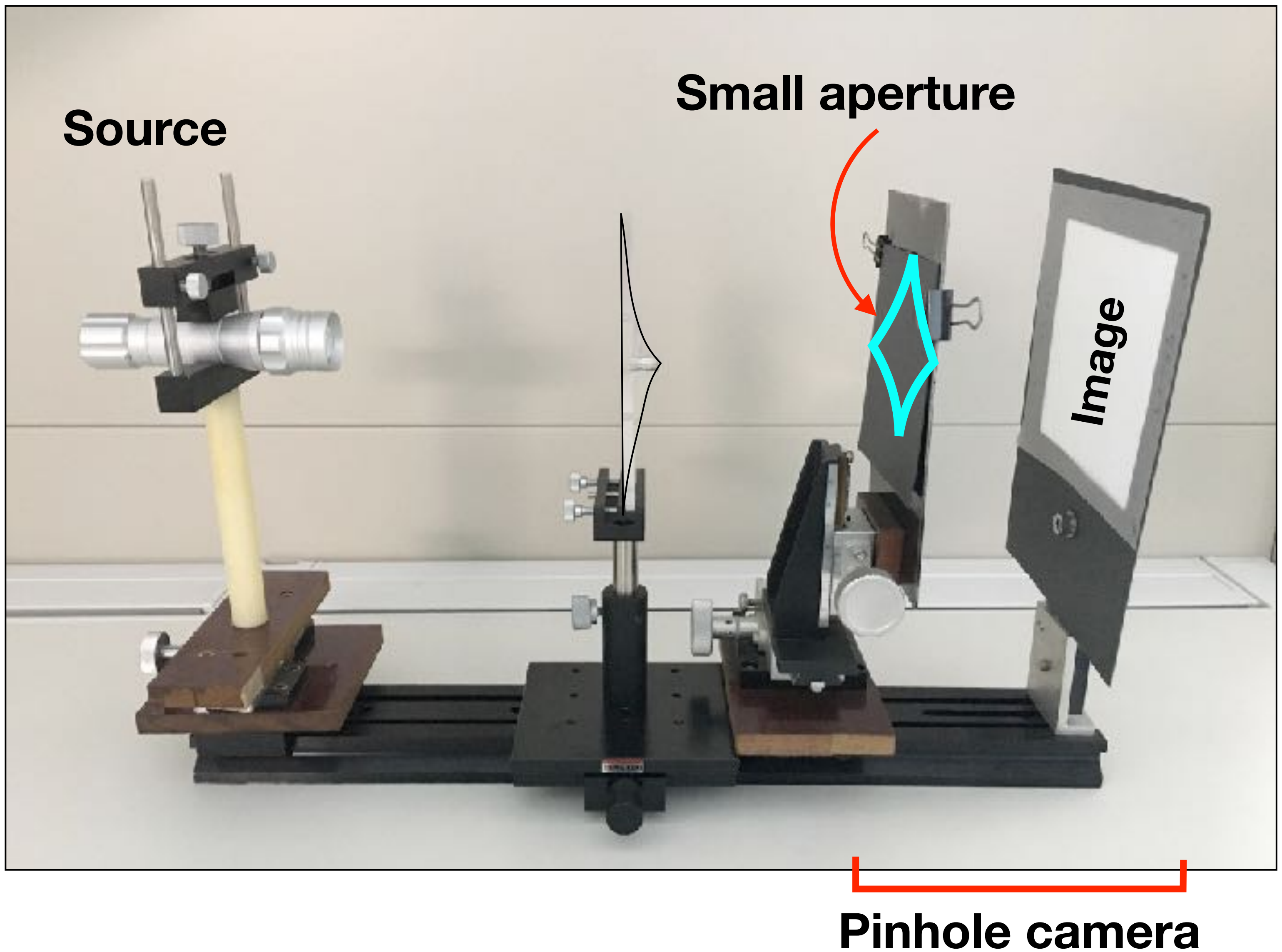
Spherical symmetric case

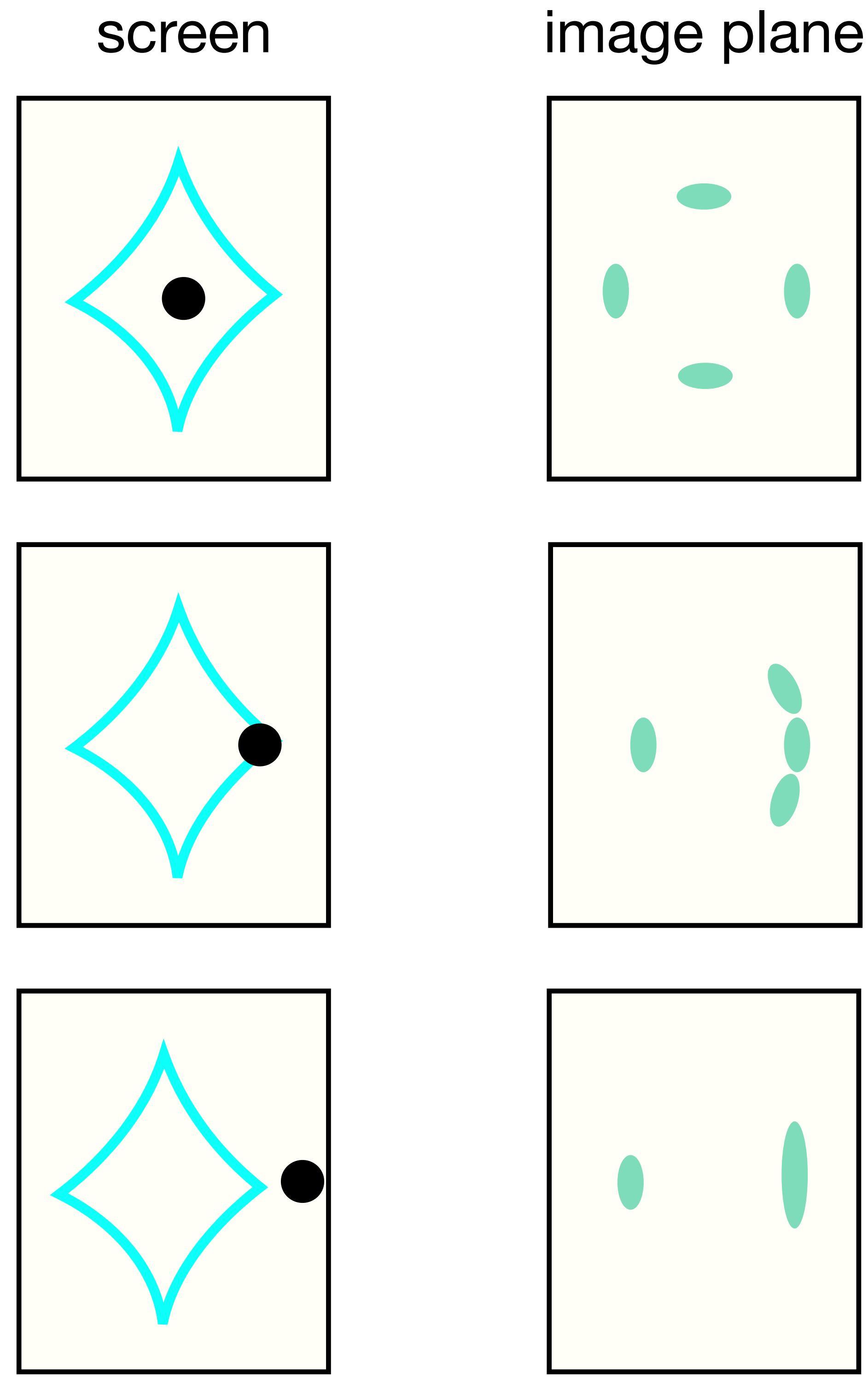


Non-spherical case

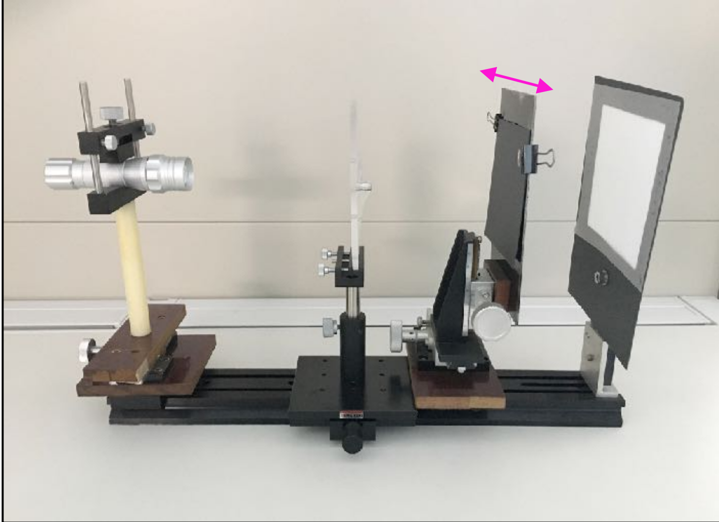


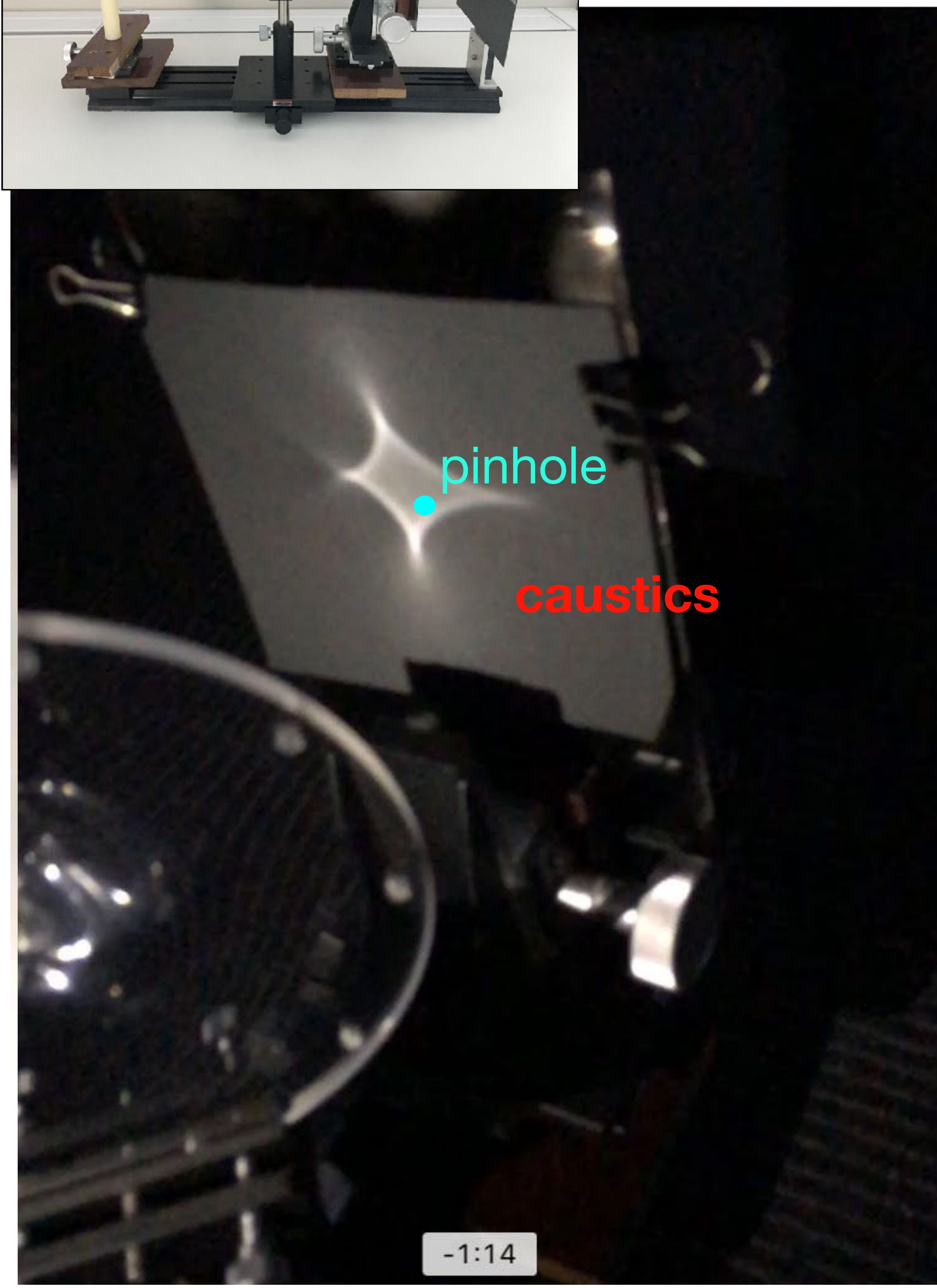
Caustics & # of image



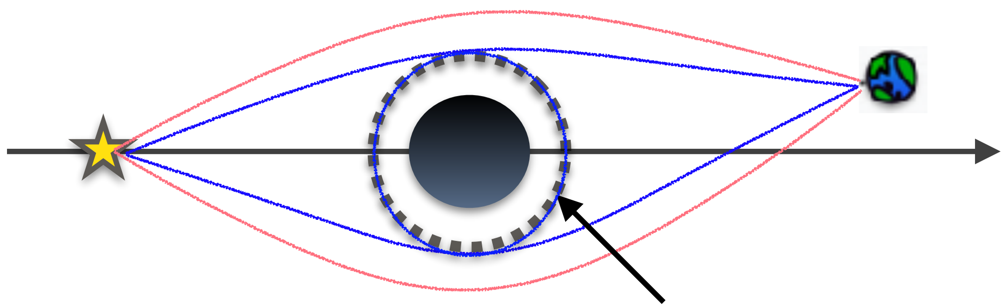


Non-spherical case

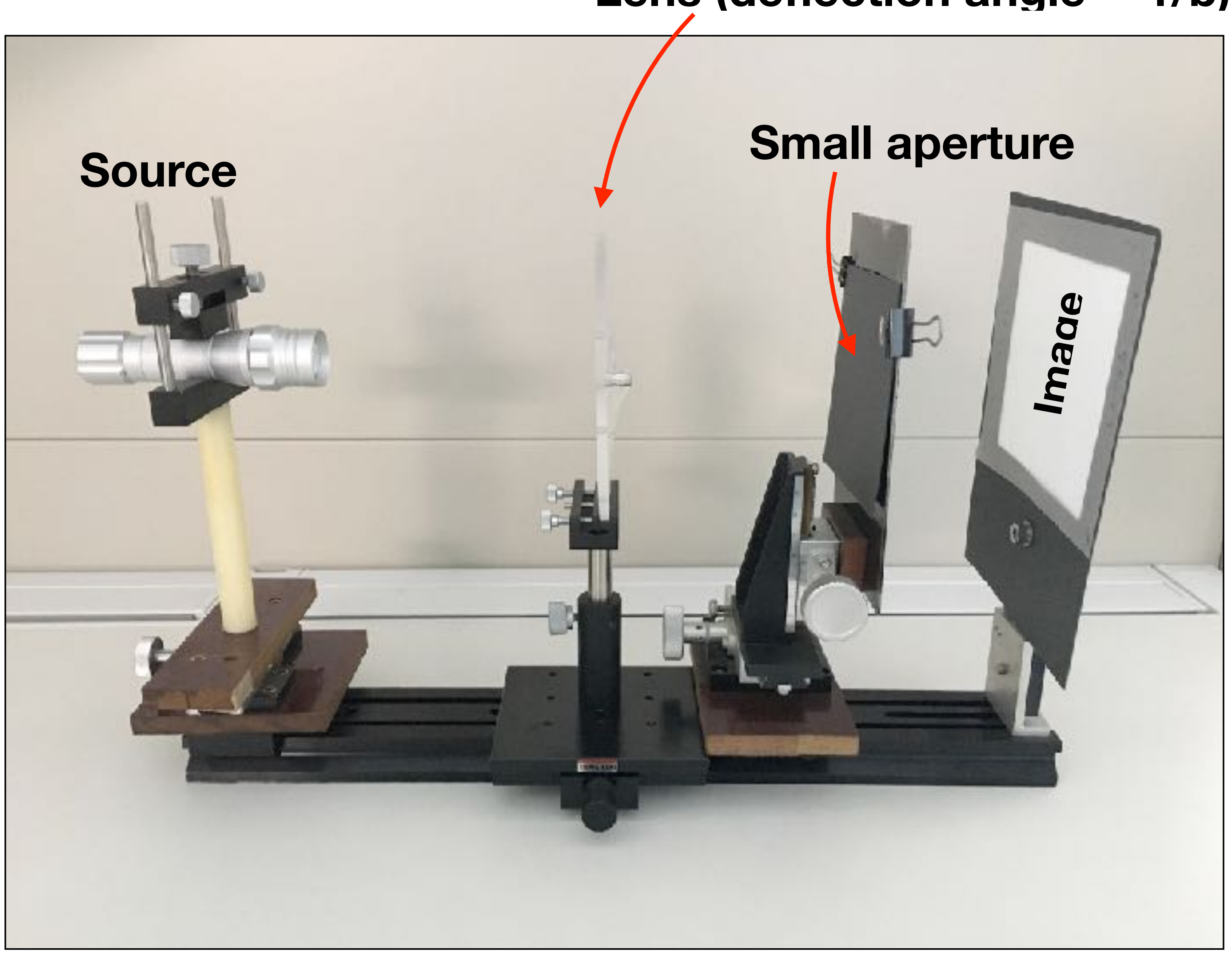




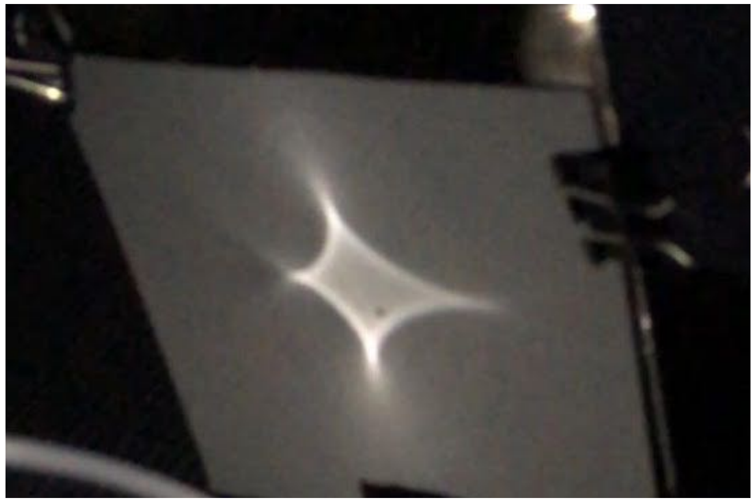
Kerr BH = Non-spherical lens
+ unstable circular orbit



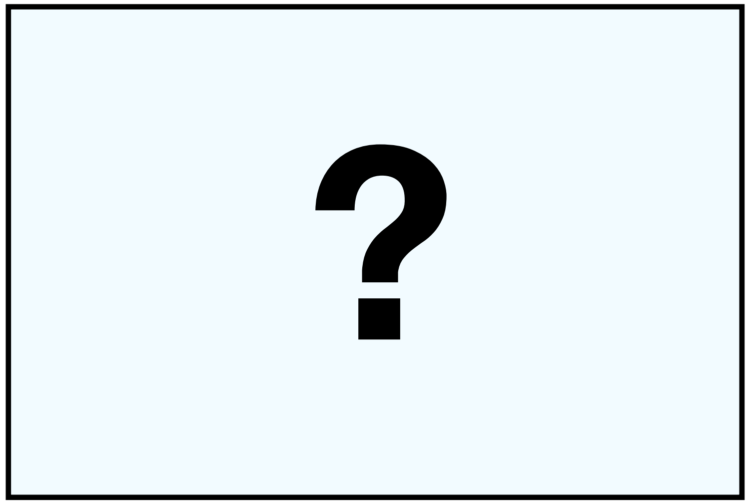
No unstable circular orbits in this model



caustics ―― mode



caustics ―― mode



Kerr BH = Non-spherical lens
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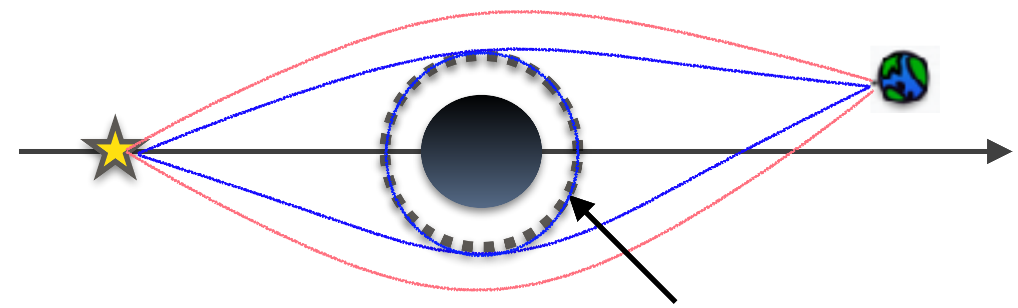
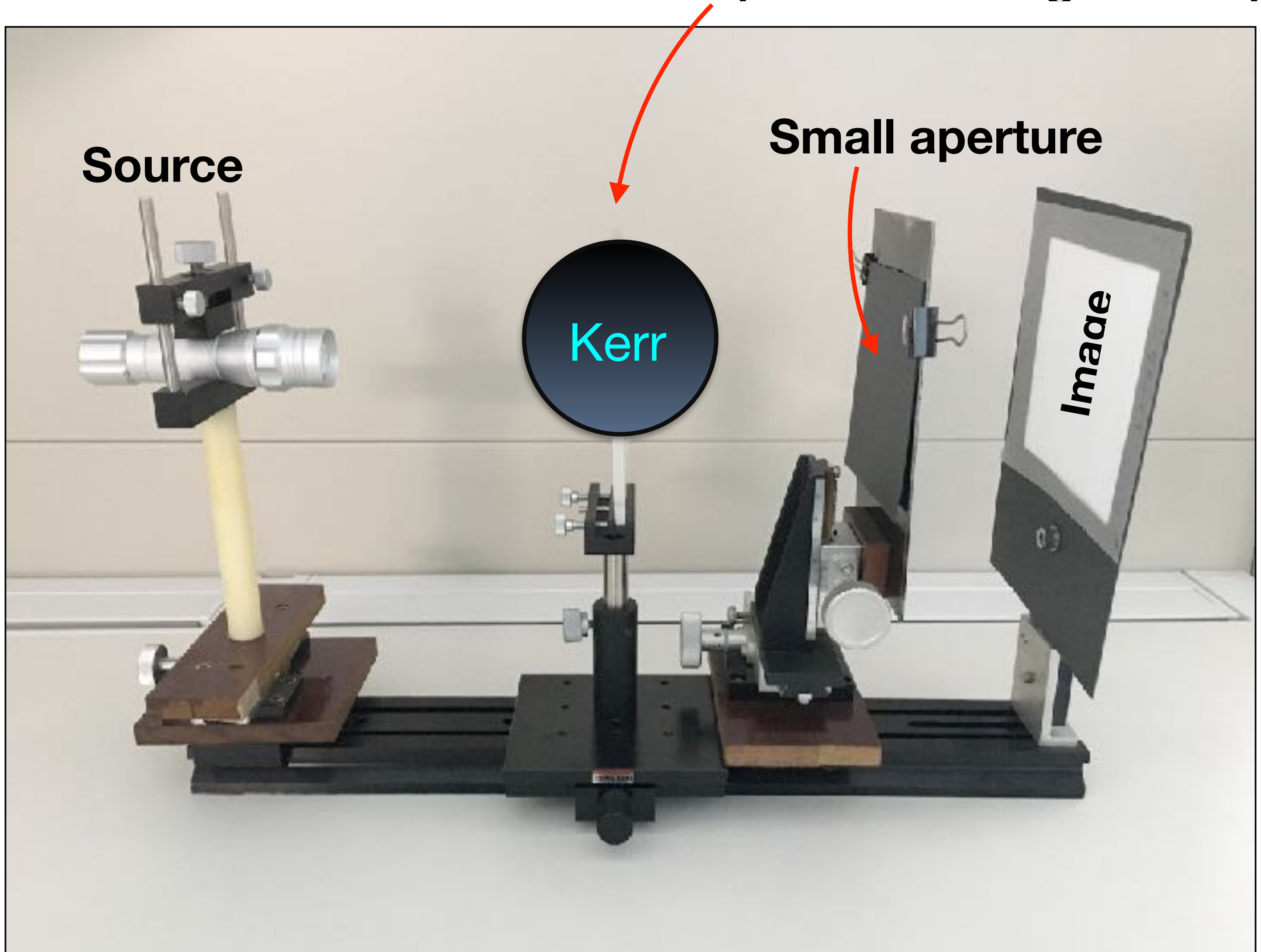
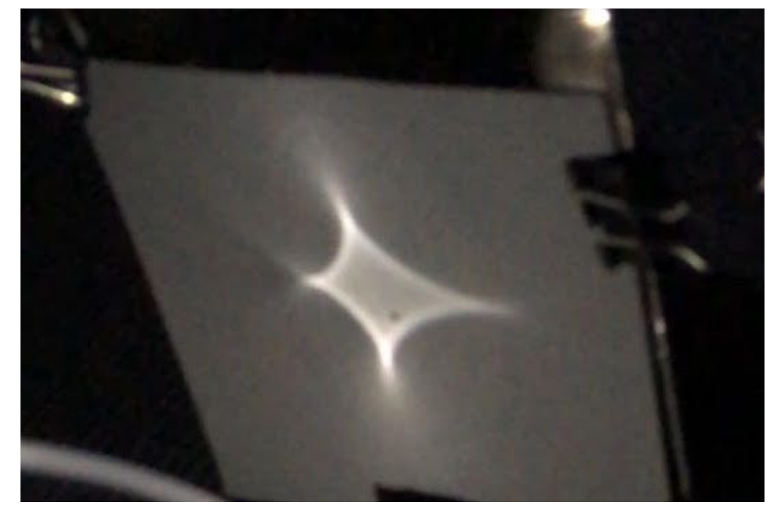


No unstable circular orbits in this model

Lens (deflection angle $\propto 1/b$)



caustics (red line) mode

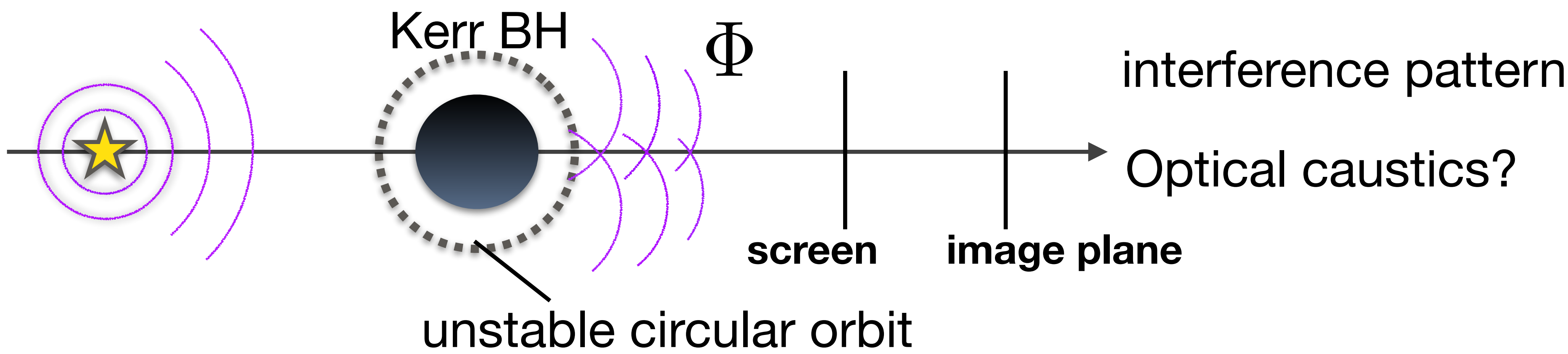


caustics (blue line) mode

?

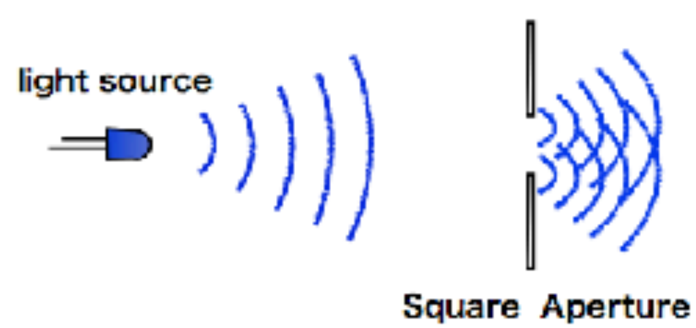
Our goal

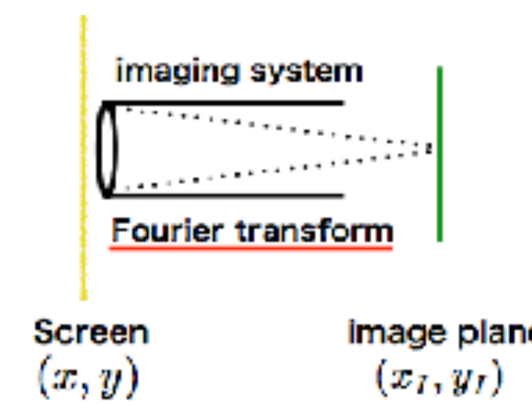
- Wave scattering by a Kerr BH



- Imaging (Wave optics)

2D Fourier transform = imaging

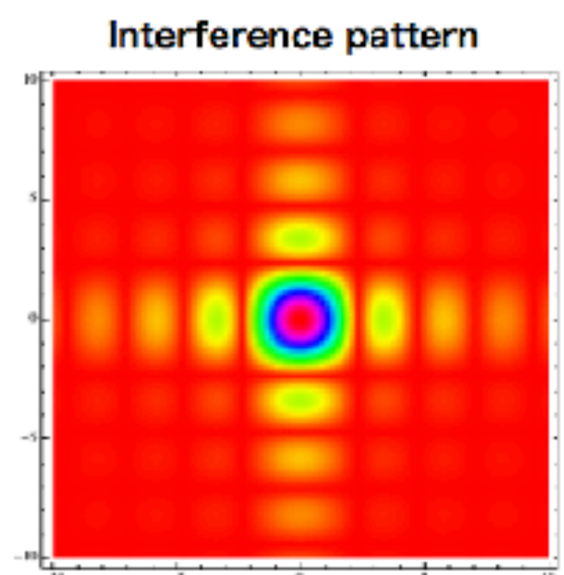


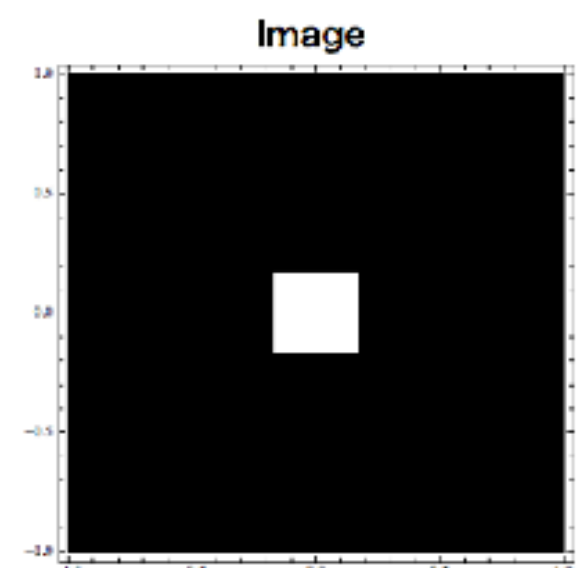


- Unstable Circular Orbit

Black Hole Shadow (image)

Effect of the unstable circular orbit in the scattered wave.





1. Introduction & Motivation

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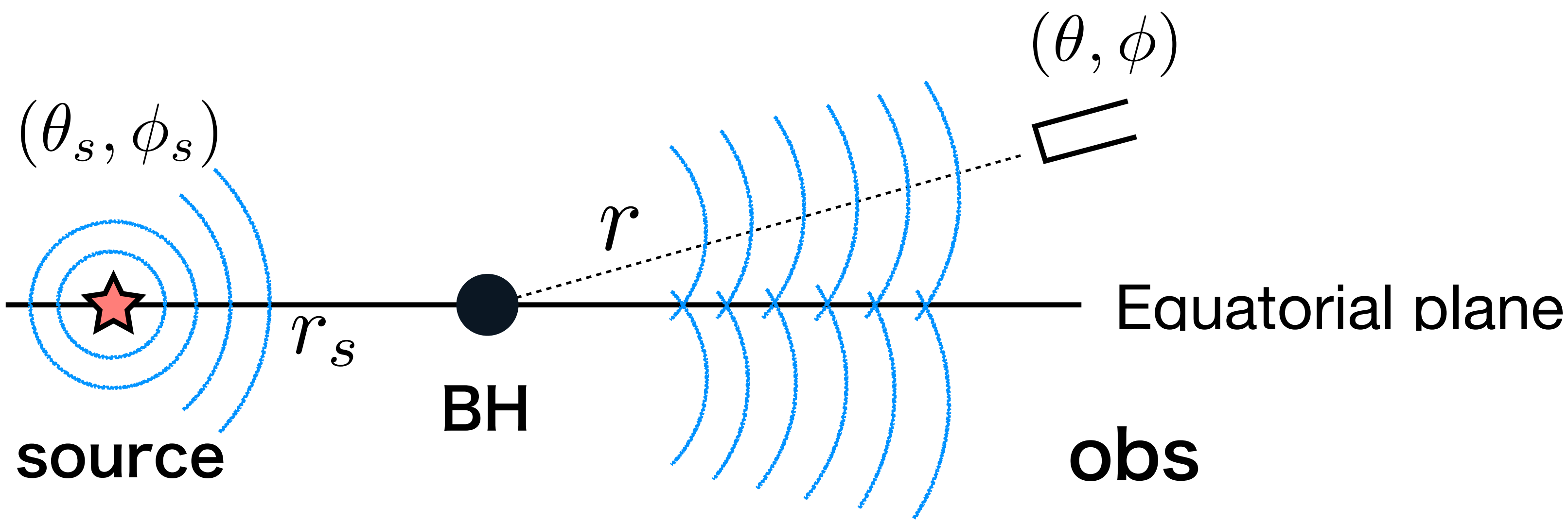
3. Interference patterns & Images

Setup

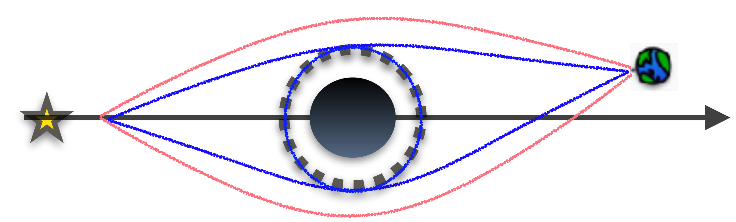


- Kerr spacetime (Boyer-Lindquist)

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{A\sin^2\theta}{\Sigma}d\phi^2$$

$$\Delta = r^2 - 2Mr + a^2 \ , \ \Sigma = r^2 + a^2\cos^2\theta \ , \ A = (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta$$

- short wavelength $M\omega \gg 1 \quad (M \gg \lambda)$



interference

- scalar wave , monochromatic
- stationary point source

$$\Box\Phi(x, x_s) = S \qquad S = -\frac{e^{-i\omega t}}{\sqrt{-g}}\delta^3(\boldsymbol{x} - \boldsymbol{x}_s)$$

$$\Phi(x, x_s) = G(\boldsymbol{x}, \boldsymbol{x}_s)e^{-i\omega t}$$

$$-\omega^2 g^{tt}G - 2\omega g^{t\phi}\partial_\phi G + \frac{1}{\sqrt{-g}}\partial_j\left(\sqrt{-g}g^{jk}\partial_k G\right) = -\frac{1}{\sqrt{-g}}\delta^3(\boldsymbol{x} - \boldsymbol{x}_s)$$

Partial wave expansion of $G(\boldsymbol{x}, \boldsymbol{x}_s)$

$$G(\boldsymbol{x}, \boldsymbol{x}_s) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\psi_{\ell m}(r, r_s)}{\sqrt{r^2+a^2}\sqrt{r_s^2+a^2}} \underline{S_{\ell m}(\theta) S^*_{\ell m}(\theta_s)} e^{im\phi} e^{-im\phi_s}$$

Spheroidal harmonics

The radial part

$$\frac{d^2\psi_{\ell m}}{dr_*^2} + Q(r; \ell, m)\psi_{\ell m} = -\underline{\delta(r - r_s)}$$

source term

$$Q(r; \ell, m) = \frac{[\omega(r^2+a^2) - ma]^2 - \Delta(A_{\ell m} + a^2\omega^2 - 2am\omega)}{(r^2+a^2)^2}$$ for $M\omega \gg 1$

To obtain $\psi_{\ell m}(r, r_s)$, we use a property of the Green function:

W :Wronskian

$$\psi_{\ell m}(r, r_s) = -\frac{u_1(r_s)\, u_2(r)}{W}\theta(r - r_s) - \frac{u_1(r)\, u_2(r_s)}{W}\theta(r_s - r),$$

where u_1 and u_2 are independent solutions of the homogeneous eq.

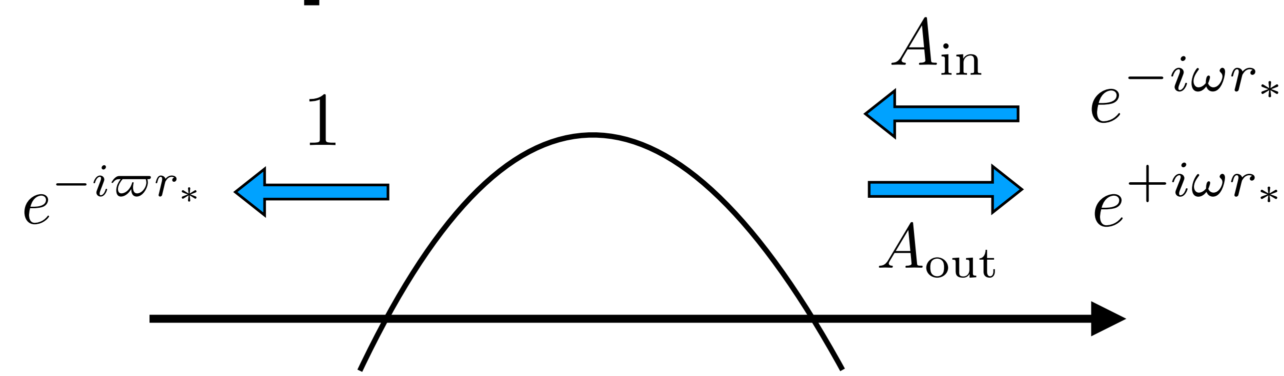
$$\frac{d^2 u_{\ell m}}{dr_*^2} + Q(r; \ell, m) u_{\ell m} = 0$$

$$u_{\ell m} \sim \sin\left(\omega r_* - \frac{\pi\ell}{2} + \underline{\underline{\delta_{\ell m}}}\right) \quad r_* \to \infty$$

phase shift

Radial equation and the phase shift

$$\frac{d^2 u_{\ell m}}{dr_*^2} + Q(r;\ell,m)u_{\ell m} = 0$$



- IN mode

$$u_{\rm IN} = \begin{cases} e^{-i\varpi r_*} & (r_* \to -\infty) \\ A_{\rm out}e^{i\omega r_*} + A_{\rm in}e^{-i\omega r_*} & (r_* \to +\infty) \end{cases}$$

purely ingoing @ horizon

- UP mode

$$u_{\rm UP} = \begin{cases} B_{\rm out}e^{i\varpi r_*} + B_{\rm in}e^{-i\varpi r_*} & (r_* \to -\infty) \\ e^{i\omega r_*} & (r_* \to +\infty), \end{cases}$$

purely outgoing @ infinity

$$e^{2i\delta_{\ell m}} \equiv (-)^{\ell+1}\frac{A_{\rm out}}{A_{\rm in}}$$

S matrix

Green function (with $r > r_s$)

$$\psi_{\ell m}(r, r_s) = -\frac{u_{\rm IN}(r_s)u_{\rm UP}(r)}{W} = \frac{i}{2\omega A_{\rm in}}u_{\rm IN}(r_s)u_{\rm UP}(r) \qquad r, r_s \gg M$$

$$\equiv \frac{e^{i\omega r_*}}{2i\omega}(-)^{\ell}\left\{e^{i\left[\omega r_{s*} - \frac{\lambda_{\ell m}}{2\omega \tilde r} + 2\delta_{\ell m}\right]} - (-)^{\ell} e^{-i\left[\omega r_{s*} - \frac{\lambda_{\ell m}}{2\omega}(1/r_s - 1/r)\right]}\right\}$$

Sum over the partial waves does not converge.

$$\lambda_{\ell m} = A_{\ell m} + a^2\omega^2$$

$$\tilde r = (1/r + 1/r_s)^{-1}$$

Fresnel diffraction

Wave scattering by a Kerr BH

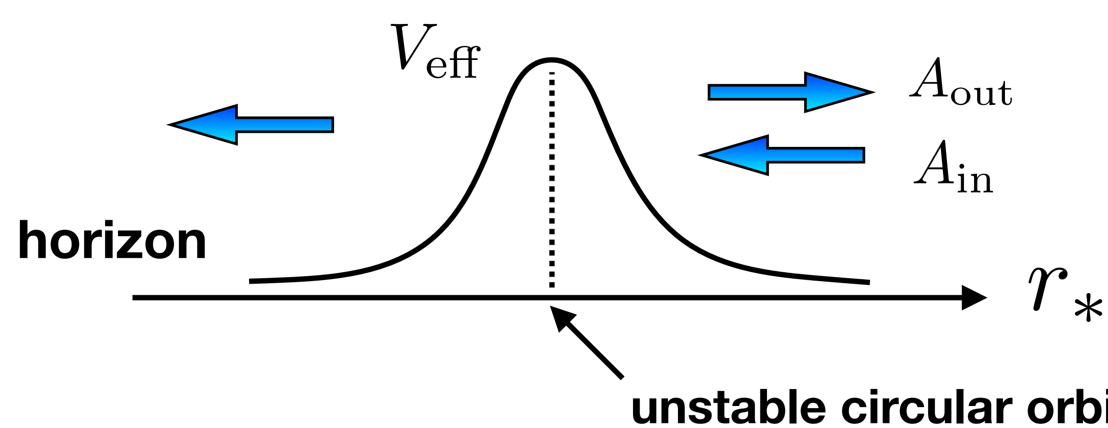
Green function

$$G(\boldsymbol{x},\boldsymbol{x}_s) \equiv \frac{e^{i\omega(r_*+r_{s*})}}{2i\omega r r_s}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(-)^{\ell}e^{i\frac{\lambda_{\ell m}}{2\omega\tilde{r}}}\underline{e^{2i\delta_{\ell m}}}Z_{\ell m}(\theta,\phi)Z^*_{\ell m}(\theta_s,\phi_s)$$

Numerical cal.

S matrix

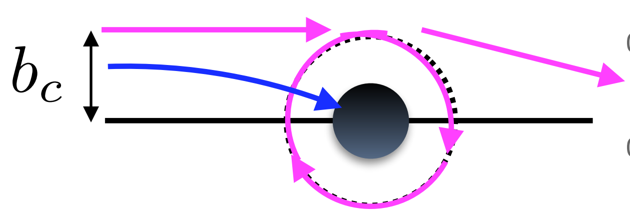
$$S = e^{2i\delta_{\ell m}} \equiv (-)^{\ell+1}\frac{A_{\rm out}}{A_{\rm in}}$$



phase shift (Schwarzschild case) $\omega M = 5$

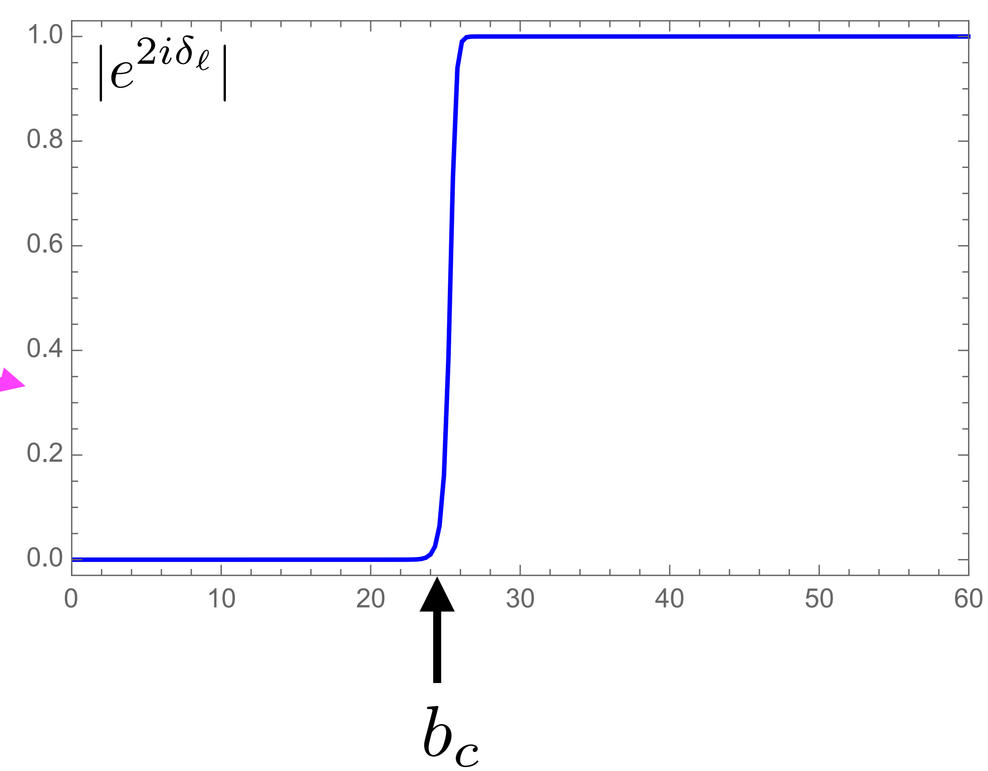
impact parameter

$$b = \ell/\omega M$$

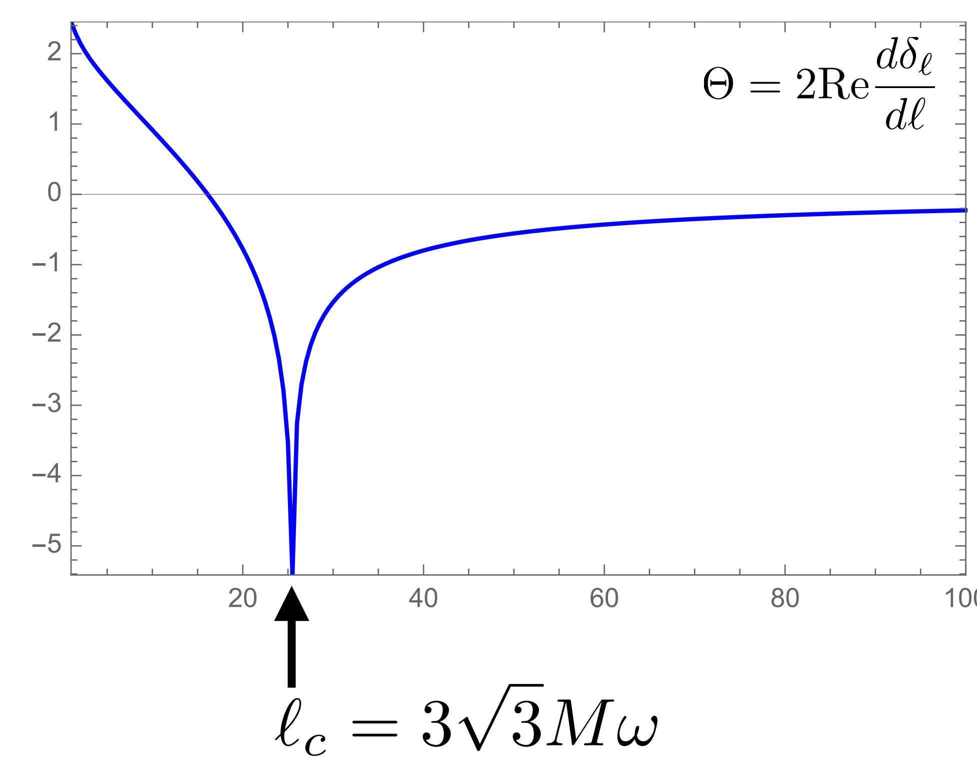


Unstable Circular Orbit

Reflection rate



Deflection angle



Phase shift

$$\frac{d^2 u_{\ell m}}{dr_*^2} + Q(r;\ell,m)u_{\ell m} = 0$$

Asymptotic form

$$u_{\ell m} = \begin{cases} e^{-i\varpi r_*} & (r_* \to -\infty) \\ A\sin\left[\omega r_* + \zeta\right] & (r_* \to +\infty), \end{cases}$$

$$\zeta = \delta_{\ell m} - \pi\ell/2$$

Prüfer method

$u_{\ell m}$ in two different forms

① $u_{\ell m} = e^{\int dr'_* P(r'_*)}$ $\quad (u/u' = P(r_*))$

$$\frac{dP}{dr_*} + P^2 + Q = 0$$

ingoing @ horizon

$P = -i\varpi \quad (r_* \to -\infty)$

$\tilde{P} = f(P)$

② $u/u' = \omega \cot\left[\omega r_* + \tilde{P}(r_*)\right]$

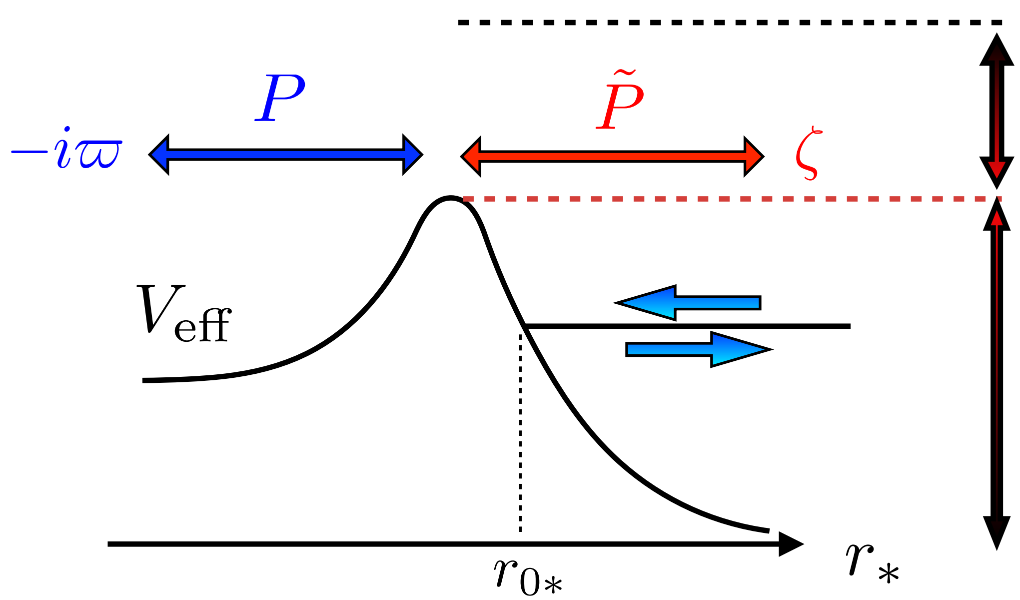
$$\frac{d\tilde{P}}{dr_*} + \left(\omega - \frac{Q}{\omega}\right)\sin^2(\omega r_* + \tilde{P}) = 0$$

$\tilde{P} = \zeta \quad (r_* \to +\infty)$

WKB method

$$\delta_{\ell m}^{\mathrm{WKB}} = \int_{r_{0*}}^{\infty} dr_* (\sqrt{Q} - \omega) + \frac{\pi\ell}{2}\left(\ell + \frac{1}{2}\right) - \omega r_{0*}$$

turning point



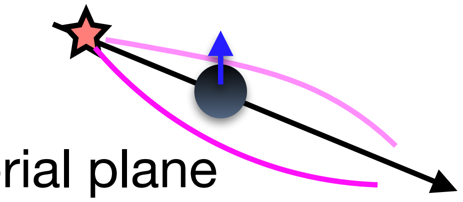
$\ell < \ell_c$ **no turning point** ⟶ **Prüfer**

$\ell > \ell_c$ **turning point** ⟶ **WKB**

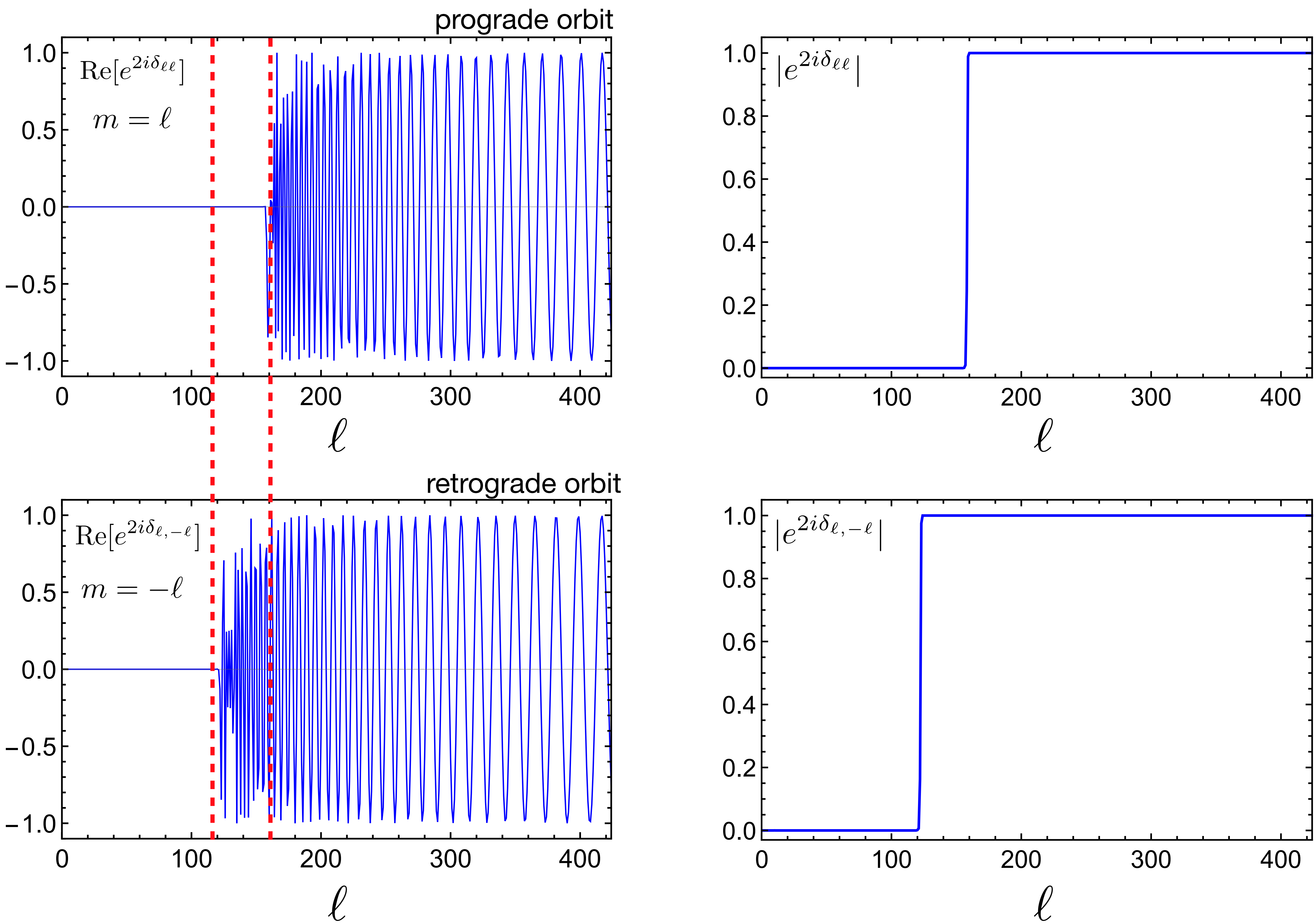
critical value

$$\ell_c = 3\sqrt{3}M\omega$$

Phase shift (Kerr case) $a = 0.6M$ $\omega M = 30$ $m = \pm\ell$



rays on the equatorial plane



Sum over the partial waves

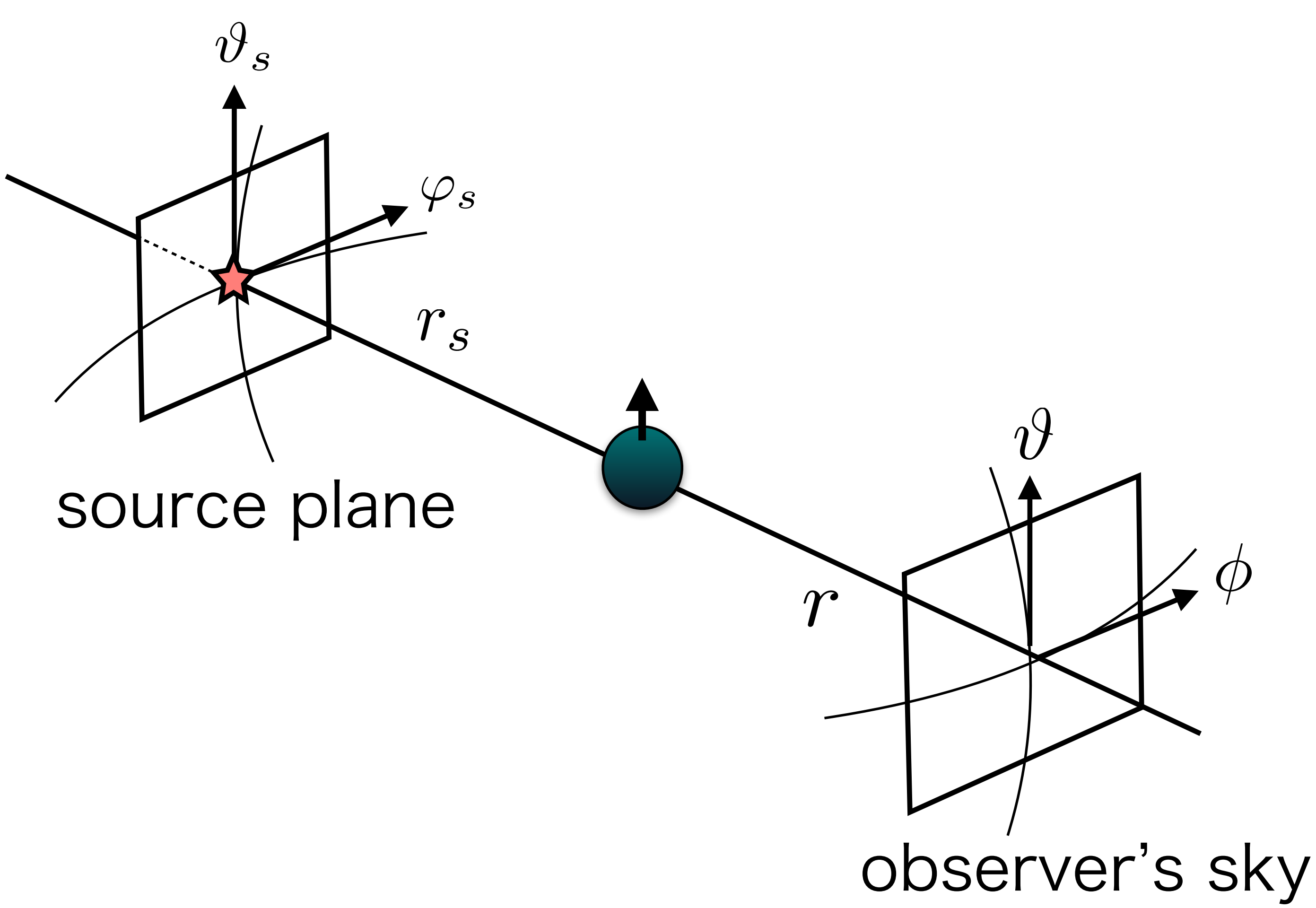
$$G(\boldsymbol{x},\boldsymbol{x}_s) \equiv \frac{e^{i\omega(r_*+r_{s*})}}{2i\omega r r_s}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(-)^{\ell} e^{i\frac{\lambda_{\ell m}}{2\omega\tilde{r}}}\, \underline{e^{2i\delta_{\ell m}}}\, Z_{\ell m}(\theta,\phi) Z^*_{\ell m}(\theta_s,\phi_s)$$

For $\omega M = 30$, $\ell_{\max} = 420$

$G(\boldsymbol{x},\boldsymbol{x}_s) \sim 160,000$ terms

equatorial plane

$10,000$ points on the obs. plane



$160,000 \times 10,000 = 1,600,000,000$ terms

It takes 2~4 days with a PC (8 cores)

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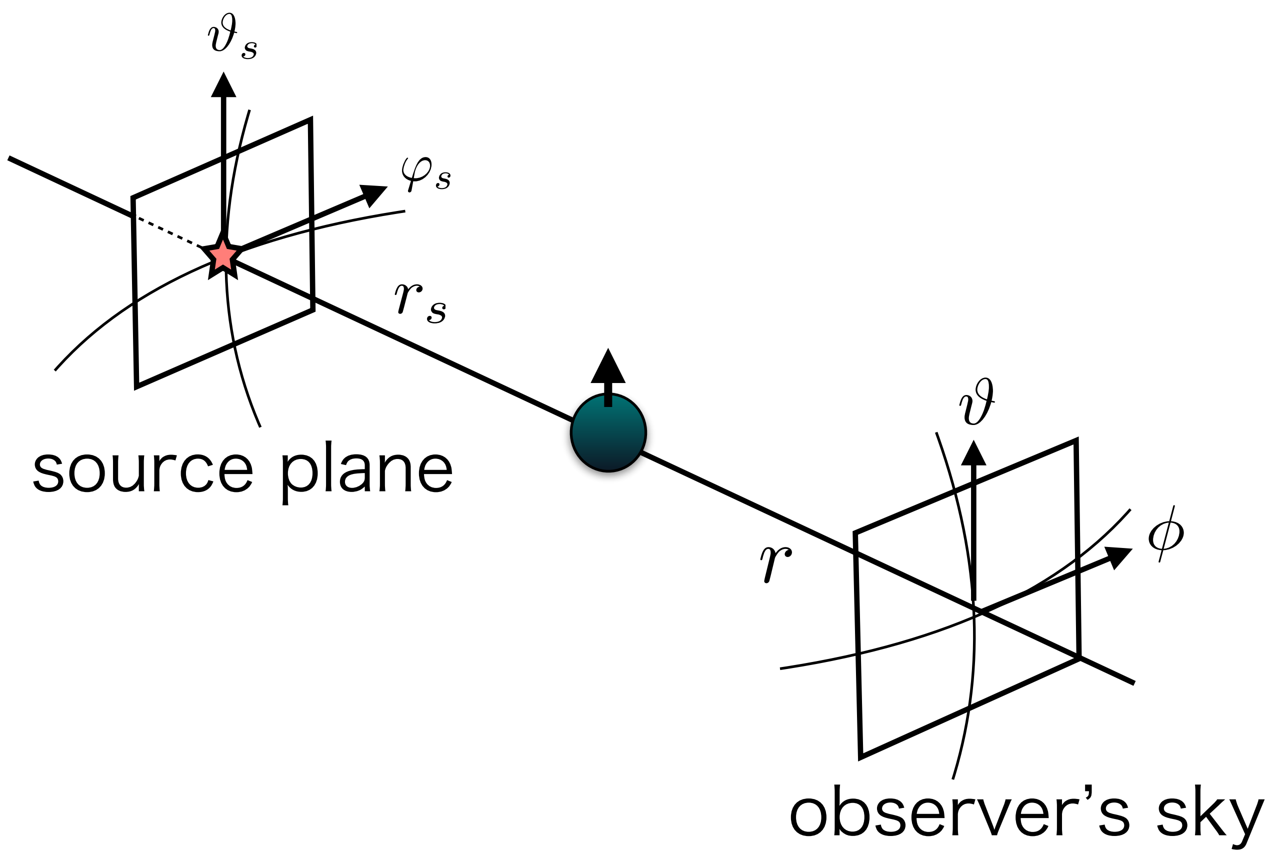
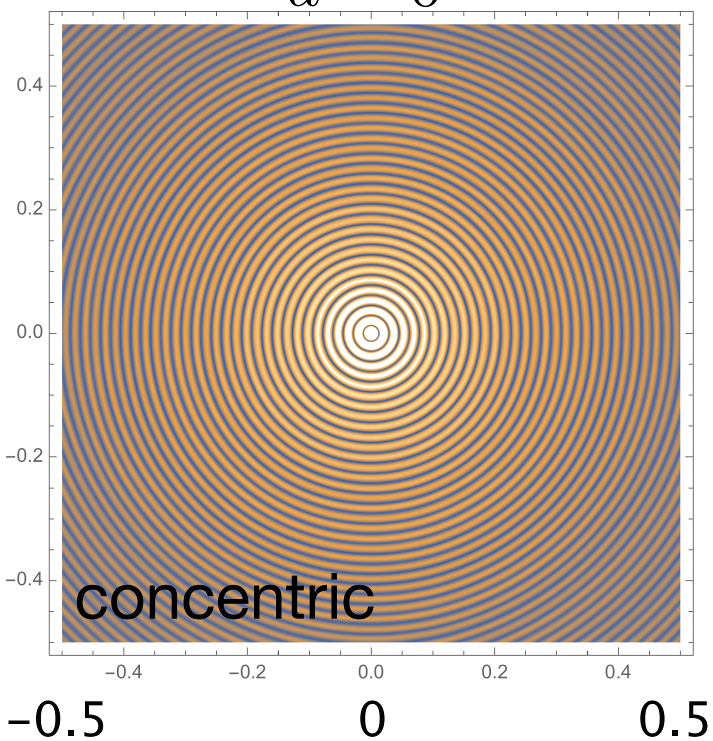
2. Wave scattering by a Kerr BH

3. Interference patterns & Images

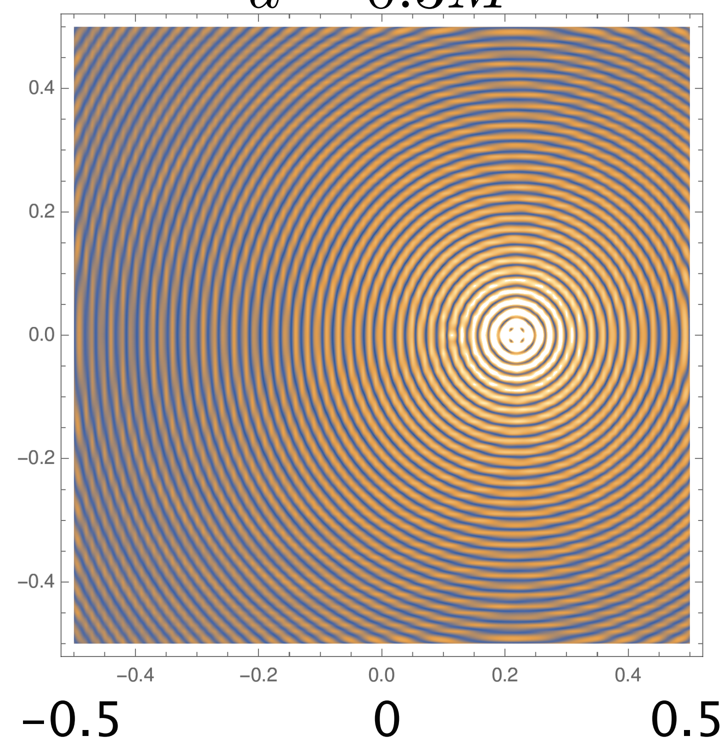


Interference patterns $\omega M = 30$

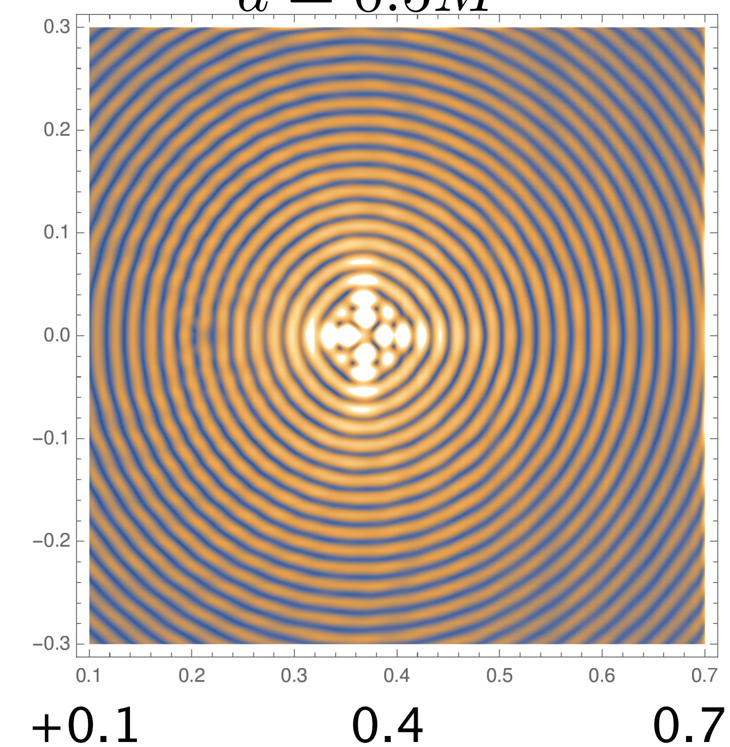
$a = 0$



$a = 0.3M$

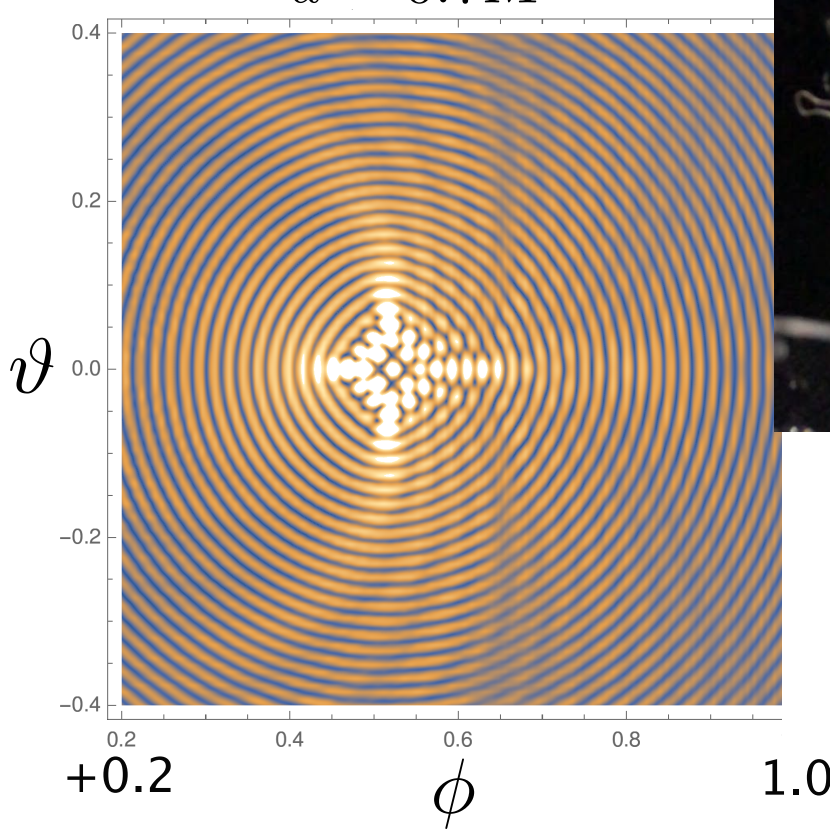


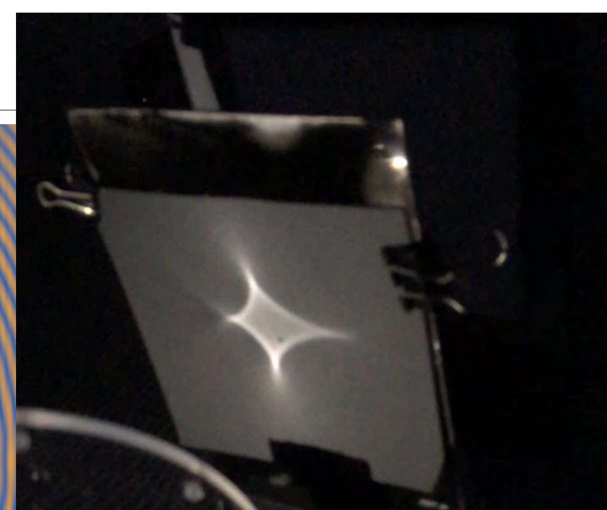
$a = 0.5M$



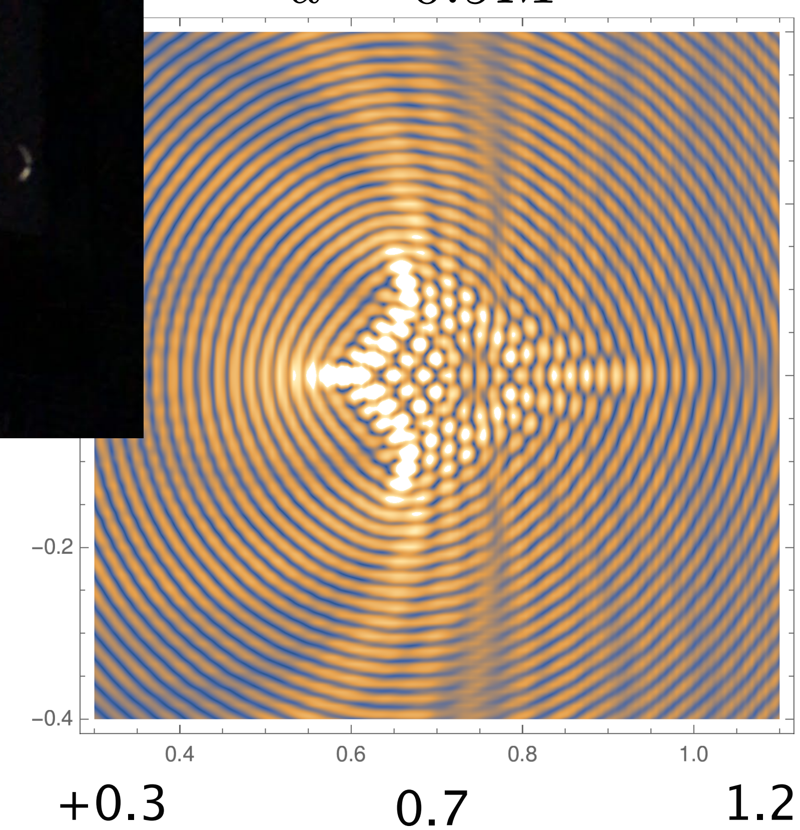
Diamond-shaped caustic

$a = 0.7M$



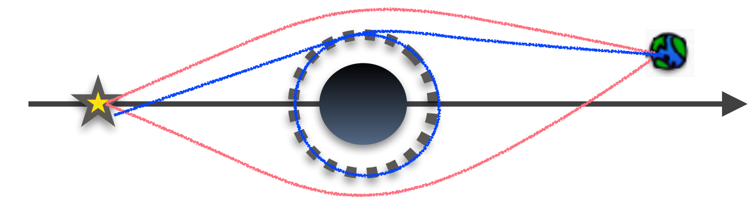


$a = 0.9M$

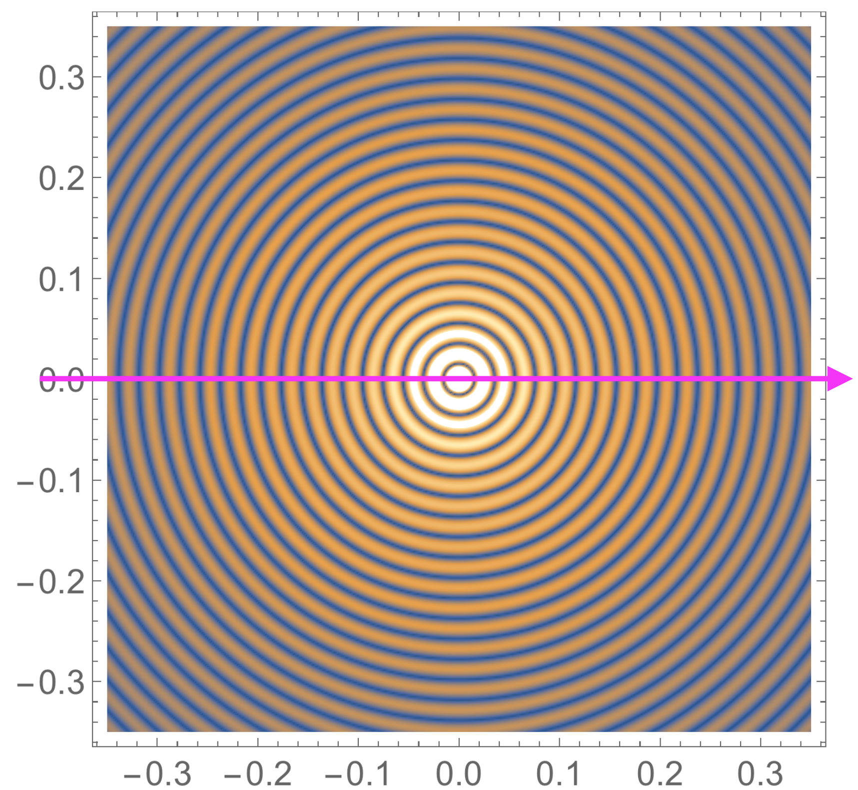


Caustics by the winding mode

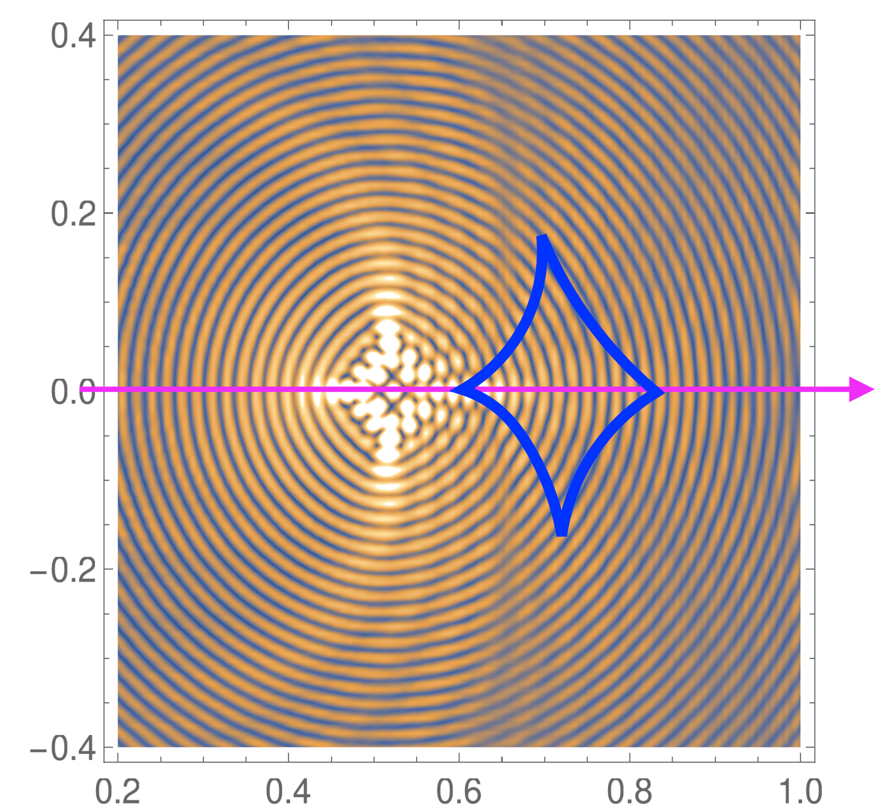
$$G = \underbrace{G^{\rm direct}}_{\ell \le \ell_c} + \underbrace{G^{\rm wind}}_{\ell > \ell_c} \qquad |G^{\rm wind}| \sim 10^{-1} |G^{\rm direct}|$$

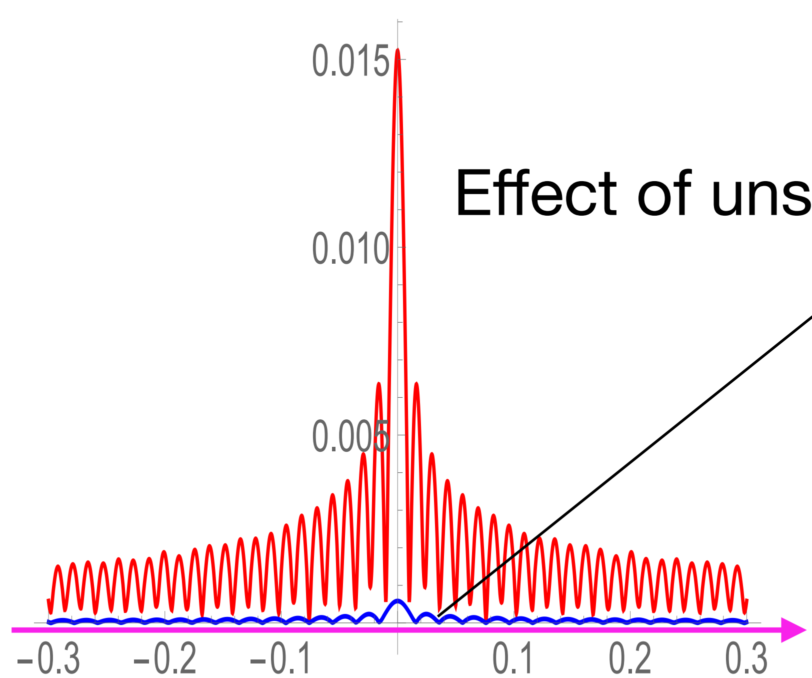


a=0



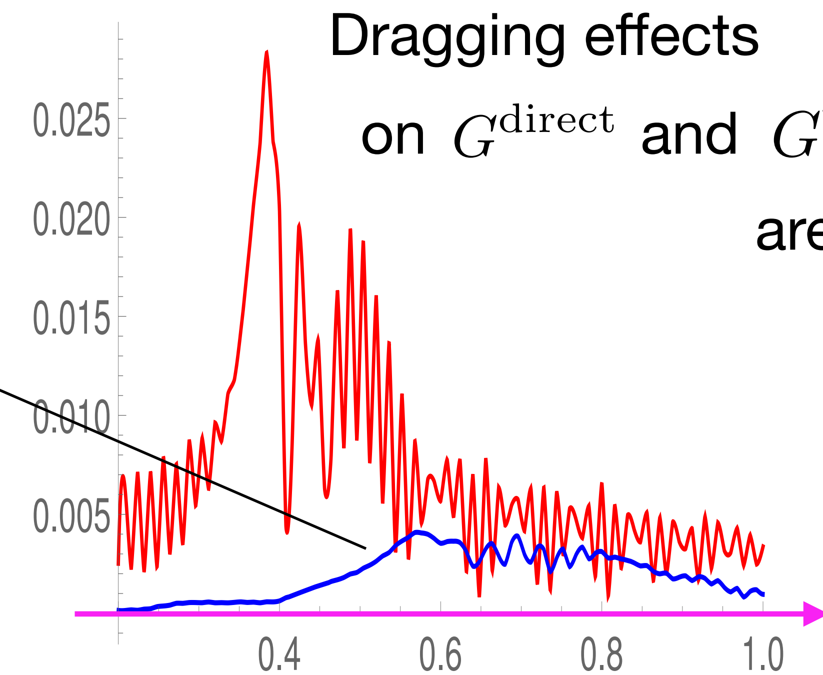
a=0.7



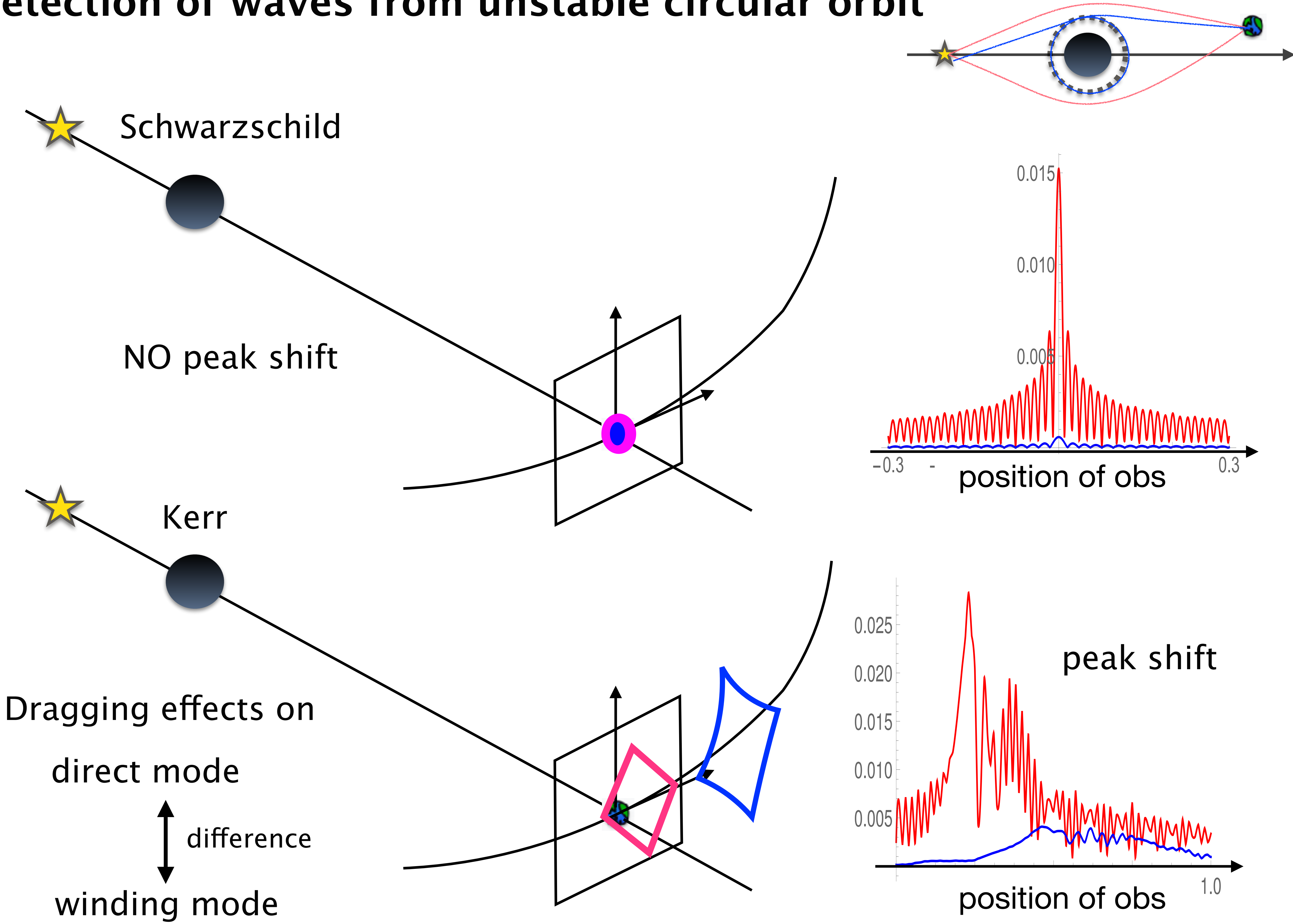


Effect of unstable circular orbit

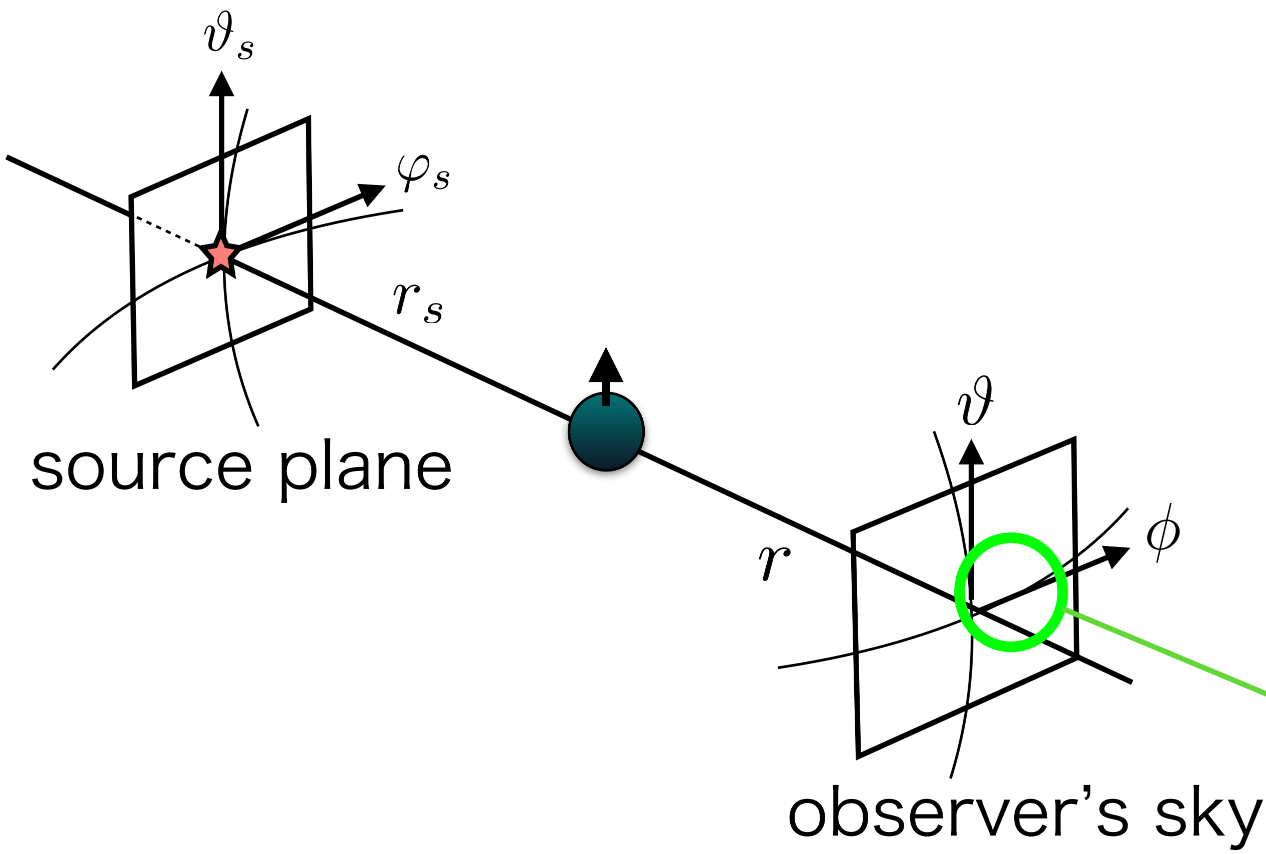
Dragging effects on $G^{\rm direct}$ and $G^{\rm wind}$ are different.



Detection of waves from unstable circular orbit



Imaging in wave optics

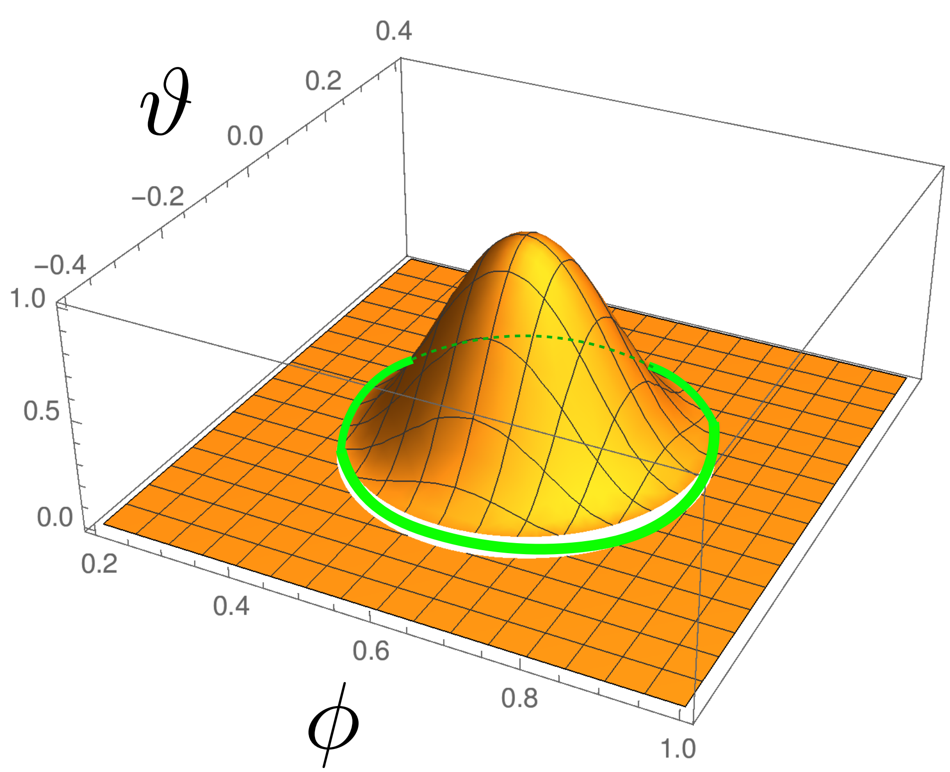


Imaging = 2D Fourier transform

$$\tilde{G} = \mathcal{F}[G(\phi, \vartheta) \times W^{\mathrm{Ham}}(\phi, \vartheta)]$$

$|\tilde{G}|$: image

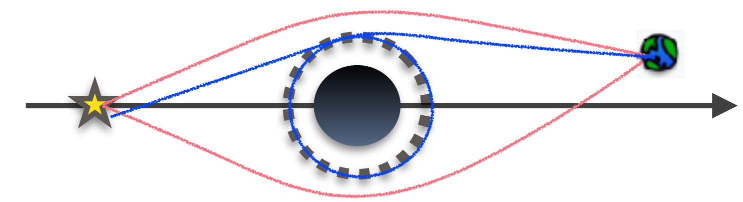
aperture an imaging system (telescope)



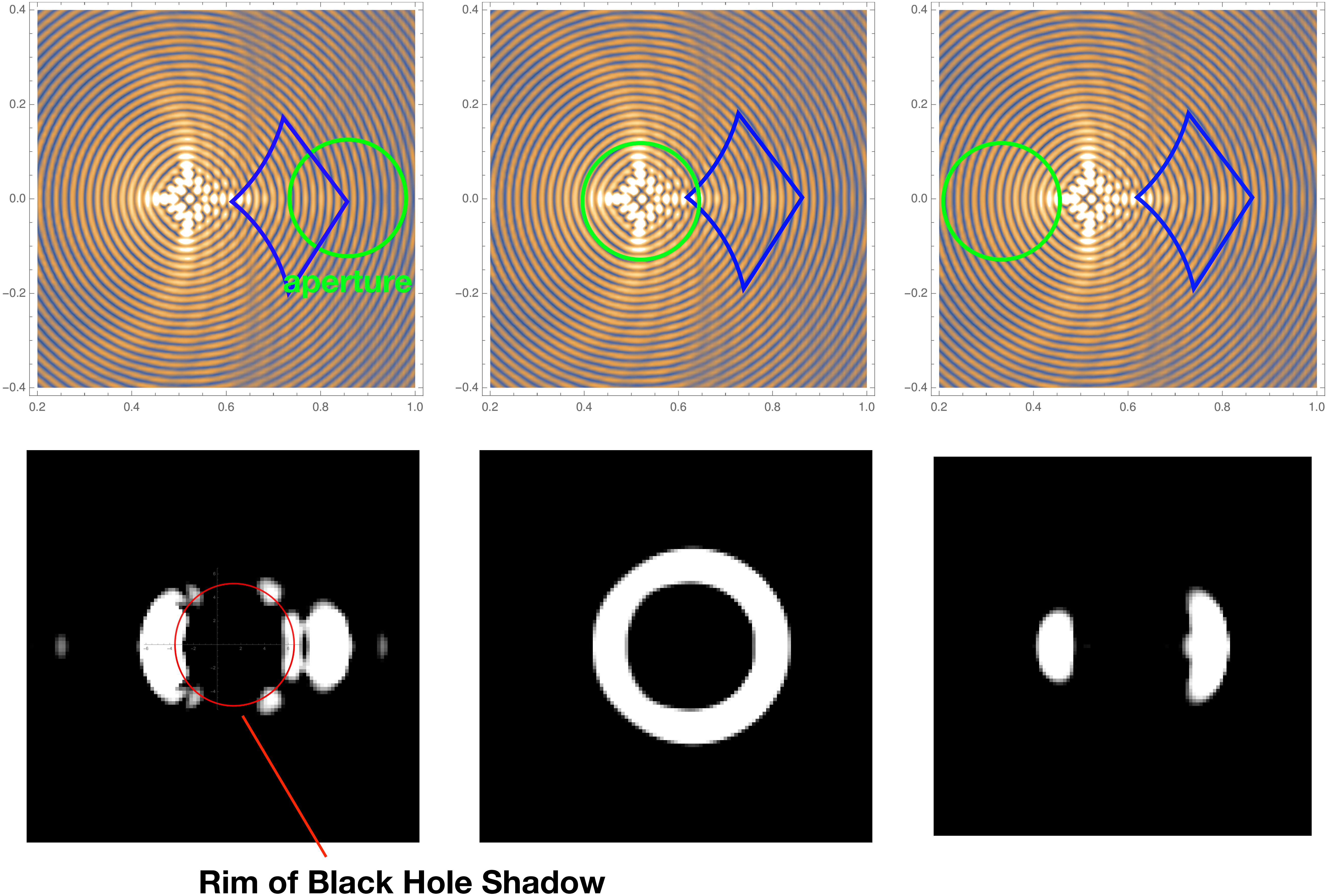
Hamming window function

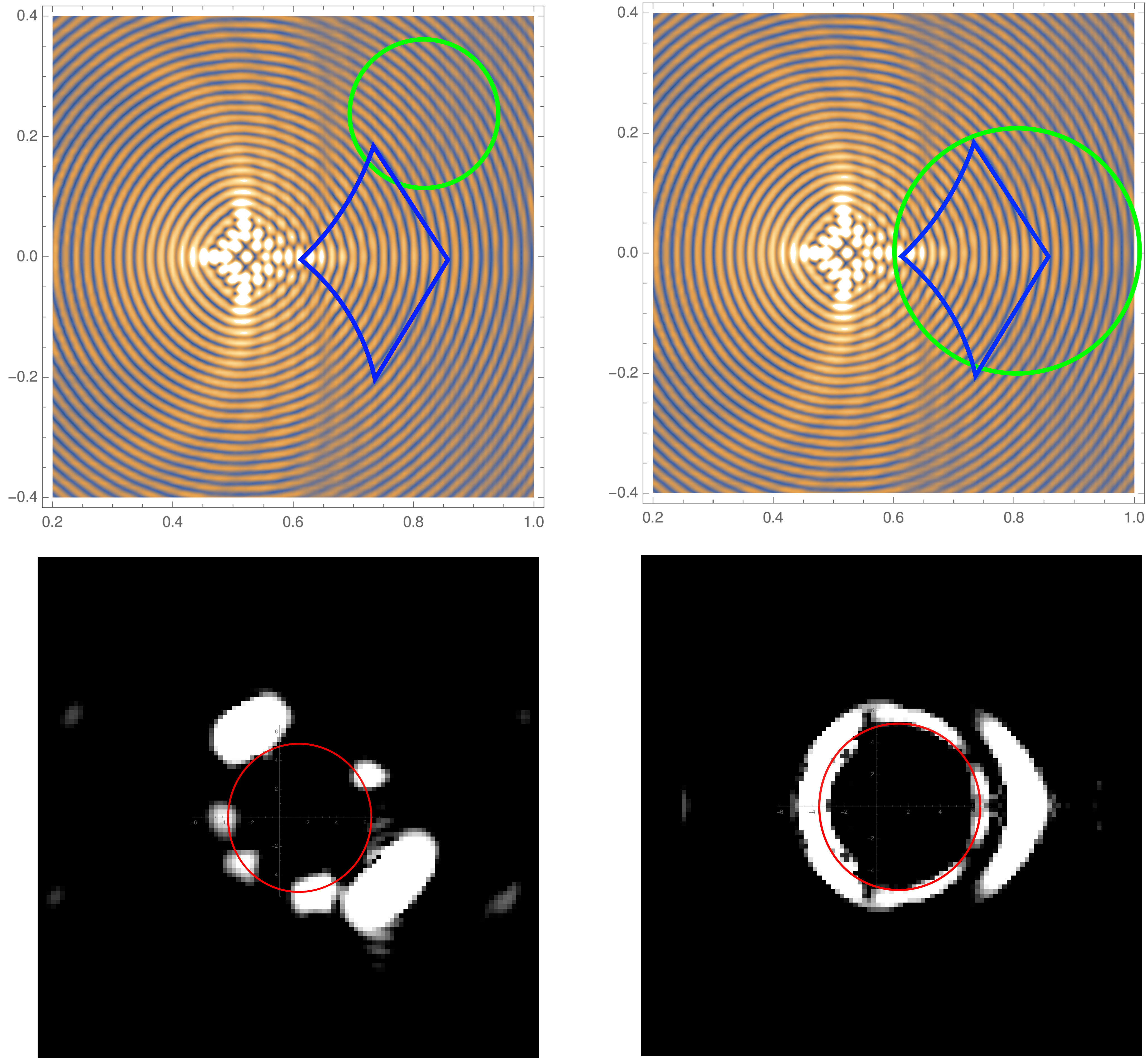
$$W^{\mathrm{Ham}}(\phi, \vartheta) = 0.54 - 0.46 \cos(2\pi\sqrt{\phi^2 + \vartheta^2})$$

The number of image & observer's position



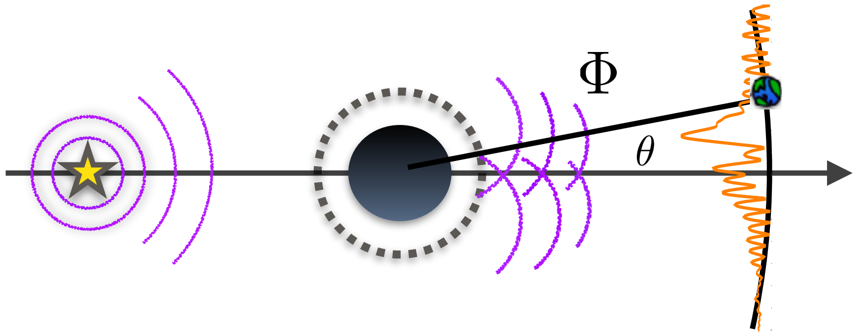
$a = 0.7M \quad M\omega = 30$





0.4
0.2
0.0
−0.2
−0.4
0.2
0.4
0.6
0.8
1.0

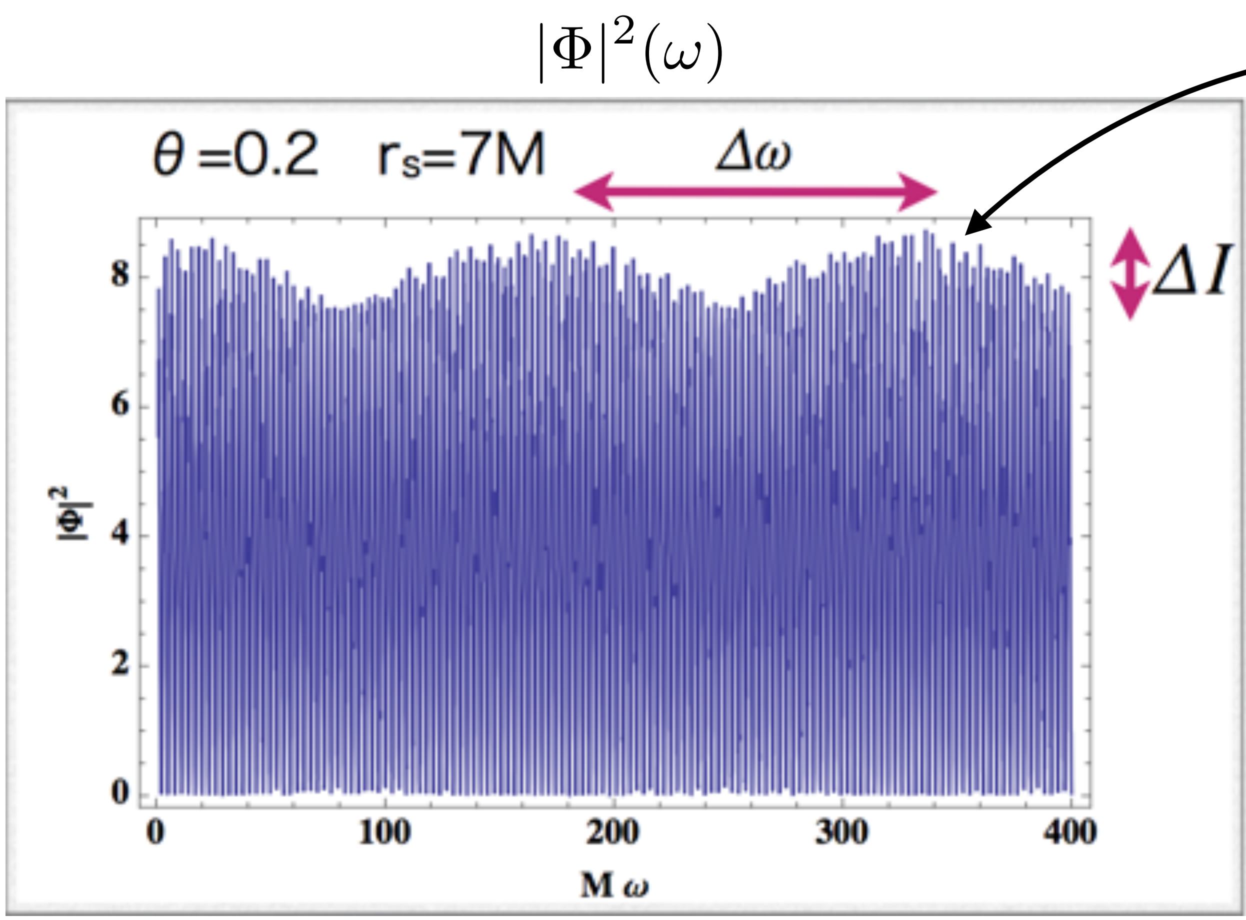
Is the interference patterns observable?



- Too short coherence time — What about pulser?
- Too much large to catch — We can use the motion of a source

Interference in the Power spectrum may be observable

short coherence time → interference in Fourier space



Modulation (effect of USCO)

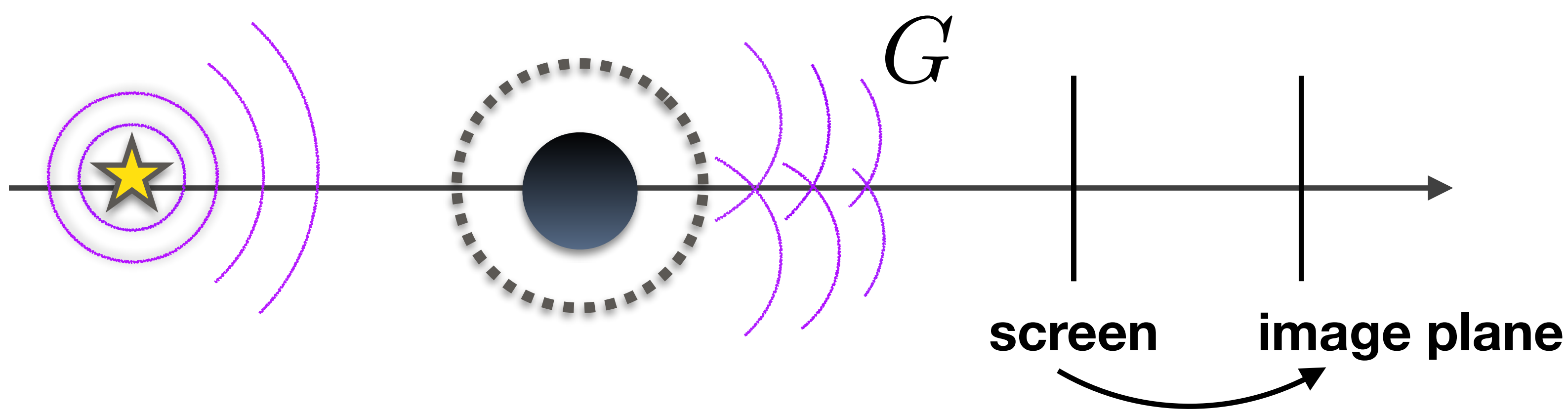
Typical freq. (Schwarzschild)

	Supermassive	Intermediate	Stellar size
mass	$10^6 M_\odot$	$10^3 M_\odot$	$10 M_\odot$
$\Delta\omega$	40Hz	40kHz	40MHz

These are sensitive to the posisions of source and obs

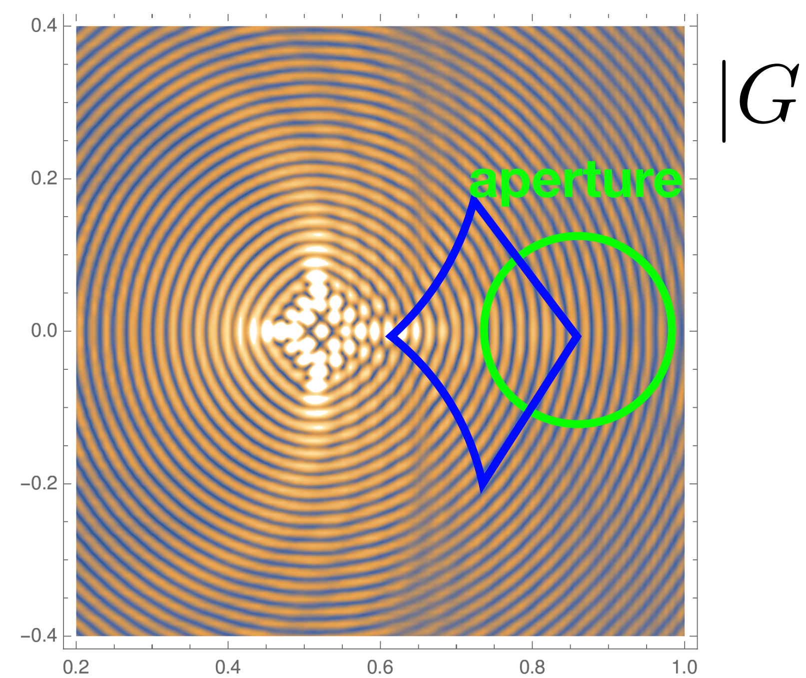
Summary

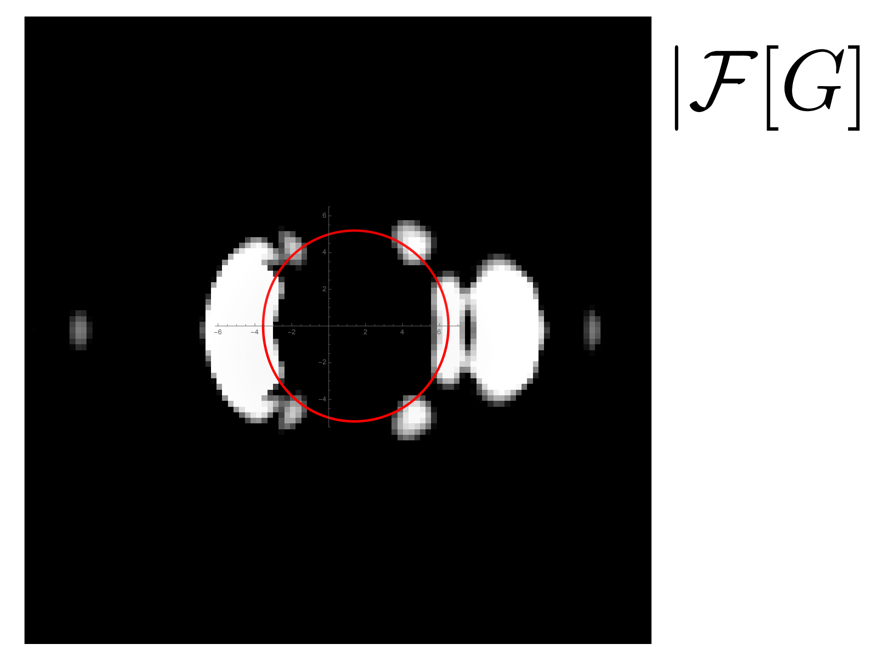
$$G(\boldsymbol{x}, \boldsymbol{x}_s) \equiv \frac{e^{i\omega(r_* + r_{s*})}}{2i\omega r r_s} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (-)^{\ell} e^{i\frac{\lambda_{\ell m}}{2\omega\tilde{r}}} e^{2i\delta_{\ell m}} Z_{\ell m}(\theta, \phi) Z^*_{\ell m}(\theta_s, \phi_s)$$



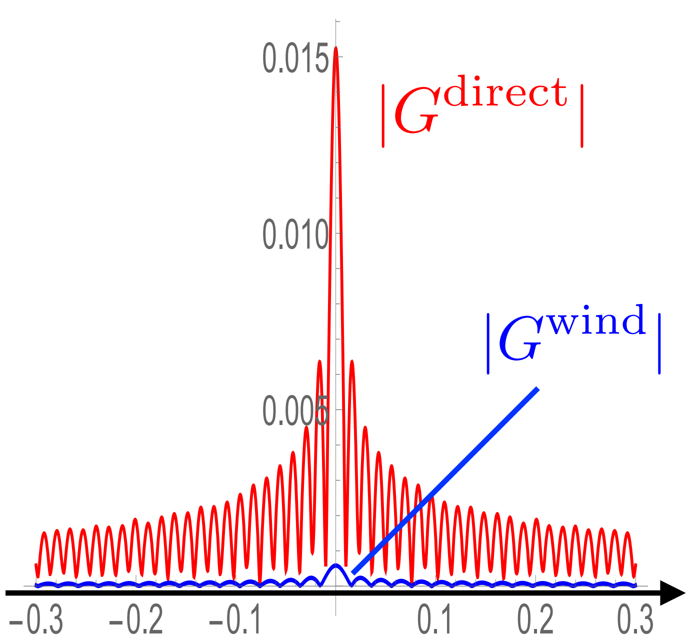
2D Fourier transform (imaging)

$$G = \underline{G^{\rm direct}} + \underline{G^{\rm wind}}$$





Schwarzschild



Kerr