NARUTO Strait in Tokushima, Japan

(Tidal whirlpool)

Analog rotating black holes in a MHD inflow

Sousuke Noda (Nagoya Univ. Japan)

Based on

“Analog rotating black holes in a magnetohydrodynamic inflow”

S.N., Yasusada Nambu, and Masaaki Takahashi

Phys. Rev. D .95, 104055 (2017)

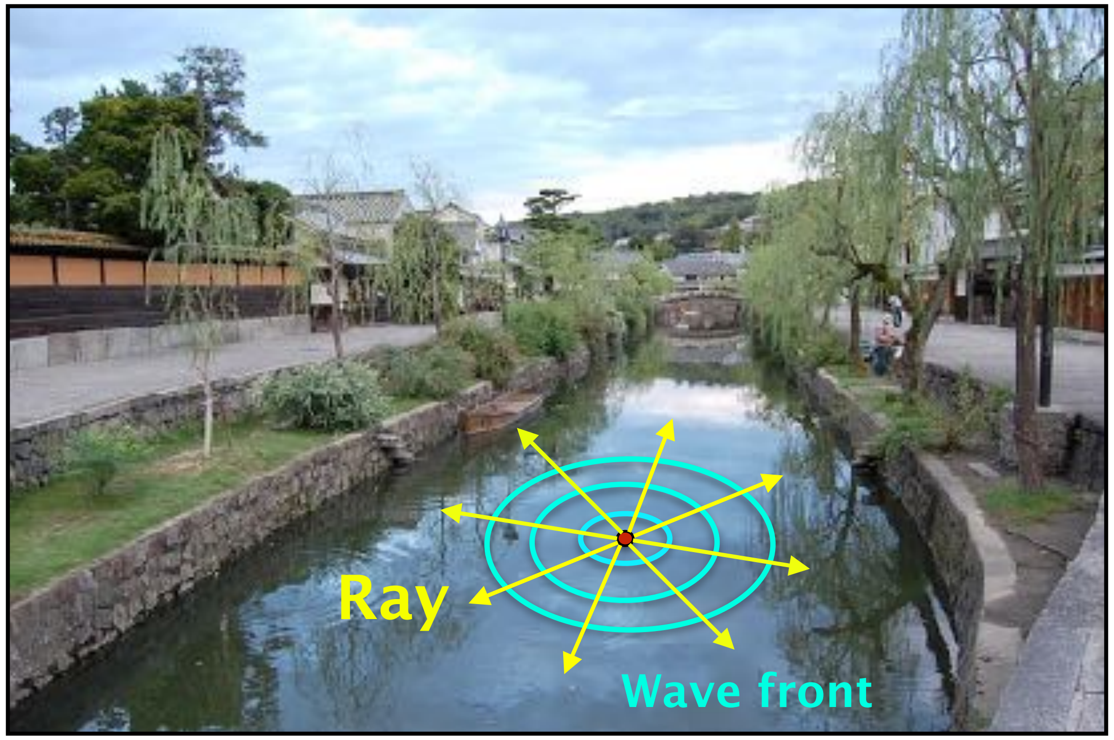
Collaborators

Yasusada Nambu (Nagoya Univ.)

Masaaki Takahashi (Aichi Edu. Univ.)

Analog geometry for acoustic waves

No flow

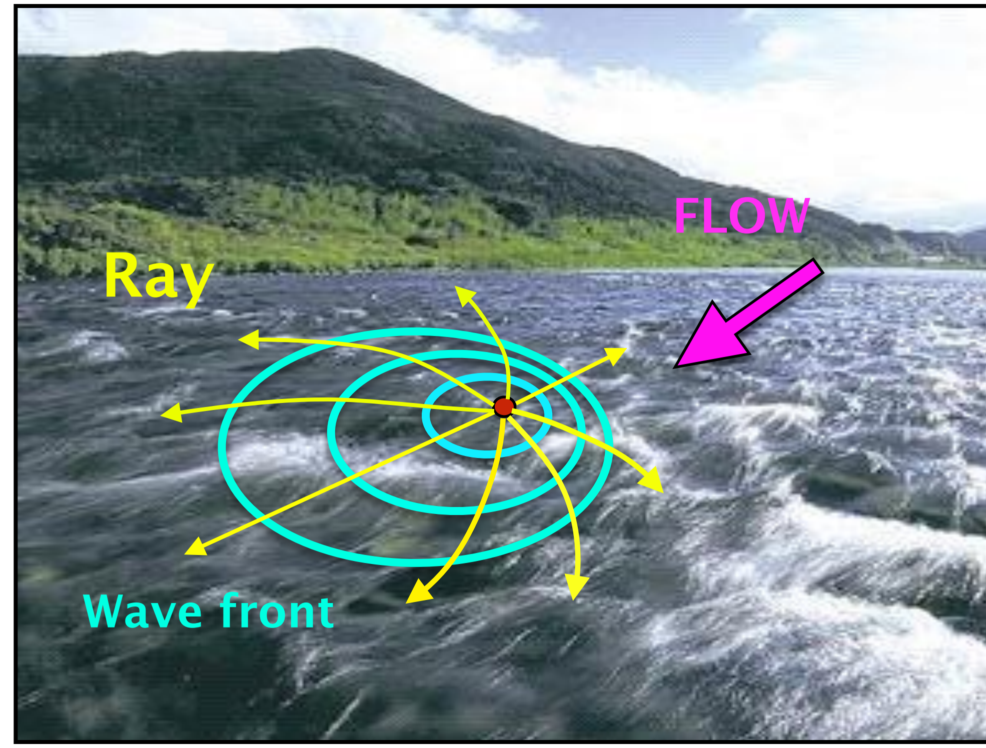


Ray's trajectory = straight line

Light rays in flat spacetime

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = 0$$

With flow



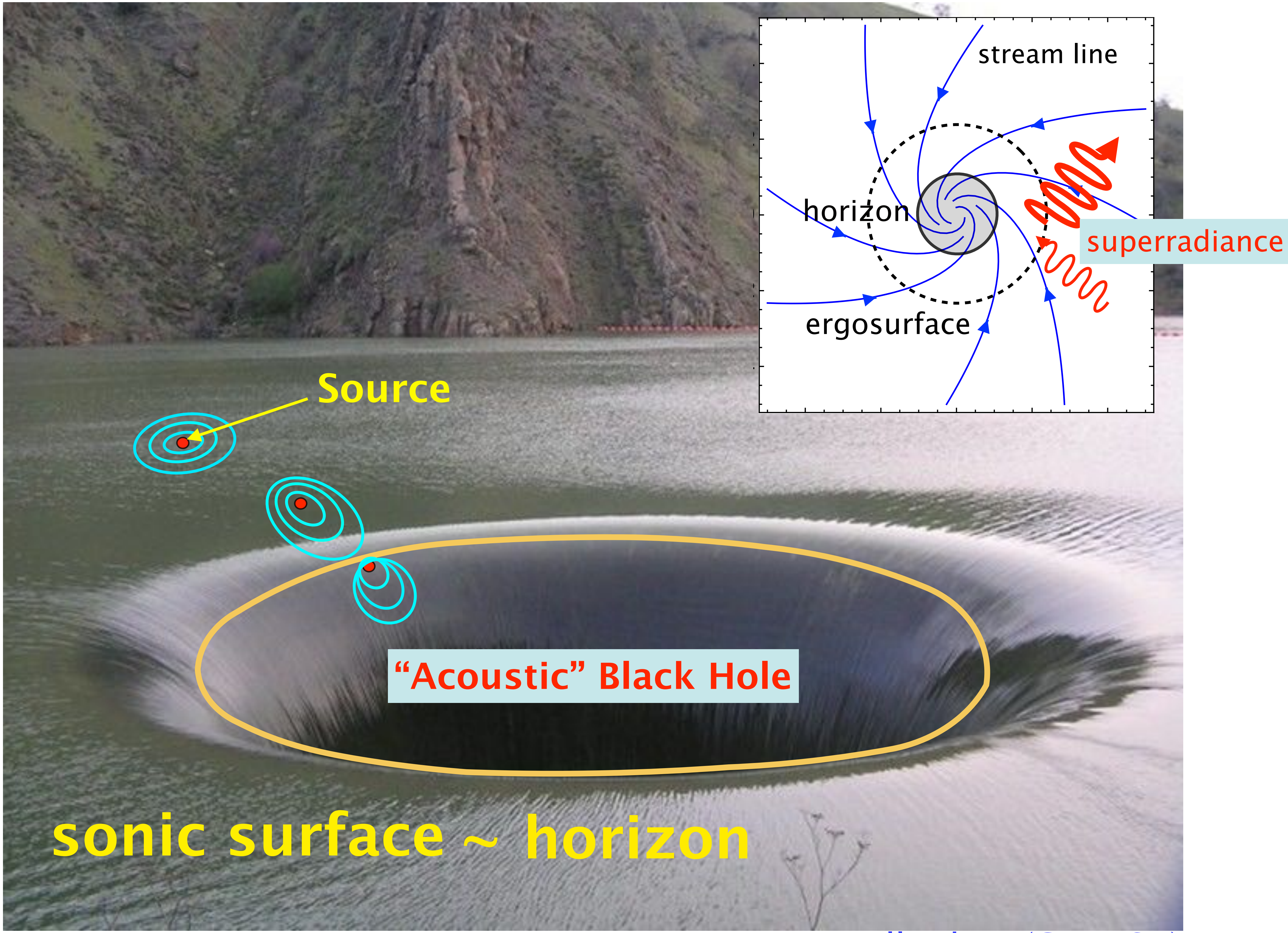
Ray's trajectory = curve

$ds^2 =$?

Light rays in curved spacetime

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = 0$$

Acoustic Black Holes



Montecello dam (CA. USA)
http://amazinglist.net/2013/03/bottomless-pit-dam-monticello-drain-hole/

Black Holes

- Solutions of Einstein eq.

Einstein eq. metric

$$G_{\mu\nu} = \kappa T_{\mu\nu} \quad \Rightarrow \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

curved spacetime

- Physical Properties

Horizon

Ergoregion (rotating BHs)

Photon sphere

No hair theorem

⋮

- Phenomena

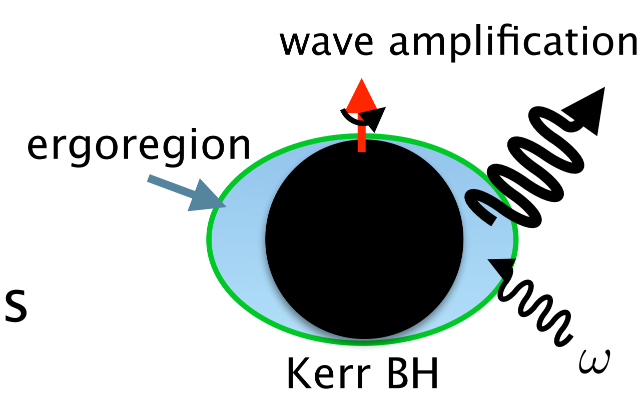
Hawking radiation

Black Hole Shadow

Quasi Normal Modes

Superradiance

⋮



superradiant condition

$$\omega < m\Omega_H$$

Acoustic Black Holes

NO !

Effective geometry for acoustic waves

Fluid Dynamics $\rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} \right] = -\nabla p + \boldsymbol{F}_{\text{ex}}$

Partially YES!

Acoustic horizon

Acoustic ergoregion

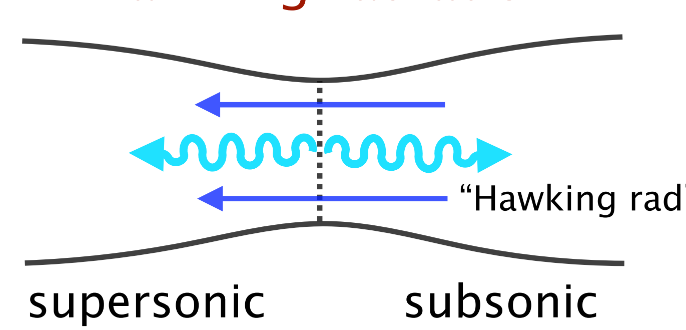
} sonic point

PhoNon sphere

~~Beautiful theorems~~

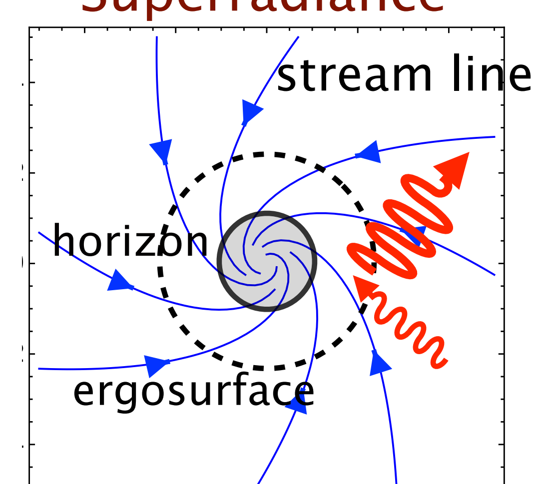
Partially YES <u>BH physics in Lab.</u>

Hawking radiation



theory: Unruh (1981)
experiment: Steinhauer (2016)

Superradiance



The motivation of our work

Acoustic BH in labs

BH physics —(sonic points / Analog model)→ Fluid dynamics

e.g.) Hawking rad,
Superradiance etc.

⇕

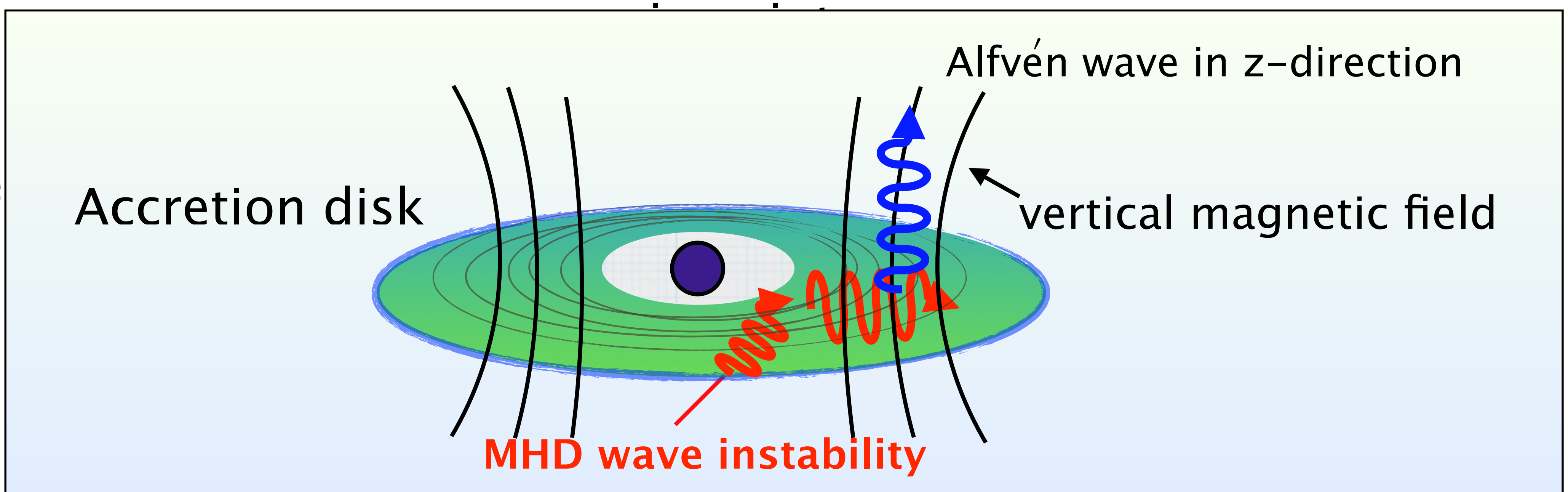
Our work

BH physics ←(Analog model)— (Astrophysical) Fluid dynamical phenomena

e.g.) Jet QPO Magnetic field

1. We discuss how to define the analog BH for MHD waves.
(This talk)
2. We “use” Black Hole physics to understand fluid phenomena.
(Future works)

The motivation of our work



Our work

(Astrophysical)

BH physics ← Analog model — Fluid dynamical phenomena

e.g.) Jet QPO Magnetic field

1. We discuss how to define the analog BH for MHD waves.

(This talk)

2. We "use" Black Hole physics to understand fluid phenomena.

(Future works)

Contents

1. **Acoustic Black Holes** (perfect fluid)

2. **Magneto-acoustic BHs** (MHD)

3. **Results**

Contents

1. Acoustic Black Holes (perfect fluid)

2. Magneto-acoustic BHs (MHD)

3. Results

Acoustic metric

- Perfect fluid (irrotational)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \ , \quad \boldsymbol{v} = -\nabla \Phi$$

velocity potential

$$\rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} \right] = -\nabla p + \boldsymbol{F}_{\rm ex} \ , \quad p = p(\rho)$$

Background — Perturbation

$$\boldsymbol{v} \to \boldsymbol{v} + \delta \boldsymbol{v} \qquad \Phi \to \Phi + \delta \Phi$$

$$\rho \to \rho + \delta \rho$$

- Wave eq. for $\delta \Phi$

$$\frac{\partial}{\partial t} \left[c_s^{-2} \rho \left(\frac{\partial \delta \Phi}{\partial t} + \boldsymbol{v} \cdot \nabla \delta \Phi \right) \right] + \nabla \cdot \left[\left\{ c_s^{-2} \rho \left(\frac{\partial \delta \Phi}{\partial t} + \boldsymbol{v} \cdot \nabla \delta \Phi \right) \right\} \boldsymbol{v} - \rho \nabla \delta \Phi \right] = 0$$

Klein–Gordon eq. in a " curved" spacetime

$$\frac{1}{\sqrt{-s}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-s}\, s^{\mu\nu} \frac{\partial \delta \Phi}{\partial x^\nu} \right) = \Box_s \delta \Phi = 0$$

Acoustic line element

$$ds^2 = s_{\mu\nu} dx^\mu dx^\nu$$

Acoustic metric

Moncrief (1980)

Unruh (1981)

Acoustic waves feel Riemannian geometry

Acoustic Horizon & ergoregion

- Acoustic line element

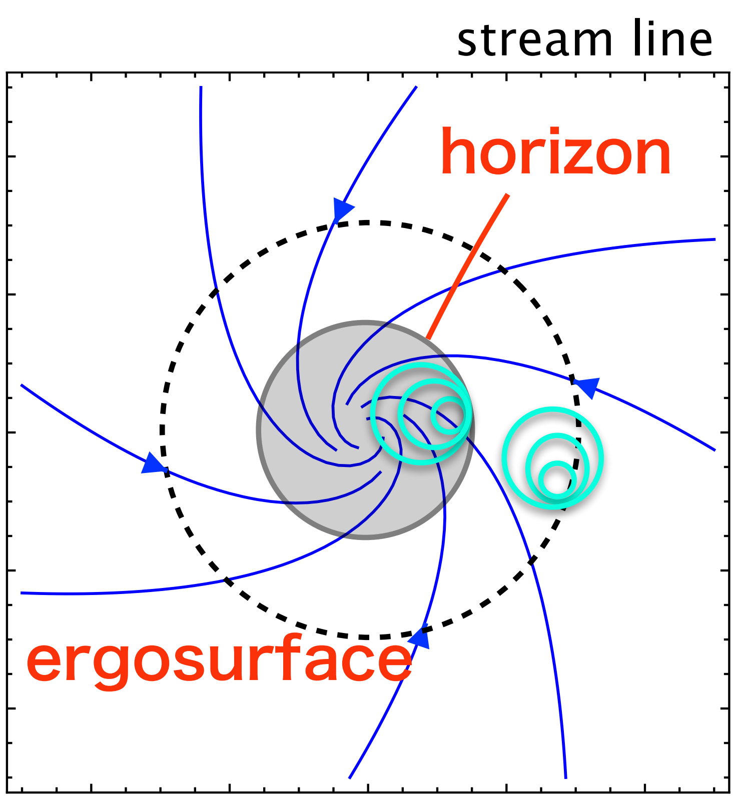
$$ds^2 = s_{\mu\nu}dx^\mu dx^\nu = \frac{\rho}{c_s}\left\{-\left[c_s^2 - \left((v^R)^2 + (v^\phi)^2\right)\right]dt^2 - 2v^\phi\, R\, d\phi\, dt + \frac{dR^2}{1-(v^R/c_s)^2} + R^2\, d\phi^2\right\}$$

For stationary & axisymmetric BG flow, there exist Killing vectors $\boldsymbol{\xi}_{(t)}$ and $\boldsymbol{\xi}_{(\phi)}$.
$\boldsymbol{v} = \boldsymbol{v}(R)$

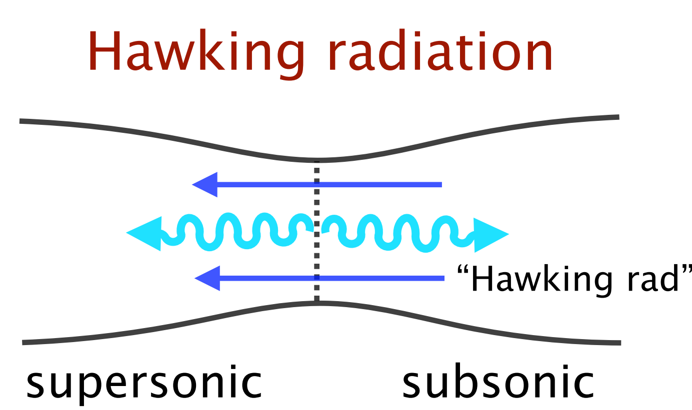
$\boldsymbol{\eta} = \boldsymbol{\xi}_{(t)} + \underline{\beta}\boldsymbol{\xi}_{(\phi)}$ (const)

$\boldsymbol{\xi}_{(t)}^2 = 0 \;\longrightarrow\; c_s = \sqrt{(v^R)^2 + (v^\phi)^2}$ Acoustic ergosurface

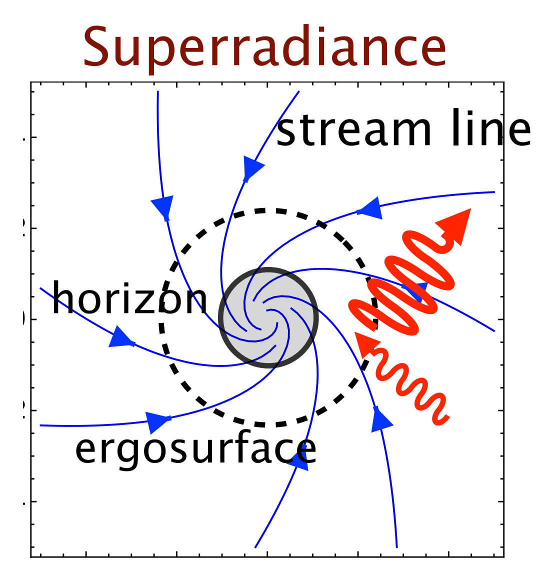
$\boldsymbol{\eta}^2 = 0 \;\longrightarrow\; c_s = v^R \quad \beta = \left.\frac{v^\phi}{R}\right|_{R_H}$ Acoustic horizon



Acoustic BH in laboratories



theory: Unruh (1981)
experiment: Steinhauer (2016)



Draining Bathtub Model

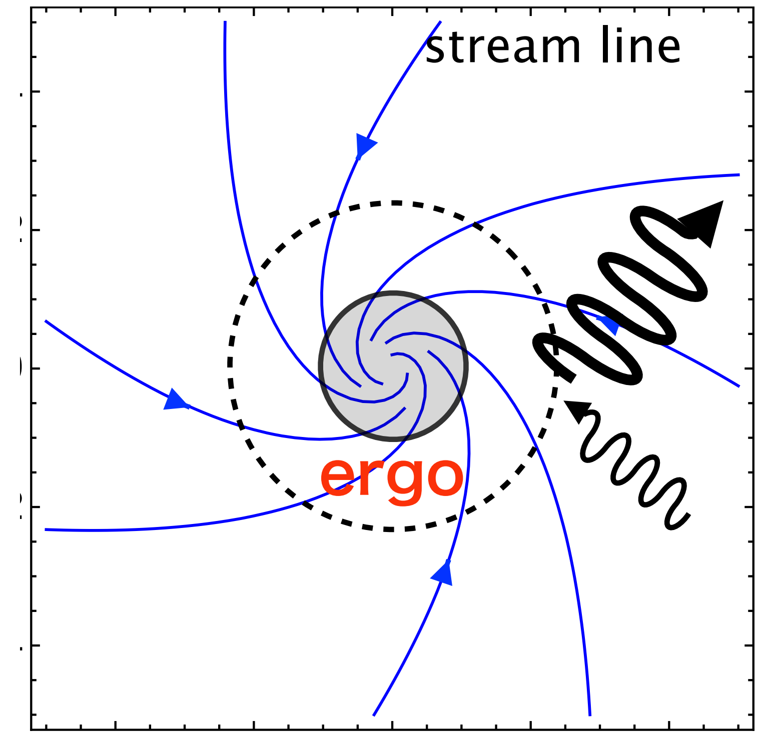
(2D)

M. Visser, Class.Quant.Grav. 15, 1767 (1998).

- Axisymmetric inflow with the sonic surface

$$v^R \propto -\frac{1}{\sqrt{R}} \quad , \quad v^\phi \propto \frac{1}{R}$$

(R, ϕ)
cylindrical coordinates



Wave equation

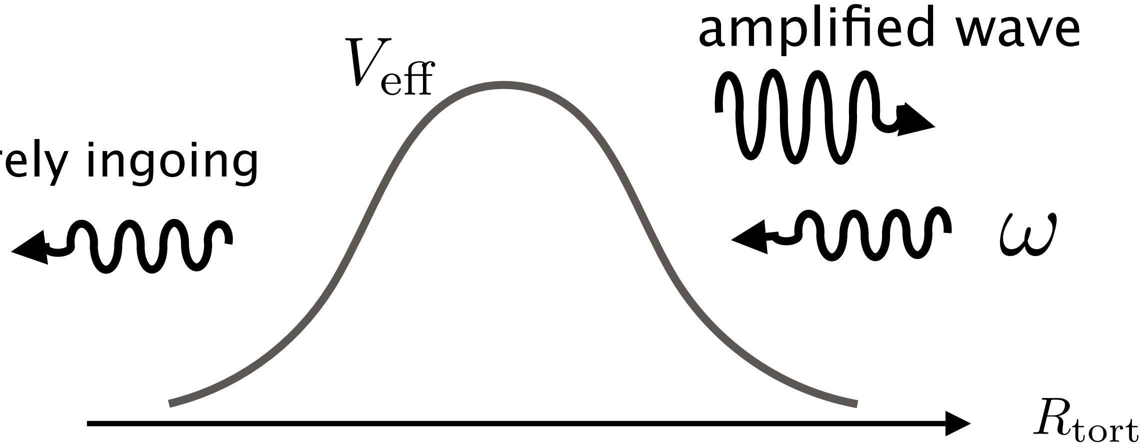
$$\Box_{s_{\mu\nu}} \delta\phi = 0 \qquad \delta\phi = \psi(R) e^{-i\omega t} e^{im\phi}$$

radial part

$$\frac{d^2\psi}{dR_{\rm tort}^2} + (\omega^2 - V_{\rm eff})\psi = 0$$

- Acoustic superradiance

Wave scattering by the acoustic BH



The Superradiant condition

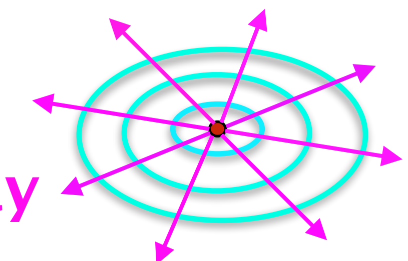
$$\omega < m \frac{v_0^\phi(R_{\rm H})}{R_{\rm H}}$$

Kerr case

$$\omega < m\Omega_{\rm H}$$

Angular velocity of the BG flow
at the acoustic horizon

Acoustic superradiance in the eikonal limit



- Eikonal approx.

Klein–Gordon eq. (WAVE) → Eikonal eq. (RAY)

$$\Box_s \delta\Phi = 0 \quad \xrightarrow{\delta\Phi \propto e^{iS}} \quad s^{\mu\nu} k_\mu k_\nu = 0 \qquad k_\mu = \frac{\partial S}{\partial x^\mu} = s_{\mu\nu}\frac{dx^\nu}{d\lambda}$$

tangent vector

We can separate S as

$$S = -\omega\, t + m\,\phi + S_R(R)$$

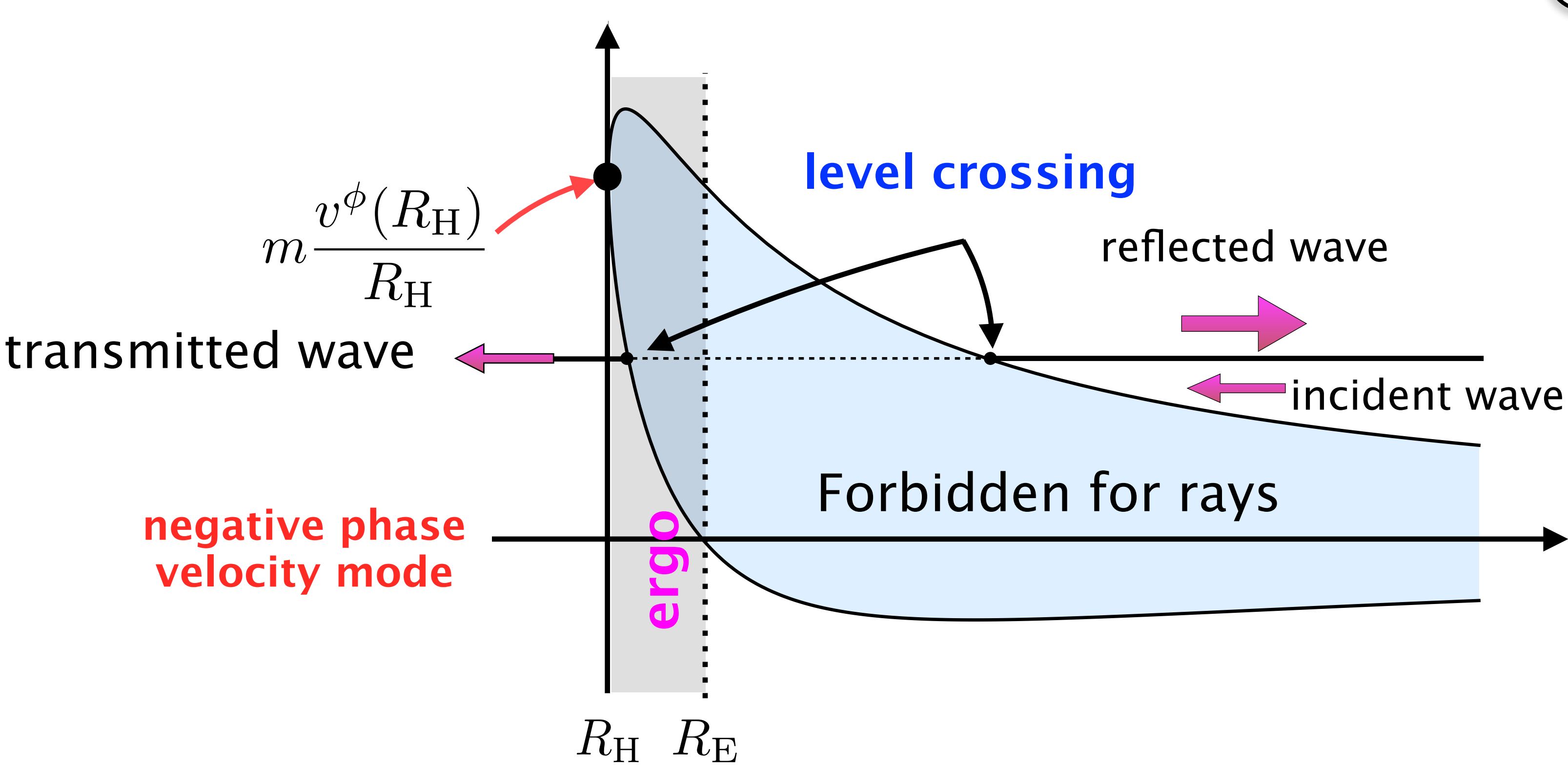
$$s^{tt}\,\omega^2 - 2 s^{t\phi}\,\omega\, m + s^{\phi\phi}\, m^2 + s^{RR}\left(\frac{dS_R}{dR}\right)^2 = 0$$

- Radial eq. & effective potential

$$\left(\frac{dR}{d\lambda}\right)^2 = \frac{1}{c_s^2}(\omega - V^+)(\omega - V^-) \geq 0$$

Effective potential

$$V^\pm = m\frac{v^\phi \pm \sqrt{c_s^2 - (v^R)^2}}{R}$$



Forbidden region for RAYS

$$V^- \leq \omega \leq V^+$$

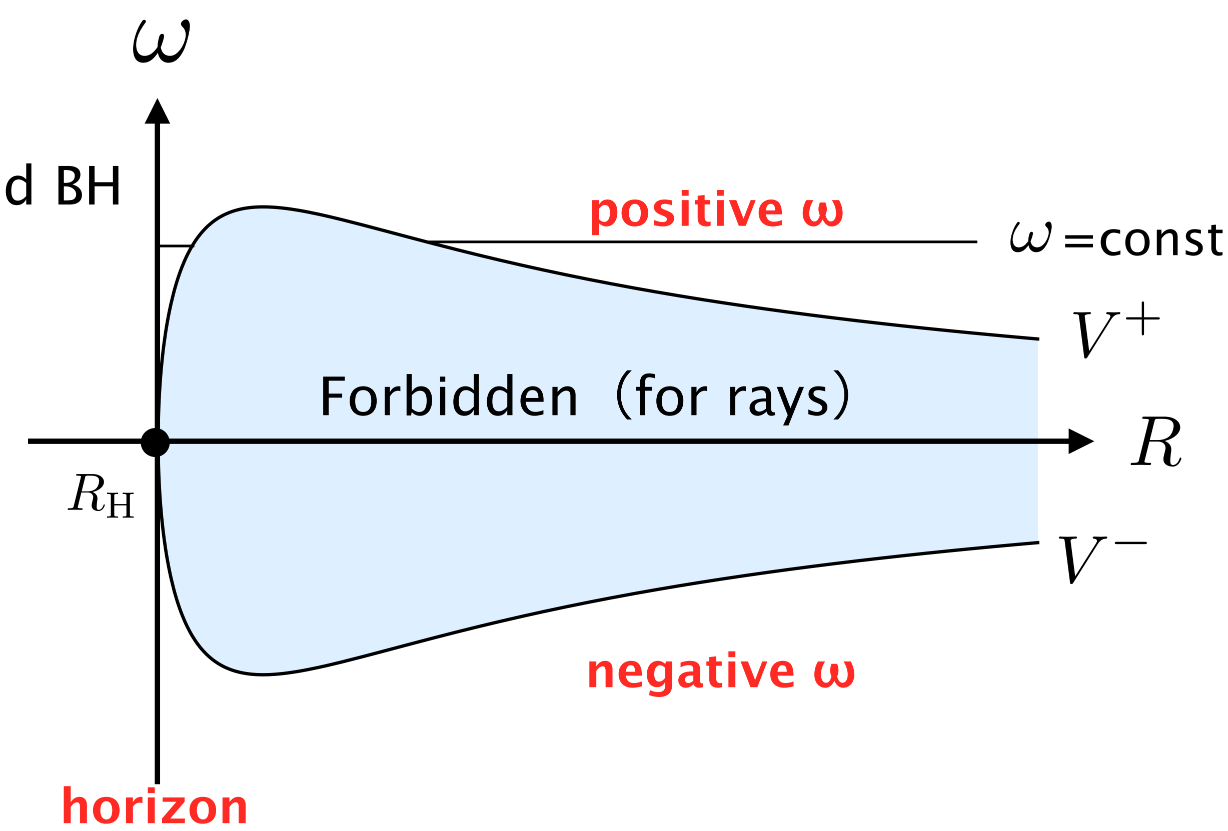
Superradiant condition

$$\omega < m\,\Omega_\mathrm{H} = m\frac{v^\phi(R_\mathrm{H})}{R_\mathrm{H}}$$

Effective potential & Acoustic superradiance

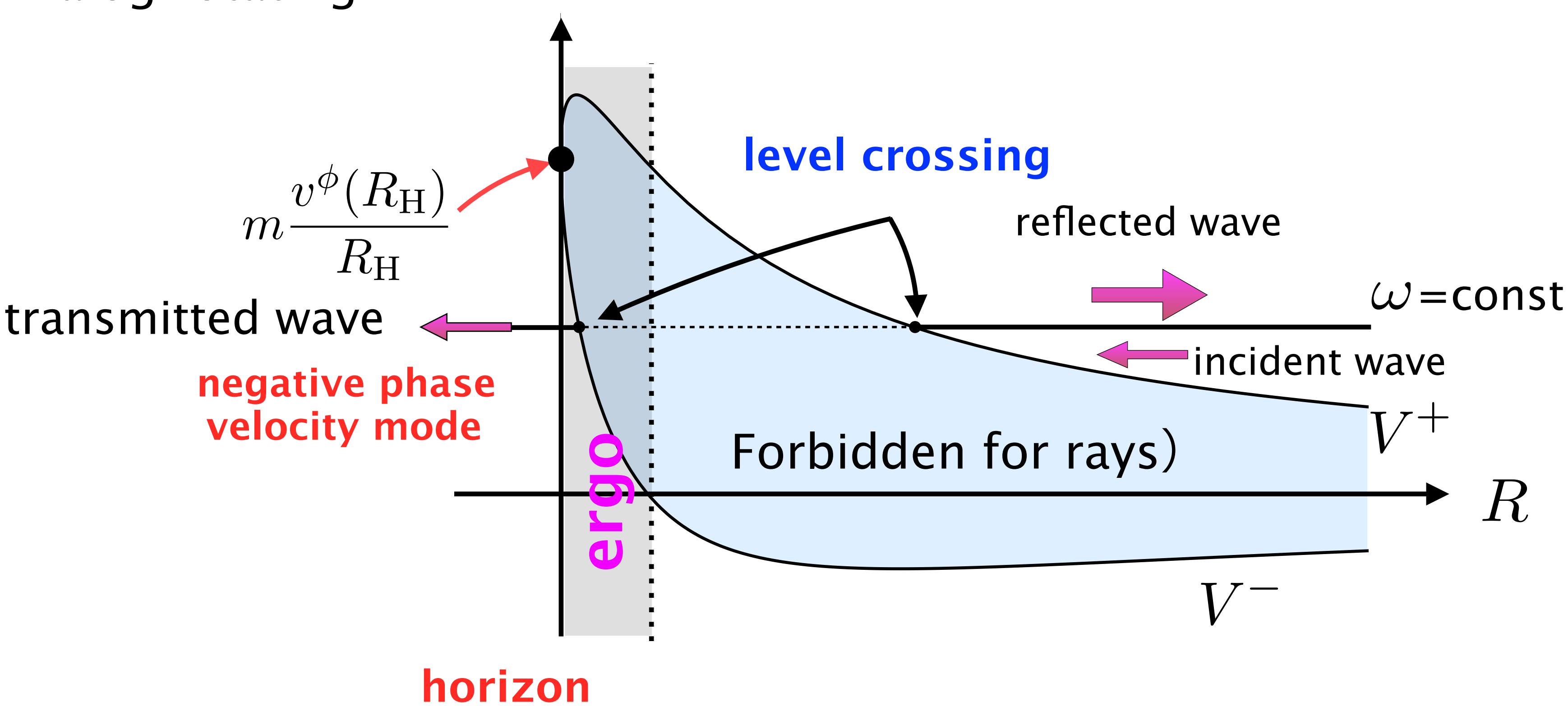
$v^\phi = 0$ case

Analog Schwarzshild BH



$v^\phi \neq 0$

Analog rotating BH



Effective potential

$$V^\pm = m\frac{v^\phi \pm \sqrt{c_s^2 - (v^R)^2}}{R}$$

Forbidden region for RAYS

$$V^- \leq \omega \leq V^+$$

Superradiant condition

$$\omega < m\,\Omega_{\rm H} = m\frac{v^\phi(R_{\rm H})}{R_{\rm H}}$$

Contents

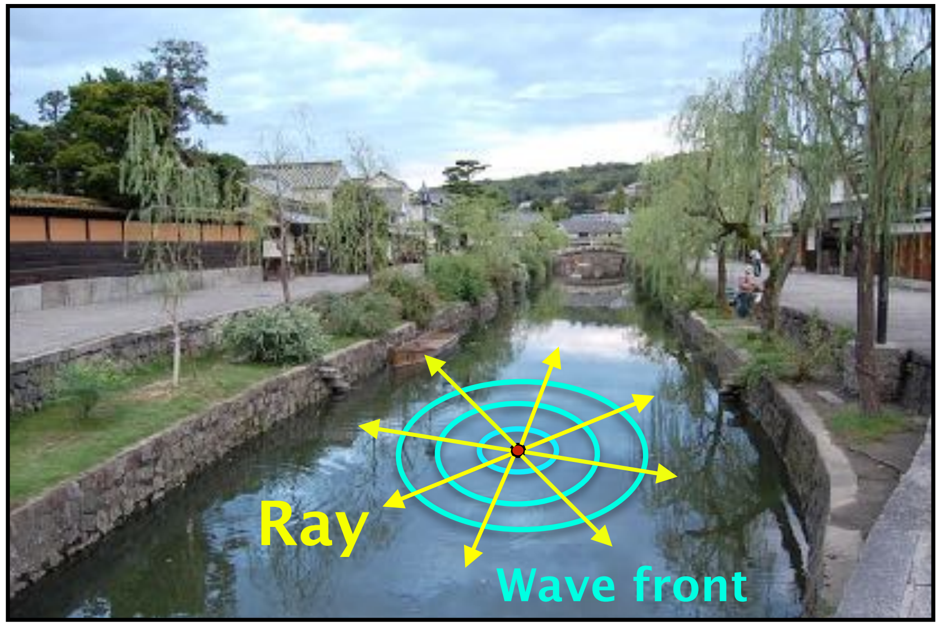
1. Acoustic Black Holes

2. Magneto-acoustic BHs

3. Summary & Future work

Difference between perfect fluid & MHD

Perfect fluids



$$ds^2 = s_{\mu\nu} dx^\mu dx^\nu$$

Riemannian geometry

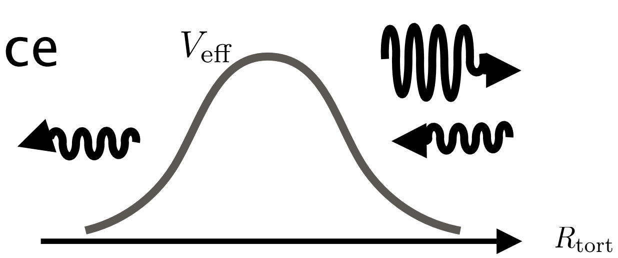
- single wave mode: $\Box_s \delta\Phi = 0$
- Isotropic propagation

  dispersion relation

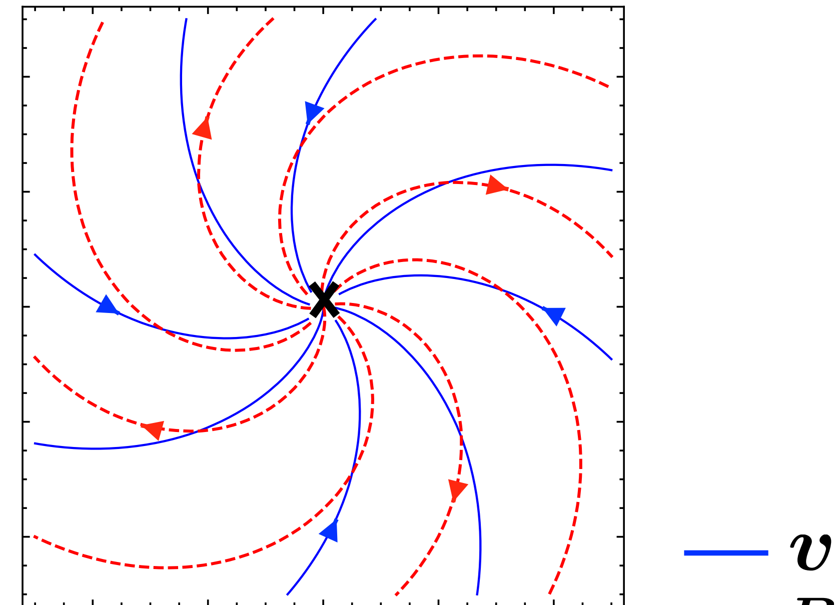
  $$\omega^2 = c_s^2 \underline{k^2} \quad \Rightarrow \quad \nabla^2 = \partial_x^2 + \partial_y^2$$

- BH-like structure : rotating BH

  Superradiance



MHD



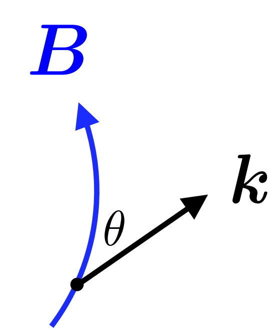
$$ds^2 = \boxed{\quad ? \quad}$$

Riemannian geometry ?????

- Multiple modes: fast & slow (2D case)
- Anisotropic propagation

  dispersion relation

  $$\omega^2 = f(k^2,\ \underline{\underline{\boldsymbol{k}\cdot\boldsymbol{B}}})$$



- rotating BH ?

Superradiant scat. of MHD waves ??

MHD wave equation (2D)

Ideal MHD

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \boldsymbol{v}) = 0 \ , \quad \frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B}) \ , \quad \nabla\cdot\boldsymbol{B} = 0$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \frac{\nabla p}{\rho} + \frac{1}{4\pi\rho}\boldsymbol{B}\times(\nabla\times\boldsymbol{B}) + \boldsymbol{F}_{\rm ex} = 0$$

Background / Perturbation

$$\boldsymbol{v} \to \boldsymbol{v} + \delta\boldsymbol{v} \qquad \rho \to \rho + \delta\rho$$

$$\boldsymbol{B} \to \boldsymbol{B} + \delta\boldsymbol{B} \qquad \boldsymbol{F}_{\rm ex} \to \boldsymbol{F}_{\rm ex}$$

MHD wave equation (2D)

Helmholtz theorem

$$\delta\boldsymbol{v} = \begin{pmatrix} \partial_x\Phi \\ \partial_y\Phi \end{pmatrix} + \begin{pmatrix} \partial_y\Psi \\ -\partial_x\Psi \end{pmatrix} = \begin{pmatrix} \partial_x & \partial_y \\ \partial_y & -\partial_x \end{pmatrix}\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}$$

Lagrange derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla$$

Alfvén velocity

$$\boldsymbol{V}_{\rm A} = \frac{1}{\sqrt{4\pi\rho}}\boldsymbol{B}$$

Wave Eq. for multiple wave modes

$$\frac{D^2}{Dt^2}\begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = \left[\begin{pmatrix} c_s^2\nabla^2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \hat{L}_1^2 & \hat{L}_1\hat{L}_2 \\ \hat{L}_1\hat{L}_2 & \hat{L}_2^2 \end{pmatrix}\right]\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}$$

$$\hat{L}_1 = (V_{\rm A})_y\,\partial_x - (V_{\rm A})_x\,\partial_y$$

$$\hat{L}_2 = (V_{\rm A})_x\,\partial_x + (V_{\rm A})_y\,\partial_y$$

How is the acoustic metric introduced ??

MHD wave & Acoustic wave

- MHD wave eq (2D)

$$\frac{D^2}{Dt^2}\begin{pmatrix}\Phi\\ \Psi\end{pmatrix} = \left[\begin{pmatrix} c_s^2\,\nabla^2 & 0 \\ 0 & 0\end{pmatrix} + \begin{pmatrix}\hat{L}_1^2 & \hat{L}_1\hat{L}_2 \\ \hat{L}_1\hat{L}_2 & \hat{L}_2^2\end{pmatrix}\right]\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}$$

$$\hat{L}_1 = (V_{\rm A})_y\,\partial_x - (V_{\rm A})_x\,\partial_y$$

$$\hat{L}_2 = (V_{\rm A})_x\,\partial_x + (V_{\rm A})_y\,\partial_y$$

$$\boldsymbol{V}_{\rm A} \propto \boldsymbol{B}$$

- $\boldsymbol{B} = 0$ case

$$\frac{1}{\sqrt{-s}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-s}\,s^{\mu\nu}\frac{\partial \Phi}{\partial x^\nu}\right) = 0 \quad \xrightarrow{\text{eikonal limit}} \quad s^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} = 0$$

Eikonal equation

Acoustic metric = coefficients of the eikonal equation

- $\boldsymbol{B} \neq 0$ case (MHD) We introduce the acoustic metric through the eikonal equation. (up to conformal factor)

Eikonal eq. for MHD waves (2D)

$$\Phi = |\Phi|\,e^{iS}, \quad \Psi = |\Psi|\,e^{iS}$$

$$\left(\frac{\partial S}{\partial t} + \boldsymbol{v}\cdot\nabla S\right)^4 - (c_s^2 + V_{\rm A}^2)|\nabla S|^2\left(\frac{\partial S}{\partial t} + \boldsymbol{v}\cdot\nabla S\right)^2 + (c_s^2\,|\nabla S|^2)(\boldsymbol{V}_{\rm A}\cdot\nabla S)^2 = 0$$

$$\longrightarrow \quad M^{\mu\nu\lambda\sigma}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu}\frac{\partial S}{\partial x^\lambda}\frac{\partial S}{\partial x^\sigma} = 0$$

Non quadratic form.....

Magneto-acoustic geometry

Eikonal eq. for 2D MHD waves

$$M^{\mu\nu\lambda\sigma}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu}\frac{\partial S}{\partial x^\lambda}\frac{\partial S}{\partial x^\sigma}=0$$

Quartic metric → NON Riemannian !

Magneto-acoustic metric

Finsler metric

If we define line element as

$$ds^2 = F(x,y) = \left(M^{\mu\nu\lambda\sigma}y_\mu y_\nu y_\lambda y_\sigma\right)^{1/4}$$

$y_\mu = \partial_\mu S$,

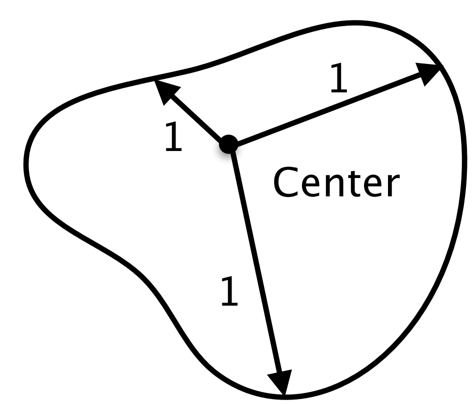
$F(x,y)$ satisfies the homogeneity condition: $F(x,\sigma y) = \sigma F(x,y)$. $\sigma = \mathrm{const}$

Finsler geometry

1. Generalization of Riemannian geometry
2. Distance depends on the direction.

⋮

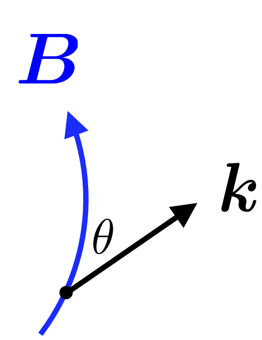
unit "circle" in a Finsler manifold



Anisotropic propagation of MHD

dispersion relation

$$\omega^2 = f(k^2,\ \boldsymbol{k}\cdot\boldsymbol{B})$$



analogue geometry → Finsler geometry

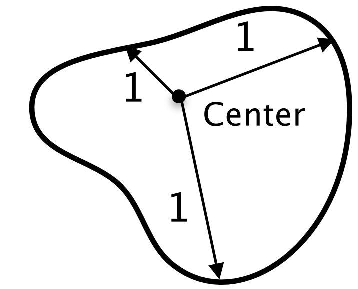
Unfortunately…..

I’m not familiar with calculations of Finsler geometry.

(for now)

Reduction of metric

The Finslerian magneto-acoustic metric $M^{\mu\nu\lambda\sigma}$



Riemannian magneto-acoustic metric $M^{\mu\nu}$

In strong (or weak) $\boldsymbol{B}$ case

$V_{\rm A}^2 \gg c_s^2$ or $V_{\rm A}^2 \ll c_s^2$

magnetic pressure $\sim V_{\rm A}^2$

gas pressure $\sim c_s^2$

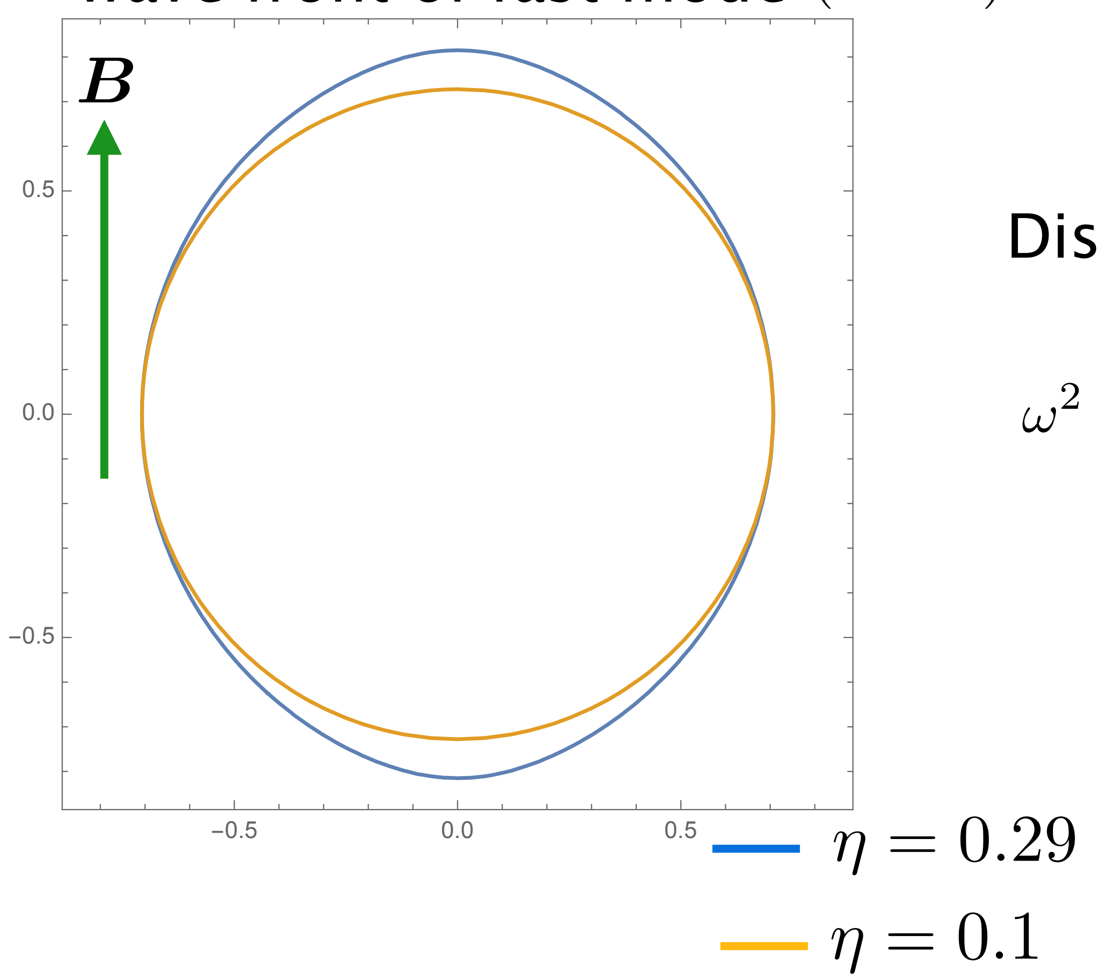
Alfvén velocity

$$\boldsymbol{V}_{\rm A} = \frac{1}{\sqrt{4\pi\rho}}\boldsymbol{B}$$

sound velocity

$$c_s = \sqrt{\partial p/\partial \rho}$$

wave front of fast mode $(\boldsymbol{v}=0)$



Dispersion relation of fast mode

$$\omega^2 = \frac{V_{\rm M}^2(k_x^2+k_y^2)}{2}\left[1+\sqrt{1-4\eta\left(\frac{\boldsymbol{b}\cdot\boldsymbol{k}}{k}\right)^2}\right] \qquad \eta \equiv \left(\frac{c_s V_{\rm A}}{V_{\rm M}^2}\right)^2 \ll 1$$

Almost isotropic propagation

of MHD waves (fast mode) for small η

Reduced Magneto-acoustic metric

- We solve the eikonal eq. for $\left(\frac{\partial S}{\partial t} + \boldsymbol{v}\cdot\nabla S\right)^2$

$\boldsymbol{b} = \boldsymbol{B}/B$

$$\left(\frac{\partial S}{\partial t} + \boldsymbol{v}\cdot\nabla S\right)^2 = \frac{V_{\rm M}^2\,|\nabla S|^2}{2}\left[1 \pm \sqrt{1 - 4\left(\frac{c_s\,V_{\rm A}}{V_{\rm M}^2}\right)^2\left(\frac{\boldsymbol{b}\cdot\nabla S}{|\nabla S|}\right)^2}\,\right]$$

+ fast mode

− slow mode

$V_{\rm M}^2 \equiv V_{\rm A}^2 + c_s^2$

$$\eta \equiv \left(\frac{c_s\,V_{\rm A}}{V_{\rm M}^2}\right)^2 \ll 1 \quad \longrightarrow \quad \sqrt{} \quad \text{can be expanded}$$

- Two eikonal eqs.

$$\left(\frac{\partial S}{\partial t} + \boldsymbol{v}\cdot\nabla S\right)^2 \approx \begin{cases} V_{\rm M}^2\left(|\nabla S|^2 - \eta\,(\boldsymbol{b}\cdot\nabla S)^2\right) & \text{fast mode} \\ \eta\,V_{\rm M}^2\,(\boldsymbol{b}\cdot\nabla S)^2 & \text{slow mode} \end{cases}$$

assign the coefficients →

$$M^{\mu\nu}_{\rm fast}\,\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} = 0 \quad , \quad M^{\mu\nu}_{\rm slow}\,\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} = 0$$

Are these “metrics” ?

$$M^{\mu\nu}_{\rm fast} = \begin{pmatrix} -1 & -v^i \\ -v^i & V_{\rm M}^2\,\delta^{ij} - \left(v^i\,v^j + \eta\,V_{\rm M}^2\,b^i b^j\right) \end{pmatrix}, \quad M^{\mu\nu}_{\rm slow} = \begin{pmatrix} -1 & -v^i \\ -v^i & -\left(v^i\,v^j - \eta\,V_{\rm M}^2\,b^i b^j\right) \end{pmatrix}, \quad i,j = 1,2$$

Fast mode $M^{\mu\nu}_{\rm fast}$

$M^{\mu\nu}_{\rm fast}$ has the inverse $(M_{\rm fast})_{\mu\nu}$ → We can define the inner product

Distance $ds^2_{\rm fast} = (M_{\rm fast})_{\mu\nu} dx^\mu dx^\nu$

- Magneto-acoustic line element

$$ds^2_{\rm fast} \propto -\left[(V_{\rm M}^2 - \boldsymbol{v}^2) - \eta\,(\boldsymbol{b}\cdot\boldsymbol{v})^2\right] dt^2 - 2\left[v_i + \eta\, b_i(\boldsymbol{b}\cdot\boldsymbol{v})\right] dt\, dx^i + (\delta_{ij} + \eta\, b_i b_j)\, dx^i dx^j$$

$$dt \to dt + \frac{v^R}{V_{\rm M}^2 - \eta\, V_{\rm M}^2 (b^R)^2 - (v^R)^2} dR \quad , \quad d\phi \to d\phi + \frac{v^R\, v^\phi + \eta\, b^R\, b^\phi\, V_{\rm M}^2}{V_{\rm M}^2 - \eta\, V_{\rm M}^2\,(b^R)^2 - (v^R)^2}\frac{dR}{R}$$

→

$$ds^2_{\rm fast} \propto -\left[(V_{\rm M}^2 - \boldsymbol{v}^2) - \eta\,(\boldsymbol{b}\cdot\boldsymbol{v})^2\right] dt^2 - 2\left[v^\phi + \eta\, b^\phi(\boldsymbol{b}\cdot\boldsymbol{v})\right] R\, dt\, d\phi + \frac{dR^2}{1 - \eta\,(b^R)^2 - (v^R/V_{\rm M})^2} + \left[1 + \eta\,(b^\phi)^2\right] R^2 d\phi^2$$

- Magneto-acoustic horizon & ergosurface

Magneto-acoustic horizon

$$(v^R)^2 = V_{\rm M}^2 - \eta V_{\rm M}^2 (b^R)^2$$

Propagation speed in the radial direction

Magneto-acoustic ergosurface

$$\boldsymbol{v}^2 = V_{\rm M}^2 - \eta(\boldsymbol{b}\cdot\boldsymbol{v})^2$$

Slow mode

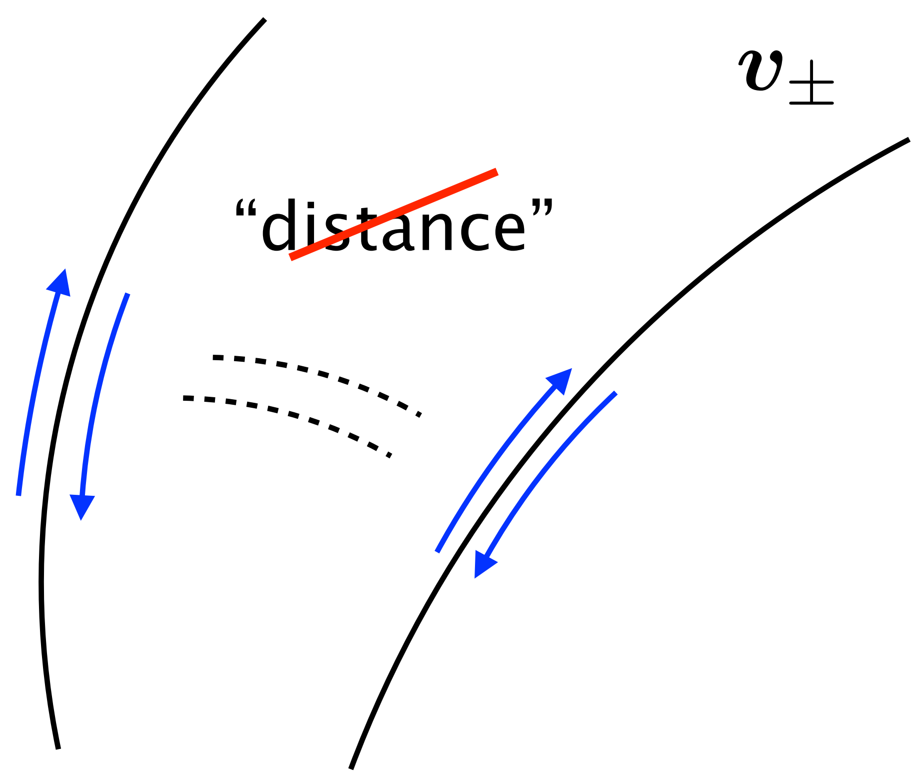
$M_{\rm slow}^{\mu\nu}$ no inverse ⇒ no inner pruduct

The eikonal eq. for the slow mode is advective equation

$$\frac{\partial S}{\partial t} + \boldsymbol{v}_{\pm} \cdot \nabla S = 0 \quad , \quad \boldsymbol{v}_{\pm} \equiv \boldsymbol{v} \pm (c_s V_{\rm A}/V_{\rm M})\boldsymbol{b}$$

BG MHD flow

Slow wave mode propagate along $\boldsymbol{v}_{\pm}$



It's impossible to introduce acoustic metric for the propagation of the slow mode.

2D Background MHD inflow

Analog rotating black holes in a MHD inflow

2D Axisymmetric MHD inflow

Basic equations & Conserved quantities

- Ideal MHD eq. (stationary)

$$\nabla \cdot (\rho \boldsymbol{v}) = 0 \quad , \quad \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = 0 \quad , \quad \nabla \cdot \boldsymbol{B} = 0$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \frac{\nabla p}{\rho} + \frac{1}{4\pi\rho}\boldsymbol{B} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{F}_{\rm ex} = 0$$

Equation of state

$$p \propto \rho^{\Gamma} \quad 1 \le \Gamma \le 4$$

axisymmetric inflow

$$\boldsymbol{v} = v^R(R)\, \boldsymbol{e}_{\hat{R}} + v^{\phi}(R)\, \boldsymbol{e}_{\hat{\phi}} \quad , \quad \boldsymbol{B} = B^R(R)\, \boldsymbol{e}_{\hat{R}} + B^{\phi}(R)\, \boldsymbol{e}_{\hat{\phi}}$$

- conserved quantities

$$\rho\, v^R R = \underline{\underline{\text{const.}}} \; < 0 \quad , \quad R B^R = \underline{\underline{\text{const.}}} \; > 0$$

inflow, outward

$$R\, v^{\phi} - \frac{B^R}{4\pi\rho\, v^R}\, R B^{\phi} = \text{const.} \equiv \underline{\underline{L}}$$

angular momentum

$$v^R B^{\phi} - v^{\phi} B^R = \text{const.} \equiv -\underline{\underline{\Omega_{\rm F}}}\, R\, B^R$$

angular velocity of the magnetic field lines

Can our BG flow satisfy η ≪ 1 ?

For entire region,

$$\eta \equiv \left(\frac{c_s V_{\rm A}}{V_{\rm M}^2}\right)^2 \ll 1$$

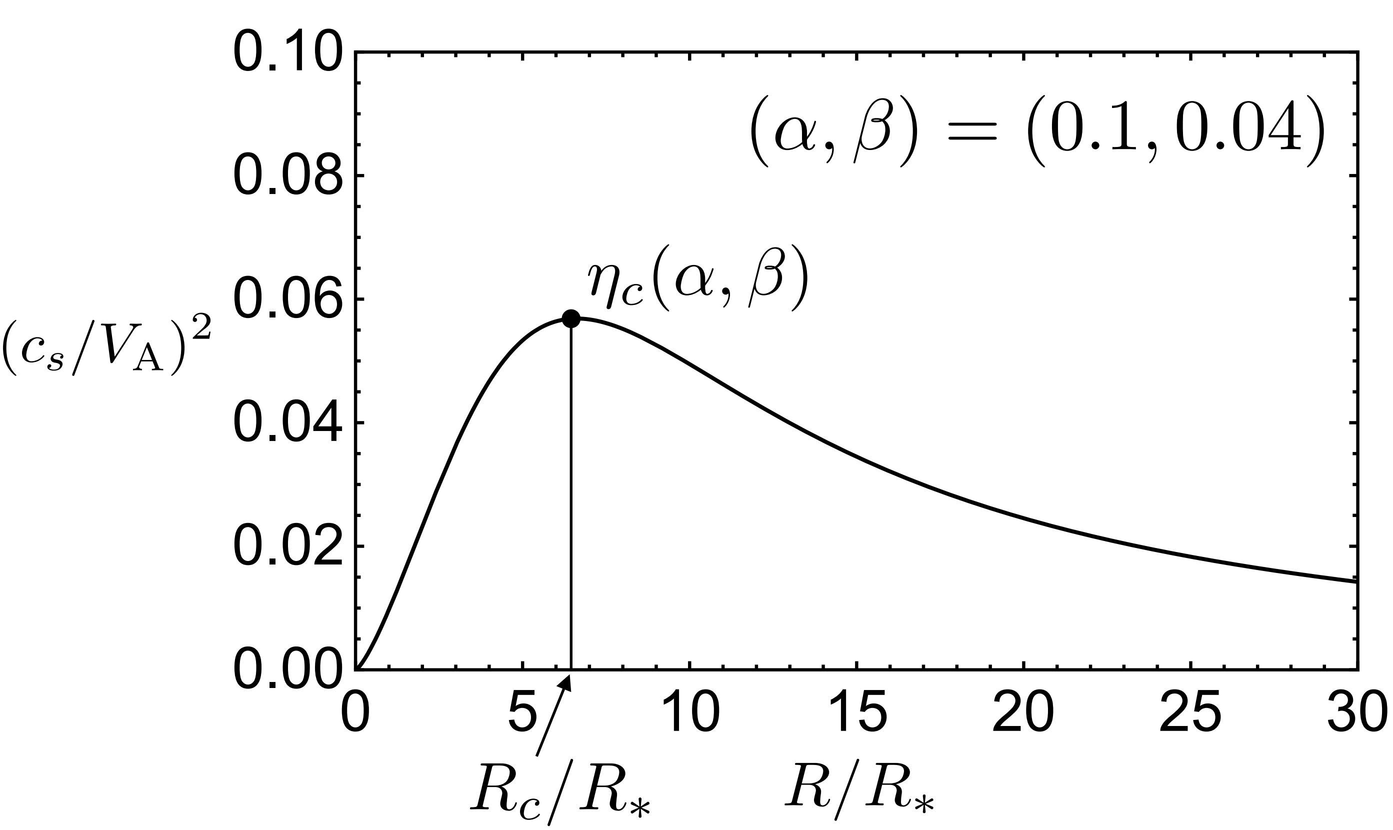
Condition for reduction of the magneto-acoustic metric

$$\left(\frac{\partial S}{\partial t} + \boldsymbol{v}\cdot\nabla S\right)^2 = \frac{V_{\rm M}^2 |\nabla S|^2}{2}\left[1 \pm \sqrt{1 - 4\left(\frac{c_s V_{\rm A}}{V_{\rm M}^2}\right)^2 \left(\frac{\boldsymbol{b}\cdot\nabla S}{|\nabla S|}\right)^2}\right]$$

mag-pressure dominated $V_{\rm A}^2 \gg c_s^2$

gas-pressure dominated $c_s^2 \gg V_{\rm A}^2$

Which is allowed for our BG flow ?



Near R=0 and far region, $V_{\rm A}^2 \gg c_s^2$

Only Mag-pressure dominated flow can be realize for BG flow under $\eta \ll 1$

$$\eta \equiv \left(\frac{c_s V_{\rm A}}{V_{\rm M}^2}\right)^2 \ll 1 \quad \Longrightarrow \quad \eta_c(\alpha, \beta) \ll 1$$

Brief summary so far..

① $\eta \equiv \left(\frac{c_s V_A}{V_M^2} \right)^2 \ll 1$ (Riemannian) Magneto-acoustic metric for the fast mode (coefficients of the eikonal eq.)

Magneto-acoustic metric

$$ds^2_{\text{fast}} \propto -\left[(V_M^2 - \boldsymbol{v}^2) - \eta\,(\boldsymbol{b}\cdot\boldsymbol{v})^2\right] dt^2 - 2\left[v^\phi + \eta\, b^\phi (\boldsymbol{b}\cdot\boldsymbol{v})\right] R\,dt\,d\phi + \frac{dR^2}{1 - \eta\,(b^R)^2 - (v^R/V_M)^2} + \left[1 + \eta\,(b^\phi)^2\right] R^2 d\phi^2$$

② Analog horizon & ergoregion for the fast wave mode

Magneto-acoustic horizon

$$(v^R)^2 = V_M^2 - \eta V_M^2 (b^R)^2$$

Magneto-acoustic ergosurface

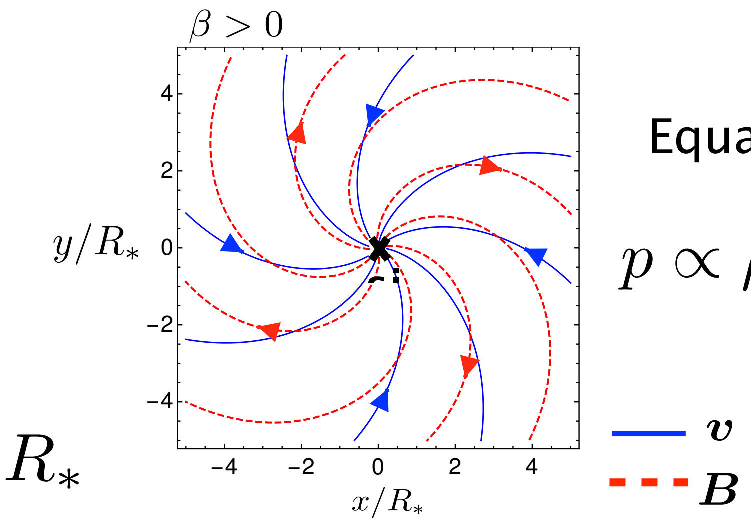
$$\boldsymbol{v}^2 = V_M^2 - \eta(\boldsymbol{b}\cdot\boldsymbol{v})^2$$

③ Background flow Magnetic-pressure dominated

Flow & wave velocities are characterized by (α, β)

$\alpha \equiv -c_s(R_*)/v_*^R > 0$ ~sound velocity @ R_*

$\beta \equiv \Omega_F R_*/v_*^R$ ~angular velocity of B @ R_*



Equation of state

$p \propto \rho^\Gamma$ $1 \le \Gamma \le 4$

Magneto-acoustic geometry

How do we examine?

Motion of magneto-acoustic ray

$$k^\mu \equiv \frac{dx^\mu}{d\lambda} = M^{\mu\nu}_{\rm fast} \frac{\partial S}{\partial x^\nu}, \quad (M_{\rm fast})_{\mu\nu} k^\mu k^\nu = 0$$

tangent vector

separation of variables

$$S = -\omega t + m\phi + S_R(R)$$

- Radial part

$$\left(\frac{dR}{d\lambda}\right)^2 = \frac{1}{V_{\rm A}^2}\left[1 - \eta\,(b^R)^2\right](\omega - V^+)(\omega - V^-)$$

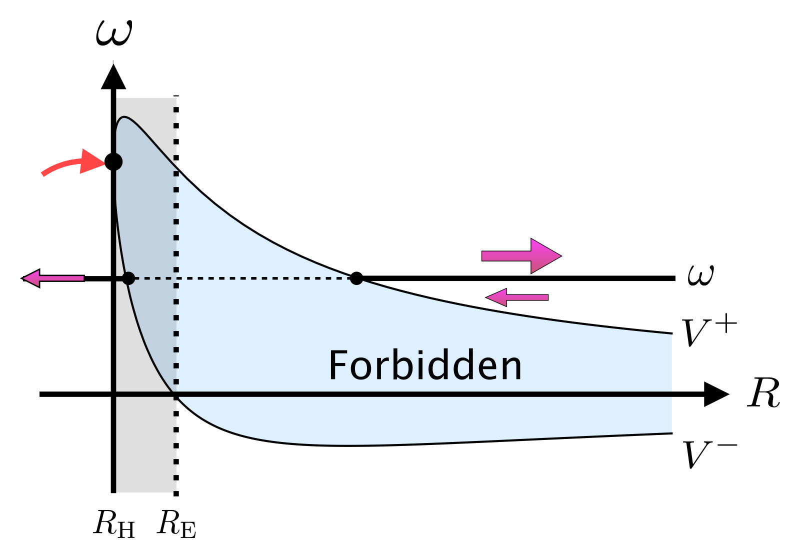
Forbidden region for RAYS

$$V^- \leq \omega \leq V^+$$

- Effective potentials

$$V^\pm = m\frac{-(M_{\rm fast})_{t\phi} \pm \sqrt{(M_{\rm fast})^2_{t\phi} - (M_{\rm fast})_{\phi\phi}(M_{\rm fast})_{tt}}}{(M_{\rm fast})_{\phi\phi}}$$

$$= \frac{m}{R}\,\frac{v^\phi + \eta b^\phi(\boldsymbol{b}\cdot\boldsymbol{v}) \pm \sqrt{V_{\rm M}^2 - (v^R)^2 - \eta\left[(v^R)^2 - (b^\phi)^2 V_{\rm M}^2\right]}}{1 + \eta\,(b^\phi)^2}$$

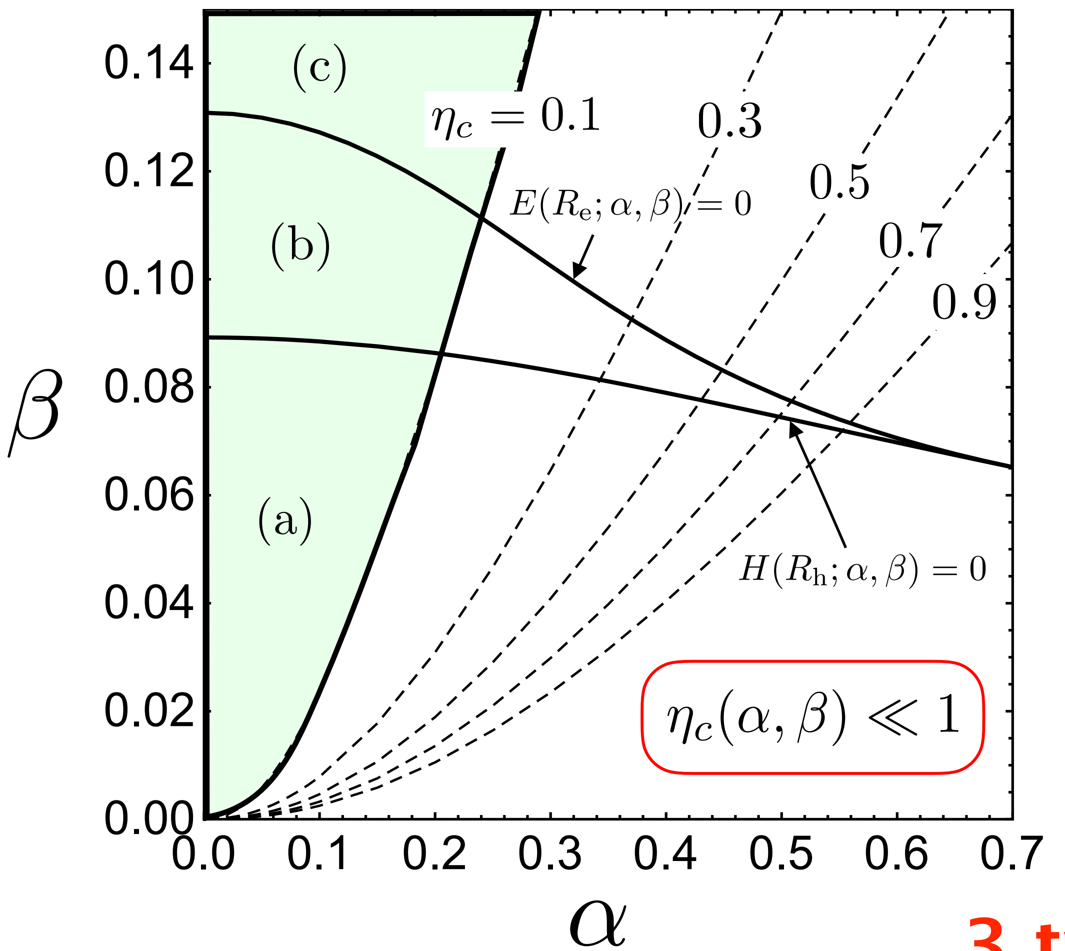


$\sqrt{\ }$ =0 Magneto-acoustic horizon

V^- =0 Magneto-acoustic ergosurface

Depending on (α,β), these condition

Classification of Magneto-acoustic geometry



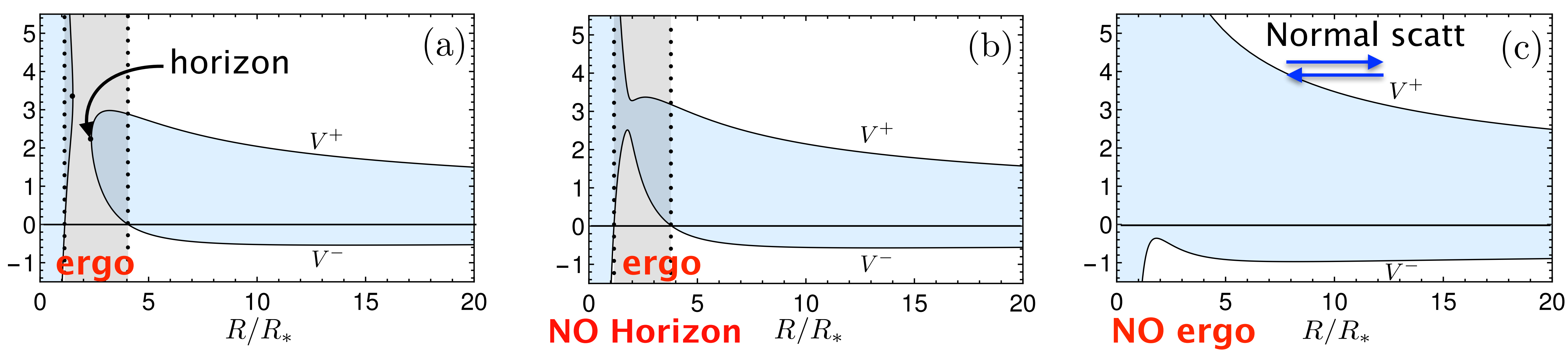
$\alpha \equiv -c_s(R_*)/v_*^R > 0$ ~sound velocity @ R_*

$\beta \equiv \Omega_F R_*/v_*^R$ ~Angular velocity of B @ R_*

Effective potential for rays

$$V^{\pm} = m\frac{-(M_{\rm fast})_{t\phi} \pm \sqrt{(M_{\rm fast})_{t\phi}^2 - (M_{\rm fast})_{\phi\phi}(M_{\rm fast})_{tt}}}{(M_{\rm fast})_{\phi\phi}}$$

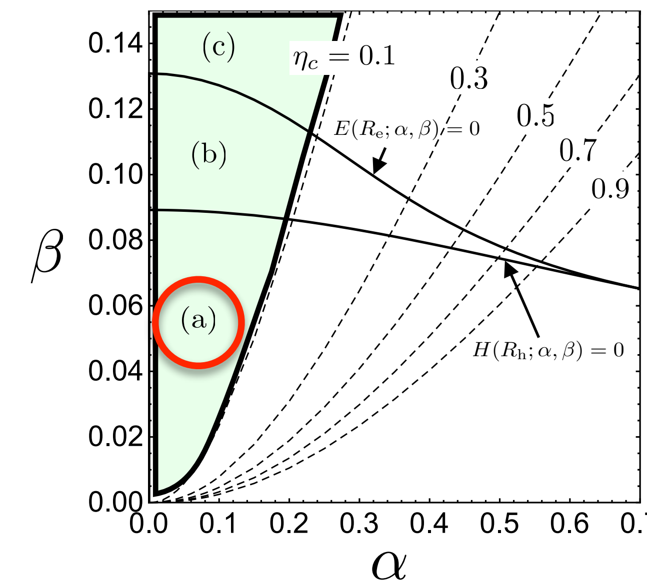
3 types of geometry



Rotating BH — Star with the ergoregion — Star

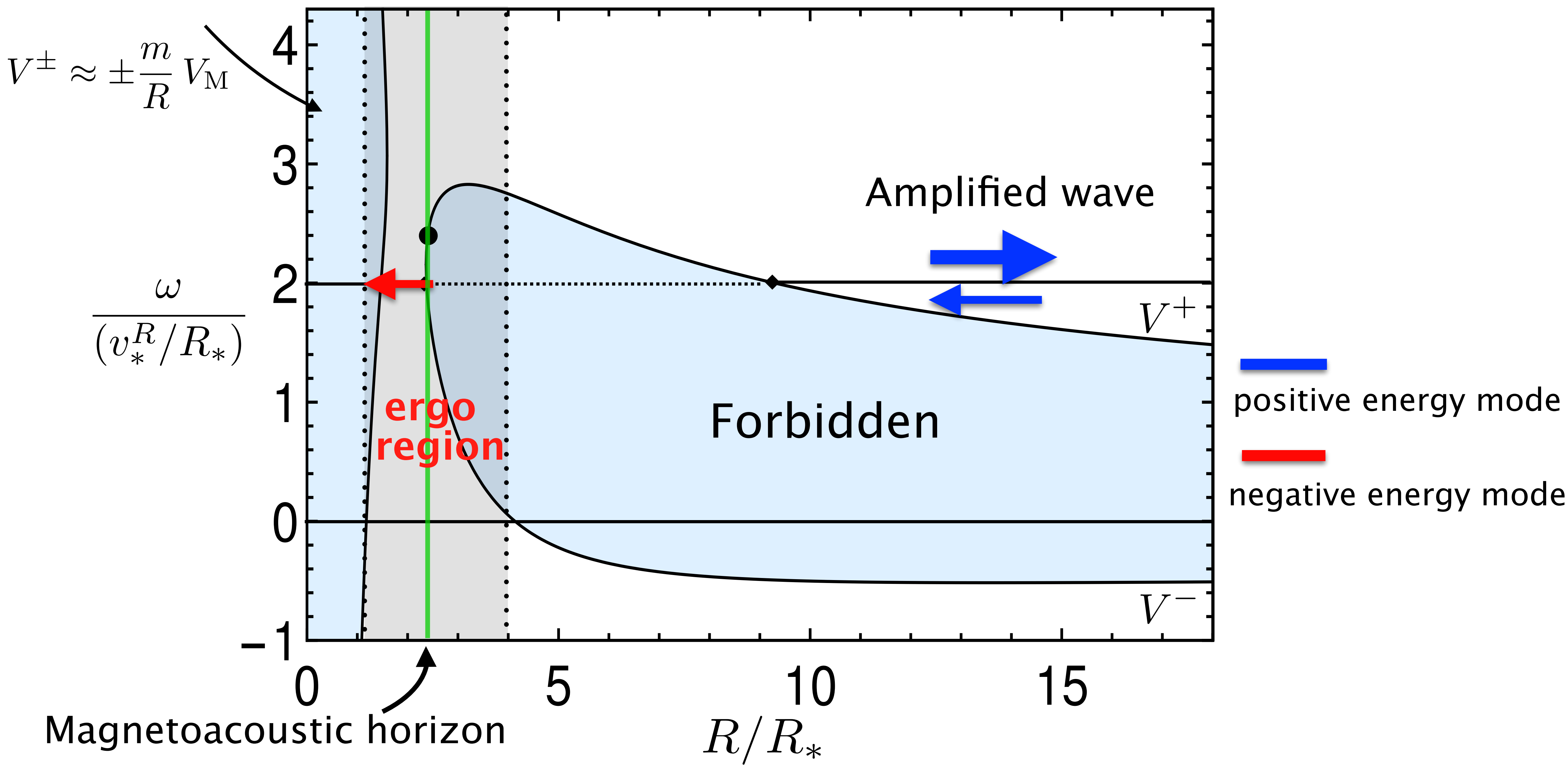
Type (a) Analogue rotating black hole

Superradiance for MHD waves



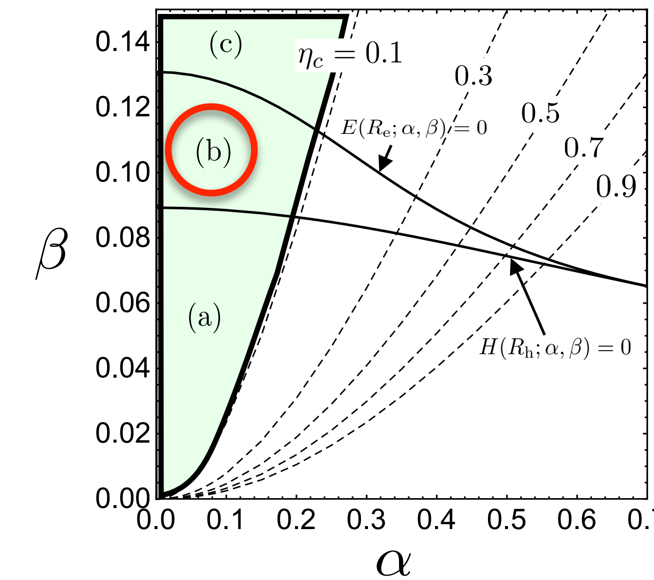
The magnetic pressure

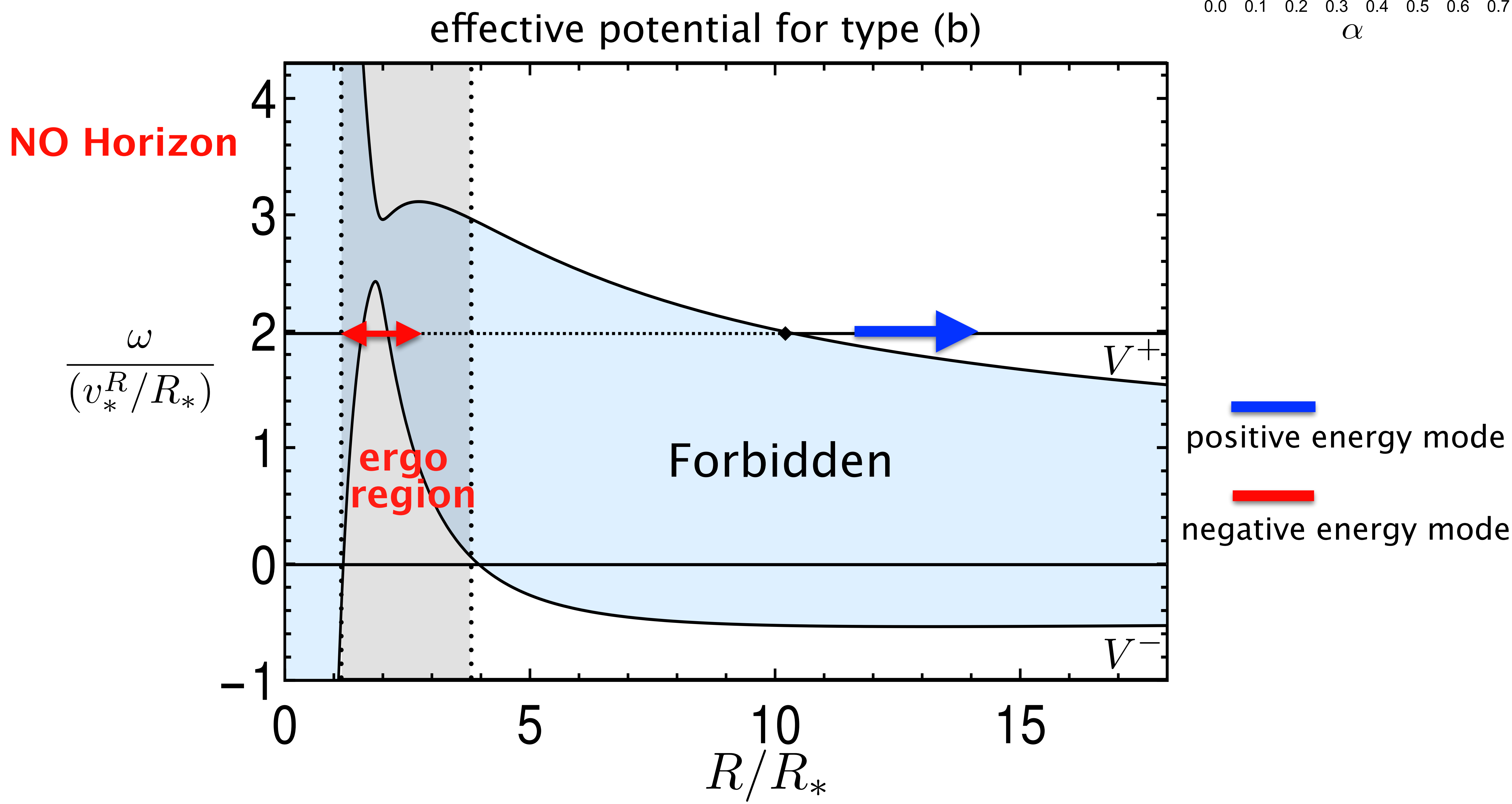
effective potential for type (a)



Type (b) Ultraspinning star

Ergoregion instability by MHD waves scattering





Applications (Future work)

POINT

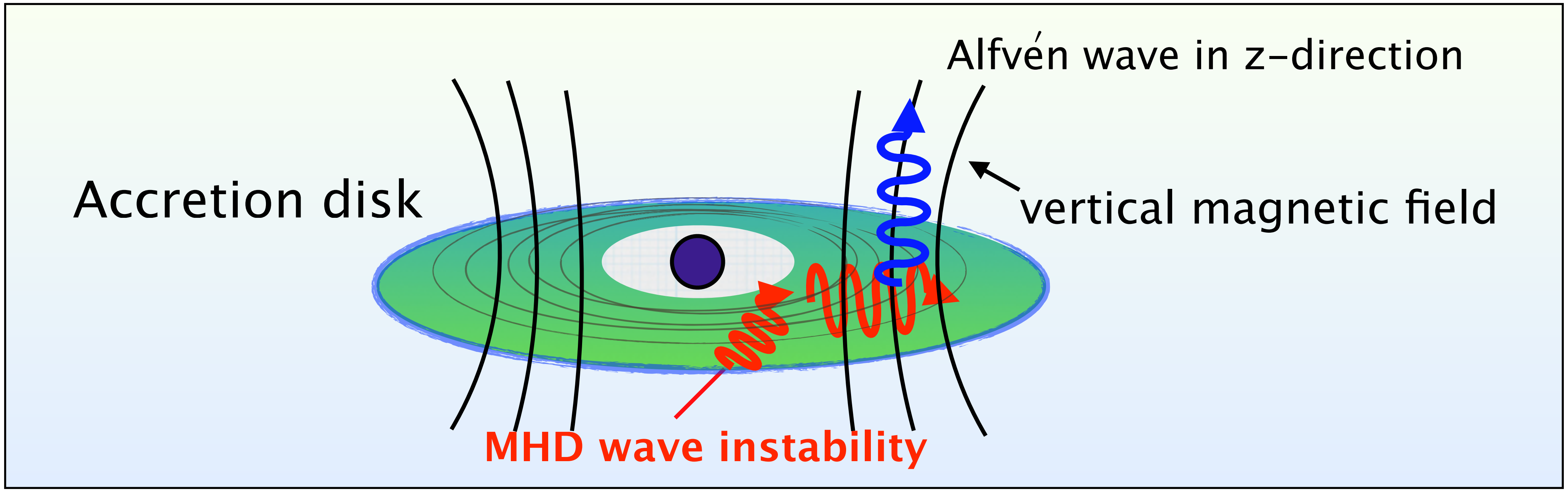
Magneto-acoustic horizon

$$(v^R)^2 = V_M^2 - \eta V_M^2 (b^R)^2$$

Magneto-acoustic ergosurface

$$\boldsymbol{v}^2 = V_M^2 - \eta (\boldsymbol{b} \cdot \boldsymbol{v})^2$$

Magneto-acoustic BH in astrophysical situations



- Central object does not have to be a BH — MHD Flow makes analog BH
- MHD-analog superradance or ergoregion instability — MHD Wave around a star
- QNMs of a Magneto-acoustic BH — peculiar motion of accretion disk ?

Radial Alfvén point

- Regularity condition

$$v^R B^\phi - v^\phi B^R = \text{const.} \equiv -\underline{\underline{\Omega_F}}\, R\, B^R$$

$$R\, v^\phi - \frac{B^R}{4\pi\rho\, v^R}\, R B^\phi = \text{const.} \equiv \underline{\underline{L}}$$

$\Rightarrow$

$$v^\phi = \Omega_F\, R\, \frac{M_A^2\, L\, R^{-2}\, \Omega_F^{-1} - 1}{M_A^2 - 1}$$

$$B^\phi = \frac{\Omega_F R B^R}{v^R} \frac{M_A^2(\underline{L R^{-2} - \Omega_F^{-1}})}{M_A^2 - 1}$$

When Radial Mach number

$M_A^2(R) \equiv \left(\frac{v^R}{V_A^R}\right)^2$ becomes unit $R = R_*$, v^ϕ and B^ϕ are singular.

Radial Alfvén point

$$\boxed{L = \Omega_F R_*^2}$$

Regularity condition @ R_*

$\Rightarrow$

$$v^\phi = \frac{\Omega_F\, R_*}{v_*^R} \frac{v^R - (R/R_*) v_*^R}{M_A^2 - 1}$$

$$B^\phi = -\frac{B^R\, \Omega_F}{v_*^R R_*} \frac{R^2 - {R_*}^2}{M_A^2 - 1}$$

RHS is written by v^R and B^R

$RB^R = \text{const.} \;\; > 0$

$\longrightarrow$ We just solve v^R.

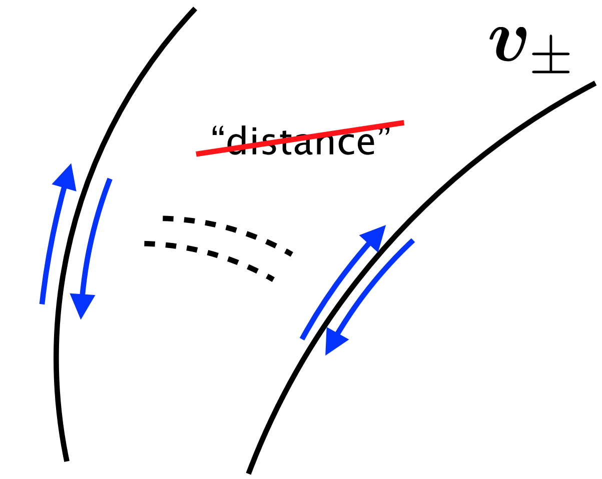
Summary

- Magneto-acoustic metric for the fast mode $\eta \equiv \left(\frac{c_s V_A}{V_M^2}\right)^2 \ll 1$

Magneto-acoustic metric

$$ds^2_{\rm fast} \propto -\left[(V_M^2 - \boldsymbol{v}^2) - \eta\,(\boldsymbol{b}\cdot\boldsymbol{v})^2\right] dt^2 - 2\left[v^\phi + \eta\, b^\phi(\boldsymbol{b}\cdot\boldsymbol{v})\right] R\,dt\,d\phi + \frac{dR^2}{1-\eta\,(b^R)^2 - (v^R/V_M)^2} + \left[1+\eta\,(b^\phi)^2\right] R^2 d\phi^2$$

no metric for slow mode



- Analog horizon & ergoregion for the fast wave mode

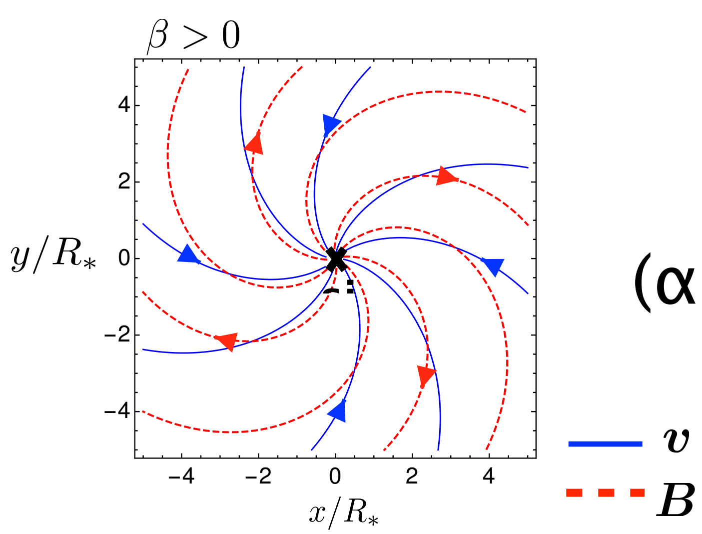
Magneto-acoustic horizon

$$(v^R)^2 = V_M^2 - \eta V_M^2 (b^R)^2$$

Magneto-acoustic ergosurface

$$\boldsymbol{v}^2 = V_M^2 - \eta(\boldsymbol{b}\cdot\boldsymbol{v})^2$$

- Analog superradiance & ergoregion instability for MHD waves



(α, β)

Analog	wave phenomena
rotating BHs	MHD wave superradiance
ultraspinning star	ergoregion instability
ratating star	normal scatteing

BACK UP

Equation for v^R

Euler's equation

$$v^R \frac{dv^R}{dR} - \frac{v^\phi}{R} + \frac{1}{\rho}\frac{dp}{dR} + \frac{1}{4\pi\rho}\frac{B^\phi}{R}\frac{d}{dR}(RB^\phi) + F_{\rm ex}^R = 0$$

← Eq. for v^R and $F_{\rm ex}^R$

① Give $F_{\rm ex}^R$ and solve this equation for v^R

✓ ② There exists one $F_{\rm ex}^R$ for a given v^R → draining bathtub type inflow

$$v^R = v_*^R \left(\frac{R}{R_*}\right)^{-1/2}$$

Background flow & propagation velocities

- Background MHD flow

$$v^R = v_*^R \left(\frac{R}{R_*}\right)^{-1/2}, \quad v^\phi = -\beta\, v_*^R \left[\left(\frac{R}{R_*}\right)^{1/2} + \left(\frac{R}{R_*}\right)^{-1/2} + 1\right]$$

$$B^R = B_*^R \frac{R_*}{R}, \quad B^\phi = -\beta B_*^R \left[1 + \left(\frac{R}{R_*}\right)^{1/2}\right]\left[1 + \left(\frac{R}{R_*}\right)^{-1}\right]$$

Equation of state

$$p \propto \rho^\Gamma \quad 1 \le \Gamma \le 4$$

$$\boxed{\beta \equiv \Omega_F R_* / v_*^R}$$

~ang velo of B @ R_*

- Wave velocities

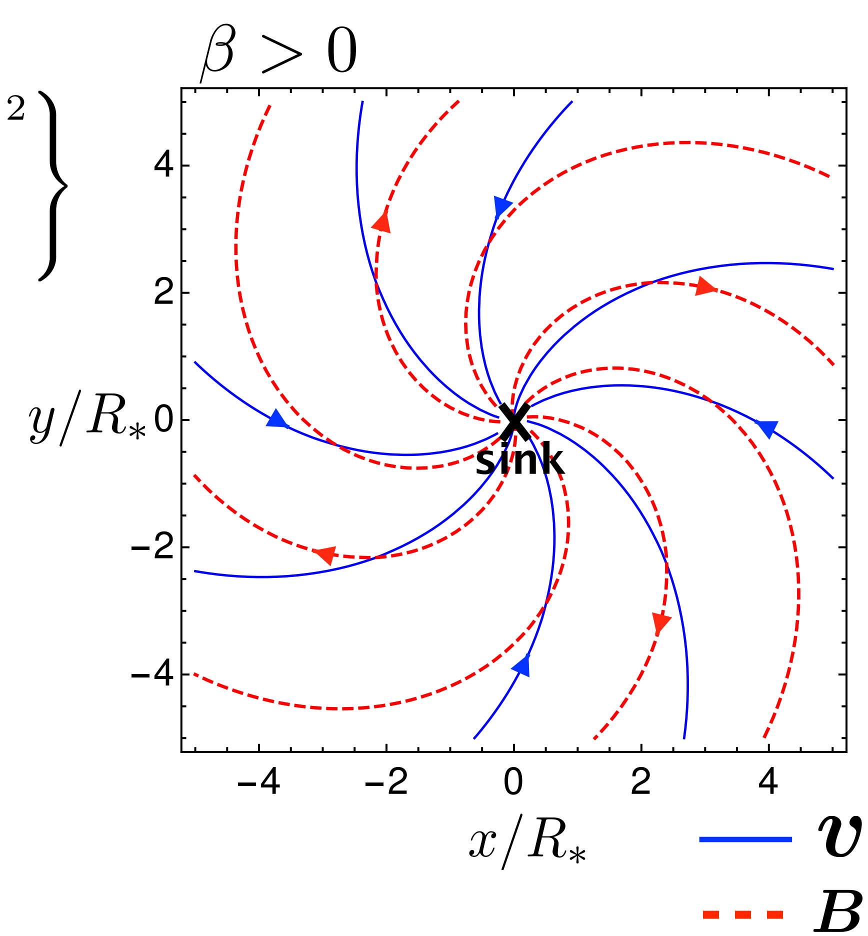
Alfvén velocity

$$V_A^2 = (v_*^R)^2 \left\{ \left(\frac{R}{R_*}\right)^{-3/2} + \beta^2 \left(\frac{R}{R_*}\right)^{1/2} \left[1 + \left(\frac{R}{R_*}\right)^{1/2}\right]^2 \left[1 + \left(\frac{R}{R_*}\right)^{-1}\right]^2 \right\}$$

$$c_s^2 = \alpha^2 (v_*^R)^2 \left(\frac{R}{R_*}\right)^{-(\Gamma-1)/2}, \quad V_M^2 = V_A^2 + c_s^2$$

$$\boxed{\alpha \equiv -c_s(R_*)/v_*^R > 0}$$

~sound velo @ R_*



BG flow & Velocies are characterized by α and β

Acoustic superradiance

- Wave equation for the acoustic disturbance

$$\frac{1}{\sqrt{-s}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-s}s^{\mu\nu}\frac{\partial\delta\Phi}{\partial x^\nu}\right)=0$$

Separation of variable

$$\delta\Phi=\frac{\psi(R)}{R^{(7-\Gamma)/16}}e^{i(-\omega\,t+m\,\phi)}$$

$$m=\pm 0,\quad \pm 1,\cdots$$

radial eq.

$$-\frac{d^2\psi}{dR_{\rm tort}^2}+V_{\rm eff}(R;\omega,m,\Gamma)\psi=0$$

effective potential

$$V_{\rm eff}=-\frac{1}{c_s^2}\left(\omega-\frac{m\,v^\phi}{R}\right)^2-g(R)\left[\frac{(7-\Gamma)(9+\Gamma)}{4096}\frac{g(R)}{R^2}-\frac{7-\Gamma}{16R}\frac{dg(R)}{dR}-\frac{m^2}{R^2}\right]$$

$$g(R) = 1-(v^R/c_s)^2$$

$$\partial/\partial R_{\rm tort}=g(R)\cdot\partial/\partial R$$

WKB sol

$$\psi=\begin{cases}\exp\left(-i\int^{R_{\rm tort}}\frac{1}{c_s}\left(\omega-\frac{mv^\phi}{R}\right)dR_{\rm tort}\right) & \text{for}\quad R\sim R_{\rm H}\\ C_{\rm in}\,\exp\left(-i\int^{R_{\rm tort}}\frac{\omega}{c_s}dR_{\rm tort}\right)+C_{\rm out}\,\exp\left(i\int^{R_{\rm tort}}\frac{\omega}{c_s}dR_{\rm tort}\right) & \text{for}\quad R\sim R_{\rm f}\gg R_H\end{cases}$$

Conservation of Wronskian

scattering with amplification $\left|\frac{C_{\rm out}}{C_{\rm in}}\right|>1$

$$\left|\frac{C_{\rm out}}{C_{\rm in}}\right|^2+\frac{c_s(R_{\rm H})}{c_s(R_{\rm f})}\cdot\frac{\omega-m\,\Omega_{\rm H}}{\omega}\left|\frac{1}{C_{\rm in}}\right|^2=1$$

reflection rate

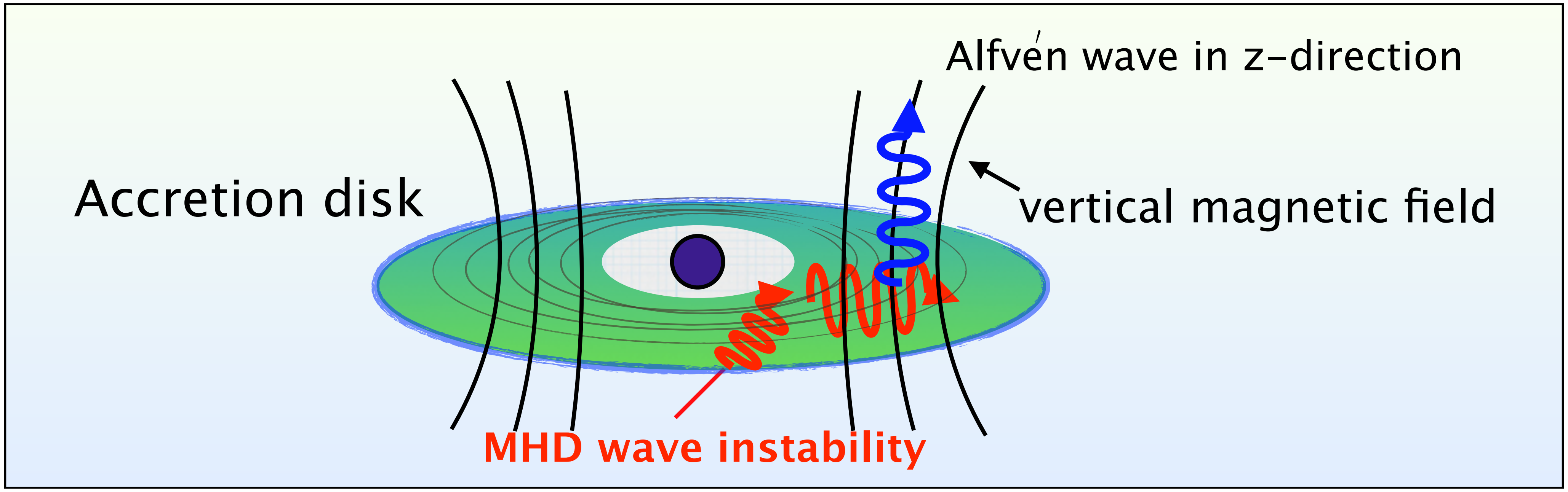
transmission rate

Superradiant condition

$$\omega<m\,\Omega_{\rm H}=m\frac{v^\phi(R_{\rm H})}{R_{\rm H}}$$

Future work ①

- The magnetoacoustic BHs in 3D, Astrophysical situation



- Magneto-acoustic BH (or geometry)を考える上では、中心天体はBHでなくても良い。
- MHD-superradanceやergoregion instabilityとJETなどの天体現象は？
- ３次元にすると、Alfven waveも出てくる。

MHD wave方程式

$$\frac{D^2 \delta \boldsymbol{v}}{Dt^2} - c_s^2 \nabla(\nabla \cdot \delta \boldsymbol{v}) + \boldsymbol{V}_{\mathrm{A}} \times \nabla \times (\nabla \times (\delta \boldsymbol{v} \times \boldsymbol{V}_{\mathrm{A}})) = 0 \quad \Longrightarrow \quad \begin{aligned} \omega_{\mathrm{f,s}} &= \omega(k)_{\mathrm{f,s}} \\ \omega_{\mathrm{Al}} &= \omega(k)_{\mathrm{Al}} \end{aligned}$$

Future work ③

eikonalでは波の増幅率等は計算できない

完全流体の場合

音波方程式 → $$\frac{1}{\sqrt{-s}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-s}s^{\mu\nu}\frac{\partial\delta\Phi}{\partial x^\nu}\right)=0$$

速度ポテンシャル

MHDの場合

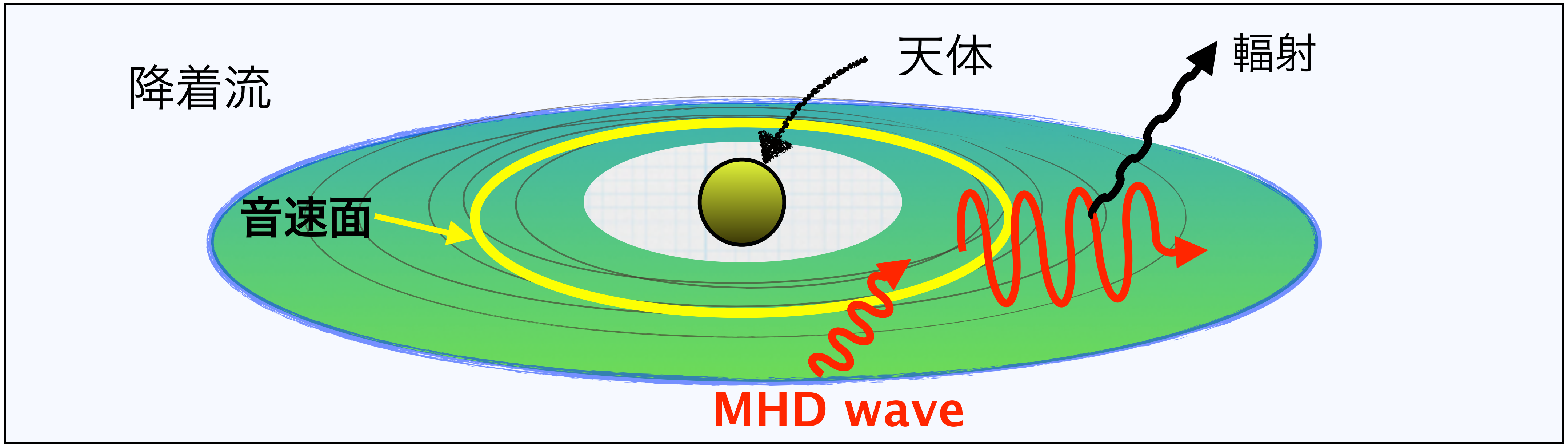
MHD wave方程式 → KG型 ????　　$\boldsymbol{B}=\nabla\alpha\times\nabla\beta$

磁気優勢 or 流体優勢　　Clebsch ポテンシャルを使う？

もしできたら。。。

Magneto-acoustic BH の Quasi Normal Modes や Bombの計算

$\omega_{\rm QNM}$ や $\omega_{\rm Bomb}$ との輻射の関係は？　円盤振動学？



疑問点、質問

MHD flowをAcoustic BHとして見てみると、不安定性が起こるか否か

$$(v^R)^2 = V_{\rm M}^2 - \eta V_{\rm M}^2 (b^R)^2$$ Magneto-acoustic horizon

(MHD waveが動径方向に出ようとしても出られない面)

$$\boldsymbol{v}^2 = V_{\rm M}^2 - \eta(\boldsymbol{b}\cdot\boldsymbol{v})^2$$ Magneto-acoustic ergosurface

Background flowがこれらを満たすかで決まった。（少なくとも今回のflowでは）

horizonやergoの存否

質問

流体の方程式をそのまま解いて、MHDの計算をする場合にも
上記の条件は重要なものなのか？　不安定性の条件は？

疑問

Magneto-ergoregion instabiliry ⟷ MRIなどの不安定性

Analogue BHの不安定性はBG flowの不安定性

Superradiant condition

Klein-Gordon eq. (WAVE)

$$\Box_s \delta\Phi = 0 \quad , \quad \delta\phi = \psi(R) e^{-i\omega t} e^{im\phi}$$

$$\frac{d\psi}{dR_{\mathrm{tort}}} + (\omega^2 - V_{\mathrm{eff}})\psi = 0$$

Asymptotic sol

$$\psi = \begin{matrix} e^{-i\left(\omega - \frac{mv^\phi}{R_{\mathrm{H}}}\right) R_{\mathrm{tort}}} \\ C_{\mathrm{in}} e^{-i\omega R_{\mathrm{tort}}} + C_{\mathrm{out}} e^{i\omega R_{\mathrm{tort}}} \end{matrix}$$

Conservation of Wronskian

reflection rate — transmission rate

$$\left| \frac{C_{\mathrm{out}}}{C_{\mathrm{in}}} \right|^2 + \frac{c_s(R_{\mathrm{H}})}{c_s(R_{\mathrm{f}})} \cdot \frac{\omega - m\,\Omega_{\mathrm{H}}}{\omega} \left| \frac{1}{C_{\mathrm{in}}} \right|^2 = 1$$

$$\left| \frac{C_{\mathrm{out}}}{C_{\mathrm{in}}} \right| > 1 \quad \Longleftrightarrow \quad \omega < m\,\Omega_{\mathrm{H}} = m \frac{v^\phi(R_{\mathrm{H}})}{R_{\mathrm{H}}}$$

Superradiant condition in the eikonal limit

Klein–Gordon eq. (WAVE)

$$\Box_s \delta\Phi = 0 \quad , \quad \delta\phi = \psi(R) e^{-i\omega t} e^{im\phi}$$

$$\frac{d\psi}{dR_{\rm tort}} + (\omega^2 - V_{\rm eff})\psi = 0$$

Asymptotic sol

$$\psi = \begin{cases} e^{-i\left(\omega - \frac{m v^\phi}{R_{\rm H}}\right) R_{\rm tort}} \\ C_{\rm in} e^{-i\omega R_{\rm tort}} + C_{\rm out} e^{i\omega R_{\rm tort}} \end{cases}$$

Conservation of Wronskian

reflection rate — transmission rate

$$\left|\frac{C_{\rm out}}{C_{\rm in}}\right|^2 + \frac{c_s(R_{\rm H})}{c_s(R_{\rm f})} \cdot \frac{\omega - m\,\Omega_{\rm H}}{\omega} \left|\frac{1}{C_{\rm in}}\right|^2 = 1$$

$$\left|\frac{C_{\rm out}}{C_{\rm in}}\right| > 1 \quad \Longleftrightarrow \quad \omega < m\,\Omega_{\rm H} = m\frac{v^\phi(R_{\rm H})}{R_{\rm H}}$$

Eikonal eq. (RAY)

$$s^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = 0 \quad , \quad \frac{\partial S}{\partial x^\mu} = s_{\mu\nu} \underline{\frac{dx^\nu}{d\lambda}}$$

tangent vector of RAYS

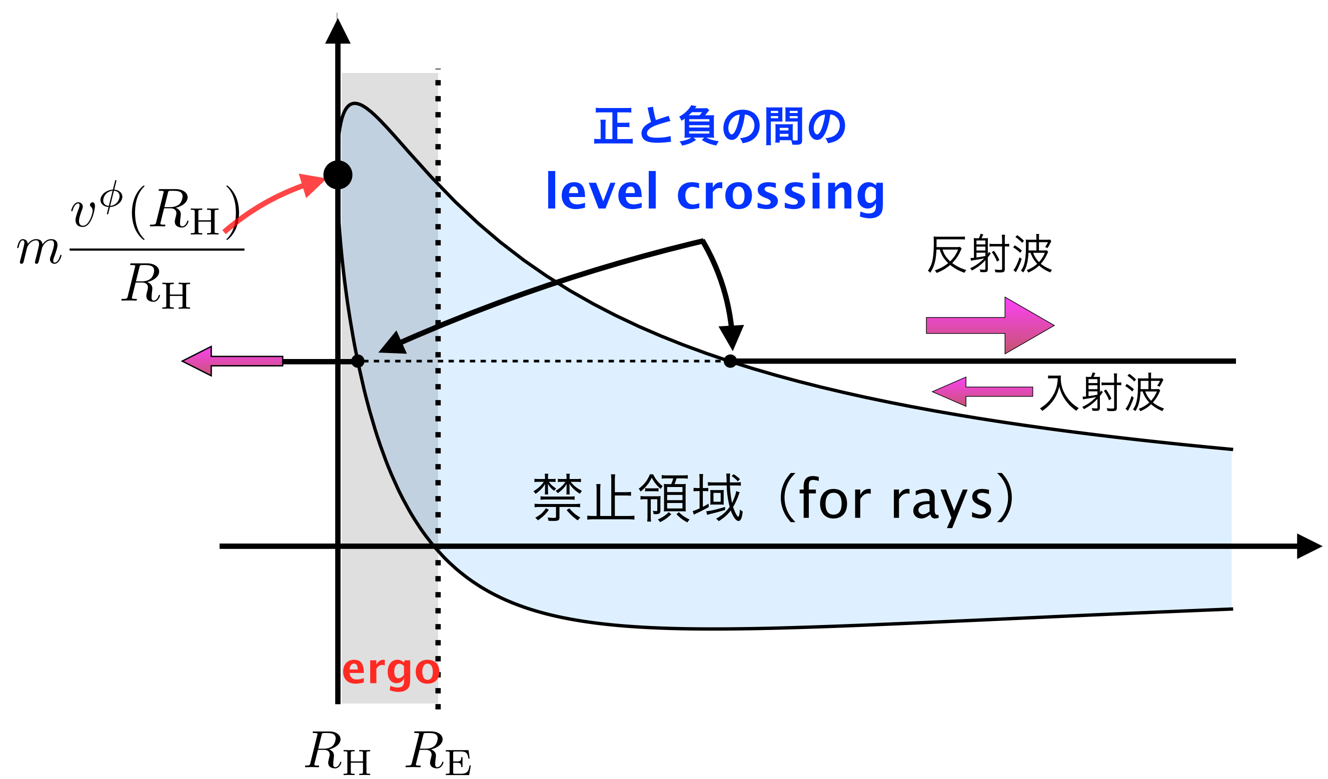
$$S = -\omega\, t + m\, \phi + S_R(R)$$

$$\left(\frac{dR}{d\lambda}\right)^2 = \frac{1}{c_s^2} (\omega - V^+)(\omega - V^-) \geq 0$$

$$V^\pm = m \frac{v^\phi \pm \sqrt{c_s^2 - (v^R)^2}}{R}$$

Forbidden regio

$$V^- \leq \omega \leq V$$



Black Holes

- Solutions of Einstein eq.

Einstein eq. metric

$$G_{\mu\nu} = \kappa T_{\mu\nu} \Rightarrow ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

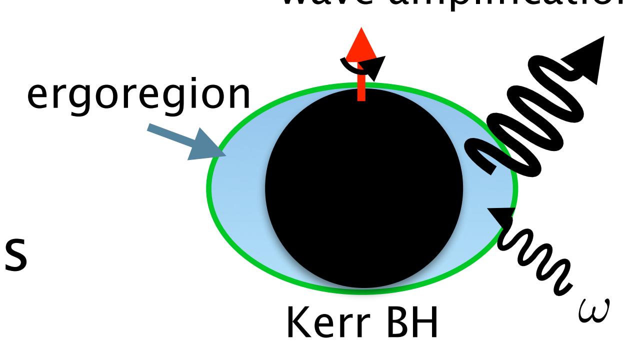
curved spacetime

- Physical Properties

Horizon
Ergoregion (rotating BHs)
Photon sphere
No hair theorem
⋮

- Phenomena

Hawking radiation
Black Hole Shadow
Quasi Normal Modes
Superradiance
⋮



superradiant condition

$$\omega < m\Omega_H$$

Acoustic Black Holes

NO !

Effective geometry for acoustic waves

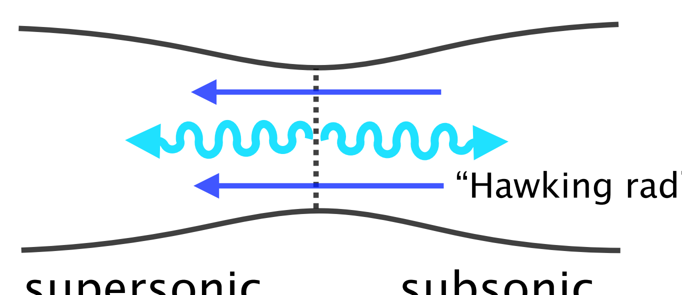
Fluid Dynamics $\rho\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = -\nabla p + \boldsymbol{F}_{\rm ex}$

Partially YES!

Acoustic horizon
Acoustic ergoregion } sonic point
PhoNon sphere
~~Beautiful theorems~~

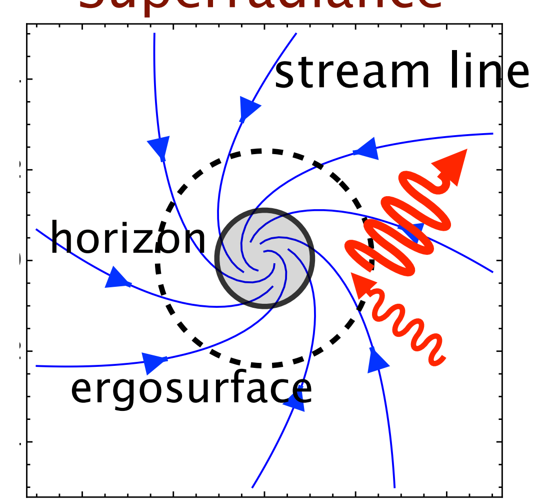
Partially YES

Hawking radiation



theory: Unruh (1981)
experiment: Steinhauer (2016)

Superradiance

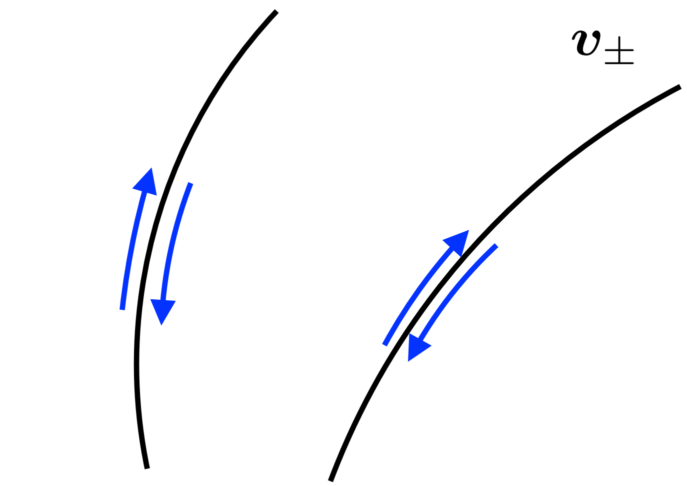


Slow mode

eikonal方程式

$$\frac{\partial S}{\partial t} + \boldsymbol{v}_{\pm} \cdot \nabla S = 0 \ , \quad \boldsymbol{v}_{\pm} = \boldsymbol{v} \pm (c_s V_{\mathrm{A}}/V_{\mathrm{M}})\boldsymbol{b} \approx \boldsymbol{v} \pm c_s \boldsymbol{b}$$

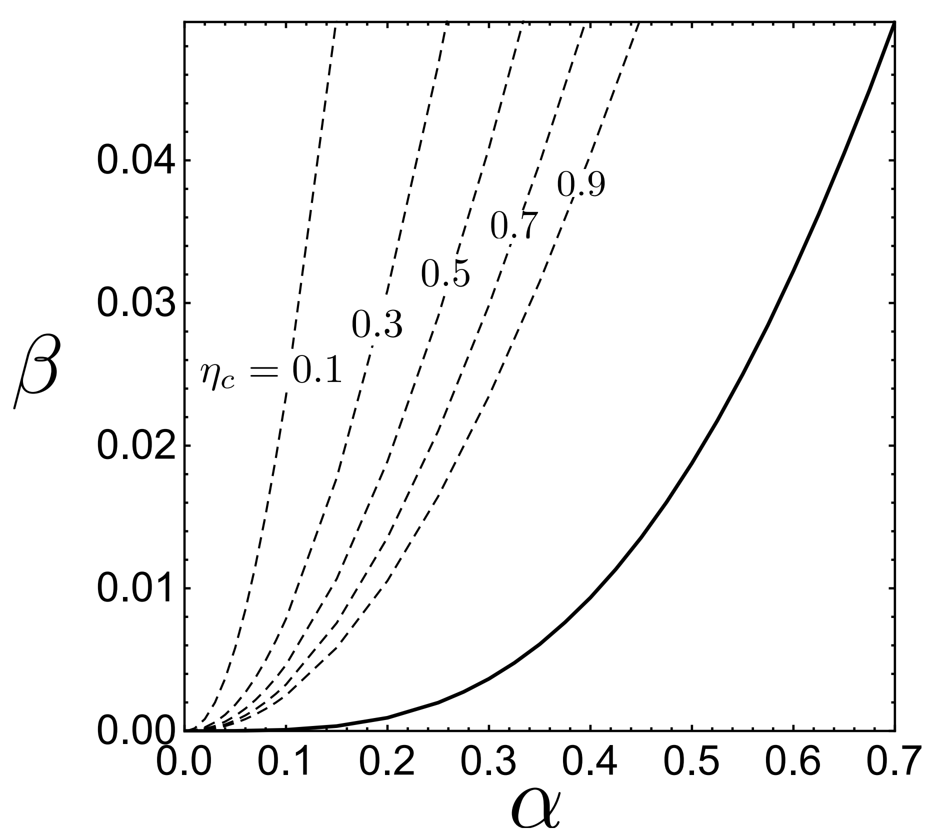
BG MHD flow



$\boldsymbol{v}_{\pm}$ に縛られて伝播する。

R成分の符号

$$v_{+}^{R} = v^{R} + c_s b^{R} = |v_{*}^{R}| \left(\frac{R}{R_*}\right)^{-1/2} \left[-1 + \frac{\alpha}{\sqrt{1+\beta^2\left(1+\sqrt{R/R_*}\right)^2 (1+R/R_*)^2}} \left(\frac{R}{R_*}\right)^{(3-\Gamma)/4} \right]$$



内向きか外向きかどちらに伝播するか？

今考えている(α,β)の範囲内では $v_{\pm}^{R} < 0$

全領域で内向きでhorizonのような境界はない