Spherically symmetric solutions of Massive gravity in the Decoupling Limit

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based on arXiv:0901.0393 + work in progress

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Outline of the presentation

- Introduction & definitions
- The Goldstone boson formalism and the Decoupling Limit
- Spherically symmetric solutions in the Decoupling Limit
- Conclusion

Introduction

Context & Motivations

- Modified gravity theories - a way to get late time acceleration of the Universe and (maybe) solve the Cosmological Constant problem.

➡ Naive idea: give a mass to the graviton with $m \sim 1/H_0$

- Pathologies: non-bounded by below Hamiltonian and ghosts [Boulware & Deser '72; Creminelli, Nicolis, Papucci, Trincherini '05; Deffayet & Rombouts '05], singular solutions (?) [Damour et al. '03], but interesting toy model for more realistic theories.

- Massive gravity = basic ingredient for models with extra-dimensions (like DGP):

  - in 5D, the gravity is massless,
  - viewed from the 4D brane, the gravity is mediated by a tower a massive Kaluza-Klein gravitons with no massless mode.

What is Massive gravity?

- The quadratic action for the massive graviton:

f: background metric (often flat) $\quad H_{\mu\nu}$: spin 2 excitation over f

$$S = \frac{M_P^2}{2}\int d^4x \left(\underbrace{\text{“}H\partial^2 H + \ldots\text{''}}_{\text{Kinetic term}} - \frac{m^2}{4}[\underbrace{H_{\mu\nu}H^{\mu\nu} - (H^\mu_\mu)^2}_{\text{Pauli-Fierz Mass term}}]\right) + \int d^4x\, \frac{1}{2}\underbrace{T_{\mu\nu}H^{\mu\nu}}_{\text{Matter coupling}}$$

- Non-linear completion: $\quad$ dynamical metric $g = f + H$

$$S = \frac{M_P^2}{2}\int d^4x \left(\sqrt{-g}R[g] \qquad\qquad \right) + S_m[g]$$

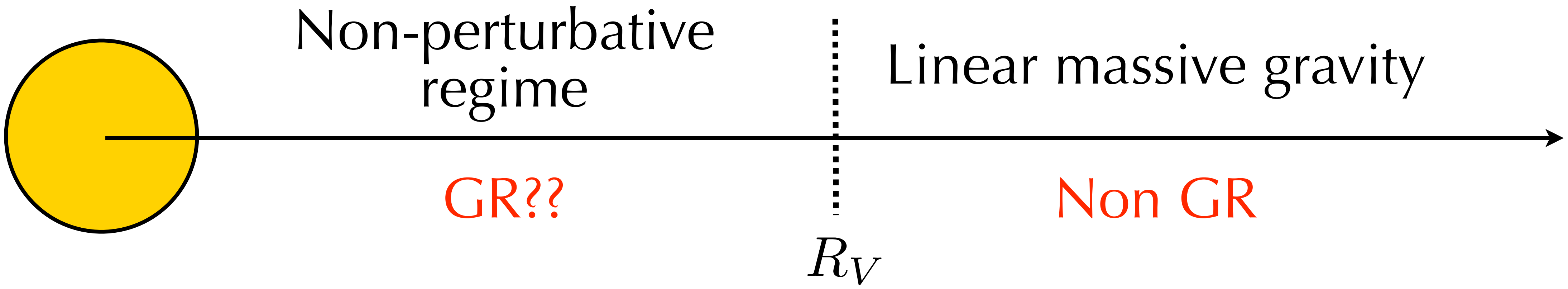
Scalar density

- Examples:

$$\mathcal{V}^{(BD)}[\mathbf{g}^{-1}\mathbf{f}] = \sqrt{-f}\; H_{\mu\nu}H_{\sigma\tau}\left(f^{\mu\sigma}f^{\nu\tau} - f^{\mu\nu}f^{\sigma\tau}\right)$$ *Boulware and Deser ‘72*

$$\mathcal{V}^{(AGS)}[\mathbf{g}^{-1}\mathbf{f}] = \sqrt{-g}\; H_{\mu\nu}H_{\sigma\tau}\left(g^{\mu\sigma}g^{\nu\tau} - g^{\mu\nu}g^{\sigma\tau}\right)$$ *Arkani-Hamed, Georgi and Schwartz ‘03*

Static Spherically Symmetric solutions



- New scale: the Vainshtein radius $R_V = \left(R_S\, m^{-4}\right)^{1/5}$ *Vainshtein '72*

➡ Is it possible to find a solution regular everywhere?

- Our approach is to study these questions in a specific limit: the decoupling limit

$$\begin{array}{r|l} M_P \to \infty & \Lambda \equiv (M_P m^4)^{1/5} \sim const \\ m \to 0 & T_{\mu\nu}/M_P \sim const \end{array}$$

Arkani-Hamed, Georgi and Schwartz '03

➡ Are there regular solutions in the DL?

➡ To what extend does the DL encode the physics of the full system?

The Goldstone Picture & the Decoupling Limit

The Stuckelberg mechanism (I)

Goal: to separate explicitly the various degrees of freedom (tensor, vector, scalar) of a massive field.

Example: *the Proca's field* $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu$ breaks gauge invariance

- Field redefinition $A_\mu \to A_\mu - \partial_\mu B$

$$\Rightarrow \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2(A_\mu - \partial_\mu B)(A^\mu - \partial^\mu B)$$

New gauge invariance:
$$\begin{aligned} A_\mu &\to A_\mu + \partial_\mu \Lambda \\ B &\to B + \Lambda \end{aligned}$$

- "Unitary gauge": $B = 0$
- "Longitudinal gauge": $\partial_\mu A^\mu = 0$

$$\Rightarrow \mathcal{L} = \underbrace{-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - \frac{m^2}{2}A_\mu A^\mu}_{\text{2 DOF}} - \frac{m^2}{2}\underbrace{\partial_\mu B \partial^\mu B}_{\text{1 DOF}}$$

The Stuckelberg mechanism (II)

- Massive spin-2 graviton: in the action

$$S = \frac{M_P^2}{2}\int d^4x \,\left(\sqrt{-g}R[g] - \frac{m^2}{4}\mathcal{V}\left[\mathbf{g}^{-1}\mathbf{f}\right]\right) + S_m[g],$$

replace
$$\begin{aligned} f_{\mu\nu}(x) &\rightarrow f_{\mu\nu}(x) = \partial_\mu X^A(x)\partial_\nu X^B(x) f_{AB}\left(X(x)\right) \\ g_{\mu\nu}(x) &\rightarrow g_{\mu\nu}(x) \end{aligned}$$

the action is now invariant under both
$$\begin{aligned} X^A &\rightarrow X'^A \\ x^\mu &\rightarrow x'^\mu \end{aligned}$$

- "Unitary gauge": $X_0^A(x) \equiv \delta_\mu^A x^\mu$

- In non-unitary gauge, introduce the "Goldstone boson" π :

$$X^A(x) = X_0^A(x) + \pi^A(x).$$

The Goldstone boson expansion

- Scalar-vector decomposition: $\pi^A(x) = f^{AB}\left(A_B + \partial_B \phi\right)$

- In terms of $h_{\mu\nu} = g_{\mu\nu} - f_{\mu\nu}$, it corresponds to the redefinition:

$$\begin{aligned} h_{\mu\nu} \;\rightarrow\; H_{\mu\nu} \;\; &= \;\; h_{\mu\nu} - \partial_\mu A_\nu - \partial_\nu A_\mu - 2\partial_\mu\partial_\nu\phi \\ &\quad -\partial_\mu A_\sigma \partial_\nu A^\sigma - \partial_\mu\partial_\sigma\phi\,\partial_\nu\partial^\sigma\phi \\ &\quad -\partial_\nu A^\sigma \partial_\mu\partial_\sigma\phi - \partial_\mu A^\sigma\partial_\nu\partial_\sigma\phi \end{aligned}$$

Mixing

$$\begin{aligned} \Rightarrow S = \frac{M_P^2}{8}\int d^4x \Big\{ & 2h^{\mu\nu}\partial_\mu\partial_\nu h - 2h^{\mu\nu}\partial_\nu\partial_\sigma h^\sigma_\mu + h^{\mu\nu}\Box h_{\mu\nu} - h\Box h \\ & + m^2\left[h^2 - h_{\mu\nu}h^{\mu\nu} - F_{\mu\nu}F^{\mu\nu} - 4(h\partial A - h_{\mu\nu}\partial^\mu A^\nu) - 4(h\Box\phi - h_{\mu\nu}\partial^\mu\partial^\nu\phi)\right]\Big\} \\ + \int d^4x\, \frac{1}{2} T_{\mu\nu}h^{\mu\nu} & \end{aligned}$$

Shift: $h_{\mu\nu} = \hat{h}_{\mu\nu} - m^2\eta_{\mu\nu}\phi$ and gauge fixing ➡ demixing

The Decoupling Limit

- Form of the kinetic term:

$$S \supset \int d^4x \Big\{ M_P^2 \hat{h} \Box \hat{h} + ... + M_P^2 m^2 A \Box A + ... + M_P^2 m^4 \phi \Box \phi + ... \Big\}$$

- Canonical normalization:

$$\begin{aligned} \tilde{h}_{\mu\nu} &= M_P \hat{h}_{\mu\nu}, \\ \tilde{A}^\mu &= M_P m A^\mu, \\ \tilde{\phi} &= M_P m^2 \phi. \end{aligned}$$

- Dominant higher order term: $\dfrac{(\partial^2 \tilde{\phi})^3}{\Lambda^5}$ with $\Lambda = \left(m^4 M_P\right)^{1/5}$

- In order to concentrate on these terms, take the Decoupling Limit:

$$\boxed{\begin{aligned} M_P &\rightarrow \infty \\ m &\rightarrow 0 \\ \Lambda &\sim const \\ T_{\mu\nu}/M_P &\sim const \end{aligned}}$$

Action for $\tilde{\phi}$ in the Decoupling Limit

- The action for the scalar sector:

$$S = \frac{1}{2}\int d^4x \left\{\frac{3}{2}\tilde{\phi}\Box\tilde{\phi} + \frac{1}{\Lambda^5}\left[\alpha\,(\Box\tilde{\phi})^3 + \beta\,(\Box\tilde{\phi}\;\tilde{\phi}_{,\mu\nu}\;\tilde{\phi}^{,\mu\nu})\right] - \frac{1}{M_P}T\tilde{\phi}\right\}$$

- Equation of Motion:

$$\nabla_\mu \left\{3\Lambda^5\nabla^\mu\tilde{\phi} + 3\alpha\nabla^\mu\left(\Box\tilde{\phi}\right)^2 + \beta\nabla^\mu\left(\tilde{\phi}_{;\delta\gamma}\right)^2 + 2\beta\nabla^\nu\left(\Box\tilde{\phi}\tilde{\phi}^{;\mu}{}_{\nu}\right)\right\} = \frac{\Lambda^5}{M_P}T$$

Can be integrated!

Spherically Symmetric case:

$$3\,\frac{\tilde{\phi}'}{R} + \frac{2}{\Lambda^5}\left\{3\alpha\left(-4\frac{\tilde{\phi}'^2}{R^4} + 2\frac{\tilde{\phi}'\tilde{\phi}''}{R^3} + 2\frac{\tilde{\phi}''^2}{R^2} + 2\frac{\tilde{\phi}'\tilde{\phi}^{(3)}}{R^2} + \frac{\tilde{\phi}''\tilde{\phi}^{(3)}}{R}\right) + \right.$$
$$\left. +\beta\left(-6\frac{\tilde{\phi}'^2}{R^4} + 2\frac{\tilde{\phi}'\tilde{\phi}''}{R^3} + 4\frac{\tilde{\phi}''^2}{R^2} + 2\frac{\tilde{\phi}'\tilde{\phi}^{(3)}}{R^2} + 3\frac{\tilde{\phi}''\tilde{\phi}^{(3)}}{R}\right)\right\}$$
$$= -\frac{1}{R^3}\int_0^R d\tilde{R}\;\tilde{\rho}\left(\tilde{R}\right)\;\tilde{R}^2$$

The Vainshtein mechanism

- Schematic equation of motion for $\tilde{\phi}$:

$$\cancel{\Box\tilde{\phi}} + \text{''}\frac{1}{\Lambda^5}\Box\left(\Box\tilde{\phi}\right)^2\text{''} = \frac{T}{M_P}$$

Strong
coupling

- Einstein equations:

$$\tilde{G}_{\mu\nu} + \text{``}\cancel{\Box\tilde{\phi}}\text{``} = \frac{T}{M_P}$$

GR!

Spherically symmetric solutions in the Decoupling Limit

The EoM in the Decoupling Limit

- New rescaled variables: $\xi \equiv \frac{R}{R_V} \quad , \quad w \equiv -\Lambda^5 R_V^2 \times 2\frac{\tilde{\phi}'}{R}$
- Equations of Motion:

Linear term

$$3\frac{\tilde{\phi}'}{R} + \frac{2}{\Lambda^5}\Bigg\{3\alpha\left(-4\frac{\tilde{\phi}'^2}{R^4} + 2\frac{\tilde{\phi}'\tilde{\phi}''}{R^3} + 2\frac{\tilde{\phi}''^2}{R^2} + 2\frac{\tilde{\phi}'\tilde{\phi}^{(3)}}{R^2} + \frac{\tilde{\phi}''\tilde{\phi}^{(3)}}{R}\right) +$$
$$+\beta\left(-6\frac{\tilde{\phi}'^2}{R^4} + 2\frac{\tilde{\phi}'\tilde{\phi}''}{R^3} + 4\frac{\tilde{\phi}''^2}{R^2} + 2\frac{\tilde{\phi}'\tilde{\phi}^{(3)}}{R^2} + 3\frac{\tilde{\phi}''\tilde{\phi}^{(3)}}{R}\right)\Bigg\}$$
$$= -\frac{1}{R^3}\int_0^R d\tilde{R}\,\tilde{\rho}\left(\tilde{R}\right)\tilde{R}^2$$

Source term

Q=quadratic, second order differential operator

➡ Equation for w:

$$\boxed{2\,Q(w) + \frac{3}{2}w = \frac{1}{\xi^3}}$$

$$2Q(w) + \frac{3}{2}\,w = \frac{1}{\xi^3}$$

close to source

far from source

Vainshtein solution,

$$2Q(w) + \cancel{\frac{3}{2}w} = \frac{1}{\xi^3}$$

$$w \propto \frac{1}{\sqrt{\xi}}, \quad (\nu = -\lambda)$$

perturbative regime,

$$\cancel{2Q(w)} + \frac{3}{2}\,w = \frac{1}{\xi^3}$$

$$w \to \frac{2}{3\xi^3}, \quad (\nu = -2\lambda)$$

Another solution!

$$2Q(w) + \cancel{\frac{3}{2}}w = \cancel{\frac{1}{\xi^3}}$$

$$Q(w) = 0 \quad \Rightarrow \quad w \propto \xi^{p_{1,2}}, \quad (\nu = -\lambda)$$

Cauchy problem and initial conditions

Is it enough to know the asymptotic behavior at infinity?

NOTE: EOM is singular at $\xi = \infty$

$$2Q(w) + \frac{3}{2}\,w = \frac{1}{\xi^3}$$
$$w(\infty) \to \frac{2}{3\xi^3}$$

$\Rightarrow$ existence, uniqueness ?

YES (Toy example 1)

$$\begin{cases} w'' + w = \frac{1}{\xi} \\ w(\infty) \to \frac{1}{\xi} \end{cases}$$

$$\begin{aligned} w &= \frac{1}{\xi} - \frac{2}{\xi^3} + \ldots \\ &+ \cancel{C_1 \cos\xi} + \cancel{C_2 \cos\xi} \end{aligned}$$

NO (Toy example 2)

$$\begin{cases} -w'' + w = \frac{1}{\xi} \\ w(\infty) \to \frac{1}{\xi} \end{cases}$$

$$\begin{aligned} w &= \frac{1}{\xi} - \frac{2}{\xi^3} + \ldots \\ &+ C_1 \exp(-\xi) + \cancel{C_2 \exp\xi} \end{aligned}$$

Example 1: the BD potential (I)

$$\mathcal{V}^{(BD)}[\mathbf{g^{-1}f}] = \sqrt{-f}\; h_{\mu\nu} h_{\sigma\tau} \left(f^{\mu\sigma} f^{\nu\tau} - f^{\mu\nu} f^{\sigma\tau} \right)$$

Boulware, Deser'72

$$\boxed{2\left(\frac{\dot{w}^2}{4} + \frac{w\ddot{w}}{2} + 2\frac{w\dot{w}}{\xi}\right) + \frac{3}{2}w = \frac{1}{\xi^3}}$$

- Asymptotic behavior at infinity: linear theory

$$\cancel{2\left(\frac{\dot{w}^2}{4} + \frac{w\ddot{w}}{2} + 2\frac{w\dot{w}}{\xi}\right)} + \frac{3}{2}w = \frac{1}{\xi^3} \qquad \Rightarrow w(\xi) \sim w_\infty(\xi) \equiv \frac{2}{3\,\xi^3}$$

- Linearization around $w_\infty(\xi)$

$$w = w_\infty + \delta w \;\Rightarrow\; \delta w'' + \frac{\delta w'}{\xi} + \frac{9}{4}\xi^3 \delta w = -\frac{3}{\xi^5}$$

~~two oscillatory modes~~ ➡ unique solution

Example 1: the BD potential (II)

$$\boxed{2\left(\frac{\dot{w}^2}{4}+\frac{w\ddot{w}}{2}+2\frac{w\dot{w}}{\xi}\right)+\frac{3}{2}w=\frac{1}{\xi^3}}$$

- Small distance behavior: no Vainshtein scaling

$$2\left(\frac{\dot{w}^2}{4}+\frac{w\ddot{w}}{2}+2\frac{w\dot{w}}{\xi}\right)+\cancel{\frac{3}{2}w}=\frac{1}{\xi^3} \quad \Rightarrow \text{try} \quad w(\xi)\sim\frac{A}{\sqrt{\xi}}$$

Imaginary solution!

- Another scaling is possible:

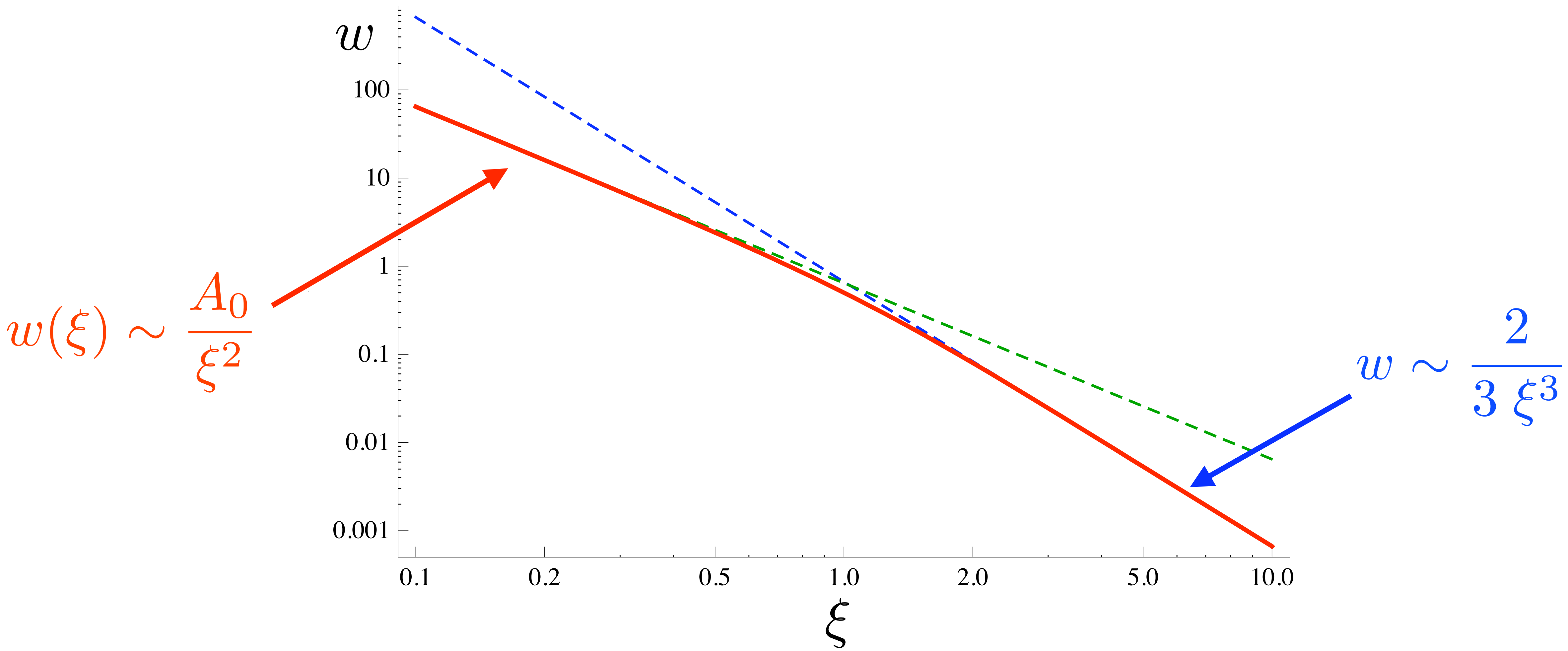
$$2\left(\frac{\dot{w}^2}{4}+\frac{w\ddot{w}}{2}+2\frac{w\dot{w}}{\xi}\right)+\cancel{\frac{3}{2}w}=\cancel{\frac{1}{\xi^3}} \quad \Leftrightarrow \quad Q(w)=0 \quad \Leftrightarrow \quad w(\xi)\sim\frac{A}{\xi^2}$$

2 free constants

$$w(\xi)=\frac{A_0}{\xi^2}+\frac{3A_0B_0+\ln\xi}{3A_0}\,\xi-\frac{3}{8}\,\xi^2+\dots$$

Example 1: the BD potential (III)

$$2\left(\frac{\dot{w}^2}{4}+\frac{w\ddot{w}}{2}+2\frac{w\dot{w}}{\xi}\right)+\frac{3}{2}w=\frac{1}{\xi^3}$$



➡ Unique regular solution (A_0 and B_0 fixed)

Example 2: the AGS potential (I)

$$\mathcal{V}^{(AGS)}[\mathbf{g}^{-1}\mathbf{f}] = \sqrt{-g}\, h_{\mu\nu}h_{\sigma\tau}\left(g^{\mu\sigma}g^{\nu\tau} - g^{\mu\nu}g^{\sigma\tau}\right)$$

Arkani-Hamed, Georgi, Schwartz'72

$$\boxed{-2\left(\frac{\dot{w}^2}{4} + \frac{w\ddot{w}}{2} + 2\frac{w\dot{w}}{\xi}\right) + \frac{3}{2}w = \frac{1}{\xi^3}}$$

- Asymptotic behavior at infinity: linear theory

$$\cancel{2\left(\frac{\dot{w}^2}{4} + \frac{w\ddot{w}}{2} + 2\frac{w\dot{w}}{\xi}\right)} + \frac{3}{2}w = \frac{1}{\xi^3} \qquad \Rightarrow w(\xi) \sim w_\infty(\xi) \equiv \frac{2}{3\,\xi^3}$$

- Linearization around $w_\infty(\xi)$

$$w = w_\infty + \delta w \;\Rightarrow\; \delta w'' + \frac{\delta w'}{\xi} - \frac{9}{4}\xi^3\delta w = -\frac{3}{\xi^5}$$

1 decreasing mode
+ 1 ~~exploding mode~~

➡ solutions isn't unique?

NB: the exploding mode makes the numerical integration tricky!

Example 2: the AGS potential (II)

$$\boxed{-2\left(\frac{\dot{w}^2}{4}+\frac{w\ddot{w}}{2}+2\frac{w\dot{w}}{\xi}\right)+\frac{3}{2}w=\frac{1}{\xi^3}}$$

- Small distance behavior: there is Vainshtein solution

$$2\left(\frac{\dot{w}^2}{4}+\frac{w\ddot{w}}{2}+2\frac{w\dot{w}}{\xi}\right)+\cancel{\frac{3}{2}w}=\frac{1}{\xi^3} \quad\Rightarrow\quad w(\xi)=\sqrt{\frac{8}{9\,\xi}}$$

free constant

$$w(\xi)=\sqrt{\frac{8}{9\,\xi}}+B_0\,\xi^{-\frac{5}{4}+\frac{3\sqrt{5}}{4}}-\frac{3\left(-5+\sqrt{5}\right)}{8\sqrt{2}\left(-4+\sqrt{5}\right)}B_0^2\,\xi^{-2+\frac{3\sqrt{5}}{2}}+\frac{6\xi^2}{31}+\ldots$$

- Another scaling is possible:

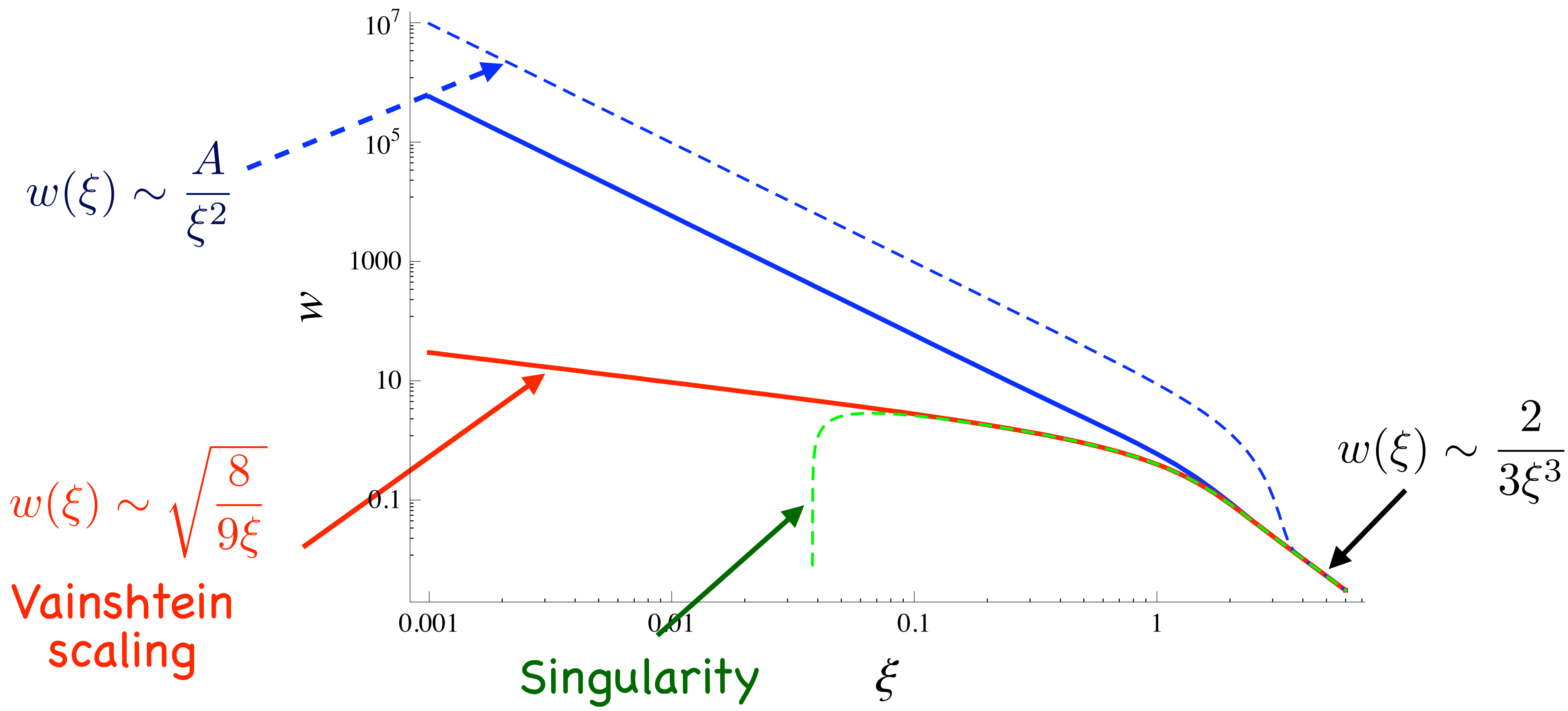
$$2\left(\frac{\dot{w}^2}{4}+\frac{w\ddot{w}}{2}+2\frac{w\dot{w}}{\xi}\right)+\cancel{\frac{3}{2}w}=\cancel{\frac{1}{\xi^3}} \quad\Leftrightarrow\quad Q(w)=0 \quad\Leftrightarrow\quad w(\xi)\sim\frac{A}{\xi^2}$$

2 free constants

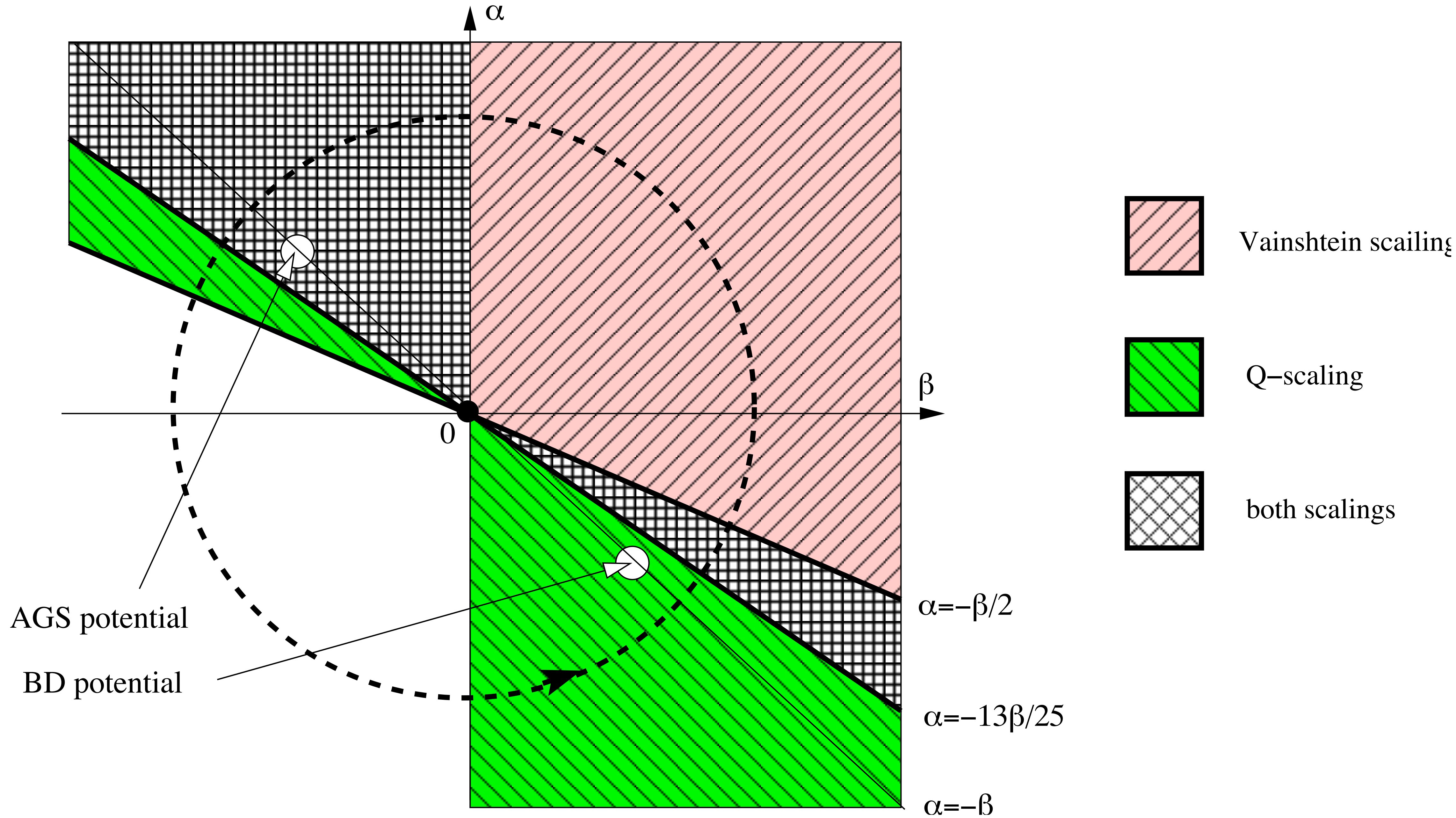
$$w(\xi)=\frac{A_0}{\xi^2}+\frac{3A_0B_0-\ln\xi}{3A_0}\,\xi+\frac{3}{8}\,\xi^2+\ldots$$

Example 2: the AGS potential (III)

NUMERICS



General case of α and β



Conclusion

Conclusion

- The decoupling limit, which focuses on the relevant higher order terms, is an efficient way to classify the different models, and study them.

- We were able to find regular solutions, with two different scalings: Vainshtein and Q-scalings.

- The Vainshtein mechanism works in the DL.

- What remains to be done: see to what extent such solutions can be generalized to the full system (work under progress).