09.01.2012

KINETIC EQUATION IN DE SITTER SPACE

E.T.Akhmedov, P.Burda and A.Sadofyev

(ITEP, MIPT, Moscow)

BASED ON ARXIV:1110.2257 AND TO APPEAR

I. dS space:

$$X_0^2 - X_i^2 = -R^2 \to -1, \quad i = 1, \dots D$$
$$ds_{D+1}^2 = dX_0^2 - dX_i^2$$

Global coordinates (cover entire dS space):

$$X_0 = \sinh t$$
$$X_i = \omega_i \cosh t, \quad \omega_i^2 = 1,$$
$$ds^2 = dt^2 - \cosh^2 t \, d\Omega_{D-1}^2$$

Planar coordinates in the expanding Poincare patch (half of dS space):

$$X_0 = \sinh t + \frac{\vec{x}^2}{2} e^t$$
$$X_D = -\cosh t + \frac{\vec{x}^2}{2} e^t$$
$$X_a = e^t x_a, \quad a = 1, \dots, D-1, \quad \vec{x} = (x_1, \dots, x_{D-1})$$

The metric on expanding Poincare patch ($X_0 \geq X_D$):

$$ds^2 = dt^2 - e^{2\,t}\, d\vec{x}^2 = a(\eta)\, \left[d\eta^2 - d\vec{x}^2\right]$$
$$a(\eta) = \frac{1}{\eta}, \quad \eta = e^{-t}$$

$\eta \to \infty$ — **past,** $\eta \to 0$ — **future**

Contracting Poincare patch is similar:

$$X_0 = \sinh t - \frac{\vec{x}^2}{2}\, e^{-t}$$
$$X_D = \cosh t - \frac{\vec{x}^2}{2}\, e^{-t}$$
$$X_a = e^{-t}\, x_a, \quad a = 1, \ldots, D-1, \quad \vec{x} = (x_1, \ldots, x_{D-1})$$

The metric on the contracting Poincare patch:

$$ds^2 = dt^2 - e^{-2\,t}\, d\vec{x}^2 = a(\eta)\, \left[d\eta^2 - d\vec{x}^2\right]$$
$$a(\eta) = \frac{1}{\eta}, \quad \eta = e^{t}$$

$\eta \to 0$ — **past,** $\eta \to \infty$ — **future**

II. Large IR corrections in dS space:

We are going to consider 4D real minimally coupled scaler theory

$$L = \sqrt{|g|}\,\left[\frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2} - \frac{\lambda}{3}\,\phi^3\right] \tag{1}$$

In particular, loop corrections to the two–point function:

$$\langle\phi(\eta_1,\vec{p})\,\phi(\eta_2,-\vec{p})\rangle \equiv \int e^{i\vec{p}\vec{x}}\,\langle\phi(\eta_1,0)\,\phi(\eta_2,\vec{x})\rangle \tag{2}$$

Typical corrections are $\propto \lambda^2 \log(p\,\sqrt{\eta_1\eta_2})$

as $p\sqrt{\eta_1\eta_2} \to 0$**. We need to sum them up**

Dyson–Schwinger equation reduces in the IR limit to the Boltzman's kinetic equation

III. Interacting fields in dS

Klein–Gordon equation:

$$\left[a^{-4}\,\partial_\eta\, a^2\,\partial_\eta - \frac{\Delta}{a^2} + m^2\right]\,\phi(\eta, x) = 0.$$

$\phi_k(\eta, x) = g_k(\eta)\, e^{-i\,\vec{k}\,\vec{x}}$**, where** $g_k(\eta) = \eta^{3/2}\, h(k\eta)/\sqrt{2}$

$h(k\eta)$ **— solution o the Bessel equation**

$$\phi(\eta, \vec{x}) = \int d^3k\,\left[a_k\, g_k(\eta)\, e^{-i\,\vec{k}\,\vec{x}} + a_k^+\, g_k^*(\eta)\, e^{i\,\vec{k}\,\vec{x}}\right],$$

IV Tree–level two–point function:

Free Hamiltonian $H(\eta) = a^2(\eta)\, \int d^3x\, T_{00}(\eta)$

$$
\begin{aligned}
H_0(\eta) &= \int d^3k\, \left[a_k^+\, a_k\, A_k(\eta) + a_k\, a_{-k}\, B_k(\eta) + c.c.\right], \\
A_k(\eta) &= \frac{a^2(\eta)}{2} \left\{ \left|\frac{dg_k}{d\eta}\right|^2 + \left[k^2 + a^2(\eta)\, m^2\right]\, |g_k|^2 \right\}, \\
B_k(\eta) &= \frac{a^2(\eta)}{2} \left\{ \left(\frac{dg_k}{d\eta}\right)^2 + \left[k^2 + a^2(\eta)\, m^2\right]\, g_k^2 \right\}.
\end{aligned}
$$

Solution of the KG equation and of $B_k = 0$ **do not coincide.**

Hence, H_0 **is not diagonal.**

E.g. Bunch–Davies harmonics

$$
g_k(\eta) \equiv \frac{\sqrt{\pi}\, \eta^{\frac{3}{2}}\, e^{-\frac{\pi\mu}{2}}}{2}\, \mathcal{H}^{(1)}_{i\mu}(k\,\eta), \quad \mu = \sqrt{m^2 - \frac{9}{4}},
$$

$\mathcal{H}^{(1)}_{i\mu}(x)$ **is the Hankel function. For them** $B_k(\eta \to \infty) \to 0$

Nonstationary situation. Then Retarded propagator:

$$D^R(\eta_1, \eta_2|\vec{r}) = \theta(\eta_1 - \eta_2)\ \langle[\phi(\eta_1, 0),\ \phi(\eta_2, \vec{r})]\rangle$$

And Keldysh propagator

$$D^K(\eta_1, \eta_2|\vec{r}) = \tfrac{1}{2}\ \langle\{\phi(\eta_1, 0),\ \phi(\eta_2, \vec{r})\}\rangle$$

We define

$$d^K(\eta_1, \eta_2|\vec{p}) = (\eta_1\, \eta_2)^{-\frac{3}{2}}\ \int d^3r\, D^K(\eta_1, \eta_2|\vec{r})\, e^{-i\,\vec{p}\,\vec{r}}$$

Tree–level

$$d_0^K(p\eta_1, p\eta_2) = \frac{1}{2}\ [h(p\eta_1)\, h^*(p\eta_2) + h^*(p\eta_1)\, h(p\eta_2)]\ \left\langle a_p\, a_p^+ \right\rangle + \\ + h(p\eta_1)\, h(p\eta_2)\ \langle a_p\, a_{-p}\rangle + h.c.,$$

If we average wrt the "vacuum" state for the given choice of harmonics:

$$\left\langle a_p^+\, a_p\right\rangle = 0, \quad \left\langle a_p\, a_{-p}\right\rangle = 0, \quad \left\langle a_p^+\, a_{-p}^+\right\rangle = 0$$

And

$$D^K(\eta_1, \eta_2, |\vec{x}-\vec{y}|) = C_1\, \left(z^2-1\right)^{-\frac{1}{2}}\, P^1_{-\frac{1}{2}+i\,\mu}(z) + \\ + C_2\, \left(z^2-1\right)^{-\frac{1}{2}}\, Q^1_{-\frac{1}{2}+i\,\mu}(z),$$

P^1_ν **and** Q^1_ν **— associated Legendre functions**

$z = 1 + \frac{(\eta_1-\eta_2)^2 - |\vec{x}-\vec{y}|^2}{2\eta_1\,\eta_2}$ **— hyperbolic distance between** $(\eta_1, \vec{x})$ **and** $(\eta_2, \vec{y})$

$C_{1,2}$ **— complex constants. Depend on the particular choice of the Harmonics.**

E.g. for the BD harmonics $C_2 = 0$ **— analytical continuation from the sphere**

<u>One–loop two–point function:</u>

$$d_1^K(p\eta_1, p\eta_2) = \frac{1}{2}\,\left[h(p\eta_1)\,h^*(p\eta_2) + h^*(p\eta_1)\,h(p\eta_2)\right]\,2\,n_p + \\ + h(p\eta_1)\,h(p\eta_2)\,\kappa_p + h.c.$$

$$n_p(\eta) \equiv \langle a_p^+\,a_p\rangle = \frac{\lambda^2}{4\,\pi^2}\int_p^{1/\eta}\frac{dk}{k}\iint_\infty^0 dx_1\,dx_2\,(x_1\,x_2)^{\frac{1}{2}}\times \\ \times h\left[\frac{p}{k}\,x_1\right]\,h^*\left[\frac{p}{k}\,x_2\right]\,h^2(x_1)\,\left[h^*(x_2)\right]^2$$

$$\kappa_p \equiv \langle a_p\,a_{-p}\rangle = \frac{\lambda^2}{2\,\pi^2}\int_p^{1/\eta}\frac{dk}{k}\int_\infty^0 dx_1\int_\infty^{x_1} dx_2\,(x_1\,x_2)^{\frac{1}{2}}\times \\ \times h^*\left[\frac{p}{k}\,x_1\right]\,h^*\left[\frac{p}{k}\,x_2\right]\,h^2(x_1)\,\left[h^*(x_2)\right]^2.$$

Even if we start with $n_p = 0$, $\kappa_p = 0$. They are generated at loops — pair creation.

Feynman diagrammatic technic does not lead to terms $\propto h(p\eta_1)\,h(p\eta_2)$ and c.c..

We are interested in the leading IR terms $p\eta_{1,2} \to 0$

For BD harmonics:

$$h(x) \approx A_+ \, x^{i\mu} + A_- \, x^{-i\mu}, \quad \text{as} \quad x \to 0$$

And

$$d^K_{0+1}\left(p\eta_1, p\eta_2\right) \approx \frac{\coth\left(\pi\mu\right)}{2\mu} \, s^{i\mu} + A_+ \, A_-^* \, \left(p\,\eta\right)^{2\,i\,\mu} \times$$
$$\times \left\{1 + \frac{\lambda^2}{2\pi^2\mu} \, \log\left(\frac{1}{p\eta}\right) \, \iint_\infty^0 dx_1 \, dx_2 \, \left(x_1 \, x_2\right)^{\frac{1}{2}} \, h^2(x_1) \, \left[h^*(x_2)\right]^2 \times \right.$$
$$\left. \times \left[\theta(x_1 - x_2) \, \left(\frac{x_1}{x_2}\right)^{i\mu} - \theta(x_2 - x_1) \, \left(\frac{x_1}{x_2}\right)^{-i\mu}\right]\right\} + c.c.$$

$s = \frac{\eta_1}{\eta_2}$, $\eta = \sqrt{\eta_1 \, \eta_2}$

From this answer we see that $n_p \sim |\kappa_p|$. Hence, BD harmonics are not suitable for the kinetic equation.

<u>Side remark:</u>

Analytical continuation from the sphere — renormalization of the mass only

$\mu \to \mu + \Delta\mu$, $\Delta\mu$ **is complex**

Then the corrections would have had the form:

$$d_1^K\left(p\eta_1, p\eta_2\right) = \frac{\coth\left(\pi\mu\right)}{2\mu}\, s^{i\mu}\, i\,\Delta\mu\,\log(s) + \\ + A_+\, A_-^*\, \left(p\,\eta\right)^{2\,i\,\mu}\, 2\,i\,\Delta\mu\,\log(p\eta) + c.c.$$

Does not coincide with that what we actually get.

One loop result for the out–Jost functions:

$$h(x) = \sqrt{\frac{\pi}{\sinh(\pi\mu)}}\, J_{i\mu}(x).$$

$h(x) \sim x^{i\mu}$, $x \to 0$

$$h(x) = \sqrt{\frac{\pi}{4\,\sinh(\pi\,\mu)\,x}}\,\left[e^{i\,x} + e^{-\pi\,\mu - i\,x}\right], \quad x \to \infty$$

The leading IR contribution to the two–point function:

$$d^K_{0+1}\,(p\eta_1, p\eta_2) \approx \frac{1}{2\,\mu}\,\left[s^{i\mu} + s^{-i\mu}\right] \times$$
$$\times \left\{1 + \frac{\lambda^2}{2\,\pi^2\,\mu}\,\log\left(\frac{1}{p\eta}\right)\,\left|\int_\infty^0 dx\,x^{\frac{1}{2}+i\,\mu}\,\left[h^2(x) - \frac{\pi\,e^{-\pi\mu}}{4\,\sinh(\pi\,\mu)\,x}\right]\right|^2\right\}$$

The κ_p is suppressed in comparison with n_p in the IR limit. Suitable for the kinetic equation.

V. The kinetic equation:

$$\frac{dn_p(\eta)}{d\log(\eta)} = \frac{\lambda^2}{\pi^2}\int_0^\infty dk\,\eta\,(k\eta)^{\frac{1}{2}}\times$$
$$\times\Big\{F_1\times\Big[(1+n_p)\,n_k\,n_{p-k} - \; n_p\,(1+n_k)\,(1+n_{p-k})\Big](\eta') +$$
$$+2\,F_2\times\Big[n_k\,(1+n_{k-p})\,(1+n_p)\;-\;(1+n_k)\,n_{k-p}\,n_p\Big](\eta') +$$
$$+\,F_3\times\Big[(1+n_k)\,(1+n_{p+k})\,(1+n_p)\;-\;n_k\,n_{p+k}\,n_p\Big](\eta')\;\Big\}$$

$$F_1 = \mathrm{Re}\Bigg(C^*\left[\frac{p}{k}\,k\eta,\;k\eta,\;\left(\frac{p}{k}-1\right)\,k\eta\right]\times$$
$$\times\int_{k\eta_0}^{k\,\eta} dy'\,(y')^{\frac{1}{2}}\,C\left[\frac{p}{k}\,y',\;y',\;\left(\frac{p}{k}-1\right)\,y'\right]\Bigg)$$

$$F_2 = \mathrm{Re}\Bigg(C^*\left[k\eta,\;\left(1-\frac{p}{k}\right)\,k\eta,\;\frac{p}{k}\,k\eta\right]\times$$
$$\times\int_{k\eta_0}^{k\eta} dy'\,(y')^{\frac{1}{2}}\,C\left[y',\;\left(1-\frac{p}{k}\right)\,y',\;\frac{p}{k}\,y'\right]\Bigg)$$

$$F_3 = \mathrm{Re}\Bigg(D^*\left[k\eta,\;\left(\frac{p}{k}+1\right)\,k\eta,\;\frac{p}{k}\,k\eta\right]\times$$
$$\times\int_{k\eta_0}^{k\eta} dy'\,(y')^{\frac{1}{2}}\,D\left[y',\;\left(\frac{p}{k}+1\right)\,y',\;\frac{p}{k}\,y'\right]\Bigg)$$

η_0 **is the moment of time when we switch on the interactions;** $C[x,y,z] = h^*(x)\,h(y)\,h(z)$**,** $D[x,y,z] = h(x)\,h(y)\,h(z)$

Solution in the expanding patch

In the limit $p\eta \to 0$ $n(p\eta) \to 0$**. Then**

$$(1+n_p)\, n_k\, n_{p-k} - n_p\,(1+n_k)\,(1+n_{p-k}) \approx -n(p\eta)$$

$$n_k\,(1+n_{k-p})\,(1+n_p) - (1+n_k)\, n_{k-p}\, n_p \approx n(k\eta)$$

$$(1+n_k)\,(1+n_{p+k})\,(1+n_p) - n_k\, n_{p+k}\, n_p \approx 1$$

Furthermore, $n(x) \gg n(y)$ **for** $y \gg x$

$$\frac{dn(x)}{d\log(x)} = \Gamma\, n(x) - \Gamma',$$

$$\Gamma = \frac{\lambda^2}{2\pi^2\,\mu} \left| \int_\infty^0 dy\, y^{\frac{1}{2}-i\,\mu} \left[h^2\,(y) - \frac{\pi\, e^{-\pi\mu}}{4\,\sinh(\pi\,\mu)\,|y|} \right] \right|^2,$$

$$\Gamma' = \frac{\lambda^2}{2\,\pi^2\,\mu} \left| \int_\infty^0 dy\, y^{\frac{1}{2}+i\,\mu} \left[h^2\,(y) - \frac{\pi\, e^{-\pi\mu}}{4\,\sinh(\pi\,\mu)\,|y|} \right] \right|^2$$

I.e. $n(p\eta) = \frac{\Gamma'}{\Gamma}\left[C\ (p\,\eta)^{\Gamma} + 1\right]$**,** $\frac{\Gamma'}{\Gamma} \approx e^{-2\,\pi\mu} \ll 1$

C **— integration constant. Depends on the initial conditions.** $C=-1$ **— above one–loop result.**

Solution in the contracting patch

The same equation as in expanding patch with the opposite relative sign between RHS and LHS.

$\eta \to \infty$ — **future.**

As $p\eta \to 0$ **we expect** $n_p(\eta)$ **to be independent of** p **and** $n(\eta) \gg 1$

$$(1 + n_p)\, n_k\, n_{p-k} - n_p\,(1 + n_k)\,(1 + n_{p-k}) \approx -n^2(\eta)$$

$$n_k\,(1 + n_{k-p})\,(1 + n_p) - (1 + n_k)\, n_{k-p}\, n_p \approx n^2(\eta)$$

$$(1 + n_k)\,(1 + n_{p+k})\,(1 + n_p) - n_k\, n_{p+k}\, n_p \approx n^2(\eta)$$

$$\frac{dn(\eta)}{d\log(\eta)} = \bar{\Gamma}\, n^2(\eta), \qquad \text{where} \quad \bar{\Gamma} \approx \frac{\lambda^2\,\mu^2}{2\,\pi^2\, m^2\, \left(m^2 - \frac{3}{2}\right)} > 0$$

$$n(\eta) \sim \frac{1}{\bar{\Gamma}\, \log\frac{\eta_0}{\eta}}$$

$\eta < \eta_0 = e^{const/\lambda^2} \gg 1$. A — **integration constant.**