EXPLOSIVE PHENOMENA IN MODIFIED GRAVITY

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Hot topics in Modern Cosmology
Spontaneous Workshop V

9 - 14 May 2011, Cargese

Abstract:

- **Observational manifestations of some models of modified gravity, which have been suggested to explain the accelerated cosmological expansion, are analyzed for gravitating systems with time dependent mass density.**

- **It is shown that if the mass density rises with time, the system evolves to the singular state with infinite curvature scalar.**

- **The corresponding characteristic time is typically much shorter than the cosmological time.**

Possible ways to explain accelerated expansion of the Universe:

- **Dark Energy**
- **Modified Gravity**

Models with the action:

$$S = \frac{m_{Pl}^2}{16\pi} \int d^4x \sqrt{-g} f(R) + S_m,$$

$m_{Pl} = 1.22 \cdot 10^{19} GeV$ **is the Planck mass,**
R **is the scalar curvature,**
S_m **is the matter action.**

Usual Einstein gravity:

$$f(R) = R$$

Modified gravity:

$$f(R) = R + F(R)$$

Pioneering papers:

- S. Capozziello, S. Carloni, A. Troisi, RecentRes. Dev. Astron. Astrophys.1, 625 (2003)
- S.M. Carroll, V. Duvvuri, M. Trodden, M.S. Turner, Phys.Rev. D 70, 043528 (2004)

$$F(R) = -\frac{\mu^4}{R},$$

μ is a small parameter with dimension of mass.

However, such a choice of $F(R)$ leads to a strong exponential instability near massive objects.

- A.D. Dolgov, M. Kawasaki, Phys. Lett. B 573, 1 (2003)

gR^2-term:

- **S. Nojiri, S. Odintsov Phys. Rev. D 68, 123512 (2003)**

Could terminate the instability with reasonably small coefficient g for sufficiently dense objects with $\rho > 1$ g/cm^3.

For the objects with smaller mass density the coefficient g would be too large and incompatible with the existing bound on the R^2-gravity.

In the present work we examine a modified gravity model with

$$F(R) = \lambda R_0 \left[\left(1 + \frac{R^2}{R_0^2} \right)^{-n} - 1 \right]$$

- **A.A. Starobinsky, JETP Lett. 86, 157 (2007)**

Here

$\lambda > 0$ **to produce an accelerated cosmological expansion,**

n **is a positive integer,**

$R_0 \sim 1/t_U^2$, **where** $t_U \approx 4 \cdot 10^{17}\mathrm{sec}$ **is the universe age.**

Cosmology with such gravitational action, as well as some other cosmological scenarios with modified gravity, were critically analyzed in recent paper:

S.A. Appleby, R.A. Battye, A.A. Starobinsky, JCAP 1006, 005 (2010); arXiv:0909.1737v2

It was shown:

- **The past singularity exists, when $R \to \infty$ at some finite time in the past.**
- **The problem can be solved by an addition to the action of R^2 - term with sufficiently small coefficient allowed by the present observational data.**

The singularity similar to that considered in the present work was first noticed in the case of cosmological evolution back to the past in:

S.A. Appleby, R.A. Battye, JCAP 0805, 019 (2008)

Despite mathematical similarities there is an important difference:

- According AB, singularity may be avoided with a certain range of initial conditions.
- In our case singularity emerges for any initial conditions.

A. Dev, D. Jain, S. Jhingan, S. Nojiri, M. Sami, I. Thongkool, Phys. Rev. D 78 083515 (2008);
I. Thongkool, M. Sami, R. Gannouji, S. Jhingan, Phys. Rev. D 80 043523 (2009);
I. Thongkool, M. Sami, S. Rai Choudhury, Phys. Rev. D 80 127501 (2009).

A.V. Frolov, Phys. Rev. Lett. 101, 061103 (2008)

- **Infinite R singularity could arise in the future, unless the initial conditions for R are not fine-tuned.**
- **The singularity appears only for certain initial conditions.**
- **All these singularities can be eliminated by an addition of R^2 - term to the action.**

In what follows we will consider a different physical situation than those discussed in the above mentioned references.

We study behavior of astronomical objects with mass density which rises with time.

Mass density is much larger than the cosmological one:

$$\rho_m \gg \rho_c\,,$$

$\rho_c \approx 10^{-29}\,\mathrm{g/cm^3}$ **is the cosmological energy density at the present time;**
$\rho_m \sim 10^{-24}\mathrm{g/cm^3}$ **is matter density of a dust cloud in a galaxy.**

The corresponding equations of motion:

$$\left(1 + F'\right) R_{\mu\nu} - \frac{1}{2}\left(R + F\right) g_{\mu\nu} + \left(g_{\mu\nu} D_\alpha D^\alpha - D_\mu D_\nu\right) F' = \frac{8\pi T^{(m)}_{\mu\nu}}{m^2_{Pl}},$$

$F' = dF/dR$,
D_μ is the covariant derivative,
$T^{(m)}_{\mu\nu}$ is the energy-momentum tensor of matter.

By taking trace over μ and ν we obtain:

$$3D^2 F' - R + RF' - 2F = T$$

In the limit $R \gg R_0$:

$$F(R) \approx -\lambda R_0 \left[1 - \left(\frac{R_0}{R}\right)^{2n}\right] .$$

We analyze temporary evolution of solutions of equation

$$3D^2F' - R + RF' - 2F = T$$

for the gravitational field of some massive object with time varying density.

We assume:

- The gravitational field of this object is weak, as is usually the case.
- Correspondingly, the background metric is approximately flat.
- The covariant derivatives can be replaced by the usual flat space ones.

Substituting expression for $F(R)$ at large R, we obtain:

$$(\partial_t^2 - \Delta)R - (2n+2)\frac{\dot{R}^2 - (\nabla R)^2}{R} + \frac{R^2}{3n(2n+1)}\left(\frac{R^{2n}}{R_0^{2n}} - (n+1)\right)$$
$$- \frac{R^{2n+2}}{6n(2n+1)\lambda R_0^{2n+1}}(R+T) = 0\,.$$

Expressing R through

$$F' = -2n\lambda\left(\frac{R_0}{R}\right)^{2n+1}$$

and introducing the new notation

$$w = -F',$$

We get the equation for w, which takes the simple form describing an unharmonic oscillator:

$$(\partial_t^2 - \Delta) w + U'(w) = 0\,.$$

Potential $U(w)$ is equal to:

$$U(w) = \frac{1}{3}(T - 2\lambda R_0)\,w + \frac{R_0}{3}\left[\frac{q^\nu}{2n\nu}w^{2n\nu} + \left(q^\nu + \frac{2\lambda}{q^{2n\nu}}\right)\frac{w^{1+2n\nu}}{1+2n\nu}\right],$$

where

$$\nu = 1/(2n+1),\ q = 2n\lambda,\ U'(w) = dU/dw.$$

Notice that infinite R corresponds to $F' = 0$ $(w = 0)$ and if F' reaches zero, it would mean that R becomes infinitely large.

It is useful to remember:

$$T \gg R_0; \quad T/R_0 \sim \rho_m/\rho_c \gg 1; \quad w \ll 1.$$

Potential U would depend upon time, if the mass density of the object changes with time:

$$T = T(t).$$

If only the dominant terms are retained and if the space derivatives are neglected, equation simplifies to:

$$\ddot{w} + T/3 - \frac{q^{\nu}(-R_0)}{3w^{\nu}} = 0\,.$$

With dimensionless quantities:

$$t = \gamma\tau, \;\; w = \beta z$$

the equation for z becomes very simple:

$$z'' - z^{-\nu} + (1 + \kappa\tau) = 0\,.$$

Here prime means differentiation with respect to τ and the trace of the energy-momentum tensor of matter is parametrized as:

$$T(t) = T_0(1 + \kappa\tau)\,.$$

Constant β is dimensionless number:

$$\beta = \gamma^2 T_0/3 = q\left(-\frac{R_0}{T_0}\right)^{2n+1}\,.$$

γ has dimension of time and determines characteristic time scale:

$$\gamma^2 = \frac{3q}{(-R_0)}\left(-\frac{R_0}{T_0}\right)^{2(n+1)} .$$

γ may be much shorter than the universe age, t_U, due to the small factor $(R_0/T_0)^{n+1}$.

Assuming $3q \sim 1$, $R_0 \sim 1/t_U^2$, and $\rho_m = 10^{-24}\,\mathrm{g/cm^3}$ we find:

- for $n = 2$: $\gamma \approx 400$ sec.
- for $n = 3$: $\gamma \approx 0.004$ sec.

In the case of constant T ($\kappa = 0$) or very slowly varying T ($\kappa \ll 1$) the solution of equation

$$z'' - z^{-\nu} + (1 + \kappa\tau) = 0$$

is evident.

For small initial values $z(0)$ and $z'(0)$:

$z(\tau)$ oscillates near the minimum of the potential, which is situated at

$$z_{min} = (1 + \kappa\tau)^{-1/\nu}.$$

If the magnitude of $z(0)$ takes a sufficiently large value

$$z > (1-\nu)^{1/\nu},$$

such that potential

$$U(z) = z - z^{1-\nu}/(1-\nu)$$

becomes positive, at some stage $z(\tau)$ would overjump potential $U(z)$ which is equal to zero at $z = 0$.

$z(\tau)$ would reach zero, which corresponds to infinite R, and so the singularity can be reached in finite time.

Analogous situation can be realized if the initial velocity, $z'(0)$, is sufficiently large.

The singularity can be also reached in finite time even if z was initially situated at the minimum of the potential and the initial velocity $z'(0) = 0$.

It would take place if $\kappa > 0$, i.e. the energy density rises with time.

The motion of z_{min} to zero and simultaneous diminishing of the depth of the potential well make it easier for $z(\tau)$ to reach zero.

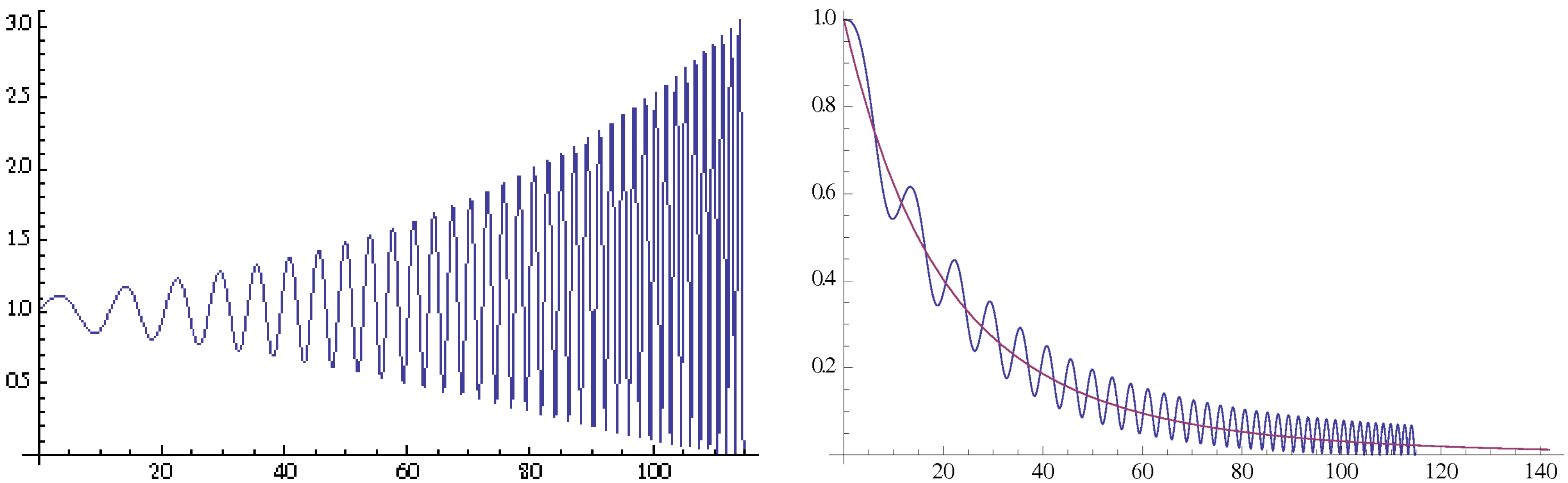

Figure 1.

Numerical solution: $n = 2,\ \kappa = 0.01,\ \rho_m/\rho_c = 10^5.$

Ratio $z(\tau)/z_{min}(\tau)$ (left) and functions $z(\tau)$ and $z_{min}(\tau)$ (right)

The initial conditions: $z(0) = 1$ and $z'(0) = 0.$

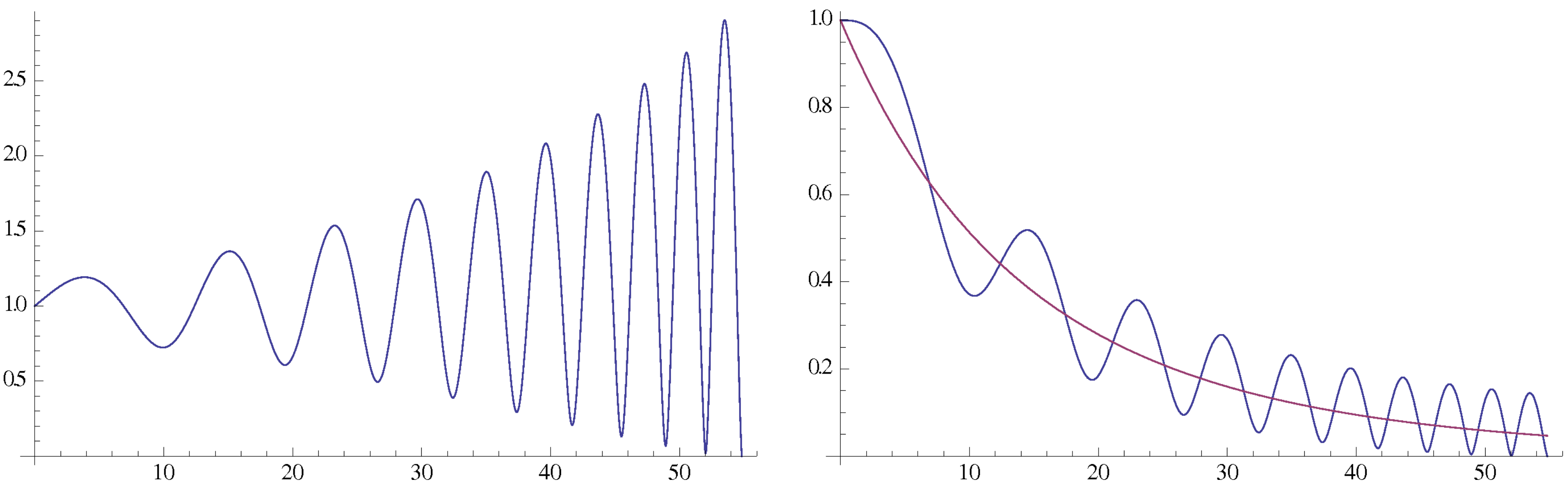

Figure 2.

Numerical solution: $n = 3,\ \kappa = 0.01,\ \rho_m/\rho_c = 10^5.$

Ratio $z(\tau)/z_{min}(\tau)$ (left) and functions $z(\tau)$ and $z_{min}(\tau)$ (right)

The initial conditions: $z(0) = 1$ and $z'(0) = 0.$

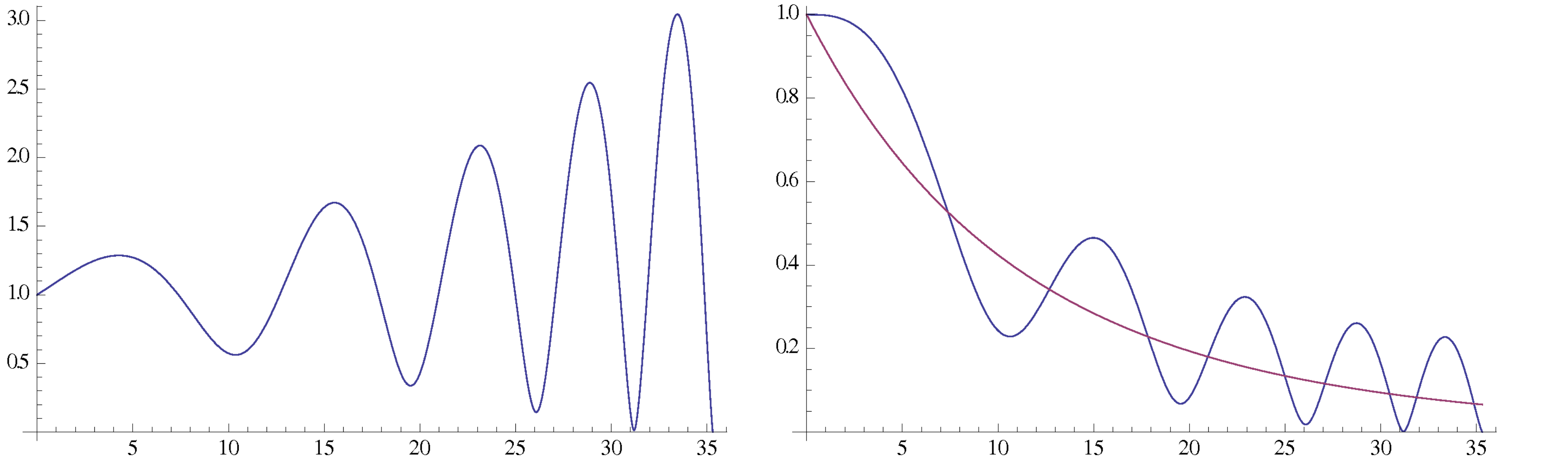

Figure 3.

Numerical solution: $n = 4,\ \kappa = 0.01,\ \rho_m/\rho_c = 10^5.$

Ratio $z(\tau)/z_{min}(\tau)$ (left) and functions $z(\tau)$ and $z_{min}(\tau)$ (right)

The initial conditions: $z(0) = 1$ and $z'(0) = 0.$

It is clearly seen:

- $z(\tau)$ **reaches zero after a finite number of oscillations around** $z_{min}(\tau)$**.**
- **When** $z_{min}(\tau)$ **shifts to smaller values, function** $z(\tau)$ **initially remains behind** $z_{min}(\tau)$**.**
- **When the displacement from the equilibrium point becomes large enough,** $z(\tau)$ **started to run after** $z_{min}(\tau)$ **with an increasing speed, then overtakes the position of the minimum, and oscillates back.**
- **After a few oscillations the retarded position of** $z(\tau)$ **happens to be above the point** $z_0(\tau)$**, where the potential is zero.**
- **The position of this point moves to smaller values with rising** $T(\tau)$**.**

Evolution of the energy density:

$$T(t) = T_0(1 + \kappa\tau) = T_0(1 + t/t_{ch})\,,$$

where

t is a physical time,

t_{ch} is the characteristic time of the variation.

Coefficient κ is expressed through t_{ch} as:

$$\kappa = \gamma/t_{ch}\,.$$

The presented cases for $\kappa = 0.01$ and $\rho_m/\rho_c = 10^5$:

- Figure 1: $n = 2$ corresponds to $t_{ch} = 4 \cdot 10^4$ sec,
- Figure 2: $n = 3$ corresponds to $t_{ch} = 0.4$ sec.

The characteristic time of the density variation can be estimated as:

$$t_{ch} \sim d/v\,,$$

where

d **is the size of the system,**
v **is the velocity of the constituent particles.**

- **Process of the collapse of the cloud: the velocity would be quite low and the characteristic time is expected to be close to the Newton free-fall time.**
- **The collision of the clouds: the velocities are typically galactic ones, about 300 km/sec.**
- **The velocity may be even larger at the collision of the supernova ejecta with galactic or intergalactic clouds.**

Another way to estimate κ:

$$\kappa = \frac{\gamma v}{d},$$

where

d is the size of the object with changing mass density or sizes of the colliding objects.

For $n > 2$ and astronomically large clouds one should expect

$$\kappa \ll 1.$$

As it is seen from the numerical calculations, the singularity is reached when

$$t \sim t_{ch}.$$

This is much shorter than the cosmological time for clouds of denser matter in galaxies or a collapsing cloud forming a star or another denser body.

Possible cases when the conditions leading to singularity can be realized:

- **collapse of gas cloud leading finally to star formation;**
- **collision of two gas clouds in a galaxy;**
- **stellar ejecta colliding with interstellar or intergalactic matter, etc.**

In the analysis of equation

$$(\partial_t^2 - \Delta)w + U'(w) = 0\,.$$

the spatial derivates have been neglected.

However, the effect of inhomogeneities can be described by an appearance of the term w/d^2 with positive coefficient.

Such a term is equivalent to an addition of an extra attractive force pushing w or z to zero, i.e. to $R \to \infty$.

Thus, in our case the inhomogeneities stimulate singularity formation in contrast to the process of structure formation due to gravitational (Jeans) instability.

Possible way to avoid singularity is to introduce R^2-term into the gravitational action:

$$\delta F(R) = -R^2/6m^2\,,$$

where m is a constant parameter with dimension of mass.

In the homogeneous case and in the limit of large ratio R/R_0 equation of motion for R is modified as:

$$\left[1 - \frac{R^{2n+2}}{6\lambda n(2n+1)R_0^{2n+1}m^2}\right]\ddot{R} - (2n+2)\,\frac{\dot{R}^2}{R} - \frac{R^{2n+2}(R+T)}{6\lambda n(2n+1)R_0^{2n+1}} = 0\,.$$

With dimensionless curvature and time

$$y = -\frac{R}{T_0}, \quad \tau_1 = t\left[-\frac{T_0^{2n+2}}{6\lambda n(2n+1)R_0^{2n+1}}\right]^{1/2}$$

equation for R is transformed into:

$$\left(1 + g y^{2n+2}\right) y'' \\ -2(n+1)\frac{(y')^2}{y} + y^{2n+2}\left[y - (1 + \kappa_1 \tau_1)\right] = 0\,,$$

where prime means derivative with respect to τ_1.

We have:

$$g = -\frac{T_0^{2n+2}}{6\lambda n(2n+1)m^2 R_0^{2n+1}} > 0\,.$$

- **The factor $(1 + gy^{2n+2})$ is always non-zero because $g > 0$.**
- **For very large m, or small g, the numerical solution demonstrates that R would reach infinity in finite time in accordance with the results presented above.**
- **Nonzero g would terminate the unbounded rise of R.**
- **To avoid too large deviation of R from the usual gravity coefficient g should be larger than or of the order of unity: $g \geq 1$.**

From the laboratory tests of gravity

- **D.J. Kapner, T.S. Cook, E.G. Adelberger, J.H. Gundlach, B.R. Heckel, C.D. Hoyle, H.E. Swanson, Phys. Rev. Lett. 98 021101 (2007)**

$$m > 10^{-2.5}\ \text{eV} \text{ and we find } n \geq 6.$$

From

- **S.A. Appleby, R.A. Battye, A.A. Starobinsky, JCAP 1006, 005 (2010); arXiv:0909.1737v2**

$$m \gg 10^{5}\ \text{GeV} \text{ and we find } n \geq 9.$$

A natural value

$$m \sim m_{Pl} \text{ and correspondingly } n \geq 12.$$

For smaller values of T_0 the bounds on n are noticeably stronger.

The frequency of oscillations in dimensionless time τ or τ_1 is typically of order of unity.

In physical time the frequency would be about:

$$\omega \sim \frac{g^{-1/2}}{t_U}\left(\frac{T_0}{R_0}\right)^{n+1} \leq m\,.$$

For example,

- for $n = 5$ and for a galactic gas cloud with $T_0/R_0 = 10^5$, the oscillation frequency would be

$$\omega \sim 10^{12}\ \mathrm{Hz} \approx 10^{-3}\ \mathrm{eV}.$$

- Higher density objects with $\rho = 1\ \mathrm{g/cm}^3$ would saturate the above bound with much higher frequency

$$\omega \sim m.$$

For denser objects the variation of T in terms of τ or τ_1 is very slow because of very small κ.

As a result the amplitude of the oscillations around the equilibrium point would be also small and possibly such oscillations are of no danger from the observational point of view.

Still it is possible that there might be intermediate cases when the oscillations would lead to observable phenomena.

We have shown:

- **The impact of the considered above versions of modified gravity on the systems with time dependent mass density could lead to the singularity $R \to \infty$ during finite time in the future.**
- **This time is typically much shorter than the cosmological one.**
- **An addition of R^2 - term could prevent from the singular behavior but at expense of quite large values of n ($n \geq 6$ or maybe $n \geq 9$) which may be at odds with the standard cosmological evolution.**

THE END

THANK YOU FOR THE ATTENTION