Conformal Invariance and Scalar-tensor formulation of General Relativity

(Weyl´s integrable space-times)

J. Fonseca Neto, C. Romero (UFPB / Brazil)

1. Introduction
2. General Relativity
3. Weyl's Integrable Manifolds
4. General Relativity as a scalar-tensor theory
5. Spherically symmetric space-time
6. Conclusion

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Introduction

1. Breakdown of general relativity theory?
   1. Weak field regime (solar system, galaxies)
   2. Strong field regime (pulsars, black holes)
   3. Cosmology (dark matter, dar energy)
2. Quest for a new theory of gravitation?
   1. New-dimensions (braneworld)
   2. Non-riemannian space-times manifolds
   3. New field equations

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General Relativity

1. Geometric framework
2. Field Equations
3. Test particle motion (geodesic postulate)
4. Matter coupling (minimal coupling)

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General Relavity – Geometric framework

1. Pseudo-Riemannian spacetime manifold
2. Lorentz Metric (lengths, angles) $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$
3. Levi-Civita connection (Christoffel symbols)

   (parallel transport, covariant derivative)

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu} \right)$$

No torsion and no non-metric

$$T^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu} = 0, \; Q_{\alpha\mu\nu} = \nabla_\alpha g_{\mu\nu} = 0$$

4. All properties determined only by the metric

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General Relavity – field equations

1. Einstein-Hilbert action

$$S = \int \sqrt{-g} R d^4 x$$

2. Einstein Equations (vacuum)

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \Leftrightarrow R_{\mu\nu} = 0$$

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General Relavity – test particle motion

1. Geodesics

$$\delta \int ds = \delta \int_{\lambda_2}^{\lambda_1} \sqrt{g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}\, d\lambda = 0$$

$$\frac{d^2x^\rho}{d\lambda^2} + \left\{ \begin{array}{c} \rho \\ \nu\alpha \end{array} \right\} \frac{dx^\nu}{d\lambda}\frac{dx^\alpha}{d\lambda} = 0$$

2. Auto-parallels (affine parameter)

$$u^\mu \nabla_\mu u^\alpha = 0$$

$$\frac{d^2x^\rho}{d\lambda^2} + \left\{ \begin{array}{c} \rho \\ \nu\alpha \end{array} \right\} \frac{dx^\nu}{d\lambda}\frac{dx^\alpha}{d\lambda} = 0$$

Obs. :same curves.

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General Relavity – test particle motion

1. Geodesic postulate. Test particle motion:
   1. Massive particles: time-like geodesics, proper time and space-time interval (-+++):

$$d\tau^2 = -ds^2 \qquad u^\rho = \frac{dx^\rho}{d\tau} \qquad g_{\mu\nu}u^\mu u^\nu = -1$$

   2. Light rays: null geodesics, causality (light cone)

$$ds^2 = 0 \qquad k^\mu = \frac{dx^\mu}{d\lambda} \qquad g_{\mu\nu}k^\mu k^\nu = 0$$

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General Relavity – matter coupling

1. **Principle of minimal couple**

   Special Relativity ---- > General Relativity

   $$\eta_{\mu\nu} = diag(-1,+1,+1,+1) \longmapsto g_{\mu\nu}$$

   $$\partial_\mu \longmapsto \nabla_\mu$$

2. **Momentum-energy tensor**

   $$\delta S_M = \delta \int \sqrt{-g} L_M d^4x = \int \sqrt{-g} T_{\mu\nu} \delta g^{\mu\nu} d^4x$$

3. **Einstein equations**

   $$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}$$

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Scalar-tensor General Relavity (GR)

1. Weyl's integrable space-time (WIST)
2. Field Equations (metric and Weyl's scalar)
3. Test particle motion
4. Matter coupling

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Scalar-tensor GR - Weyl's integrable space-times

1. Lorentz metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ $\quad g_{\mu\nu} = g_{\nu\mu}$
2. Weyl's connection (symmetric). No torsion

$$\Gamma^{\alpha}_{\mu\nu} \overset{g_{\mu\nu}}{=} \left\{ \begin{array}{c} \alpha \\ \mu\nu \end{array} \right\} + \frac{1}{2}\left(\psi_\nu \delta^\alpha_\mu + \psi_\mu \delta^\alpha_\nu - g_{\mu\nu}\psi^\alpha\right)$$

3. Non-metricity. Weyl's vector ψ_μ

$$\nabla_\nu g_{\lambda\mu} = \partial_\nu g_{\lambda\mu} - \Gamma^{\beta}_{\lambda\nu} g_{\beta\mu} - \Gamma^{\beta}_{\nu\mu} g_{\beta\lambda} = -\psi_\nu g_{\lambda\mu}$$

4. No second-clock effect. Weyl's scalar ψ

$$\psi_\mu = \partial_\mu \psi$$

5. All properties determined by both $g_{\mu\nu}$ and ψ

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Scalar-tensor GR - Weyl's integrable space-times

1. **Weyl's transformations** $x = (x^\mu) = (x^0, x^1, x^2, x^3)$

$$\tilde{g}_{\mu\nu}(x) = e^{\sigma(x)} g_{\mu\nu}(x) \qquad \tilde{\psi}(x) = \psi(x) - \sigma(x)$$

2. **Curvature of WIST and Weyl's transformations**

$$R_{\mu\nu} = \hat{R}_{\mu\nu} - \left(\psi_{;\mu\nu} - \frac{1}{2} g_{\mu\nu} \hat{\Box}\psi\right) - \frac{1}{2}\left(g_{\mu\nu}\partial_\alpha\psi\partial^\alpha\psi - \partial_\mu\psi\partial_\nu\psi\right)$$

$$R = \hat{R} - 3\hat{\Box}\psi - \frac{3}{2}\partial_\mu\psi\partial^\mu\psi$$

$$\widetilde{R}_{\mu\nu} = R_{\mu\nu}$$

$$\tilde{R} = e^{-\sigma} R$$

obs. (hat): $\hat{R} = R(g)\ ,\ \hat{R}_{\mu\nu} = R_{\mu\nu}(g)\ (only\ metric)$

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Scalar-tensor GR – field equations

1. Conformal invariant action ($g_{\mu\nu}$ and ψ)

$$S = \int \sqrt{-g} e^{\psi} R d^4x$$

2. Field equations (Weyl's scalar and metric)

$$R = \hat{R} - 3\hat{\Box}\psi - \frac{3}{2}\partial_\mu\psi\partial^\mu\psi = 0$$

$$G_{\mu\nu} = \hat{G}_{\mu\nu} - \left(\psi_{;\mu\nu} - g_{\mu\nu}\hat{\Box}\psi\right) + \frac{1}{2}\left(\partial_\mu\psi\partial_\nu\psi + \frac{1}{2}g_{\mu\nu}\partial_\alpha\psi\partial^\alpha\psi\right) = 0$$

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Scalar-tensor GR – test particle motion

1. Weyl's geodesics (auto-parallels)
2. Massive particles
3. Proper-time (affine parameter)
4. Lagrangia

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Scalar-tensor GR – test particle motion

1. Light motion. Weyl's null geodesics (auto-paralles)

$$g_{\mu\nu}k^{\mu}k^{\nu} = g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = 0$$

$$k^{\sigma}\nabla_{\sigma}^{(g,\psi)}k^{\rho} = \frac{dk^{\rho}}{d\lambda} + \Gamma^{\rho}{}_{\sigma\mu}k^{\sigma}k^{\mu} = 0$$

2. Metric null geodesics are Weyl's null geodesics reparametrized. Same light-cones

$$k^{\sigma}\nabla_{\sigma}^{(g,\psi)}k^{\rho} = \frac{dk^{\rho}}{d\lambda} + \hat{\Gamma}^{\rho}{}_{\mu\nu}k^{\mu}k^{\nu} - \psi^{\rho}g_{\sigma\mu}k^{\mu}k^{\sigma} + k^{\rho}\frac{d\psi}{d\lambda} = 0$$

$$\left(\frac{d\sigma}{d\lambda}\right)^{2}\left(\frac{d^{2}x^{\rho}}{d\sigma^{2}} + \hat{\Gamma}^{\rho}{}_{\mu\nu}\frac{dx^{\mu}}{d\sigma}\frac{dx^{\nu}}{d\sigma}\right) = -\left[\frac{d^{2}\sigma}{d\lambda^{2}} + \left(\frac{d\sigma}{d\lambda}\right)\frac{d\psi}{d\lambda}\right]\frac{dx^{\rho}}{d\sigma} = 0$$

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Scalar-tensor GR – test particle motion

1. Massive test particles. Proper-time (affine parameter) and Weyl's time-like geodesics (auto-paralles).

$$d\tau^2 = -e^{\psi} ds^2$$

$$g_{\mu\nu} u^{\mu} u^{\nu} = g_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} = -e^{-\psi}$$

$$\delta \int d\tau = \delta \int \sqrt{e^{\psi} g_{\alpha\beta} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\beta}}{d\tau}} = 0$$

$$u^{\sigma} \nabla_{\sigma}^{(g,\psi)} u^{\rho} = \frac{du^{\rho}}{d\tau} + \Gamma^{\rho}_{\sigma\mu} u^{\sigma} u^{\mu} = 0$$

Weyl's time-like geodesics are different from metric time-like geodesics.

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Scalar-tensor GR – conformal killing vectors

1. Symmetries.Conformall killing vectors.

$$k = \frac{\partial}{\partial\lambda} = k^\alpha \partial_\alpha \qquad \mathcal{L}_k g_{\mu\nu} = -k^\alpha \psi_\alpha g_{\mu\nu} = -\frac{d\psi}{d\lambda} g_{\mu\nu}$$

Invariant under Weyl's transformations

2. Constants of motion $\quad u^\mu = \frac{dx^\mu}{d\tau}$

$$C_{(k)} = k_\nu u^\nu = g_{\mu\nu} k^\mu u^\nu \qquad \frac{d}{d\tau} C_{(k)} = -\left(\frac{d\psi}{d\tau}\right) C_k \qquad C_{(k)}(\tau) = C_{(k)}(\tau_0) e^{-\int_{\tau_0}^{\tau} d\psi}$$

$$W_{(k)} = e^\psi g_{\mu\nu} k^\mu u^\nu \qquad \frac{d}{d\tau} W_{(k)} = e^\psi \left[\frac{d\psi}{d\tau} C_{(k)} + \frac{d}{d\tau} C_{(k)}\right] = 0$$

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Scalar-tensor GR – weak field limit

1. **Weak field slow motion limit.**

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu} \qquad \psi = \epsilon\psi_0$$

2. **Test particle motion**

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \qquad \frac{d^2\vec{r}}{dt^2} = -\frac{\epsilon}{2}c^2\nabla(h_{00} + \psi_0)$$

3. **Newtonian potential**

$$U = \frac{\epsilon c^2}{2}(h_{00} + \psi_0)$$

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Scalar-tensor GR – Einstein and Weyl´s frames

1. Einstein and Weyl frames (gauges)

Several equivalent descriptions of the gravitational field related by Weyl's transformations

1. Einstein frame (gauge) - unique

$$\tilde{g}^{E}_{\mu\nu} = e^{\psi} g_{\mu\nu} = inv, \; \tilde{\psi}^{E} = 0$$

2. Weyl's frame (gauge) - several

$$(g^{(1)}_{\mu\nu}, \psi^{(1)}), \ldots, (g^{(n)}_{\mu\nu}, \psi^{(n)})$$

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Static Spherically symmetric WIST

1. Static spherically symmetric solution
2. Conformal killing vectors. Const. of motion
3. Gravitational redshift
4. Precession of the perihelium
5. Einstein frame (gauge)

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Static Spherically symmetric WIST - solution

1. Field equations

$$R_{\mu\nu} = \hat{R}_{\mu\nu} - \left(\psi_{;\mu\nu} - \frac{1}{2}g_{\mu\nu}\hat{\Box}\psi\right) - \frac{1}{2}\left(g_{\mu\nu}\partial_\alpha\psi\partial^\alpha\psi - \partial_\mu\psi\partial_\nu\psi\right) = 0$$

2. Solution

$$\psi(r) = \ln\left(1 - \frac{rs}{r}\right)$$

$$ds^2 = -dt^2 + \frac{r^2\,dr^2}{(r-rs)^2} + \frac{r^3\,d\theta^2}{r-rs} + \frac{r^3\sin(\theta)^2\,d\phi^2}{r-rs}$$

$rs = 2m$ $\quad r > 2m$

3. Weak fied limit

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Static Spherically symmetric WIST – conf. Kill. vec.

1. Conformal killing vectors

$$\xi^\mu = (1,0,0,0) \qquad \eta^\mu = (0,0,0,1)$$

2. Constants of motion. Energy and momentum angular

$$g_{00}\left(\frac{dt}{d\tau}\right)^2 + g_{11}\left(\frac{dr}{d\tau}\right)^2 + g_{22}\left(\frac{d\theta}{d\tau}\right)^2 + g_{33}\left(\frac{d\phi}{d\tau}\right)^2 = -e^\psi$$

$$E = e^\psi \xi_\alpha u^\alpha = e^\psi g_{\alpha\beta} u^\alpha \xi^\beta = e^\psi g_{00} u^0 \xi^0 \qquad E = e^\psi g_{00}\frac{dt}{d\tau}$$

$$L = e^\psi \eta_\alpha u^\alpha = e^\psi g_{\alpha\beta} u^\alpha \eta^\beta = e^\psi g_{33} u^3 \eta^3 \qquad L = e^\psi g_{33}\frac{d\phi}{d\tau}$$

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Static Spherically symmetric WIST - properties

1. Gravitational redshift. Static observers

$$d\tau_A = \sqrt{e^{\psi(r_A)} g_{00}(r_A)} dt \qquad d\tau_A = \sqrt{e^{\psi(r_A)}} dt$$

$$d\tau_A = \sqrt{\frac{e^{\psi(r_A)}}{e^{\psi(r_B)}}} d\tau_B$$

$$\frac{\Delta\nu}{\nu} = \frac{\sqrt{e^{\psi(r_A)}} - \sqrt{e^{\psi(r_B)}}}{\sqrt{e^{\psi(r_A)} e^{\psi(r_B)}}} \qquad \frac{\Delta\nu}{\nu} = m\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

Equal to Schwarzschild space-time.

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Static Spherically symmetric WIST - properties

1. Orbit equation

$\theta = \frac{1}{2}\pi$ $\quad \dot{t} = dt/d\tau$ $\quad \dot{r} = dr/d\tau$ $\quad \dot{\phi} = d\phi/d\tau$ $\quad g_{00}\dot{t}^2 + g_{11}\dot{r}^2 + g_{33}\dot{\phi}^2 = e^{-\psi}$

$$\frac{e^{-2\psi}E^2}{g_{00}} + g_{11}\dot{r}^2 + \frac{e^{-2\psi}L^2}{g_{33}} = -e^{-\psi} \qquad u = \frac{1}{r} \qquad \frac{E^2}{g_{00}} + \frac{g_{11}L^2}{g_{33}^2 u^4}\left(\frac{du}{d\phi}\right)^2 + \frac{L^2}{g_{33}} = 0$$

$$E^2 - L^2\left(\frac{du}{d\phi}\right)^2 - (1-2mu)u^2L^2 = 0 \qquad \frac{d^2u}{d\phi^2} + u - 3mL^2u^2 = 0$$

2. Precession of the perihelium

Equal to Schwarzschild

$$\delta\phi_{\text{prec}} = \frac{6\pi G}{c^2}\frac{M}{a\left(1-\epsilon^2\right)}$$

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Static Spherically symmetric WIST

Einstein frame (gauge)

Performing a Weyl's transformation to the Einstein frame (gauge) where the Weyl's scalar is zero, the solution reduces to Schwarzschild solution.

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Merci beaucoup