Topics in de Sitter and anti de Sitter QFT

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1997: a revolutionary year for physics

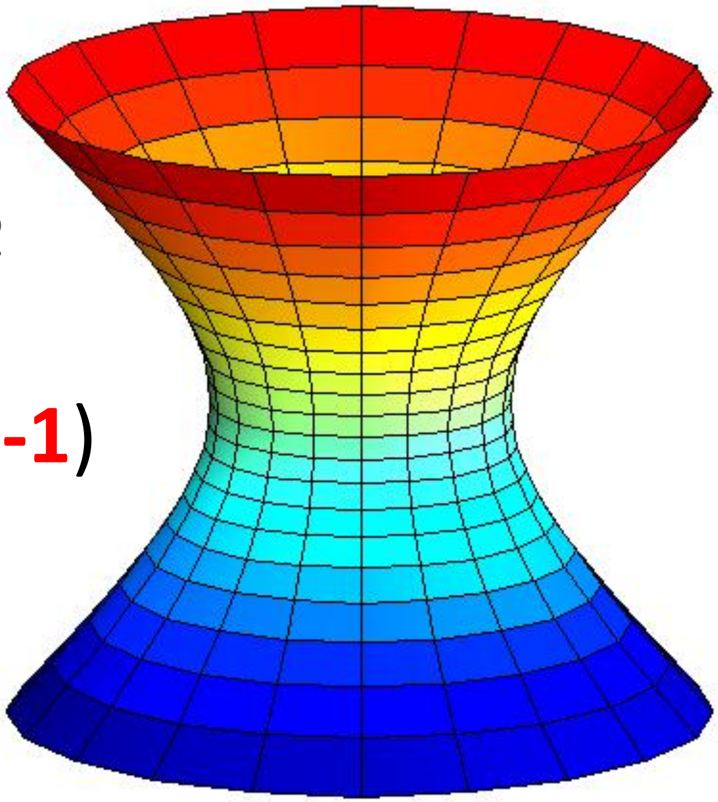
- A. G. Riess et al., "Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant", Astronomical Journal 116, 1009 (1998).
- J. Maldacena, "The Large N Limit Of Superconformal Field Theories and Supergravity", Adv. Theor. Math. Phys. 2 (1998) 231.

The shape of our universe

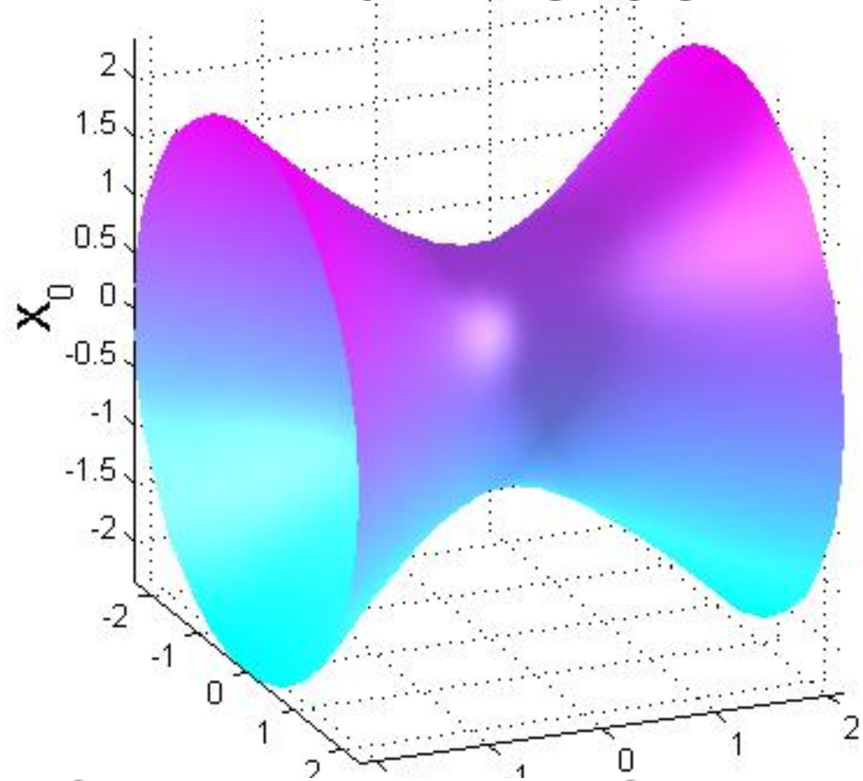
$$X_0^2 - X_1^2 - \ldots X_d^2 = -R^2$$

$M^{(d+1)}$: $\eta_{\mu\nu}$ = diag(**1,-1,……,-1**)

$$G = SO(1,d)$$



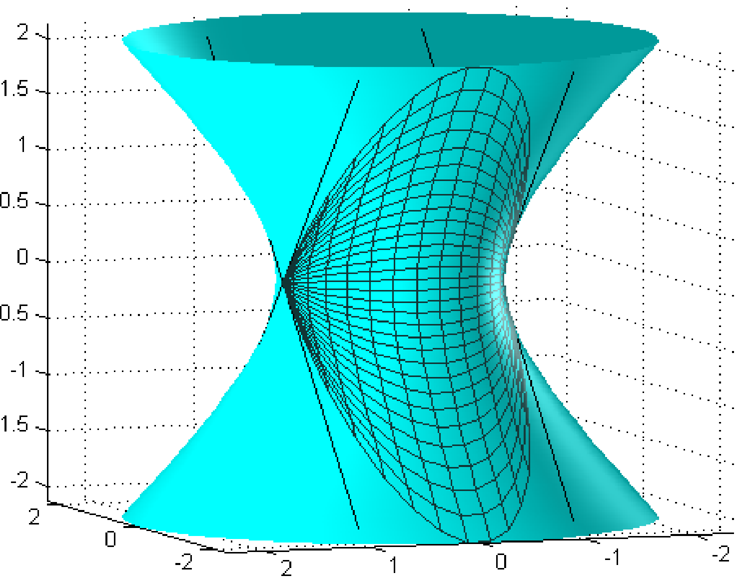
Strings and particles: the anti de Sitter universe

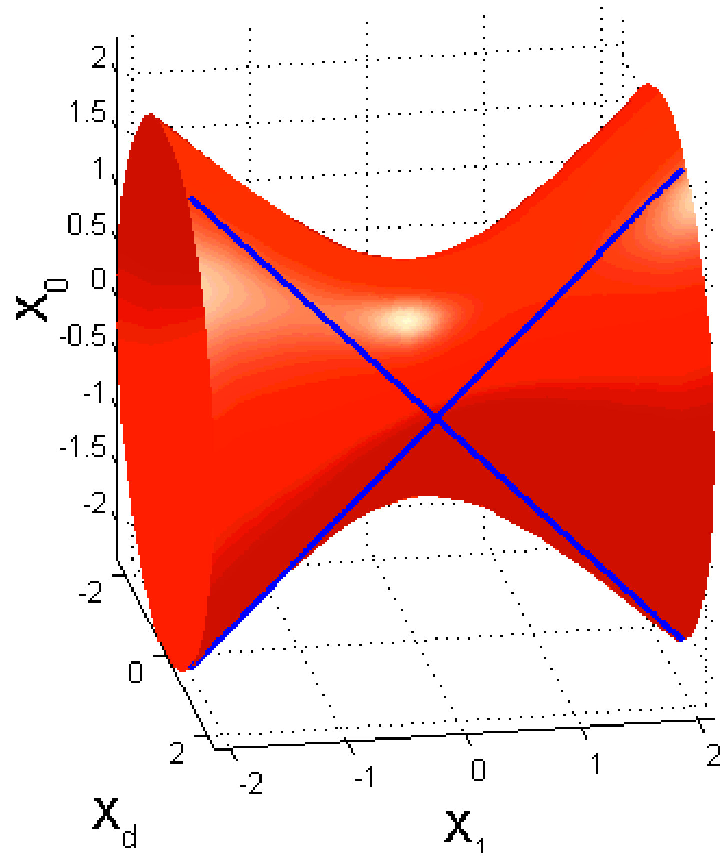


$$X_0^2 - X_1^2 - \ldots - X_{d-1}^2 + X_d^2 = R^2$$

$E^{(2,d-1)}$: $\eta_{\mu\nu}$ = diag(**1,-1,……,-1,1**) $SO(2, d-1)$

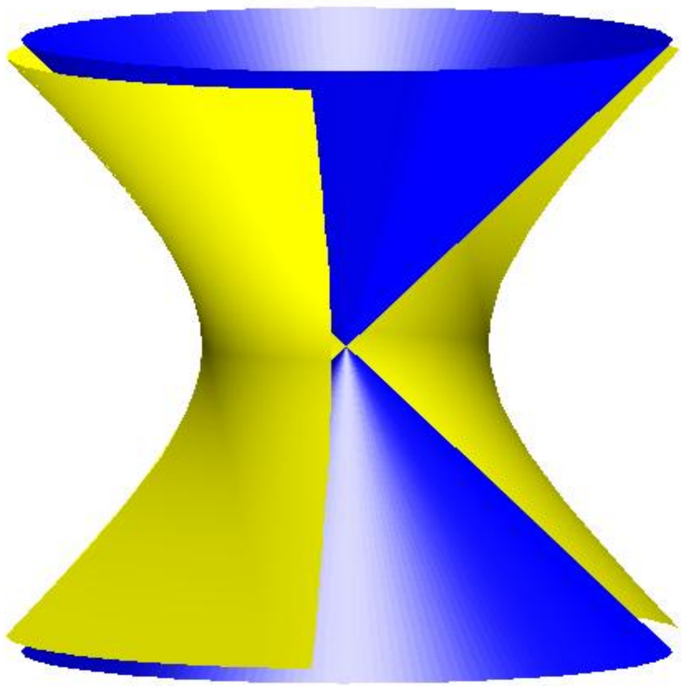
dS and AdS QFT: What Are the Problems?





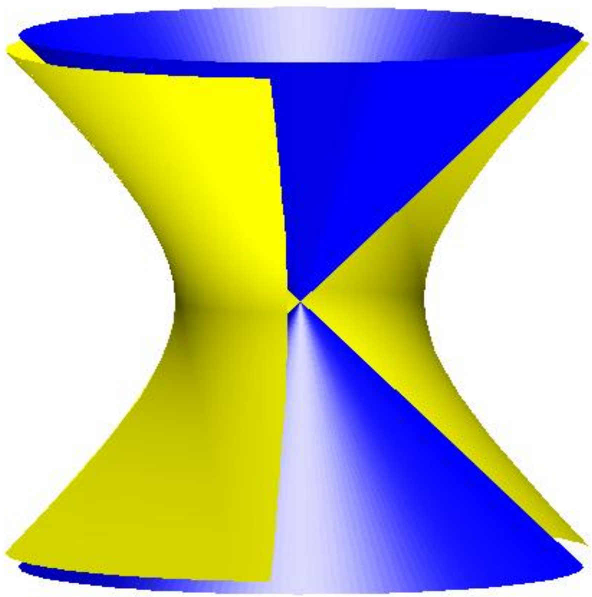
The asymptotic cone

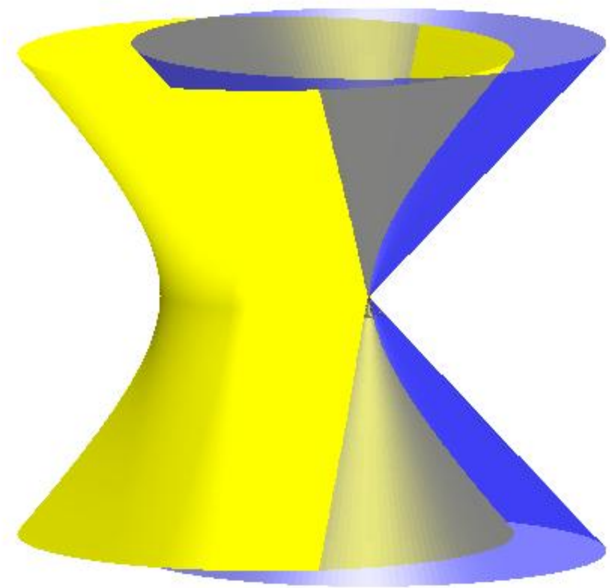
$$\{\xi_0^2 - \xi_1^2 - \ldots - \xi_d^2 = 0\}$$



$M^{(d+1)}$: $\eta_{\mu\nu}$ = diag(**1**,**-1**,**……**,**-1**)

The asymptotic cone: causal structure

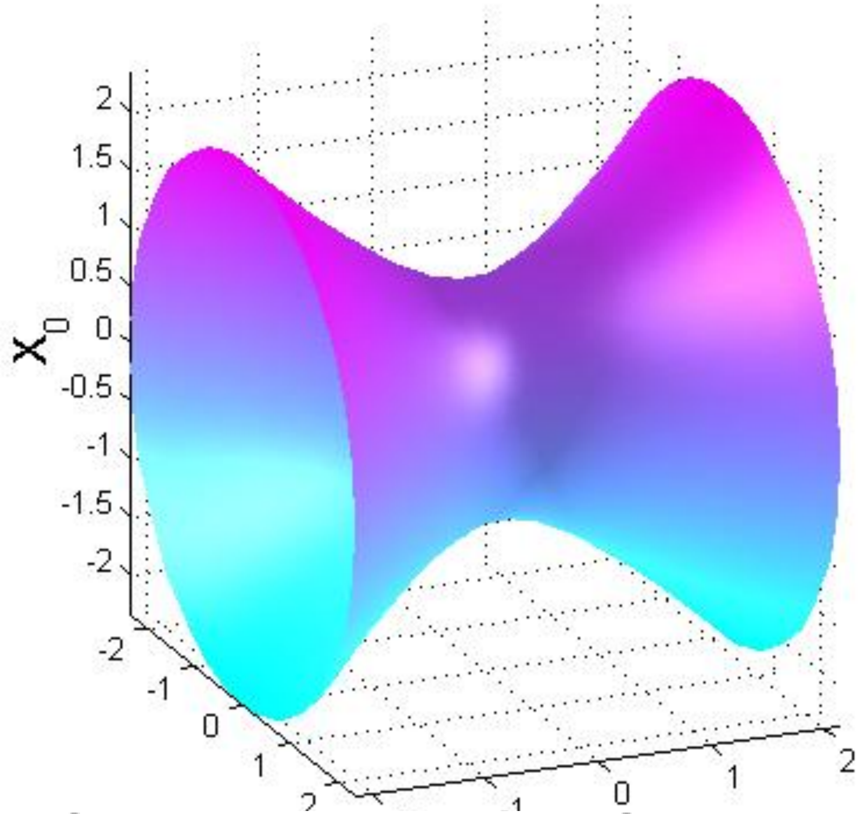




X, Y are spacelike separated iff $(X-Y)^2 < 0$ ($X - Y$ is outside the cone)

$$(X-Y)^2 = X^2 + Y^2 - 2X \cdot Y = -2R^2 - 2X \cdot Y$$

The anti de Sitter universe

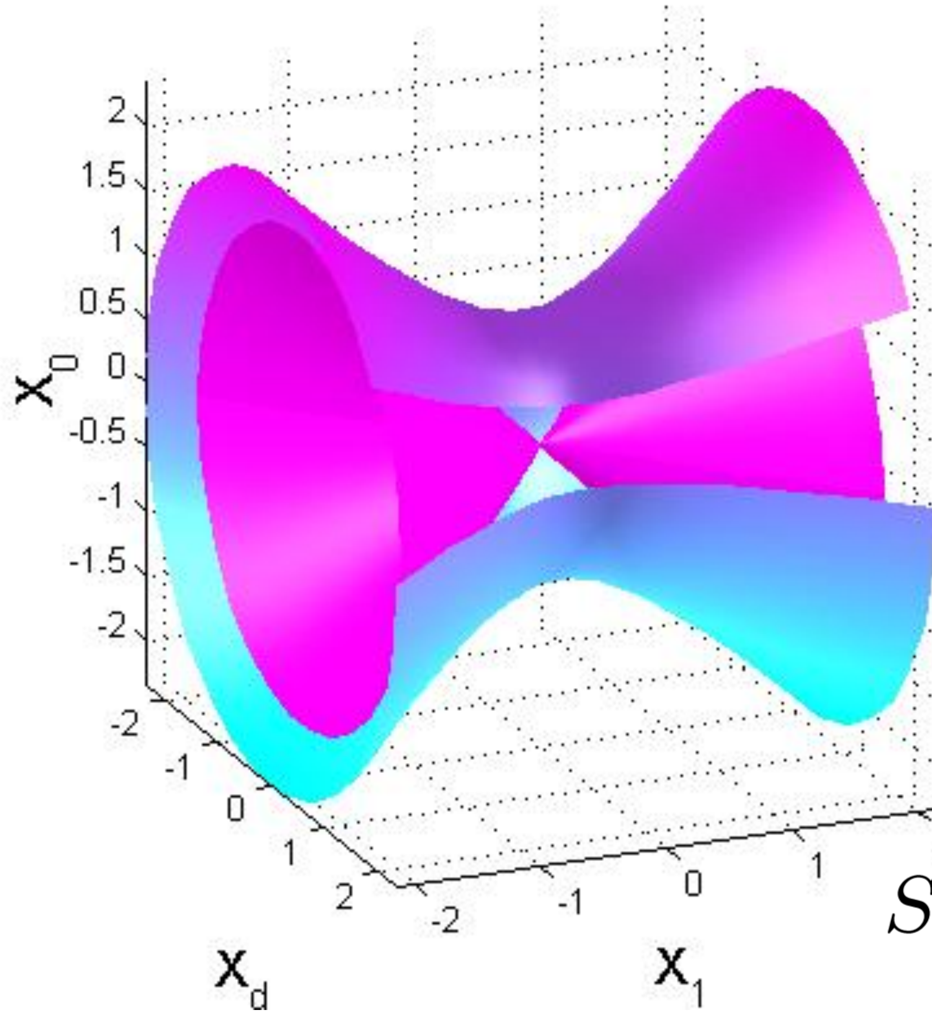


$$X_0^2 - X_1^2 - \ldots - X_{d-1}^2 + X_d^2 = R^2$$

$E^{(2,d-1)}$: $\eta_{\mu\nu}$ = diag(**1,-1,……,-1,1**) $SO(2, d-1)$

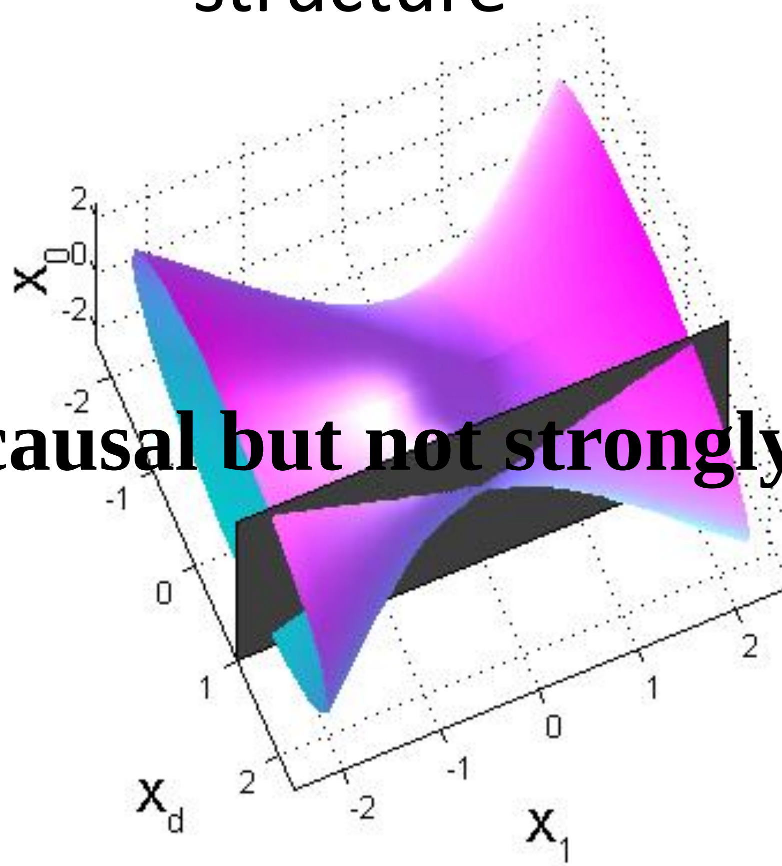
The asymptotic cone

$$\{\xi_0^2 - \xi_1^2 - \ldots - \xi_{d-1}^2 + \xi_d^2 = 0\}$$



$SO(2, d-1)$

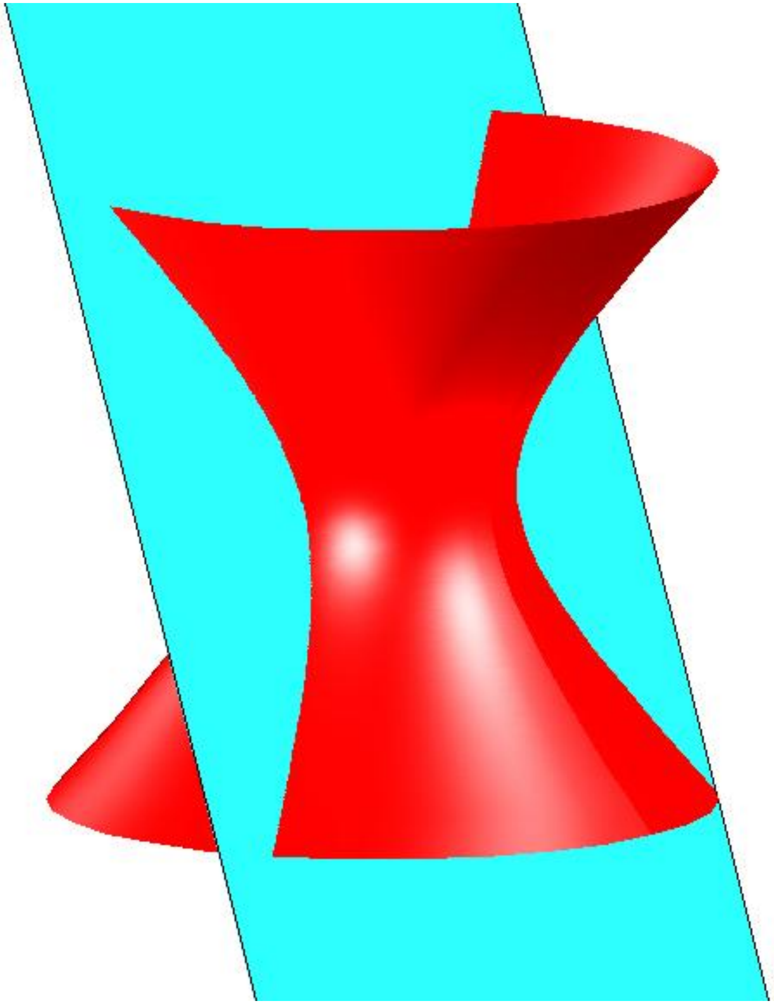
The asymptotic cone provides the causal structure



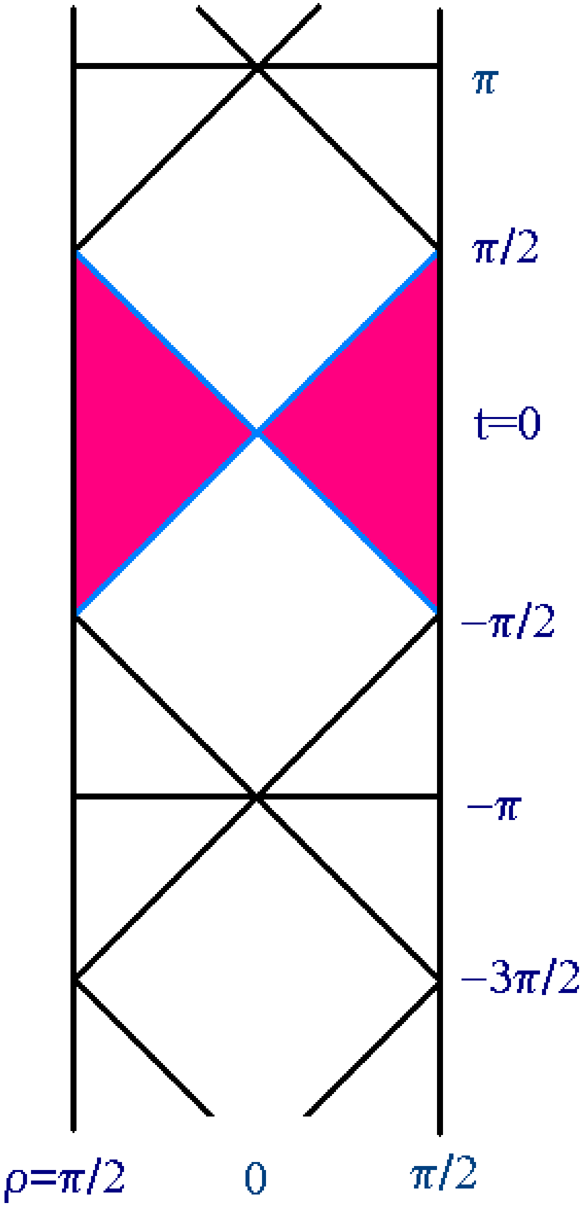
AdS: causal but not strongly causal

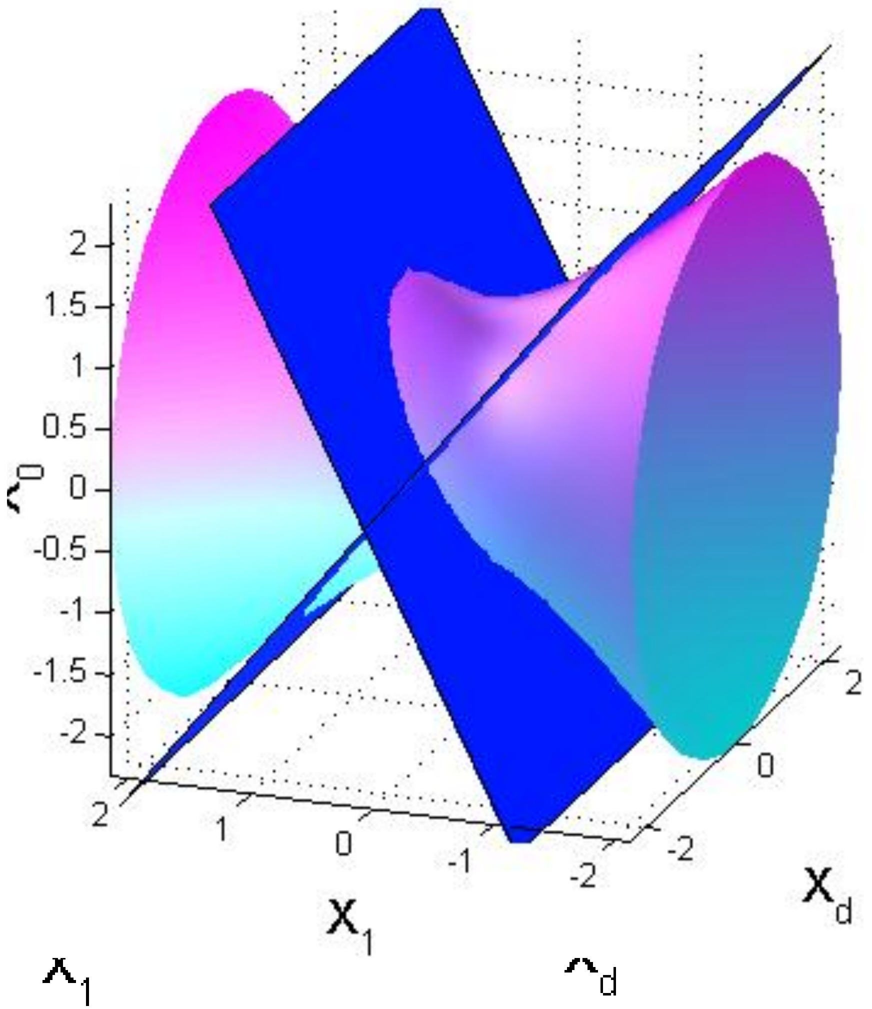
X e Y on AdS are spacelike separated iff $(X-Y)^2 < 0$
(in the ambient space sense)

ds timelike geodesics

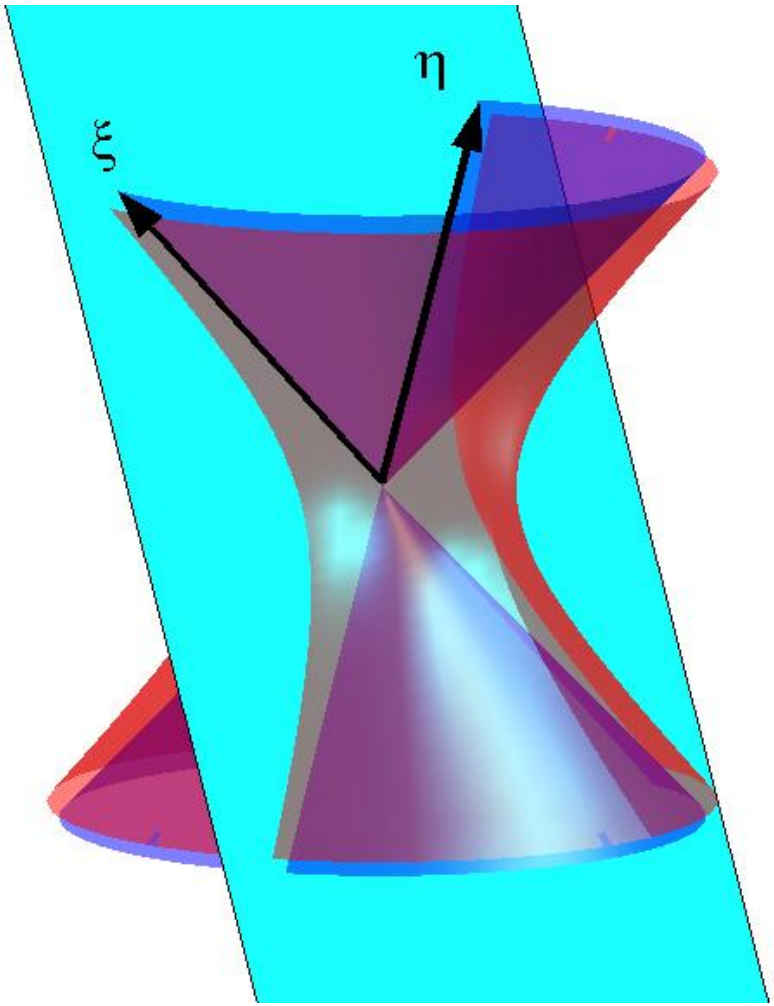


AdS: timelike geodesics





The asymptotic cone as the de Sitter momentum space



Geodesics: de Sitter

$$X_\mu(\tau) = \frac{R}{\sqrt{2\xi \cdot \eta}} \left(\xi_\mu e^{\frac{c\tau}{R}} - \eta_\mu e^{-\frac{c\tau}{R}} \right)$$

Minkowski

$$x_\mu(\tau) = x_\mu(0) + \frac{p_\mu \tau}{mc}$$

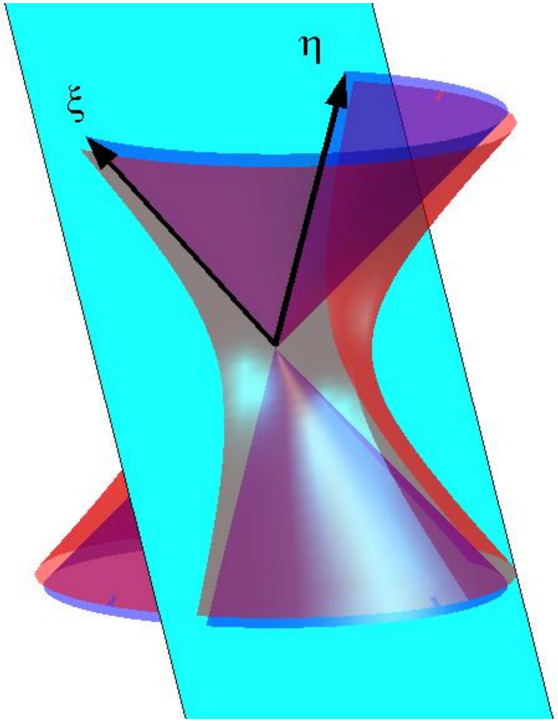
$$X_\mu(0) = \frac{R}{\sqrt{2\xi \cdot \eta}} (\xi_\mu - \eta_\mu)$$

$$X(\tau) = X(0) e^{-\frac{c\tau}{R}} + \frac{kR\xi}{m} \sinh \frac{c\tau}{R}.$$

Conserved quantities

$$X_\mu(\tau) = \frac{R}{\sqrt{2\xi \cdot \eta}} \left(\xi_\mu e^{\frac{c\tau}{R}} - \eta_\mu e^{-\frac{c\tau}{R}} \right)$$

$$K_{\xi,\eta} = mc \frac{\xi \wedge \eta}{\xi \cdot \eta}$$



Energy

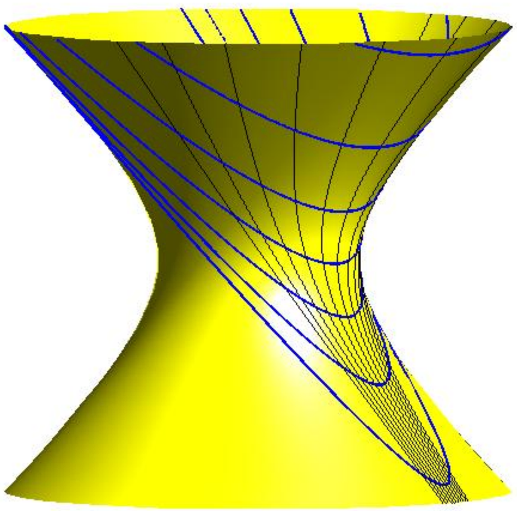
- In special relativity the energy of a particle is measured relative to an arbitrary given Lorentz frame, being the zero component of a four-vector.
- This picture does not extend to the de Sitter case where frames are defined only locally.
- The maximal symmetry of the de Sitter universe allows for the energy of a pointlike particle to be defined relative to just one reference massive free particle understood conventionally to be at rest (a sharply localized observer)
- The energy of the free particle (ξ,η) with respect to the reference geodesic (u,v) is defined as follows

$$E = E_{(\xi,\eta)}(u,v) = -\frac{c\,K_{(\xi,\eta)}(u,v)}{u \cdot v} = E_{(u,v)}(\xi,\eta) \ .$$

Energy

Geodesics parameterized by (ξ, X) and (u, Y)

$$E = mc^2 \frac{(u \cdot X)(\xi \cdot Y) - (X \cdot Y)(\xi \cdot u)}{(\xi \cdot X)(u \cdot Y)}$$

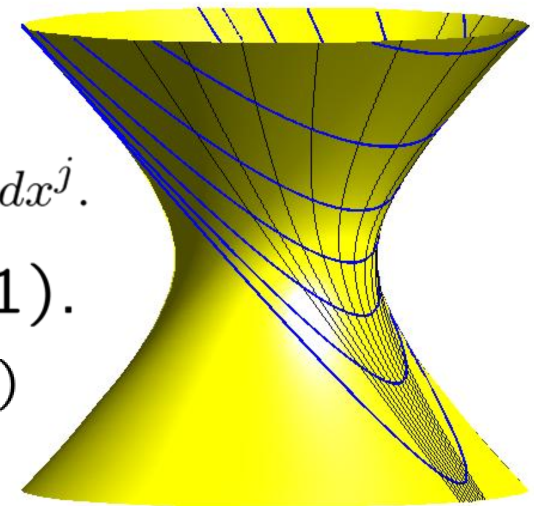


$$\begin{cases} Y_0 &= R \sinh \frac{t}{R} + \frac{1}{2R} e^{\frac{t}{R}} |\vec{x}|^2 \\ Y_1 &= \exp\left(\frac{t}{R}\right) x_1 \\ Y_2 &= \exp\left(\frac{t}{R}\right) x_2 \\ Y_3 &= \exp\left(\frac{t}{R}\right) x_3 \\ Y_4 &= R \cosh \frac{t}{R} - \frac{1}{2R} e^{\frac{t}{R}} |\vec{x}|^2 \end{cases}$$

Geodesics observer at rest at the origin

$$Y = (0, 0, 0, 0, R), \quad u = \lambda(1, 0, 0, 0, 1)$$

Energy



$$ds^2 = c^2dt^2 - e^{2ct/R}\delta_{ij}dx^idx^j = c^2dt^2 - a^2(t)\delta_{ij}dx^idx^j.$$

$Y = (0,0,0,0,R)$ and $u = \lambda(1,0,0,0,1)$.

Energy of the particle parameterized by (ξ, X) as seen by te reference observer (u, Y)

$$E = mc^2\frac{(u\cdot X)(\xi\cdot Y) - (X\cdot Y)(\xi\cdot u)}{(\xi\cdot X)(u\cdot Y)} = \frac{kc^2}{R}(\xi^0X^4 - \xi^4X^0).$$

$$X_\mu(\tau) = \frac{R}{\sqrt{2\xi\cdot\eta}}\left(\xi_\mu e^{\frac{c\tau}{R}} - \eta_\mu e^{-\frac{c\tau}{R}}\right) \qquad \xi = \frac{m}{kR}\left(X(0) + \frac{R}{c}\frac{dX(0)}{d\tau}\right).$$

$$E = mc^2\frac{dt}{d\tau} - \frac{c}{R}x^ip_i = \frac{mc^2}{\sqrt{1 - a^2(t)\frac{v^2}{c^2}}} - \frac{c}{R}x^ip_i$$

$$v^i = \frac{dx^i}{dt}, \qquad p_i = -me^{2ct/R}\frac{dx^i}{d\tau} = -\frac{ma^2(t)v^i}{\sqrt{1 - a^2(t)\frac{v^2}{c^2}}}.$$

Energy

- E can be interpreted as the correct de Sitter energy of the particle is confirmed by noting that it is the conserved quantity associated to the invariance of the particle action under time translation.
- Indeed, since in flat coordinates the spatial distances dilate in the course of time by the exponential factor the expression of an infinitesimal symmetry under time evolution is

$$t \longrightarrow t + \epsilon,$$
$$x^i \longrightarrow x^i - \frac{c}{R} x^i \epsilon.$$

- The action

$$S = -mc \int \sqrt{1 - e^{\frac{2ct}{R}} \frac{v^i v^j}{c^2} \delta_{ij}} \; dt$$

- is invariant and E can be obtained from S by standard methods

Classical scattering

$$b_1 + b_2 \longrightarrow c_1 + \ldots + c_M$$

Problem: find the outgoing momenta (ξ_f, η_f) given the ingoing ones (χ_i, ζ_i).

$$X_\mu(\tau) = \frac{R}{\sqrt{2\xi \cdot \eta}} \left(\xi_\mu e^{\frac{c\tau}{R}} - \eta_\mu e^{-\frac{c\tau}{R}} \right)$$

$$K_i = m_i c \, \frac{\chi_i \wedge \zeta_i}{\chi_i \cdot \zeta_i} \qquad K_f = m_f c \, \frac{\xi_f \wedge \eta_f}{\xi_f \cdot \eta_f}$$

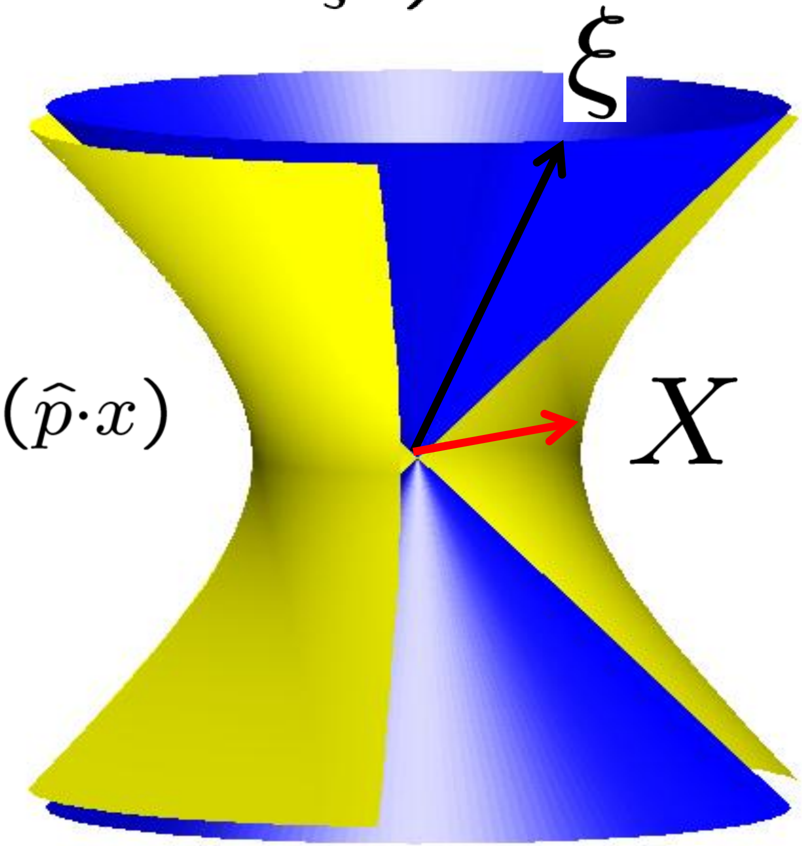
$$\sum_{i=1}^{2} K_i = \sum_{f=1}^{M} K_f, \qquad \frac{\chi_i - \zeta_i}{\sqrt{\chi_i \cdot \zeta_i}} = X = \frac{\xi_f - \eta_f}{\sqrt{\xi_f \cdot \eta_f}} \; ,$$

de Sitter plane waves

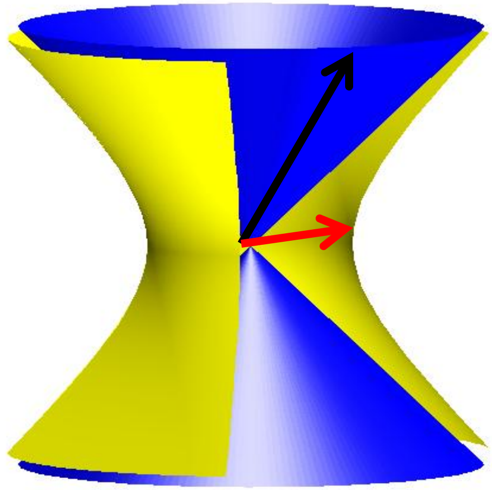
$$\psi_\lambda(X,\xi) = (X\cdot\xi)^\lambda =$$
$$= (X^0\xi^0 - X^1\xi^1 - \ldots - X^d\xi^d)^\lambda$$

$$\lambda \in \mathbf{C}, \quad \xi^2 = 0$$

$$\psi(x,p) = e^{ip\cdot x} = e^{im(\widehat{p}\cdot x)}$$



de Sitter plane waves



$$\Box(X\cdot\xi)^\lambda = \lambda(\lambda+d-1)(X\cdot\xi)^\lambda$$

Involution:

$$\lambda \longrightarrow \bar\lambda = -\lambda-(d-1)$$

$$\lambda+\bar\lambda = -(d-1)$$

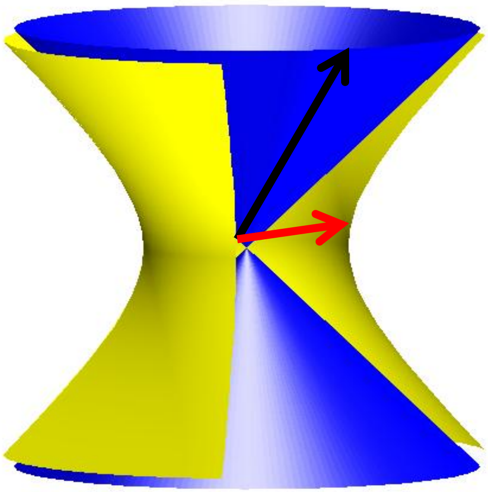
$$\Box(X\cdot\xi)^{-\lambda-d+1} = (-\lambda-d+1)(-\lambda)(X\cdot\xi)^{-\lambda-d+1}$$

Scalar waves with (complex) squared mass:

$$m^2 = \lambda\bar\lambda$$

$$\left(\Box+\lambda\bar\lambda\right)(X\cdot\xi)^\lambda = 0, \qquad \left(\Box+\lambda\bar\lambda\right)(X\cdot\xi)^{\bar\lambda} = 0$$

Principal de Sitter waves



$$\lambda = -\tfrac{d-1}{2} + i\nu\,, \qquad \nu \in \mathbf{R}$$

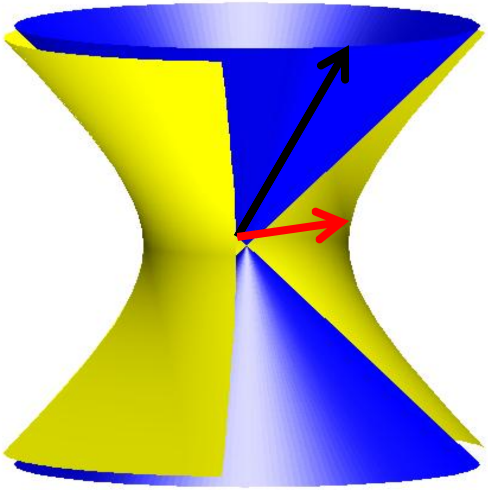
$$\psi_\lambda(X,\xi) = (X\cdot\xi)^{-\frac{d-1}{2}+i\nu}$$

$$\bar\lambda = -\lambda - (d-1) = -\tfrac{d-1}{2} - i\nu$$

$$\psi_{\bar\lambda}(X,\xi) = (X\cdot\xi)^{-\frac{d-1}{2}-i\nu} = \overline{\psi_\lambda(X,\xi)}$$

$$m^2 = \lambda\bar\lambda = |\lambda|^2 = \left(\tfrac{d-1}{2}\right)^2 + \nu^2 > \left(\tfrac{d-1}{2}\right)^2$$

Complementary de Sitter waves



$$\lambda = -\frac{d-1}{2} + \nu\,, \qquad \nu \in \mathbf{R}$$

$$\psi_\lambda(X,\xi) = (X\cdot\xi)^{-\frac{d-1}{2}+\nu}$$

These waves do not oscillate!

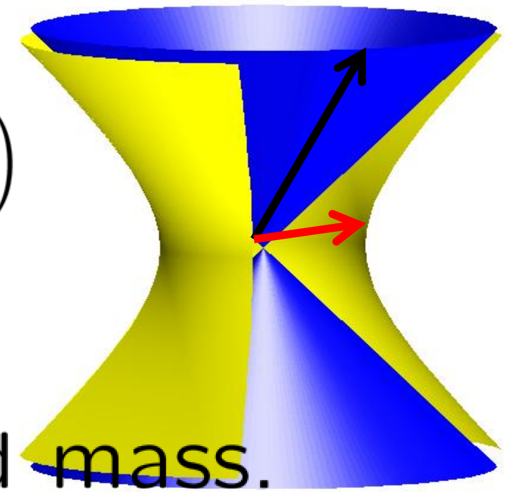
$$\bar\lambda = -\lambda - (d-1) = -\frac{d-1}{2} - \nu$$

$$\psi_{\bar\lambda}(X,\xi) = (X\cdot\xi)^{-\frac{d-1}{2}-\nu} \neq \overline{\psi_\lambda(X,\xi)} = \psi_\lambda(X,\xi)$$

$$m^2 = \lambda\bar\lambda = \left(\frac{d-1}{2}\right)^2 - \nu^2$$

$$-\left(\frac{d-1}{2}\right) < \nu < \left(\frac{d-1}{2}\right)$$

Discrete de Sitter waves



$$\lambda = -\tfrac{d-1}{2} + \nu\,,\ \ \nu \in \mathbf{R},\ \ |\nu| > \left(\tfrac{d-1}{2}\right)$$

$$\psi_\lambda(X,\xi) = (X\cdot\xi)^{-\frac{d-1}{2}+\nu}$$

have real but negative squared mass.

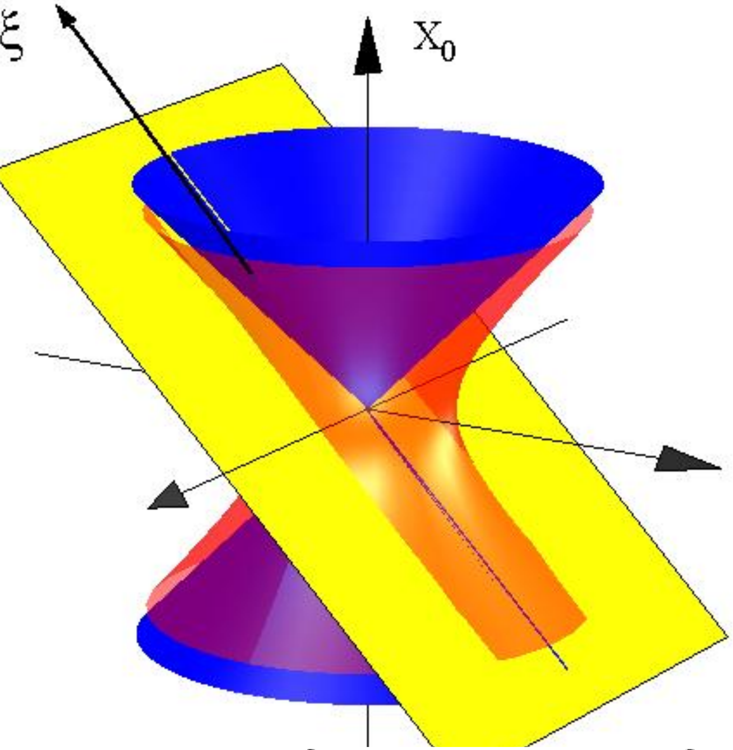
$$m^2 = \lambda\bar{\lambda} = \left(\tfrac{d-1}{2}\right)^2 - \nu^2 < 0$$

dS Tachyons?

$(X\cdot\xi)^n$ and $(X\cdot\xi)^n \log(X\cdot\xi)$, n integer

$(X\cdot\xi)^{-n-d+1}$ and $(X\cdot\xi)^{-n-d+1}\log(X\cdot\xi)$

The plane waves are however irregular



$$\psi_\lambda(X,\xi) = (X\cdot\xi)^\lambda$$

$$X \in dS : (X\cdot\xi) = 0$$

$$(X\cdot\xi)^\lambda \to |X\cdot\xi|^\lambda \left(a(\lambda)\theta(X\cdot\xi) + b(\lambda)\theta(-X\cdot\xi)\right)$$

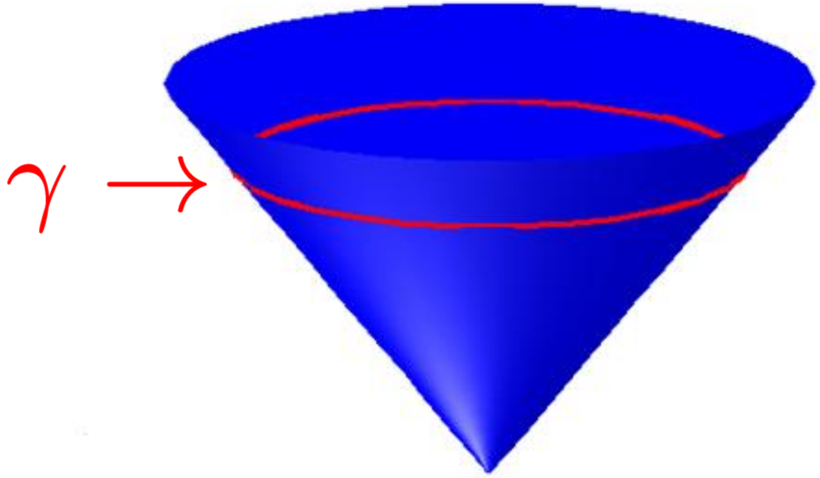
Choice of a and b: go to QFT!

skip

dS: construction of two-point functions

$$W(x_1, x_2) = \int e^{-ip\cdot x_1} e^{ip\cdot x_2} \theta(p^0)\delta(p^2 - m^2) dp$$

$$F_{\lambda,\gamma}(X_1, X_2) = \int (X_1 \cdot \xi)^{\lambda} (\xi \cdot X_2)^{-\lambda-d+1} d\mu_\gamma(\xi)$$



$$\left(\Box_{X_i} + \lambda\bar{\lambda}\right) F_\lambda(X_1, X_2) = 0,$$

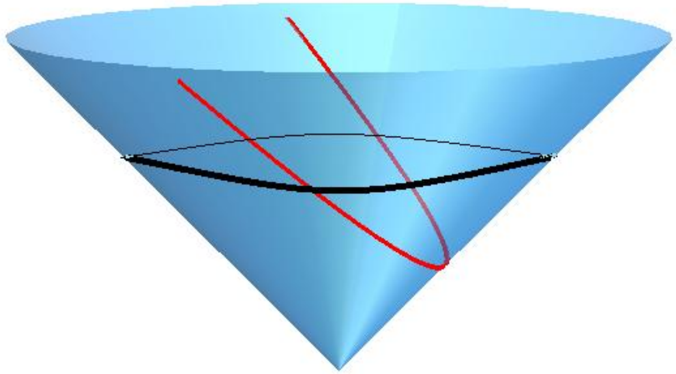
They are dS invariant

Plane waves are homogeneous functions of ξ

$$\psi(X, a\xi) = (X \cdot a\xi)^{\lambda} = a^{\lambda}\psi(X, \xi)$$

$$(X_1 \cdot a\xi)^{\lambda}(a\xi \cdot X_2)^{\bar{\lambda}} = a^{\lambda+\bar{\lambda}}(X_1 \cdot \xi)^{\lambda}(\xi \cdot X_2)^{\bar{\lambda}}$$

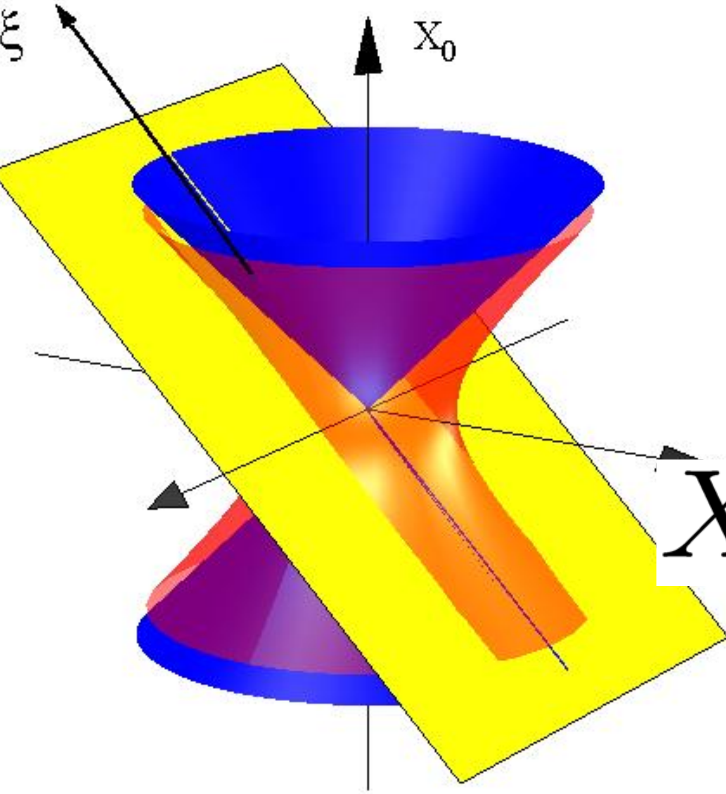
$$\lambda + \bar{\lambda} = -(d-1)$$



$$F_{\lambda,\gamma}(X_1, X_2) = \int (X_1 \cdot \xi)^{\lambda}(\xi \cdot X_2)^{-\lambda-d+1} d\mu_{\gamma}(\xi)$$

$$F_{\lambda}(gX_1, gX_2) = F_{\lambda}(X_1, X_2) = F_{\lambda}(X_1 \cdot X_2)$$

How to chose the right coefficients?



$$\psi(X,\xi) = (X\cdot\xi)^\lambda$$

$$X \in dS : (X\cdot\xi) = 0$$

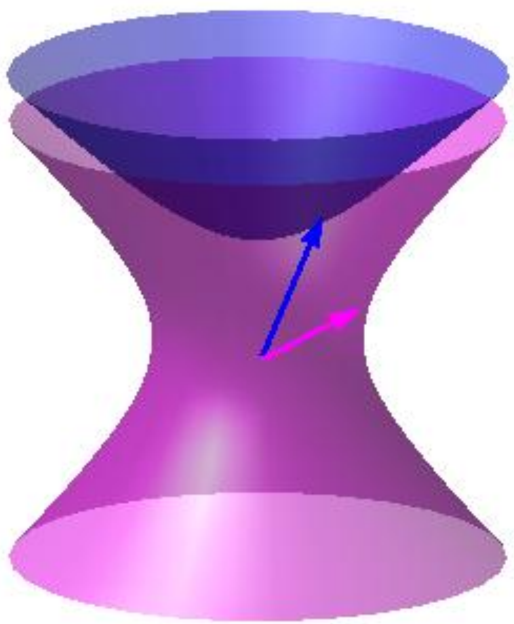
$$(X\cdot\xi)^\lambda \to |X\cdot\xi|^\lambda \left(a(\lambda)\theta(X\cdot\xi) + b(\lambda)\theta(-X\cdot\xi)\right)$$

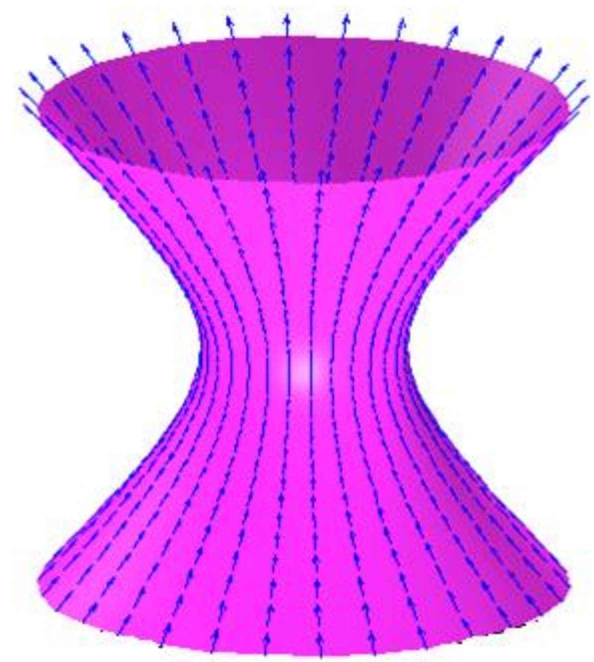
de Sitter tubes

$$dS^c = Z_0^2 - Z_1^2 - \ldots Z_d^2 = -R^2$$

$$Z = X + iY, \quad X^2 - Y^2 = -R^2 \quad X \cdot Y = 0$$

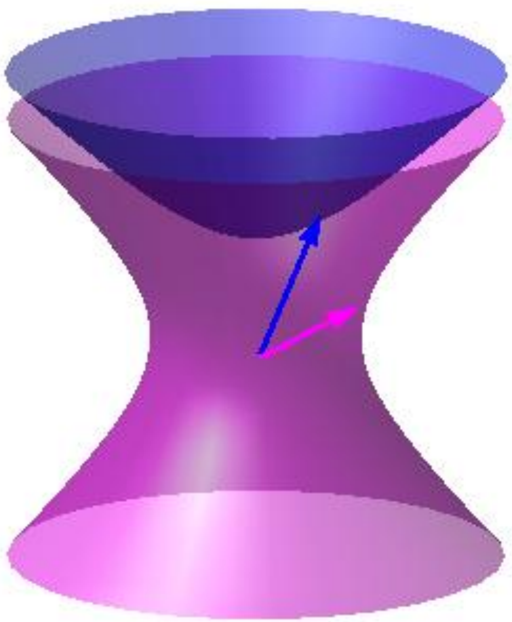
$\mathcal{T}^+ = Y$ in the forward cone.

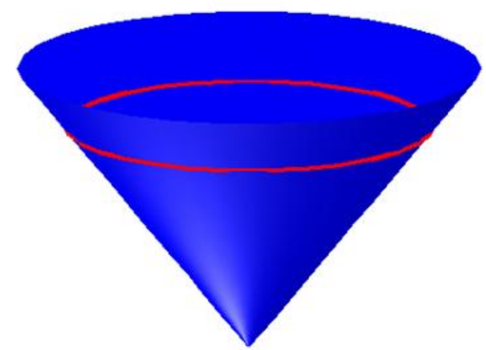




$\mathcal{T}^- = Y$ in the backward cone.

$\psi_\lambda(Z,\xi) = (Z\cdot\xi)^\lambda$ is globally well defined in both the past and future tubes because the imaginary part $Y\cdot\xi$ is always positive (negative) for $Z\in\mathcal{T}^+$ (alternatively $Z\in\mathcal{T}^+$)





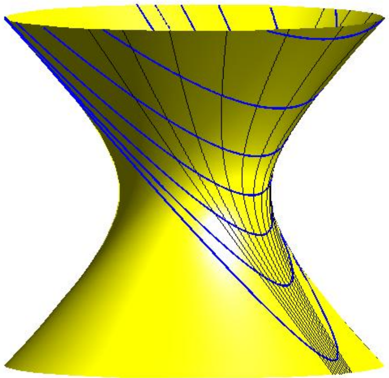
Boundary values on the reals:

$$(X\cdot\xi)^\lambda_\pm \to |X\cdot\xi|^\lambda\Big(\theta(X\cdot\xi) + e^{\pm i\pi\lambda}\theta(-X\cdot\xi)\Big)$$

$$\psi^{\pm}_{i\nu}(z,\xi) = (x \pm iy\cdot\xi)^{-\frac{d-1}{2}+i\nu},$$

are globally well-defined in the tubes.

$$\begin{cases} x^0 &= R\sinh\frac{t}{R} + \frac{1}{2R}\mathbf{x^2}\exp\frac{t}{R} \\ x^i &= \mathbf{x^i}\ \exp\frac{t}{R}, \\ x^d &= R\cosh\frac{t}{R} - \frac{1}{2R}\mathbf{x^2}\ \exp\frac{t}{R}. \end{cases},$$



$$\begin{cases} \xi^0 &= \frac{1}{2}(1+\eta^2) \\ \xi^i &= \eta \\ \xi^d &= \frac{1}{2}(1-\eta^2) \end{cases},$$

$$\widetilde{\psi^{\pm}_{i\nu}}(t,\mathbf{k}) = \int [\mathbf{x}(\mathbf{t}\pm\mathbf{i}\epsilon,\mathbf{x})\cdot\xi(\eta)]^{-\frac{3}{2}+\mathbf{i}\nu}\mathbf{e}^{\mathbf{ikx}}\mathbf{dx}$$

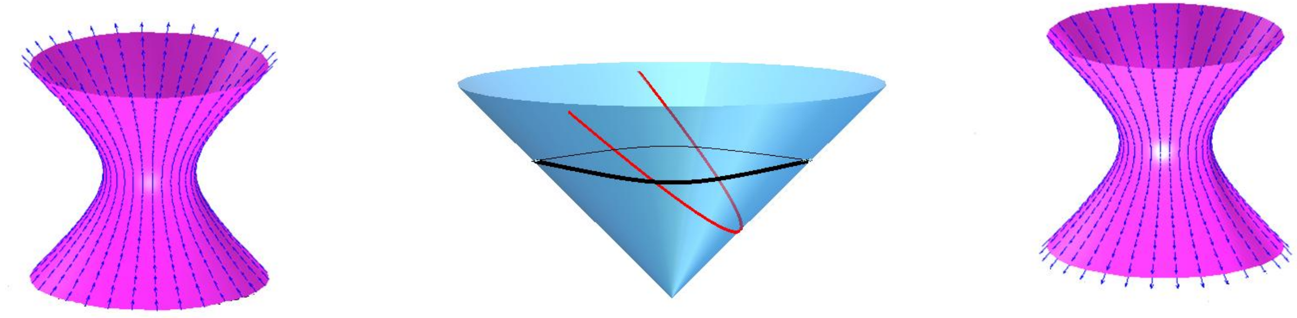
$$\widetilde{\psi^{+}_{i\nu}}(t,\mathbf{k}) = \frac{i\pi}{\Gamma\left(\frac{3}{2}-\nu\right)}(\mathbf{2}\pi\mathbf{e}^{-\mathbf{t}})^{\frac{3}{2}}\exp(\mathbf{ik}\eta)\,\mathbf{k}^{-\nu}\,\mathbf{H}^{(\mathbf{2})}_{\mathbf{i}\nu}(\mathbf{ke}^{-\mathbf{t}})$$

$$\widetilde{\psi^{-}_{i\nu}}(t,\mathbf{k}) = \frac{i\pi}{\Gamma\left(\frac{3}{2}-\nu\right)}(\mathbf{2}\pi\mathbf{e}^{-\mathbf{t}})^{\frac{3}{2}}\exp(\mathbf{ik}\eta)\,\mathbf{k}^{-\nu}\,\mathbf{H}^{(\mathbf{1})}_{\mathbf{i}\nu}(\mathbf{ke}^{-\mathbf{t}})$$

Fourier representation for Bunch-Davies aka Euclidean aka ……two-point functions

For $Z_1 \in \mathcal{T}^-$ e $Z_2 \in \mathcal{T}^+$

$$W_\lambda(Z_1, Z_2) = \int_\gamma (Z_1 \cdot \xi)^\lambda \ (\xi \cdot Z_2)^{-\lambda-(d-1)} d\mu(\xi)$$



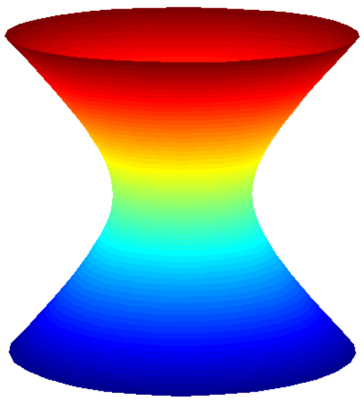
To be compared with the standard flat case:

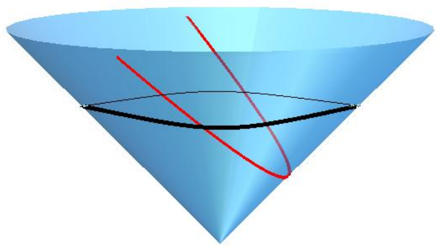
$$W(z_1 - z_2) = \int e^{-ip\cdot z_1} e^{ip\cdot z_2} \theta(p^0) \delta(p^2 - m^2) d^4p$$

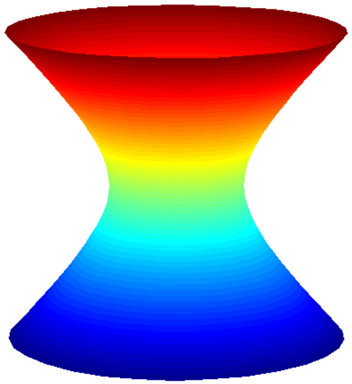
$z_1 \in T^- \ \ z_2 \in T^+$

Fourier representation for Bunch-Davies aka Euclidean on the real manifold

$$W_\lambda(X_1, X_2) = \int_\gamma |X_1 \cdot \xi|^\lambda \, |X_2 \cdot \xi|^{\bar{\lambda}} \times$$
$$\times \left(\theta(X_1 \cdot \xi) + e^{-i\pi\lambda}\theta(-X_1 \cdot \xi)\right) \left(\theta(X_2 \cdot \xi) + e^{i\pi\bar{\lambda}}\theta(-X_2 \cdot \xi)\right) d\mu(\xi)$$





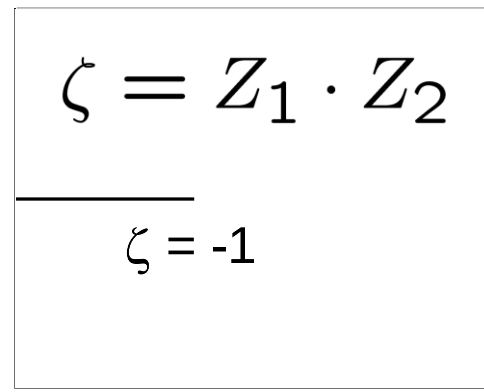


$$
\begin{aligned}
W_\lambda(Z_1,\ Z_2) &= \int_\gamma (Z_1\cdot\xi)^\lambda\ (\xi\cdot Z_2)^{-\lambda-(d-1)}d\mu(\xi) \\
&= \frac{\Gamma(-\lambda)\Gamma(\lambda+d-1)}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\ {}_2F_1\left(-\lambda,\ \lambda+d-1;\ \frac{d}{2};\ \frac{1-\zeta}{2}\right)\ . \\
&= \frac{\Gamma\left(-\lambda\right)\Gamma\left(\lambda+d-1\right)}{2(2\pi)^{\frac{d}{2}}}\ (\zeta^2-1)^{-\frac{d-2}{4}}\,P^{-\frac{d-2}{2}}_{-\lambda-\frac{d}{2}}(\zeta)
\end{aligned}
$$

a) $W(Z_2, Z_2)$ is invariant under the complex de Sitter group

$W(Z_1, Z_2)$ is maximally analytic.

The cut reflects causality



b) $W(X_1, X_2)$ is b.v.of $W(Z_1, Z_2)$ from $\mathcal{T}^-\times\mathcal{T}^+$

The permuted function $W(X_2, X_1)$

is b.v.of the same $W(Z_1, Z_2)$ from $\mathcal{T}^+\times\mathcal{T}^-$

One point tubes (AdS)

$Z = X + iY$

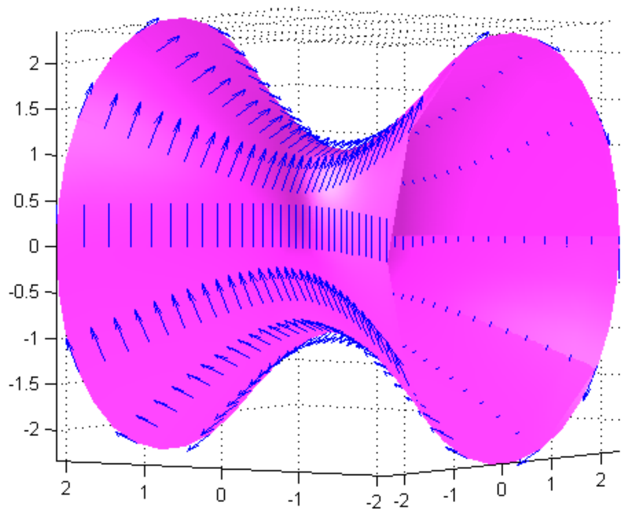
$$AdS^c = Z_0^2 - Z_1^2 - \dots Z_{d-1}^2 + Z_d^2 = +R^2$$

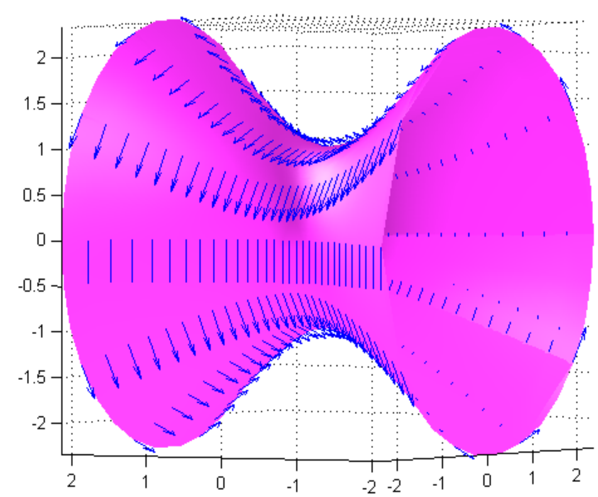
$$\mathbf{X}^2 - \mathbf{Y}^2 = R^2$$

$$\mathbf{X} \cdot \mathbf{Y} = 0$$

$$\mathcal{T}^{\rightarrow} = \left\{ \begin{array}{l} Z \in AdS^{(c)} : Y^2 > 0, \\ Y_0 X_d - Y_d X_0 > 0 \end{array} \right\}$$

$$\mathcal{T}^{\leftarrow} = \left\{ \begin{array}{l} Z \in AdS^{(c)} : Y^2 > 0, \\ Y_0 X_d - Y_d X_0 < 0 \end{array} \right\}$$



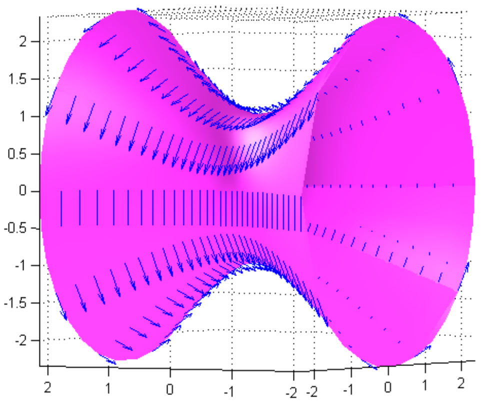


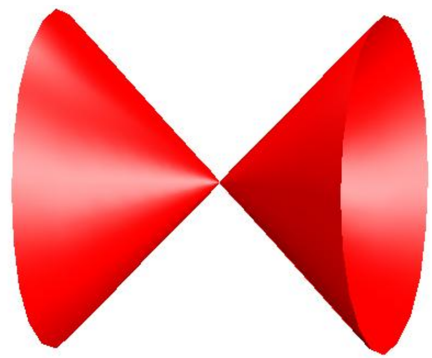
AdS Klein Gordon fields

$$\Psi(Z,\xi) = (\xi \cdot Z)^s$$

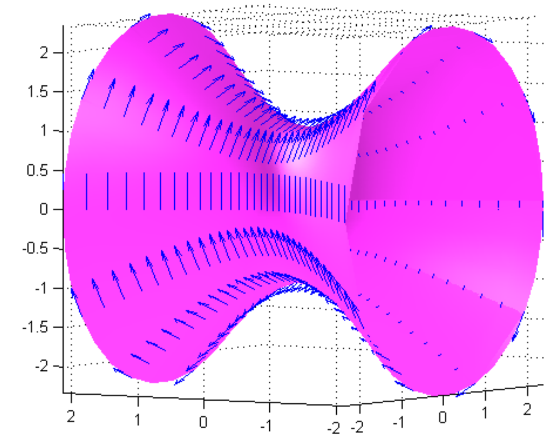
$$W(Z,Z') = \int_{\gamma(z)} (Z\cdot\xi)^{-\frac{d-1}{2}+n}(\xi\cdot Z')^{-\frac{d-1}{2}-n} d\mu(\xi)$$

$Z \in \mathcal{T}^{\leftarrow}$





$Z' \in \mathcal{T}^{\rightarrow}$



γ(z) is a (point dependent) path in the complex asymptotic cone bordered by the plane $\xi\cdot Z = 0$

What about positivity?

$$\int W_\lambda(X_1, X_2)\bar{f}(X_1)f(X_2)dX_1dX_2 \geq 0$$

For principal fields positivity is obvious

$$W_\nu(Z_1, Z_2) = \int_\gamma (Z_1 \cdot \xi)^{-\frac{d-1}{2}+i\nu}(\xi \cdot Z_2)^{-\frac{d-1}{2}-i\nu} d\mu(\xi)$$

dS-Fourier transform

$$\tilde{f}(\xi, \nu) = \int (\xi \cdot X)_+^{-\frac{d-1}{2}-i\nu} d\mu(\xi) f(x) dx$$

$$\int W_\nu(X_1, X_2)\bar{f}(X_1)f(X_2)dX_1dX_2 = \int_\gamma |\tilde{f}(\xi, \nu)|^2 d\mu(\xi) \geq 0$$

Complementary fields (inflation)

$$W_{\nu'}(Z_1, Z_2) = \int_\gamma (Z_2 \cdot \xi)^{-\frac{d-1}{2}+\nu'} (\xi \cdot Z_2)^{-\frac{d-1}{2}-\nu'} d\mu(\xi)$$

The proof is a little more difficult;
based on the following relation valid for $\nu > 0$:

$$[(X \pm iY) \cdot \xi']^{-\frac{d-1}{2}+\nu} =$$

$$= \frac{e^{\pm i\pi\nu}\Gamma\left(\frac{d-1}{2}+\nu\right)}{(2\pi)^{\frac{d-1}{2}} 2^\nu \Gamma(\nu)} \int (\xi \cdot \xi')^{-\frac{d-1}{2}+\nu} [(X \pm iY) \cdot \xi']^{-\frac{d-1}{2}-\nu} d\mu(\xi')$$

skip

De Sitter Tachyons

$$W_\lambda(Z_1,\ Z_2) = \Gamma(-\lambda)\, G_\lambda(\zeta)\ , \quad \zeta = Z_1 \cdot Z_2\,,$$

$$G_\lambda(\zeta) = \frac{\Gamma(\lambda+d-1)}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\, {}_2F_1\left(-\lambda,\ \lambda+d-1;\ \frac{d}{2};\ \frac{1-\zeta}{2}\right)\ .$$

$W_n(\zeta) = \infty$
$G_n(\zeta) = c_n {}_2F_1\left(-n,\ n+d-1;\ \frac{d}{2};\ \frac{1-\zeta}{2}\right)$ is a polynomial.

$C_n(X_1, X_2) = \lim_{\lambda\to n} C_\lambda(X_1, X_2) =$
$= \lim_{\lambda\to n}[W_\lambda(X_1, X_2) - W_\lambda(X_2, X_1)]$ exists and is nontrivial.

$$\widehat{W}_n(Z_1, Z_2) = \lim_{\lambda\to n} \Gamma(-\lambda)\ [G_\lambda(\zeta) - G_n(\zeta)]$$

$\widehat{W}_n$ has the right commutator:
$\widehat{W}_n(X_1, X_2) - \widehat{W}_n(X_2, X_1) = C_n(X_1, X_2)$

The field equation gets an anomaly

$$\widehat{W}_n(Z_1, Z_2) = \lim_{\lambda \to n} \Gamma(-\lambda) \; [G_\lambda(\zeta) - G_n(\zeta)]$$

$$[\Box - n(n+d-1)] \, \widehat{W}_n(\zeta) = G_n(\zeta),$$

$$[\Box - n(n+d-1)] \, \phi(X) = Q_n(X)$$

$$Q_n^- \, |phys\rangle = 0$$

Fourier representation

$$\widehat{W}_n(z_1, z_2) = W_n(z_1, z_2) - F_n^1(z_1, z_2) - F_n^2(z_1, z_2) + G_n(z_1,\ z_2).$$

$$\begin{aligned}
W_n(z_1, z_2) &= \int_\gamma \int_\gamma (z_1 \cdot \xi)^{1-d-n} \, (\xi \cdot \xi')^n \, \log(\xi \cdot \xi') \, (z_2 \cdot \xi')^{1-d-n} \, d\mu(\xi) d\mu(\xi') \\
F_n^1(z_1, z_2) &= \int_\gamma \int_\gamma \log(z_1 \cdot \xi)(z_1 \cdot \xi)^{1-d-n} \, (\xi \cdot \xi')^n \, (z_2 \cdot \xi')^{1-d-n} \, d\mu(\xi) d\mu(\xi') \\
F_n^2(z_1, z_2) &= \int_\gamma \int_\gamma (z_1 \cdot \xi)^{1-d-n} \, (\xi \cdot \xi')^n \, \log(z_2 \cdot \xi') \, (z_2 \cdot \xi')^{1-d-n} \, d\mu(\xi) d\mu(\xi')
\end{aligned}$$

$$\Psi \in E_n, \quad E_n = \left\{ \Psi \in C_0^\infty(X_d) \ : \int G_n(x_1 \cdot x_2) \Psi(x_2) dx_2 = 0 \right\}$$

$$\widehat{W}_n(z_1 \cdot z_2)|_{E_n \times E_n} = W_n(z_1, z_2)|_{E_n \times E_n}.$$

(Working hard one can show that the theory) is local, de Sitter invariant and positive definite

The equation of motion is obviusly anomaly-free

The positive physical space disappears in the flat limit

Remarks

For $d \geq 2$ is an even integer $z = -(z_1 - z_2)^2/4 = (1+\zeta)/2$.

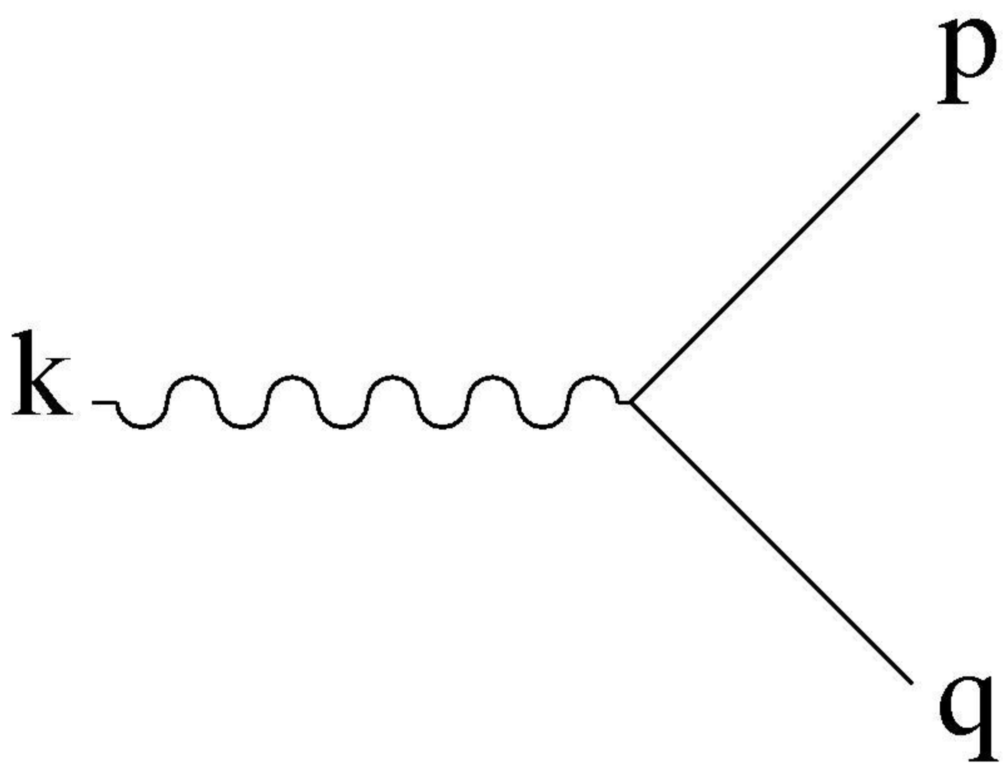
$\widehat{w}_n = z^{1-\frac{d}{2}} A(z,\ n,\ d) - \log(z) B(z,\ n,\ d) + C(z,\ n,\ d)$,

$A,\ B,\ C$ are polynomials in z

The most singular term is locally Hadamard(CCR) $(4\pi)^{-\frac{d}{2}} \Gamma\left(\frac{d}{2} - 1\right) z^{1-\frac{d}{2}}$.

A fully positive de Sitter non-invariant Allen-Folacci type quantization does not exist for $m \neq 0$. Note that the Allen-Folacci two-point function does not coincide with $\widehat{w}_n$ on the physical space.

Unstable Particles



$$\langle pq|S|k\rangle \simeq \frac{2gi(2\pi)^4}{\sqrt{8p_0q_0k_0}}\delta_4(k-p-q)$$

Interaction

Switch an interaction

$$\int \gamma\, g(x)\, \mathcal{L}(x)\, dx$$

$$\mathcal{L}(x) = \,:\phi_0(x)\phi_1(x)^{n_1}\phi_2(x)^{n_2}:$$

g(x) is an infrared cutoff . *g(x)* $\to$ 1 in the end (adiabatic limit).

Special case $\mathcal{L}(x) = \,:\phi^n(x):$

In the following

$$\mathcal{L}(x) = \,:\phi_0(x)\phi_1(x)\phi_2(x):$$

Transition probability at first order

$$\Gamma = \gamma^2 C(m_0, d)\, \rho(m_0^2; m_1, m_2) \frac{\int |F_0(v)|^2 g(v)\, dv}{\int f_0(x)\, \mathcal{W}_{m_0}(x,\ y)\, f_0(y)\, dx\, dy}$$

$$\int \mathcal{W}_{m_0}(x,\ u)\, \mathcal{W}_a(u,\ v)\, du = C(m_0,\ d)\, \delta(m_0^2 - a^2)\, \mathcal{W}_{m_0}(x,\ v).$$

$$\mathcal{W}_{m_1}(u,\ v)\mathcal{W}_{m_2}(u,\ v) = \int \rho(a^2; m_1, m_2)\, \mathcal{W}_a(u,\ v)\, da^2$$

$$F_0(v) = \int \mathcal{W}_{m_0}(v,\ x)\, f_0(x) dx$$

Projector identity

- Proiector identity: non trivial holds only for the pricipal series

$$\int w_\nu(z,\, x)\, w_{\nu'}(x,\, y)\, dx = 2\pi \coth \pi\nu \delta(\nu^2 - \nu'^2) w_\nu(z,\, y)$$

Technical intermezzo 1: computing the KL weight

Fourier (momentum space) representation of the two-point function satisfying the positivity of the energy spectrum axiom:

$$\mathcal{W}_m(u,\ v) = \frac{1}{(2\pi)^{d-1}} \int e^{-ip(u-v)} \theta(p^0) \delta(p^2 - m^2) dp$$

$$px = p^0 x^0 - p^1 x^1 - \ldots - p^{d-1} x^{d-1}$$

$$\mathcal{W}_{m_1}(u,\ v) \cdot \mathcal{W}_{m_2}(u,\ v) =$$
$$= \frac{1}{(2\pi)^{2d-2}} \int e^{-i(p_1+p_2)(u-v)} \theta(p_1^0) \delta(p_1^2 - m_1^2)\, \theta(p_2^0) \delta(p_2^2 - m_2^2)\, dp_1\, dp_2$$

Computing the KL weight

$$\rho(s;\ m_1,m_2) = (2\pi)^{1-d}\int \delta(P-p_1-p_2)\,\delta(p_1^2-m_1^2)\theta(p_1^0)\,\delta(p_2^2-m_2^2)\theta(p_2^0)\,d^dp_1\,d^dp_2\,,$$

where $P^0=\sqrt{s},\ \vec{P}=0,\ s\geq 0$.

$$\begin{aligned}
\rho(s;\ m_1,m_2) &= (2\pi)^{1-d}\int \delta(p_1^2-m_1^2)\theta(p_1^0)\,\delta\left(s-2\sqrt{s}p_1^0+m_1^2-m_2^2\right)\,d^dp_1 \\
&= \frac{(2\pi)^{1-d}\Omega_{d-1}}{2\sqrt{s}}\int_0^\infty \delta\left(r^2+m_1^2-\frac{(s+m_1^2-m_2^2)^2}{4s}\right) r^{d-2}\,dr \\
&= \frac{(2\pi)^{1-d}\Omega_{d-1}}{4\sqrt{s}}\left(\frac{(s-m_1^2-m_2^2)^2-4m_1^2m_2^2}{4s}\right)^{\frac{d-3}{2}} \\
&= \frac{(2\pi)^{1-d}\Omega_{d-1}}{4\sqrt{s}}\left(\frac{(s-(m_1+m_2)^2)(s-(m_1-m_2)^2)}{4s}\right)^{\frac{d-3}{2}}
\end{aligned}$$

$$\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}\quad \forall\ n\geq 1$$

Equal masses

$$\rho(m_0^2;\ m_1,\ m_1) = \frac{1}{2^{2d-3}\pi^{\frac{d-1}{2}}\Gamma\left(\frac{d-1}{2}\right)\, m_0}\left(m_0^2 - 4m_1^2\right)^{\frac{d-3}{2}}\ \theta(m_0^2 - 4m_1^2)$$

$$d = 4\ :\quad \rho(m_0^2;\ m_1,\ m_1)\ = \frac{1}{16\pi^2 m_0}(m_0^2 - 4m_1^2)^{\frac{1}{2}}\,\theta(m_0^2 - 4m_1^2)$$

KL weight

Evaluate the Mehler-Fock transform

$$h_d(\kappa,\nu,\lambda) = \int_1^\infty P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\kappa}(u)\, P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(u) P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\lambda}(u)(u^2-1)^{-\frac{d-2}{4}}\, du$$

which provides the Kallen-Lehmann weight

$$\rho(\kappa^2,\nu,\lambda) \;=\; \frac{\Gamma\left(\frac{d-1}{2}+i\nu\right)\Gamma\left(\frac{d-1}{2}-i\nu\right)\Gamma\left(\frac{d-1}{2}+i\lambda\right)\Gamma\left(\frac{d-1}{2}-i\lambda\right)}{2(2\pi)^{1+\frac{d}{2}}} \sinh(\pi\kappa)\; h_d(\kappa,\nu,\lambda),$$

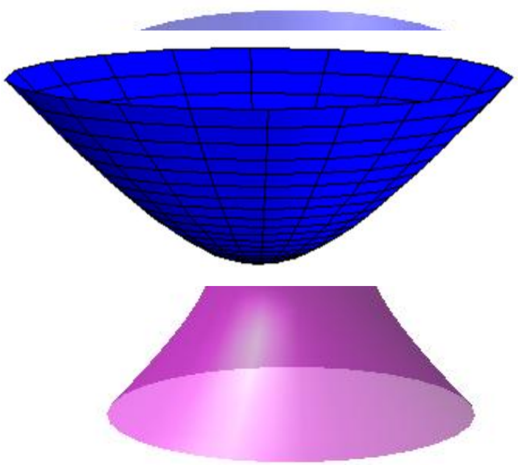
One needs something like a vectorial Fourier transform adapted to the de Sitter geometry

Another integral representation

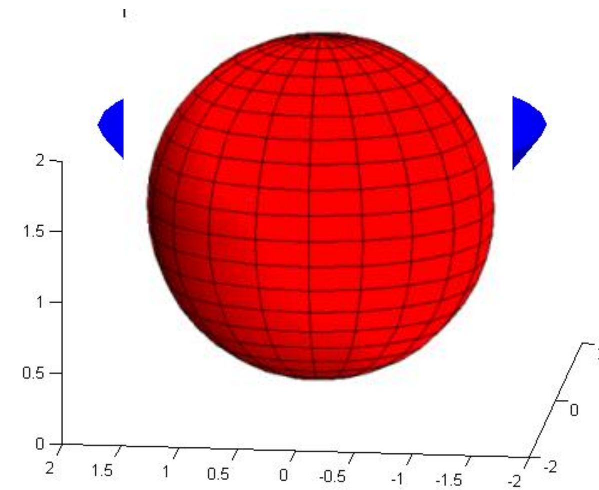
Let us evaluate the 2 pt function at purely imaginary events of the past e future tubes

$$\mathcal{W}_\nu(-iy, iy') = \frac{\Gamma(\frac{d-1}{2}+i\nu)\Gamma(\frac{d-1}{2}-i\nu)}{2^{d+1}\pi^d}\int_\gamma (y\cdot\xi)^{-\frac{d-1}{2}+i\nu}\left(\xi\cdot y'\right)^{-\frac{d-1}{2}-i\nu} d\mu_\gamma(\xi) =$$

$$= \frac{\Gamma\left(\frac{d-1}{2}+i\nu\right)\Gamma\left(\frac{d-1}{2}-i\nu\right)}{2(2\pi)^{\frac{d}{2}}}\left(\left(y\cdot y'\right)^2-1\right)^{-\frac{d-2}{4}} P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}\left(y\cdot y'\right)$$



$$y = (u, \sqrt{u^2-1}, \vec{\omega})$$
$$\xi = (1, \vec{\Omega})$$
$$y' = (1, 0, \ldots, 0)$$
$$y\cdot y' = y^0 = u \geq 1$$
$$y'\cdot\xi = 1$$

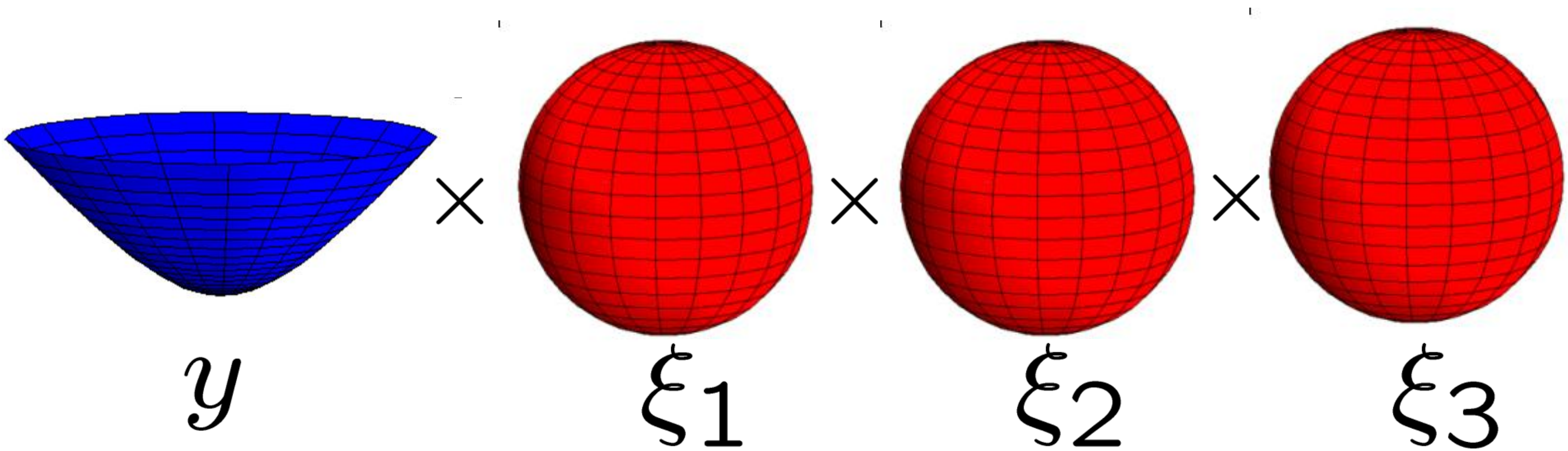


$$\left(u^2-1\right)^{-\frac{d-2}{4}} P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(u) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\gamma_0}(y\cdot\xi)^{-\frac{d-1}{2}-i\nu} d\mu_\gamma(\xi)$$

Fourier-like representation

$$h_d(\kappa,\nu,\lambda) = \int_1^\infty P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\kappa}(u)\, P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(u) P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\lambda}(u)(u^2-1)^{-\frac{d-2}{4}}\, du =$$

$$\int_{H_d}\int_{S^3_{d-1}} (y\cdot\xi_1)^{-\frac{d-1}{2}-i\kappa}\,(y\cdot\xi_2)^{-\frac{d-1}{2}-i\nu}\,(y\cdot\xi_3)^{-\frac{d-1}{2}-i\lambda}\, dy\, d\Omega_1\, d\Omega_2\, d\Omega_3$$

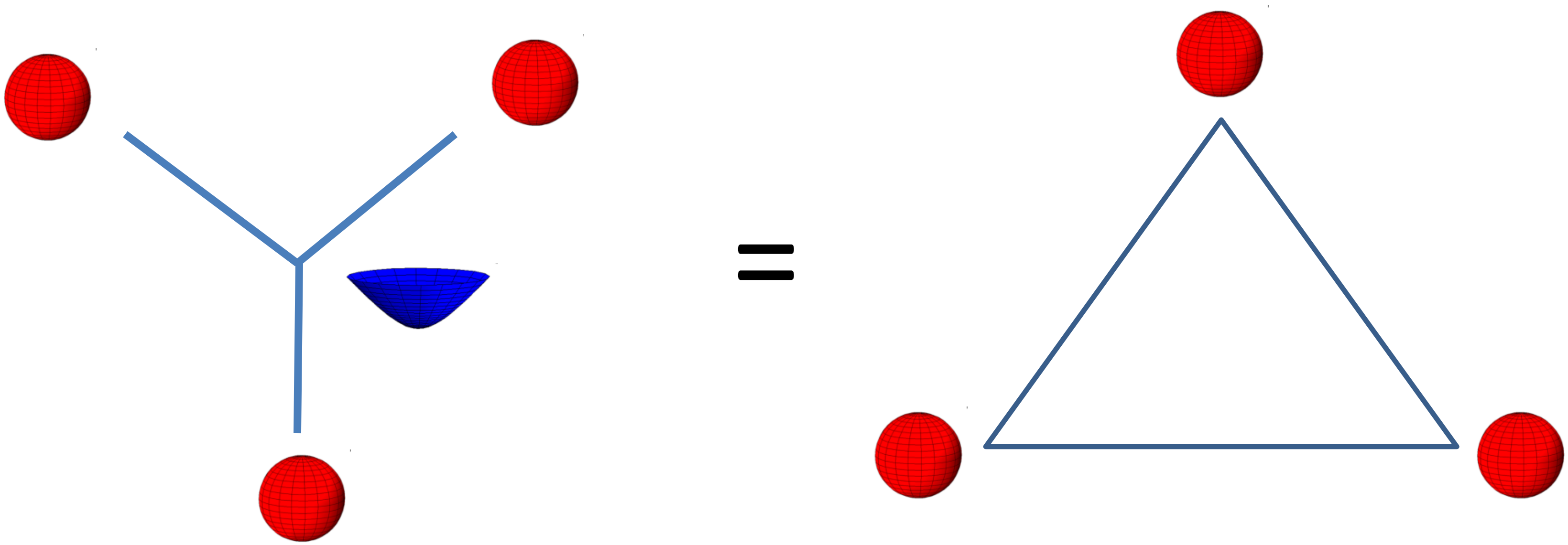


Star-triangle identity

1) Integral on the hyperboloid: a star-triangle relation

$$\int_H (y\cdot\xi_1)^{a_2+a_3}(y\cdot\xi_2)^{a_3+a_1}(y\cdot\xi_3)^{a_1+a_2}dy =$$

$$= \; c(a_1,a_2,a_3)(\xi_1\cdot\xi_2)^{a_3}(\xi_2\cdot\xi_3)^{a_1}(\xi_3\cdot\xi_1)^{a_2}$$

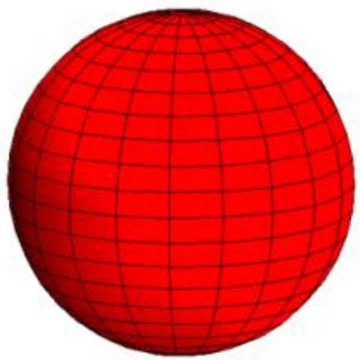


Integrate the triangle

2) Triple integral on the sphere

$$J = \int_{S^3_{d-1}} (\xi_1 \cdot \xi_2)^{a_3} (\xi_2 \cdot \xi_3)^{a_1} (\xi_3 \cdot \xi_1)^{a_2} \, d\Omega_1 \, d\Omega_2 \, d\Omega_3,$$

$$\xi_1 = (1, \vec{\Omega}_1), \quad \xi_2 = (1, \vec{\Omega}_2), \quad \xi_3 = (1, \vec{\Omega}_3)$$



$$\xi_1 \cdot \xi_2 = 1 - \vec{\Omega}_1 \cdot \vec{\Omega}_2 = \frac{1}{2}(|\vec{\Omega}_1|^2 + |\vec{\Omega}_2|^2) - \vec{\Omega}_1 \cdot \vec{\Omega}_2 = \frac{1}{2}|\vec{\Omega}_1 - \vec{\Omega}_2|^2 = \Delta_{12}^2 = r_3^2$$

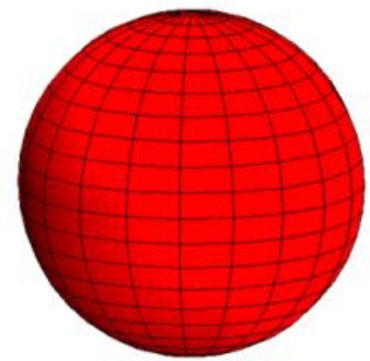
$$\int_{S^3_{d-1}} \Delta_{12}^{2a_3} \, \Delta_{23}^{2a_1} \, \Delta_{31}^{2a_2} \, d\Omega_1 \, d\Omega_2 \, d\Omega_3 =$$

$$= \int_D \rho(r_1, r_2, r_3) \, r_1^{2a_1} \, r_2^{2a_2} \, r_3^{2a_3} \, dr_1 dr_2 dr_3.$$

Integrate the triangle

$$J = \int_{S_{d-1}^3} (\xi_1 \cdot \xi_2)^{a_3} (\xi_2 \cdot \xi_3)^{a_1} (\xi_3 \cdot \xi_1)^{a_2} \, d\Omega_1 \, d\Omega_2 \, d\Omega_3,$$

$$= \int_{S_{d-1}^3} \Delta_{12}^{2a_3} \; \Delta_{23}^{2a_1} \; \Delta_{31}^{2a_2} \; d\Omega_1 \, d\Omega_2 \, d\Omega_3 =$$



$$= \int \rho(r_1, r_2, r_3) \, r_1^{2a_1} \, r_2^{2a_2} \, r_3^{2a_3} \, dr_1 dr_2 dr_3.$$

$$\rho(r_1, r_2, r_3) = \frac{4^a \omega_{d-}}{\omega} \int \rho(r_1, r_2, r_3) dr_1 dr_2 dr_3 = 1 + r_2^4 + r_3^4) - r_1^2 r_2^2 r_3^2]_+^{\frac{d-4}{2}}.$$

$J(a_1, a_2, a_3)$ are the moments of the probability density of three random points on a sphere making a triangle with sides r_1, r_2 and r_3.

A Beautiful formula

$$\int_1^\infty P_{-\frac{1}{2}+i\kappa}^{-\frac{d-2}{2}}(u)\, P_{-\frac{1}{2}+i\nu}^{-\frac{d-2}{2}}(u) P_{-\frac{1}{2}+i\lambda}^{-\frac{d-2}{2}}(u)(u^2-1)^{-\frac{d-2}{4}}\, du =$$

$$= \frac{\prod_{\epsilon,\epsilon',\epsilon''=\pm 1} \Gamma\left(\frac{d-1}{4} + \frac{i\epsilon\kappa + i\epsilon'\nu + i\epsilon''\lambda}{2}\right)}{\left[\prod_{\epsilon=\pm 1}\Gamma\left(\frac{d-1}{2} + i\epsilon\kappa\right)\right]\left[\prod_{\epsilon'=\pm 1}\Gamma\left(\frac{d-1}{2} + i\epsilon'\nu\right)\right]\left[\prod_{\epsilon''=\pm 1}\Gamma\left(\frac{d-1}{2} + i\epsilon''\lambda\right)\right]}$$

ρ never vanishes.
For $m > m_c = (d-1)/2R$ decays into heavier particeles are always possible

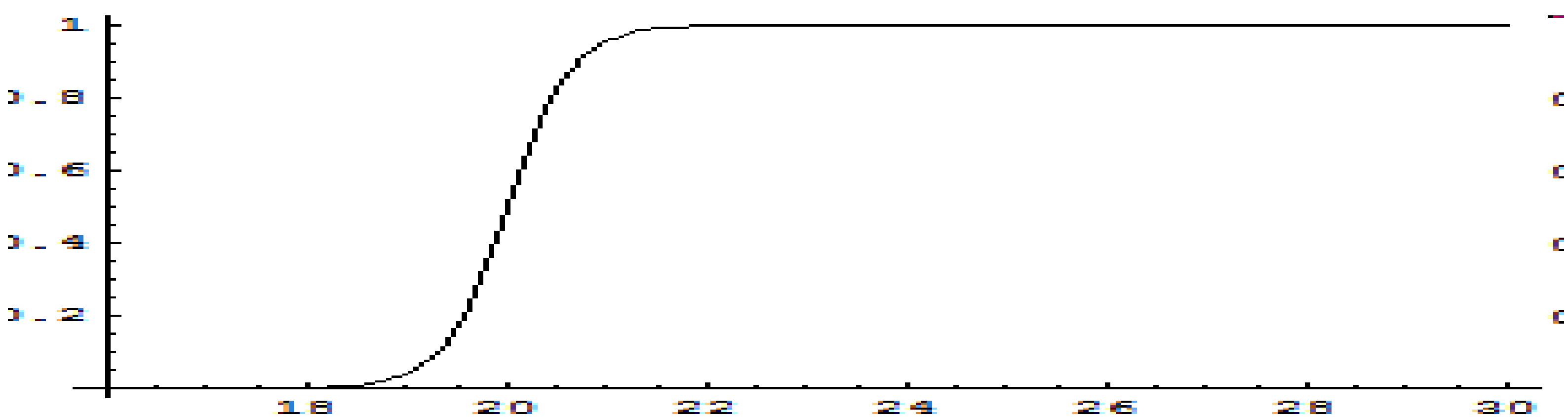
Surprisingly (for me) the Minkowskian result is recovered in the zero curvature limit by posing $\kappa = MR$, $\nu = mR$ $\lambda = m'R$:

$$\lim_{R\to\infty} \rho(\kappa^2; \nu, \lambda)\ d\kappa^2 = \rho(M^2; m, m')\ dM^2.$$

Dimension d=3

$$W_\nu(x,y)^2 = \frac{1}{4\pi}\int_0^\infty \frac{\tanh(\frac{\pi\kappa}{2})}{(\cosh\pi\kappa + \cosh 2\pi\nu)}\,\frac{\sinh\pi\kappa}{\kappa} W_\kappa(x,y)\; d\kappa^2$$

$$\Gamma(1,2) = \frac{R}{4\nu^2}\frac{\tanh(\frac{\pi\nu}{2})}{\left(1+\frac{\cosh 2\pi\kappa}{\cosh\pi\nu}\right)} \to \frac{1}{4M^2}\theta(M-2m)$$



Complementary fields. Inflation

$$W^2_\nu(z,z') = \int_{-\infty}^{\infty} \kappa d\kappa\ \rho(\kappa^2,\nu) W_\kappa(z,z')$$

$$W^2_\nu = \int_{-\infty}^{\infty} \kappa\, d\kappa\ \rho_\nu(\kappa) W_\kappa \ + \sum_{n=0}^{N-1} A_n(\nu)\, W_{i(\mu+2i\nu+2n)}$$

$$A_n(\nu) \ = \ \frac{8\pi(-1)^n}{n! 2^d \pi^{\frac{1+d}{2}} R^{d-2} \Gamma(\mu)} \frac{\Gamma(\mu+2i\nu+n)\Gamma(-2i\nu-n)}{\Gamma(\mu+2i\nu+2n)\Gamma(-\mu-2i\nu-2n)} \times \frac{\Gamma(\mu+n)\Gamma(-i\nu-n)\Gamma(\mu+i\nu+n)}{\Gamma(-i\nu-n+\frac{1}{2})\Gamma(\mu+i\nu+n+\frac{1}{2})}$$

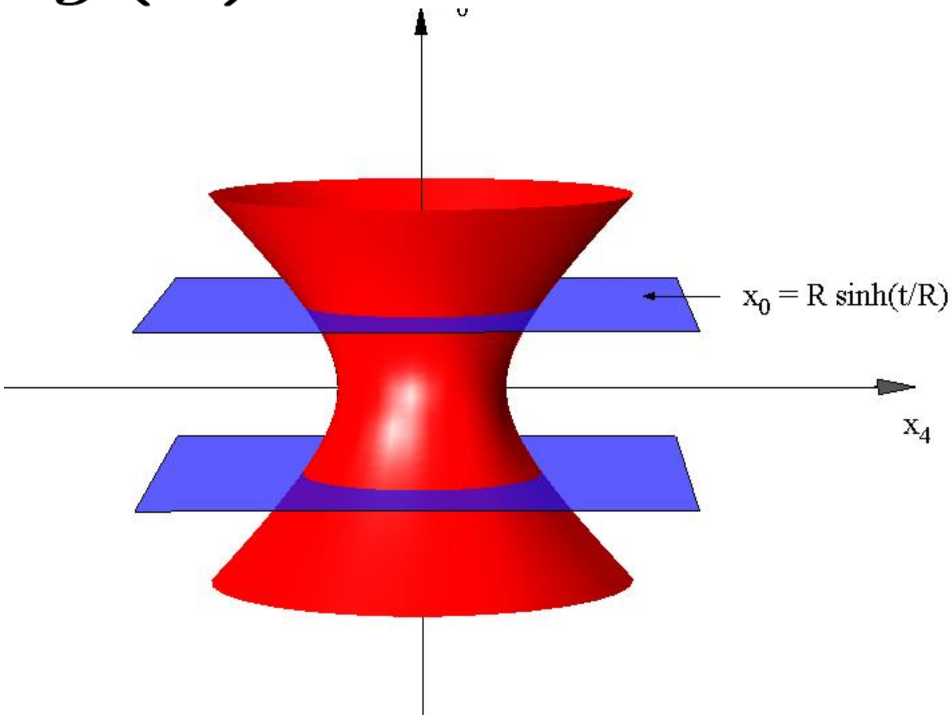
$$\mu = (d-1)/2$$

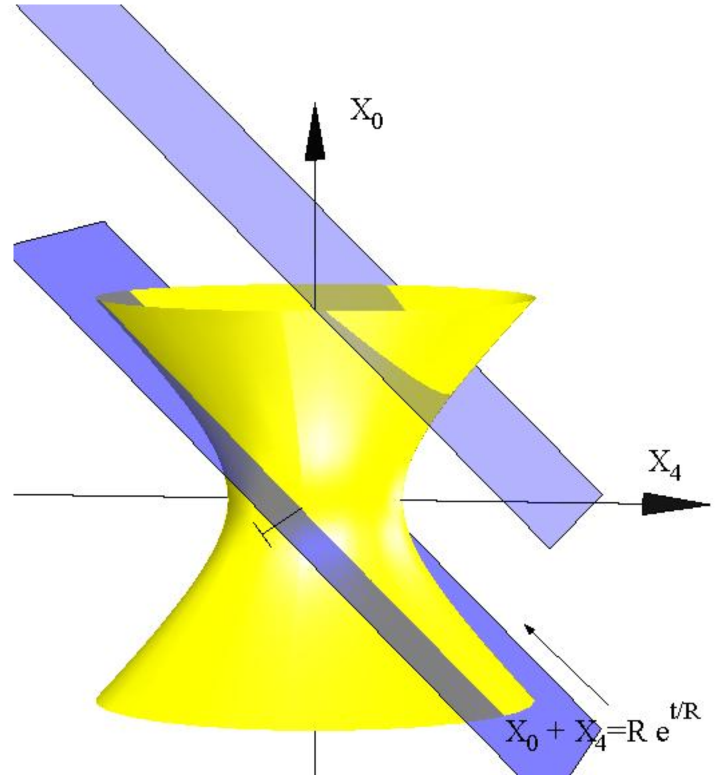
The number of discrete terms is the largest N satisfying $N < 1 + |\Im\nu| - \mu/2$, or 0 if this is negative. A particle of the complementary series with parameter $\kappa = i\beta$ can only decay into two particles with parameter $\nu = \frac{i}{2}(|\beta| + \mu + 2n)$, where n is any integer such that $0 \leq 2n < \mu - |\beta|$, and the decay is instantaneous.

A particle with mass $M \ll m_c$ can only decay into two particles of mass $m \sim M/\sqrt{2}$.

Infrared limit

$$g(x) \to 1$$





$$\lim_{T\to\infty} \frac{2\lambda^2 C(\kappa)\rho(\kappa^2,\nu)\ \int g(x)\,|F(x)|^2\,dx}{T\int \overline{f_0(x)}W_\kappa(x,\ y)\,f_0(y)\,dx\,dy} = \rho(\kappa^2,\nu)\frac{2\lambda^2\pi\,\coth(\pi\kappa)^2}{|\kappa|}$$