More about the infrared structure of the de Sitter Universe: Tachyons

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BONAPARTE

SHORT TALK

Tachyons

- What do we mean by tachyons?
- Local and covariant quantum tachyons do not exist
- De Sitter local and covariant tachyons DO exist for discrete (integer) values of the negative squared mass
- Do not know yet the physical meaning (if any)
- END OF THE SHORT TALK

Long talk

(Now you may start again to watch the screen of your laptop or surf the internet…..)

Relativistic Tachyons

- Is the trasmission of energy faster than light contradicting special relativity?

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}, \quad \vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}$$

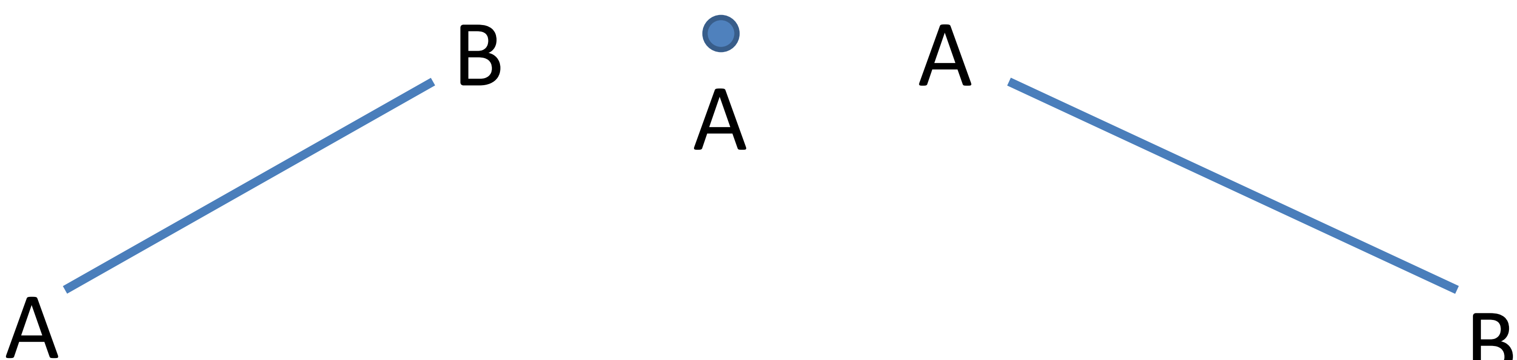
- When $v \to c$, *E* and *p* diverge.
- A classical massive particle moving $v < c$ at one time cannot move $v \geq c$ at a later time; this does not rule out other "particles" moving faster. Photons for instance do exist! But they are massless

Relativistic Tachyons II

$$E = \frac{\mu c^2}{\sqrt{\frac{v^2}{c^2} - 1}}, \quad |p| = \frac{\mu v}{\sqrt{\frac{v^2}{c^2} - 1}} \qquad m = i\mu$$

$$c^2 p^2 - E^2 = \mu^2 c^4 \qquad \frac{c|p|}{E} = \frac{v}{c} > 1$$

Feinberg's reinterpretation principle:
Emission of a positive energy tachyon = Absorption of a negative energy tachyon



Linear Quantum Field Theory

- The degrees of freedom of standard linear fields are provided by the commutator function: a bidistribution that vanishes at spacelike separated pairs

$$[\phi(x_1), \phi(x_2)] = C(x_1, x_2)\mathbf{1}$$

- On a globally hyperbolic manifold the equations of motion plus canonical initial conditions uniquely determine the commutator

$$(\Box_{x_i} + m^2)C(x_1, x_2) = 0, \qquad i = 1, 2$$

$$C(x_1, x_2)|_{x_1^0 = x_2^0} = 0,$$
$$\frac{\partial}{\partial x_2^0} C(x_1, x_2)|_{x_1^0 = x_2^0} = i\hbar\delta(\mathbf{x_1} - \mathbf{x_2}) = 0$$

Two-point functions

- A Hilbert space representation is associated to any two-point function that solves the equations of motion

$$(\Box_{x_1} + m^2)W(x_1, x_2) = (\Box_{x_2} + m^2)W(x_1, x_2) = 0$$

- The functional equation (canonical quantization)

$$W(x_1, x_2) - W(x_1, x_2) = C(x_1, x_2)$$

- And the positive-definiteness condition

$$\int W(x_1, x_2)\bar{f}(x_1)f(x_2)dx_1dx_2 \geq 0$$

- For Minkowski,dS, adS or other simmetries if unbroken: invariance:

$$W(gx_1, gx_2) = W(x_1, x_2)$$

Spectral Property

There exist a complete set of states having positive energy

equivalent to

$W(x_1, \ldots, x_n)$ is the boundary value of a function $W(z_1, \ldots, z_n)$ holomorphic in the tube $\{\mathrm{Im}(z_{k+1} - z_k) \in V^+\}$ of the complex Minkowski spacetime

Consequences

1) 2 pt functions and propagators

2) Free fields

3) Perturbation theory

4) Renormalization

Klein-Gordon field summary

$$(\Box + m^2)\phi(x) = 0$$

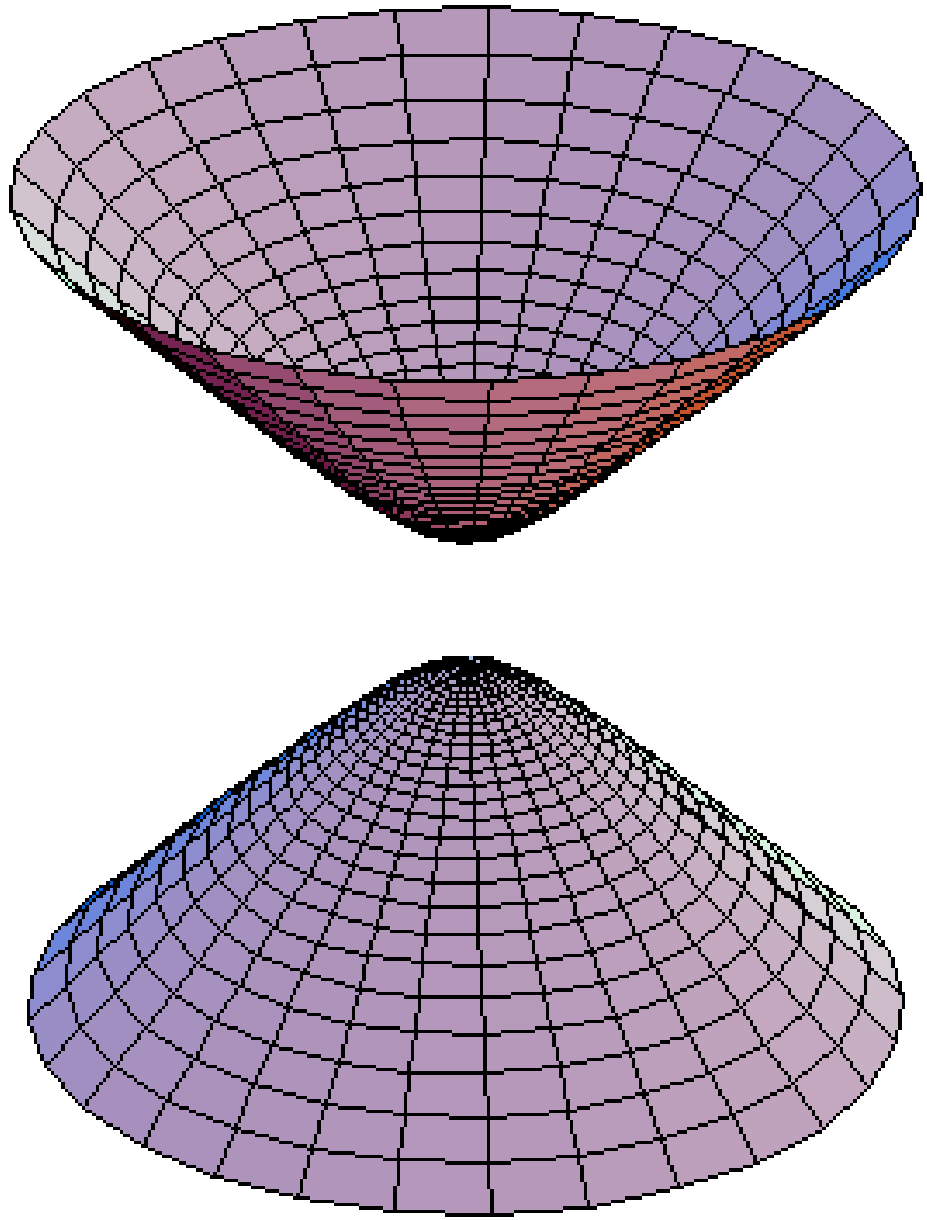
$$(\Box_{x,y} + m^2)W(x,y) = 0$$

$$\xi = x - x'$$

$$(\Box_\xi + m^2)W(\xi) = 0$$

$$(p^2 - m^2)\tilde{W}(p) = 0$$

$$\tilde{W}(p) = A\,\theta(p^0)\delta(p^2 - m^2) + B\,\theta(-p^0)\delta(p^2 - m^2)$$



Klein-Gordon bradyons

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Spectral condition

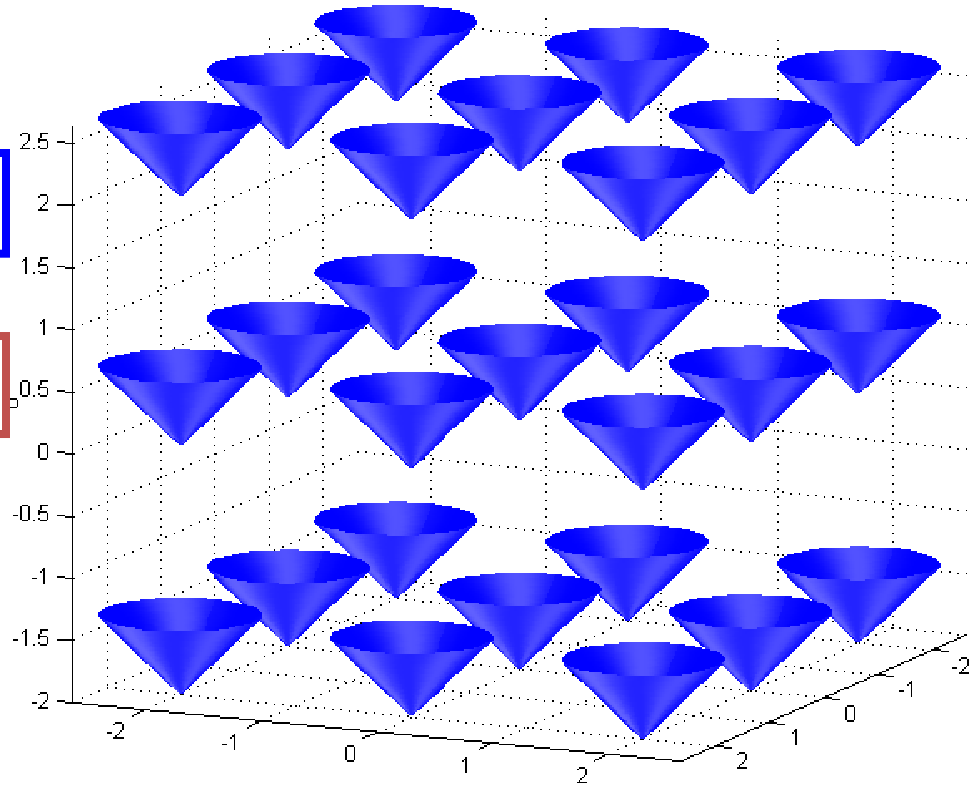
$$W(x - x') = \frac{1}{(2\pi)^d}\int e^{-ip\cdot x}e^{ip\cdot x'}\theta(p^0)\delta(p^2 - m^2)$$

The spectral condition implies that the Fourier representation is meaningful in a domain of the complex Minkowski spacetime

$$W(z-z') = \int e^{-ip\cdot z} e^{ip\cdot z'} \theta(p^0)\delta(p^2-m^2)$$

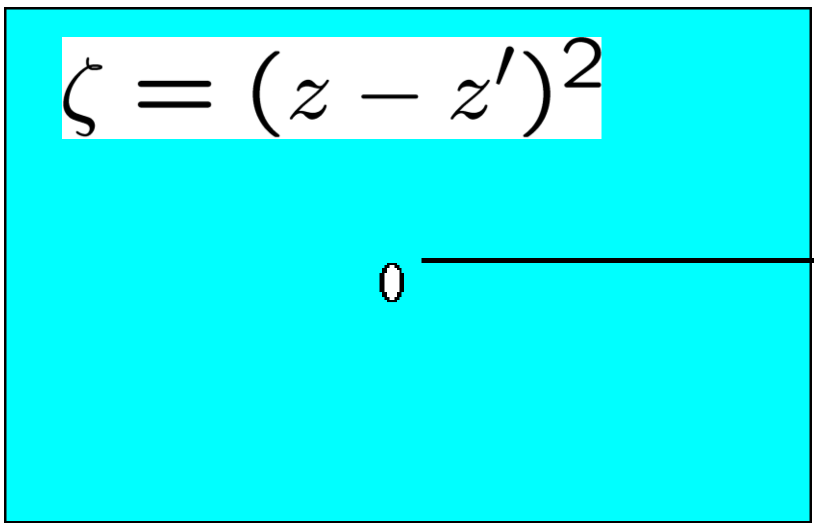
$$\boxed{z \in T^- \ (\mathrm{Im}\, z \in V^-)}$$

$$\boxed{z' \in T^+ \ (\mathrm{Im}\, z' \in V^+)}$$



Spectrum + Lorentz Inv.

$$W(z - z') = W(\zeta) \text{ is maximally analytic}$$

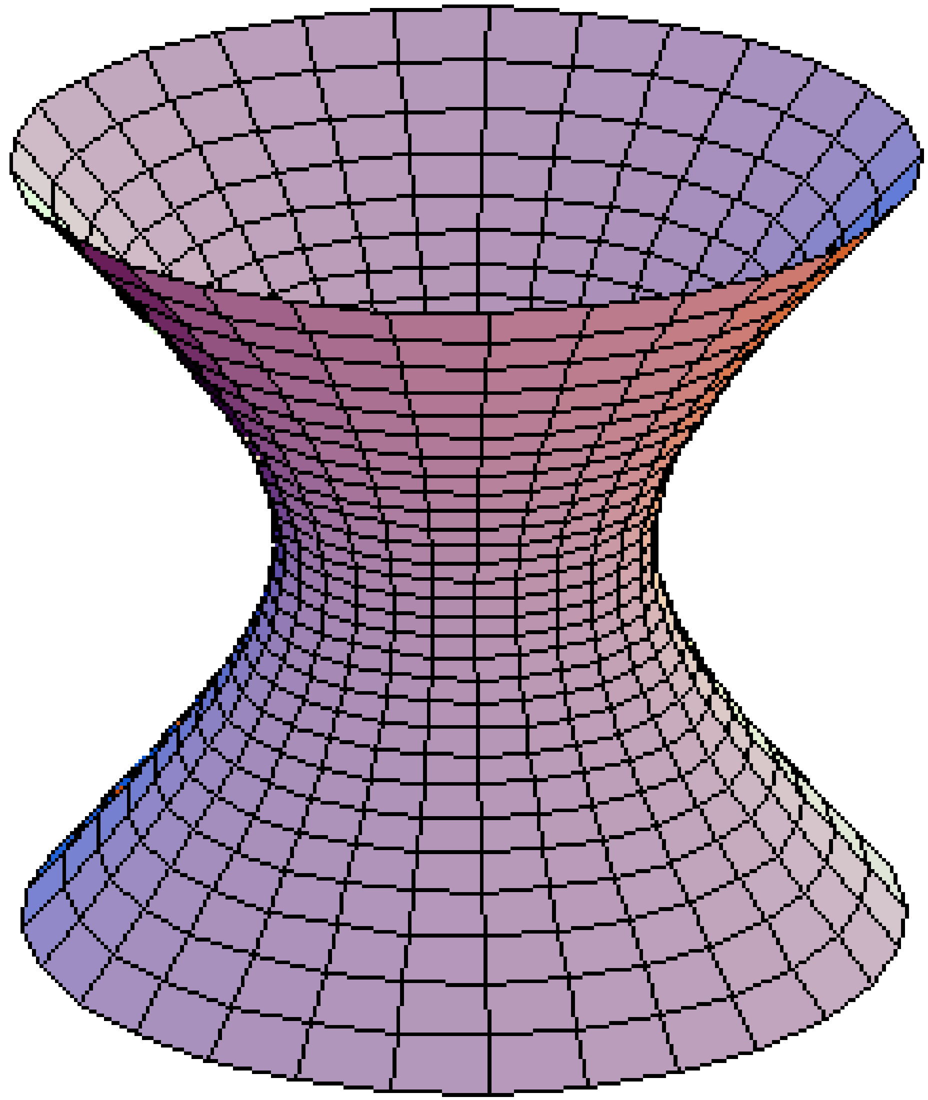


The Cut reflects Causality and the Quantum

Klein-Gordon tachyons I

$$(\Box - m^2)\phi(x) = 0 \qquad (\Box_{x,y} - m^2)W(x-y) = 0$$

$$(p^2 + m^2)\tilde{W}(p) = 0 \qquad \tilde{W}(p) = \delta(p^2 + m^2)$$



Klein-Gordon tachyons I

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$$(p^2 + m^2)\tilde{W}(p) = 0 \qquad \tilde{W}(p) = \delta(p^2 + m^2)$$

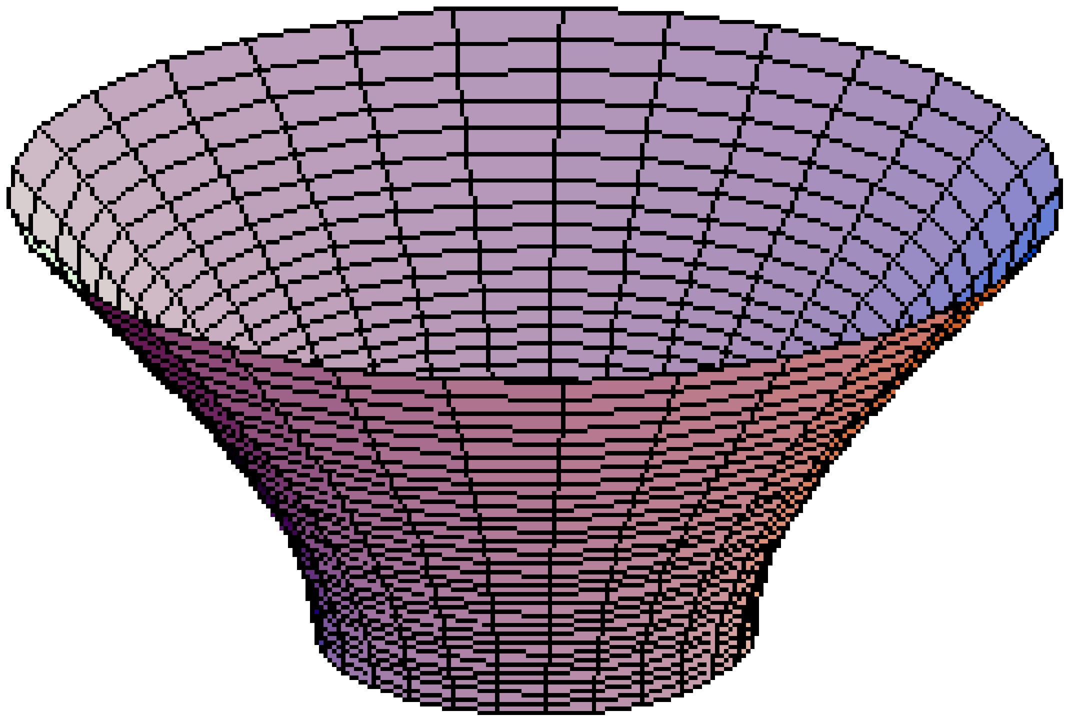
$$W(x-x') = \frac{1}{(2\pi)^d}\int e^{-ip\cdot(x-x')}\delta(p^2+m^2)$$

- The 2-p function is covariant and positive definite
- The commutator vanishes identically
- The anticommutator does not (but the quantization is not canonical)
- The energy spectrum is unbounded below

Klein-Gordon tachyons II (Feinberg)

$$(\Box - m^2)\phi(x) = 0 \qquad (p^2 + m^2)\tilde{W}(p) = 0$$

$$\tilde{W}(p) = \theta(p^0)\delta(p^2 + m^2) + \cancel{\theta(-p^0)\delta(p^2 + m^2)}$$



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$$\tilde{W}(p) = \theta(p^0)\delta(p^2 + m^2) + \cancel{\theta(-p^0)\delta(p^2 + m^2)}$$

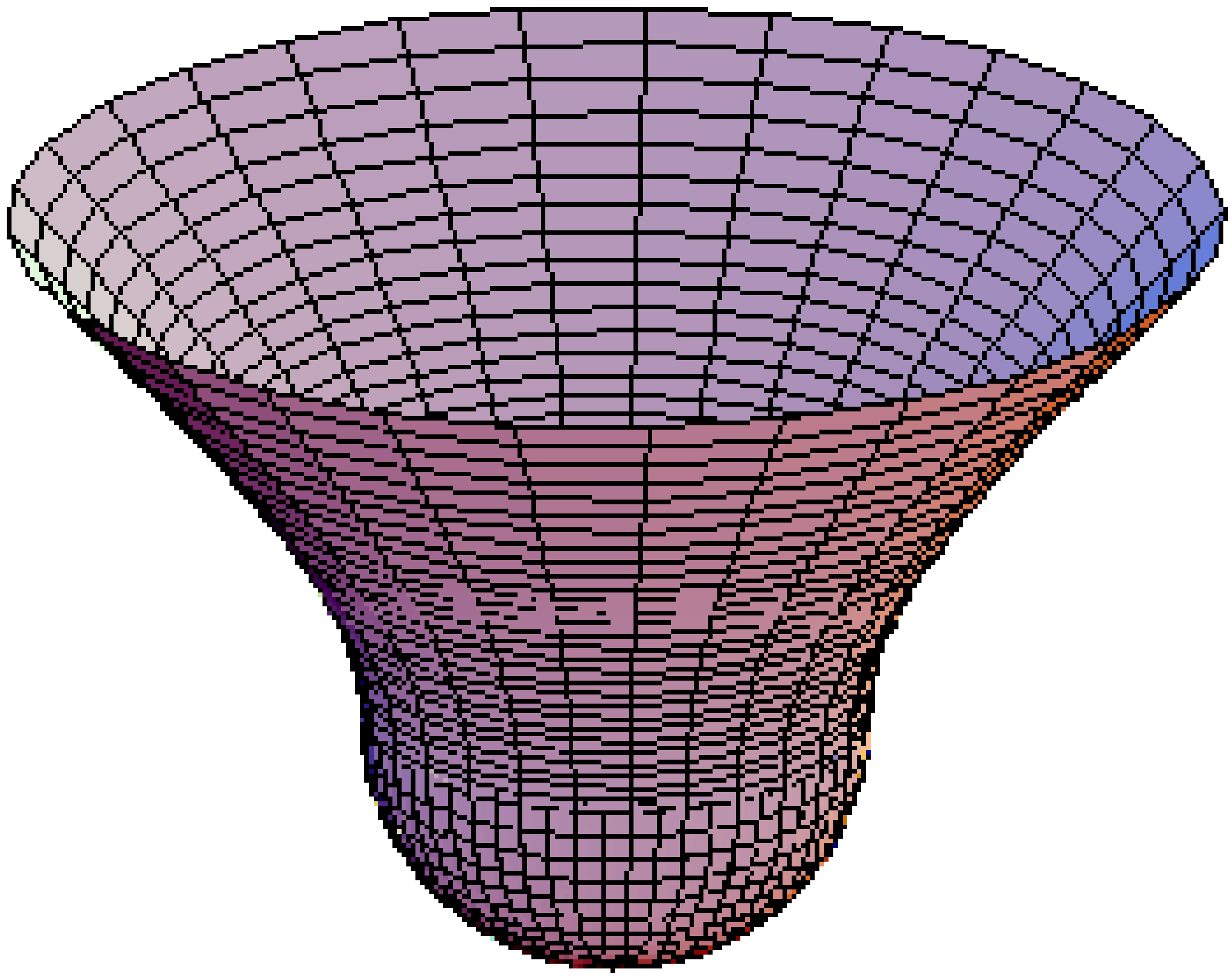
$$W(x - x') = \frac{1}{(2\pi)^d} \int e^{-ip\cdot x} e^{ip\cdot x'} \theta(p^0)\delta(p^2 + m^2)$$

- The “quantization” is not covariant but positive
- The commutator does not vanishes; it is neither canonical nor covariant
- The anticommutator is covariant but it not canonical (Feinberg invokes the name “scalar fermions” for this.)
- The energy spectrum is bounded below in every frame. Positive in the chosen frame (non-covariant spectral condition)

Klein-Gordon tachyons III (Schroer)

- Feinberg's quantization may be rendered covariant by adding a complex manifold to Feinberg's integration cycle

$$W(x-x') = \frac{1}{(2\pi)^d}\int_{p>m} e^{-i\sqrt{p^2-m^2}(x^0-x'^0)} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} \frac{d^3p}{\sqrt{p^2-m^2}}$$



Klein-Gordon tachyons III (Schroer)

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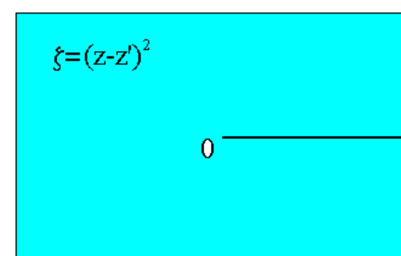
$$W(x-x') = \frac{1}{(2\pi)^d}\int_{p>m} e^{-i\sqrt{p^2-m^2}(x^0-x'^0)} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}\frac{d^3p}{\sqrt{p^2-m^2}} + \\ + \frac{i}{(2\pi)^d}\int_{p<m} e^{-\sqrt{m^2-p^2}(x^0-x'^0)} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}\frac{d^3p}{\sqrt{m^2-p^2}}$$

- The 2-point function is covariant but not positive definite. It is not a tempered distribution but grows exponentially in momentum space
- Locality holds. The theory is canonical
- There exists no Lorentz invariant physical positive subspace
- This quantization may be obtained as the analytic continuation to imaginary masses of the standard KG fields

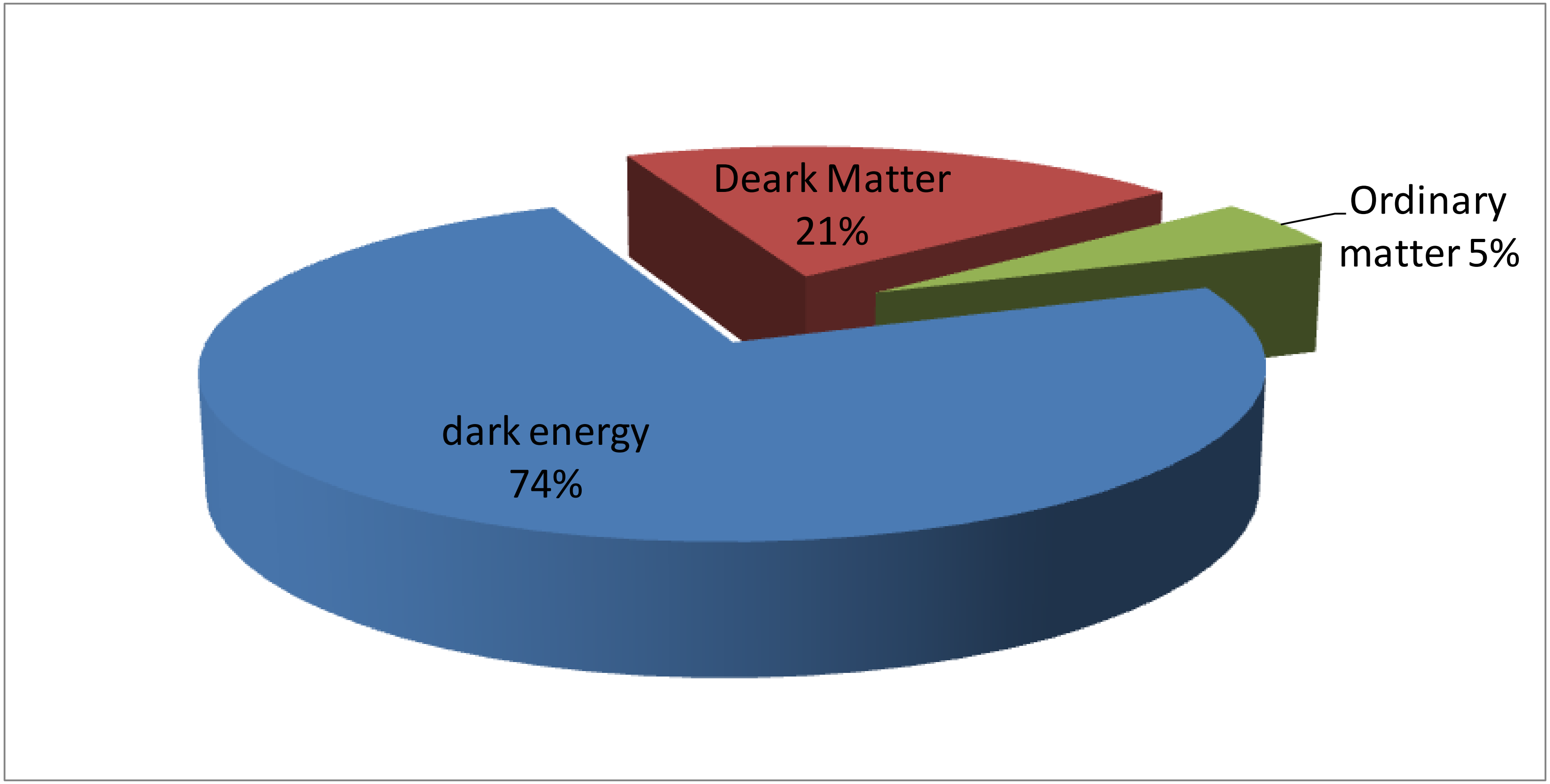


Summary

- When quantizing the Klein-Gordon tachyon it is impossible to reconcile the axioms.
- Give up either locality and covariance or positivity (and of course the spectral condition).

de Sitter

Energy content of the universe



Eq. Friedmann

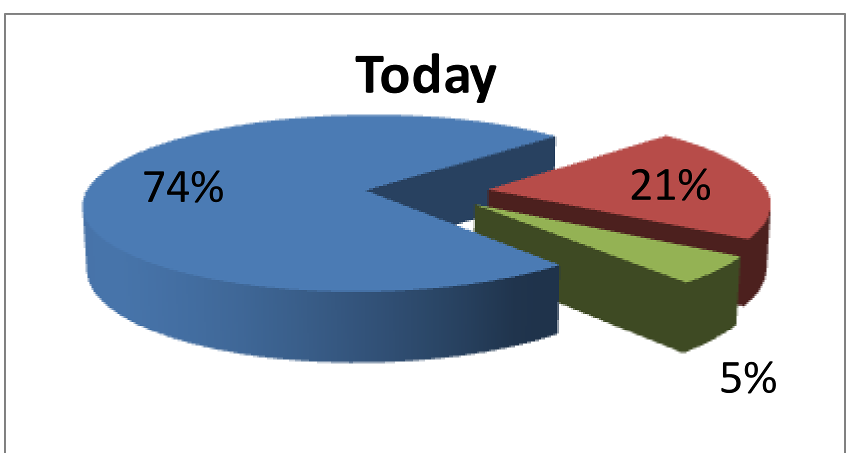
$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G(\rho_M + \rho_R + \rho_\Lambda) - \frac{K}{a^2}$$

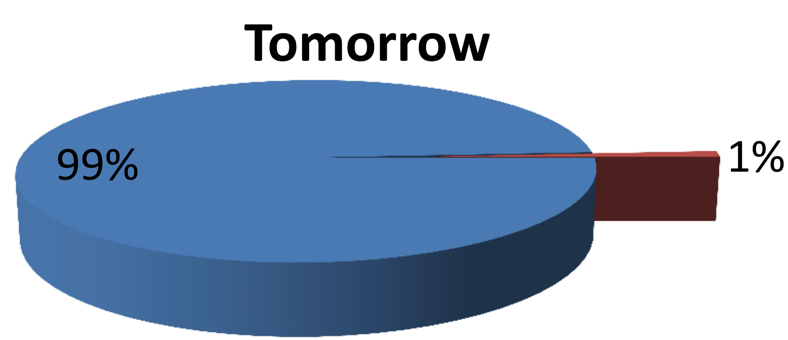
\+ State Eqs. od the various components of the cosmic fluid

$$p_M = 0, \quad p_R = \frac{1}{3}\rho_R, \quad p_\Lambda = -\rho_\Lambda,$$

= time evolution of the Hubble “constant”

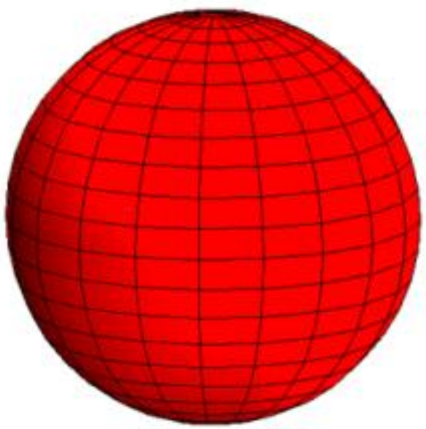
$$H^2 = \frac{\Omega_M}{a(t)^3} + \frac{\Omega_R}{a(t)^4} + \Omega_\Lambda + \frac{\Omega_K}{a(t)^2}$$





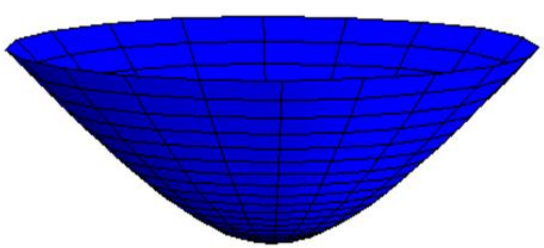
Einstein Equations

$$\ddot{a} = \frac{1}{3}\Lambda\, a \qquad \dot{a}^2 = \frac{1}{3}\Lambda\, a^2 - K$$



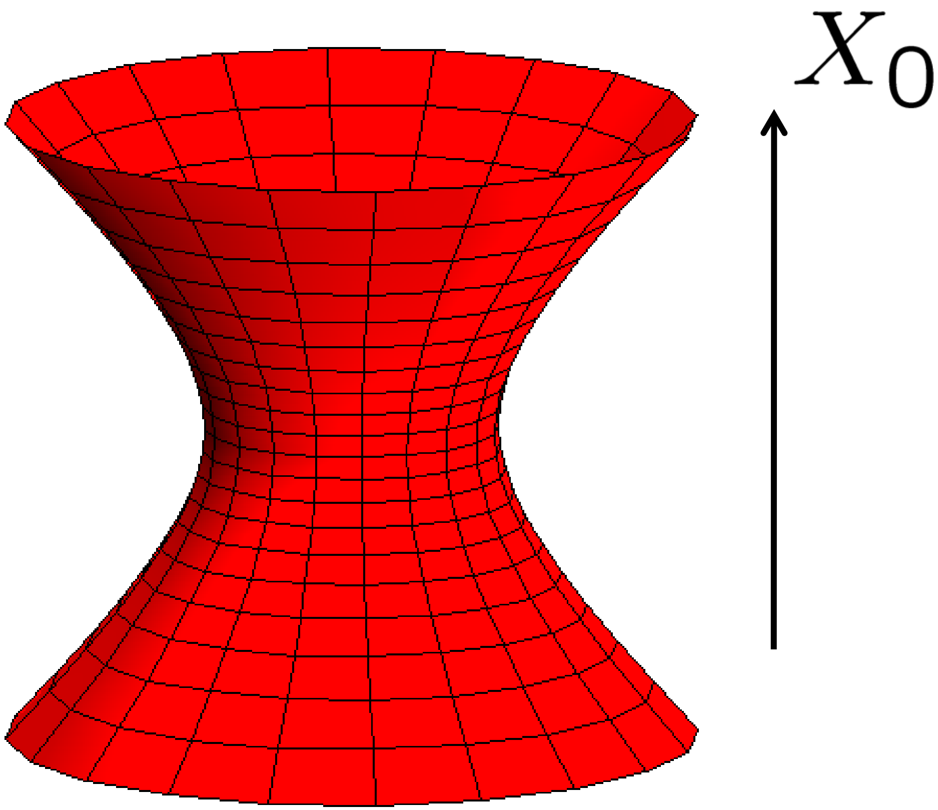
$$K = 1 \quad \rightarrow \quad a(t) = \sqrt{\frac{3}{\Lambda}}\cosh\sqrt{\frac{\Lambda}{3}}\,t$$

$$K = 0 \quad \rightarrow \quad a(t) = \exp\sqrt{\frac{\Lambda}{3}}\,t$$



$$K = -1 \quad \rightarrow \quad a(t) = \sqrt{\frac{3}{\Lambda}}\sinh\sqrt{\frac{\Lambda}{3}}\,t$$

Spherical de Sitter model



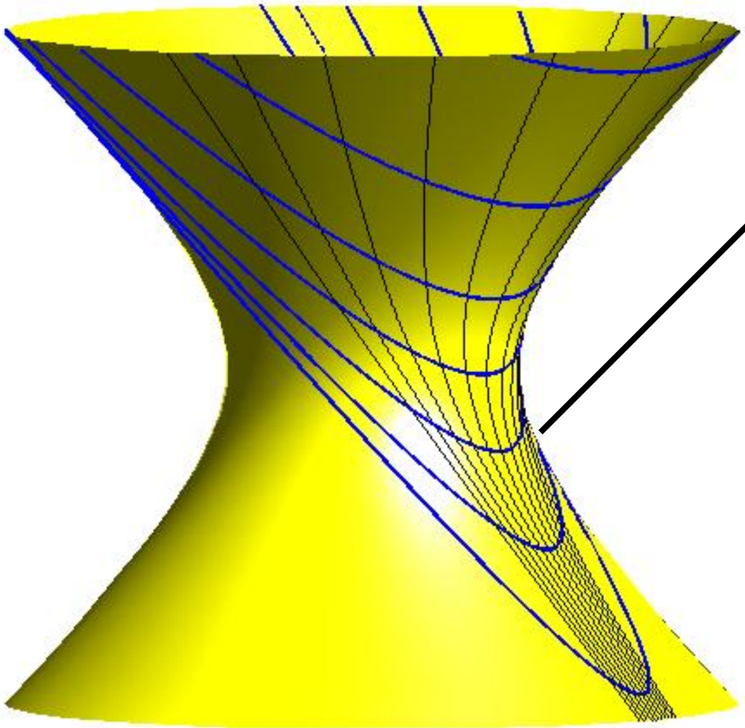
$$\begin{cases} X_0 &= R\sinh\frac{t}{R} \\ X_i &= R\cosh\frac{t}{R}\,\Omega_i \end{cases}$$

$$|\Omega|^2 = 1$$

$$R = \sqrt{\frac{3}{\Lambda}}$$

$$\begin{aligned} ds^2 &= dX_0^2 - dX_1^2 - \dots dX_4^2\Big|_{dS} = \\ &= dt^2 - R^2\cosh^2\frac{t}{R}\Big(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\Big) \end{aligned}$$

Flat de Sitter model

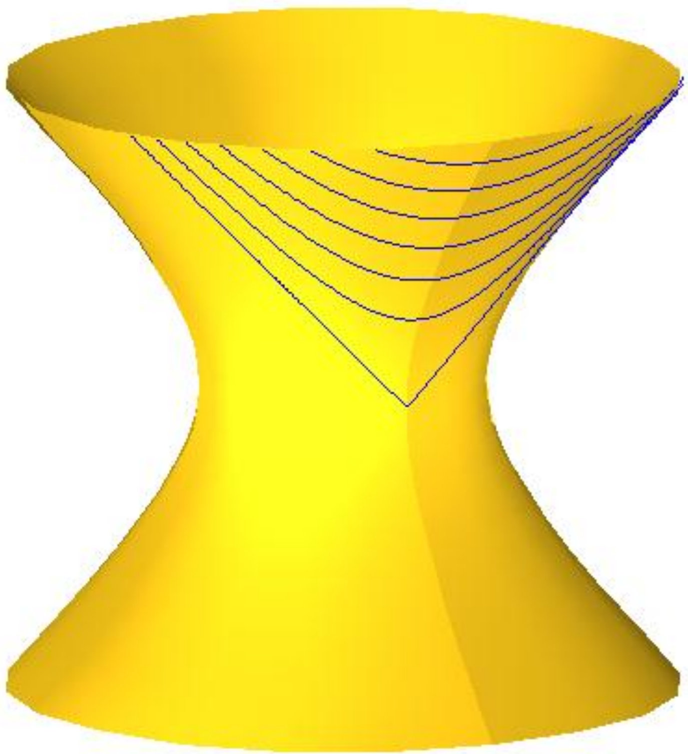


$$X_0 + X_4 = R \exp \frac{t}{R}$$

$$\begin{cases} X_0 &= R \sinh \frac{t}{R} + \frac{1}{2R} e^{\frac{t}{R}} |\vec{x}|^2 \\ X_i &= \exp\left(\frac{t}{R}\right) x_i \\ X_4 &= R \cosh \frac{t}{R} - \frac{1}{2R} e^{\frac{t}{R}} |\vec{x}|^2 \end{cases}$$

$$\begin{aligned} ds^2 &= dX_0^2 - dX_1^2 - \ldots dX_4^2 \Big|_{dS} = \\ &= dt^2 - \exp \frac{2t}{R} \left(dx_1^2 + dx_2^2 + dx_3^2\right) \end{aligned}$$

Open de Sitter model



$$\begin{cases} X_0 &= R\sinh\frac{t}{R}\cosh\chi \\ X_1 &= R\sinh\frac{t}{R}\sinh\chi\sin\theta\sin\phi \\ X_2 &= R\sinh\frac{t}{R}\sinh\chi\sin\theta\cos\phi \\ X_3 &= R\sinh\frac{t}{R}\sinh\chi\cos\theta \\ X_4 &= R\cosh\frac{t}{R} \end{cases}$$

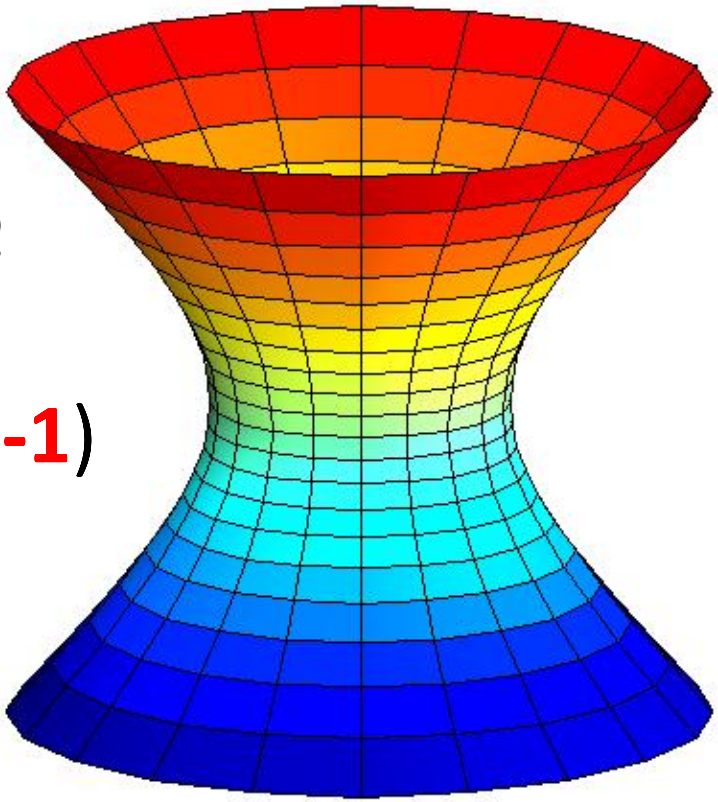
$$\begin{aligned} ds^2 &= dX_0^2 - dX_1^2 - \ldots dX_4^2\Big|_{dS} = \\ &= dt^2 - R^2\sinh^2\frac{t}{R}\Big(d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\Big) \end{aligned}$$

The shape of our universe

$$X_0^2 - X_1^2 - \dots X_d^2 = -R^2$$

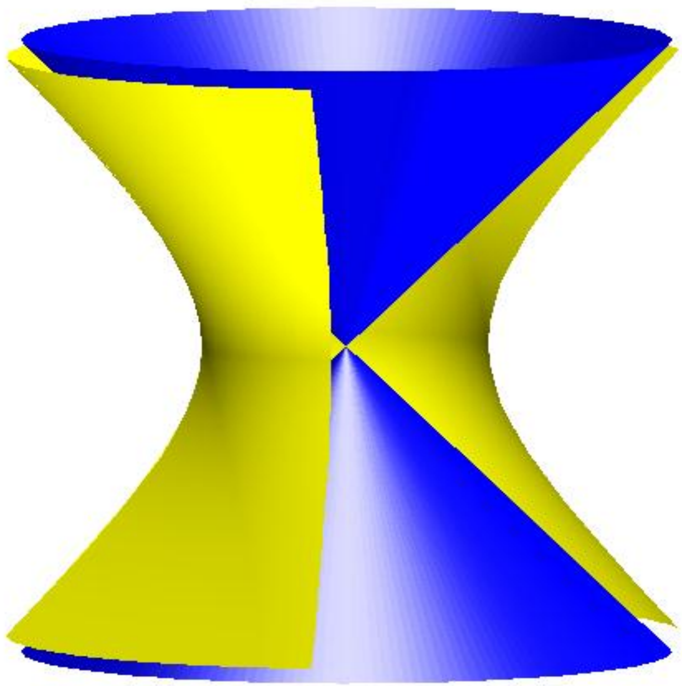
$M^{(d+1)}$: $\eta_{\mu\nu}$ = diag(**1,-1,……,-1**)

$$G = SO(1,d)$$



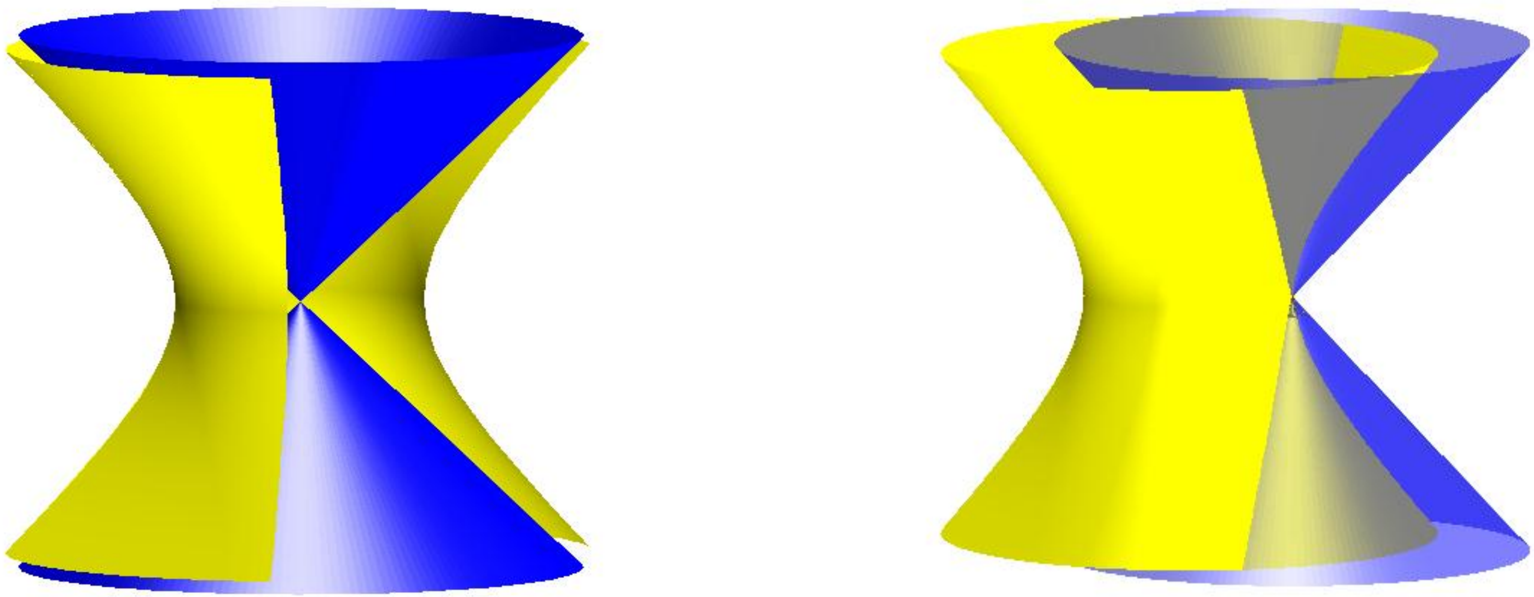
The asymptotic cone

$$\{\xi_0^2 - \xi_1^2 - \ldots - \xi_d^2 = 0\}$$



$M^{(d+1)}$: $\eta_{\mu\nu}$ = diag(**1**,**-1**,……,**-1**)

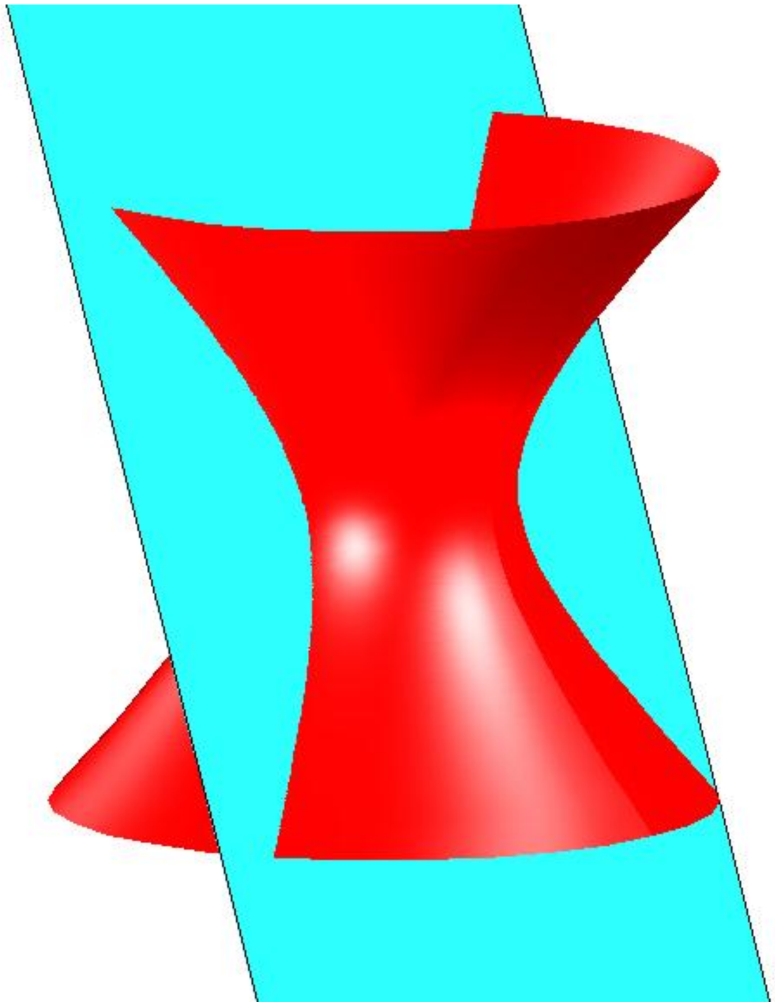
The asymptotic cone: causal structure



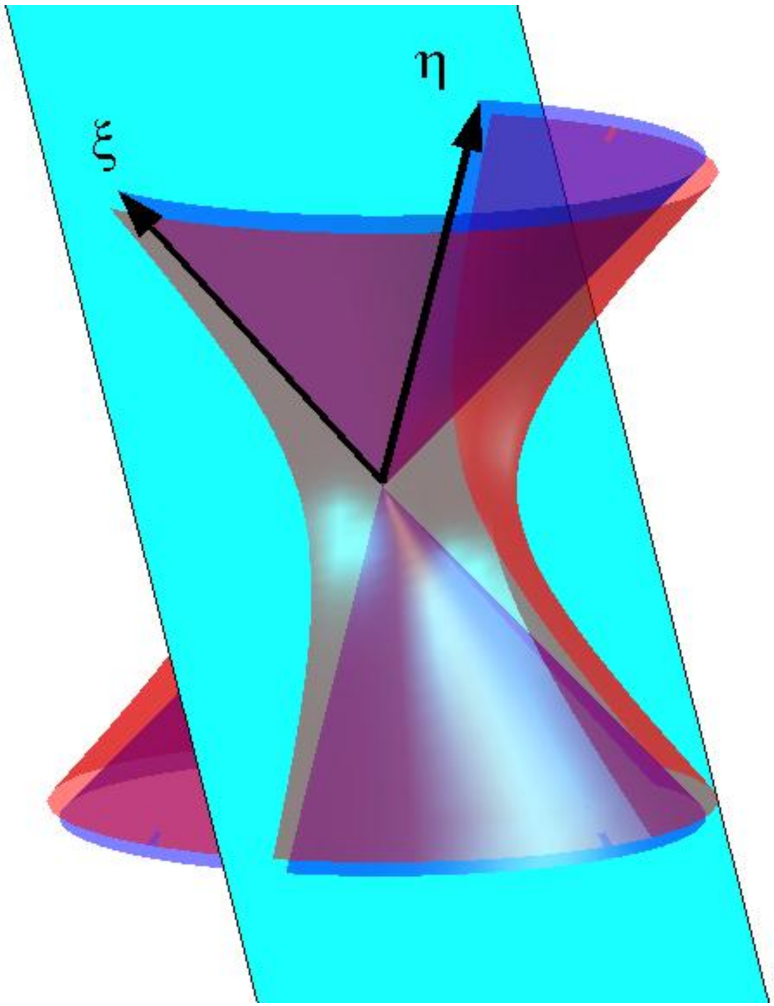
X, Y are spacelike separated iff $(X-Y)^2 < 0$ ($X-Y$ is outside the cone)

$$(X-Y)^2 = X^2 + Y^2 - 2X \cdot Y = -2R^2 - 2X \cdot Y$$

ds timelike geodesics



The asymptotic cone as the de Sitter momentum space



Geodesics: de Sitter

$$X_\mu(\tau) = \frac{R}{\sqrt{2\xi\cdot\eta}}\left(\xi_\mu e^{\frac{c\tau}{R}} - \eta_\mu e^{-\frac{c\tau}{R}}\right)$$

Minkowski

$$x_\mu(\tau) = x_\mu(0) + \frac{p_\mu \tau}{mc}$$

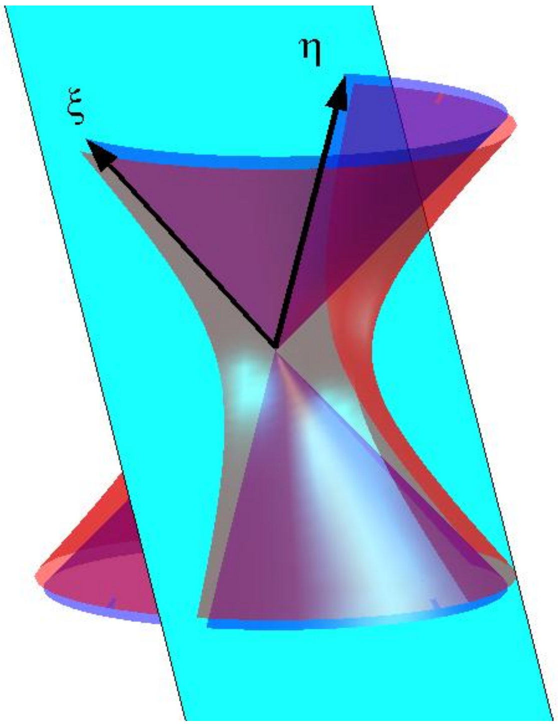
$$X_\mu(0) = \frac{R}{\sqrt{2\xi\cdot\eta}}(\xi_\mu - \eta_\mu)$$

$$X(\tau) = X(0)e^{-\frac{c\tau}{R}} + \frac{kR\xi}{m}\sinh\frac{c\tau}{R}.$$

Conserved quantities

$$X_\mu(\tau) = \frac{R}{\sqrt{2\xi\cdot\eta}}\left(\xi_\mu e^{\frac{c\tau}{R}} - \eta_\mu e^{-\frac{c\tau}{R}}\right)$$

$$K_{\xi,\eta} = mc\frac{\xi\wedge\eta}{\xi\cdot\eta}$$



Energy

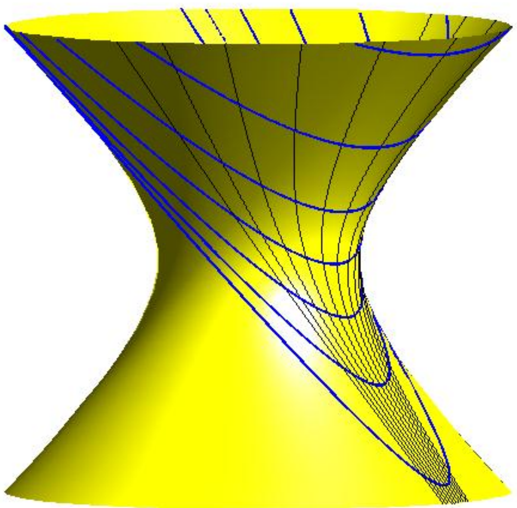
- In special relativity the energy of a particle is measured relative to an arbitrary given Lorentz frame, being the zero component of a four-vector.
- This picture does not extend to the de Sitter case where frames are defined only locally.
- The maximal symmetry of the de Sitter universe allows for the energy of a pointlike particle to be defined relative to just one reference massive free particle understood conventionally to be at rest (a sharply localized observer)
- The energy of the free particle (ξ,η) with respect to the reference geodesic (u,v) is defined as follows

$$E = E_{(\xi,\eta)}(u,v) = -\frac{c\,K_{(\xi,\eta)}(u,v)}{u\cdot v} = E_{(u,v)}(\xi,\eta)\ .$$

Energy

Geodesics parameterized by (ξ, X) and (u, Y)

$$E = mc^2 \frac{(u \cdot X)(\xi \cdot Y) - (X \cdot Y)(\xi \cdot u)}{(\xi \cdot X)(u \cdot Y)}$$

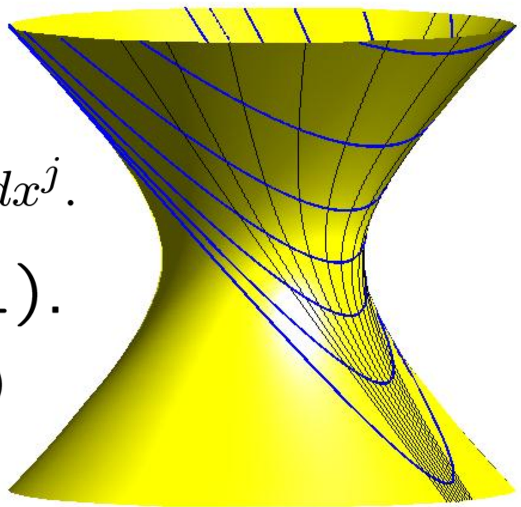


$$\begin{cases} Y_0 &= R \sinh \frac{t}{R} + \frac{1}{2R} e^{\frac{t}{R}} |\vec{x}|^2 \\ Y_1 &= \exp\left(\frac{t}{R}\right) x_1 \\ Y_2 &= \exp\left(\frac{t}{R}\right) x_2 \\ Y_3 &= \exp\left(\frac{t}{R}\right) x_3 \\ Y_4 &= R \cosh \frac{t}{R} - \frac{1}{2R} e^{\frac{t}{R}} |\vec{x}|^2 \end{cases}$$

Geodesics observer at rest at the origin

$$Y = (0, 0, 0, 0, R), \quad u = \lambda(1, 0, 0, 0, 1)$$

Energy



$$ds^2 = c^2dt^2 - e^{2ct/R}\delta_{ij}dx^idx^j = c^2dt^2 - a^2(t)\delta_{ij}dx^idx^j.$$

$Y = (0,0,0,0,R)$ and $u = \lambda(1,0,0,0,1)$.

Energy of the particle parameterized by (ξ, X) as seen by te reference observer (u, Y)

$$E = mc^2\frac{(u\cdot X)(\xi\cdot Y) - (X\cdot Y)(\xi\cdot u)}{(\xi\cdot X)(u\cdot Y)} = \frac{kc^2}{R}(\xi^0X^4 - \xi^4X^0).$$

$$X_\mu(\tau) = \frac{R}{\sqrt{2\xi\cdot\eta}}\left(\xi_\mu e^{\frac{c\tau}{R}} - \eta_\mu e^{-\frac{c\tau}{R}}\right) \qquad \xi = \frac{m}{kR}\left(X(0) + \frac{R}{c}\frac{dX(0)}{d\tau}\right).$$

$$E = mc^2\frac{dt}{d\tau} - \frac{c}{R}x^ip_i = \frac{mc^2}{\sqrt{1 - a^2(t)\frac{v^2}{c^2}}} - \frac{c}{R}x^ip_i$$

$$v^i = \frac{dx^i}{dt}, \qquad p_i = -me^{2ct/R}\frac{dx^i}{d\tau} = -\frac{ma^2(t)v^i}{\sqrt{1 - a^2(t)\frac{v^2}{c^2}}}.$$

Energy

- E can be interpreted as the correct de Sitter energy of the particle is confirmed by noting that it is the conserved quantity associated to the invariance of the particle action under time translation.
- Indeed, since in flat coordinates the spatial distances dilate in the course of time by the exponential factor the expression of an infinitesimal symmetry under time evolution is

$$
\begin{aligned}
t &\longrightarrow t + \epsilon, \\
x^i &\longrightarrow x^i - \frac{c}{R} x^i \epsilon.
\end{aligned}
$$

- The action

$$S = -mc \int \sqrt{1 - e^{\frac{2ct}{R}} \frac{v^i v^j}{c^2} \delta_{ij}} \; dt$$

- is invariant and E can be obtained from S by standard methods

Classical scattering

$$b_1 + b_2 \longrightarrow c_1 + \ldots + c_M$$

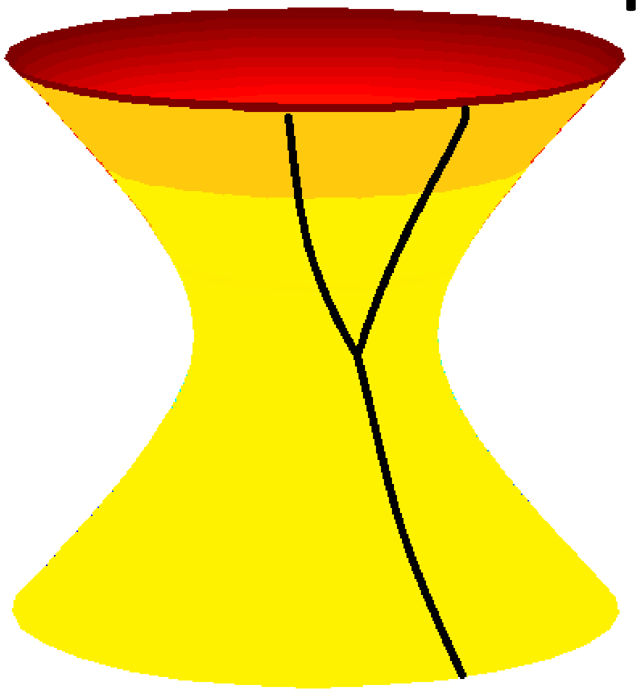
Problem: find the outgoing momenta (ξ_f, η_f) given the ingoing ones (χ_i, ζ_i).

$$X_\mu(\tau) = \frac{R}{\sqrt{2\xi \cdot \eta}} \left(\xi_\mu e^{\frac{c\tau}{R}} - \eta_\mu e^{-\frac{c\tau}{R}} \right)$$

$$K_i = m_i c \, \frac{\chi_i \wedge \zeta_i}{\chi_i \cdot \zeta_i} \qquad K_f = m_f c \, \frac{\xi_f \wedge \eta_f}{\xi_f \cdot \eta_f}$$

$$\sum_{i=1}^{2} K_i = \sum_{f=1}^{M} K_f, \qquad \frac{\chi_i - \zeta_i}{\sqrt{\chi_i \cdot \zeta_i}} = X = \frac{\xi_f - \eta_f}{\sqrt{\xi_f \cdot \eta_f}} \;,$$

Example: particle decay



$$m_1 \longrightarrow \mu_1 + \mu_2 \ ,$$

$$\xi = \frac{1}{\sqrt{2}}\left(\frac{m_1^2+\mu_1^2-\mu_2^2}{2m_1\mu_1}, \mp\frac{\sqrt{(m_1^2-\mu_1^2-\mu_2^2)^2-4\mu_1^2\mu_2^2}}{2m_1\mu_1}, 1\right)$$

$$\eta = \frac{1}{\sqrt{2}}\left(\frac{m_1^2-\mu_1^2+\mu_2^2}{2m_1\mu_2}, \pm\frac{\sqrt{(m_1^2-\mu_1^2-\mu_2^2)^2-4\mu_1^2\mu_2^2}}{2m_1\mu_2}, 1\right)$$

$$m_1 > \mu_1 + \mu_2 .$$

Mass subadditivity
will not hold in the quantum description

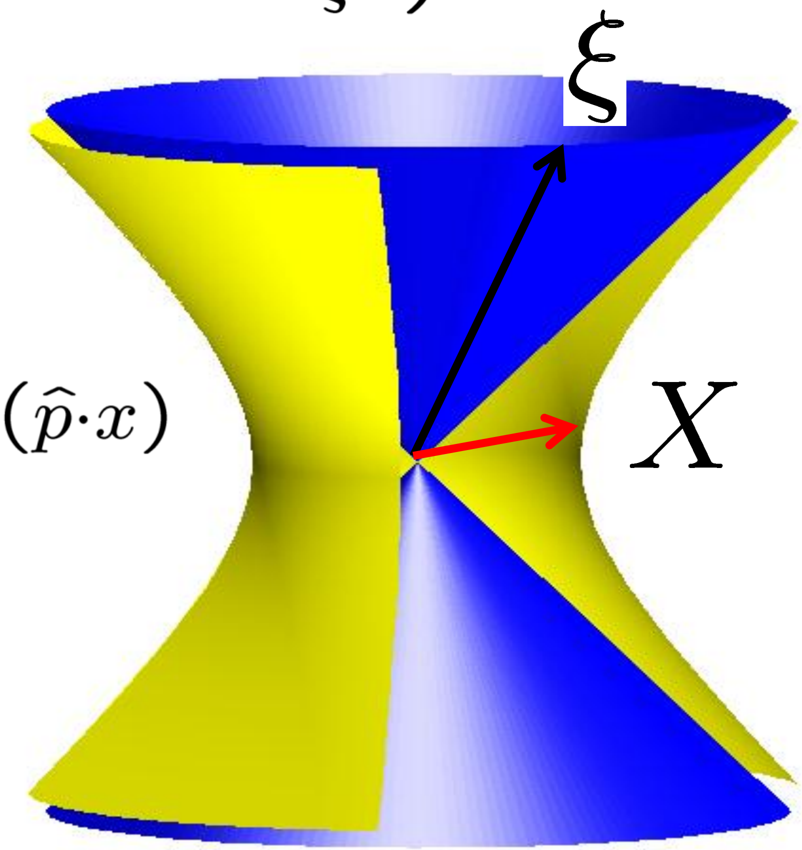
de Sitter plane waves

$$\psi_\lambda(X,\xi) = (X\cdot\xi)^\lambda =$$

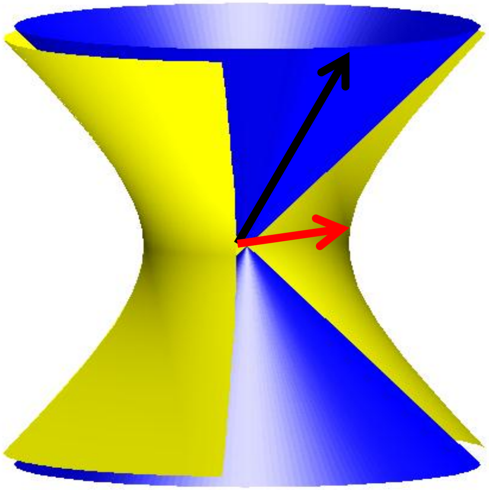
$$= (X^0\xi^0 - X^1\xi^1 - \ldots - X^d\xi^d)^\lambda$$

$$\lambda \in \mathbf{C}, \quad \xi^2 = 0$$

$$\psi(x,p) = e^{ip\cdot x} = e^{im(\widehat{p}\cdot x)}$$



de Sitter plane waves



$$\Box(X\cdot\xi)^{\lambda} = \lambda(\lambda+d-1)(X\cdot\xi)^{\lambda}$$

Involution:

$$\lambda \longrightarrow \bar{\lambda} = -\lambda-(d-1)$$

$$\lambda+\bar{\lambda} = -(d-1)$$

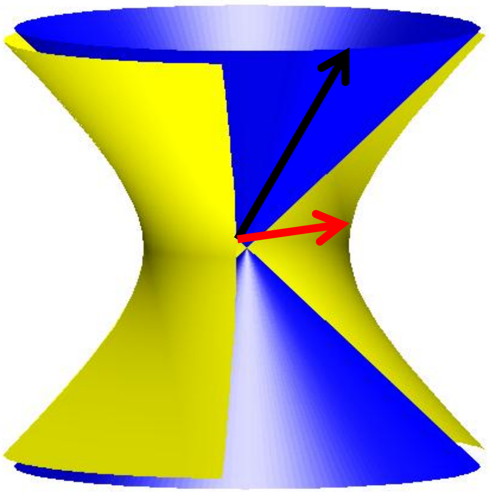
$$\Box(X\cdot\xi)^{-\lambda-d+1} = (-\lambda-d+1)(-\lambda)(X\cdot\xi)^{-\lambda-d+1}$$

Scalar waves with (complex) squared mass:

$$m^2 = \lambda\bar{\lambda}$$

$$\left(\Box+\lambda\bar{\lambda}\right)(X\cdot\xi)^{\lambda} = 0, \qquad \left(\Box+\lambda\bar{\lambda}\right)(X\cdot\xi)^{\bar{\lambda}} = 0$$

Principal de Sitter waves



$$\lambda = -\frac{d-1}{2} + i\nu\,, \qquad \nu \in \mathbf{R}$$

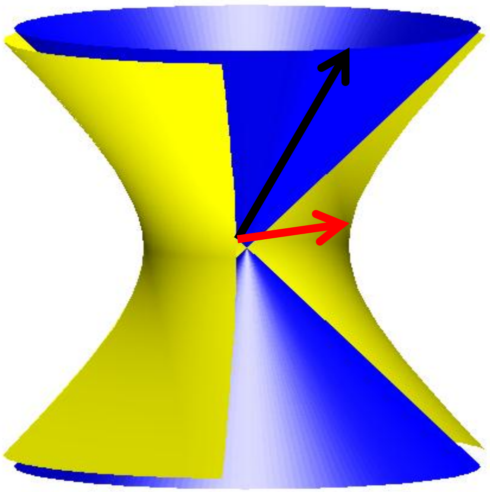
$$\psi_\lambda(X,\xi) = (X\cdot\xi)^{-\frac{d-1}{2}+i\nu}$$

$$\bar\lambda = -\lambda - (d-1) = -\frac{d-1}{2} - i\nu$$

$$\psi_{\bar\lambda}(X,\xi) = (X\cdot\xi)^{-\frac{d-1}{2}-i\nu} = \overline{\psi_\lambda(X,\xi)}$$

$$m^2 = \lambda\bar\lambda = |\lambda|^2 = \left(\frac{d-1}{2}\right)^2 + \nu^2 > \left(\frac{d-1}{2}\right)^2$$

Complementary de Sitter waves



$$\lambda = -\frac{d-1}{2} + \nu\,, \qquad \nu \in \mathbf{R}$$

$$\psi_\lambda(X,\xi) = (X\cdot\xi)^{-\frac{d-1}{2}+\nu}$$

These waves do not oscillate!

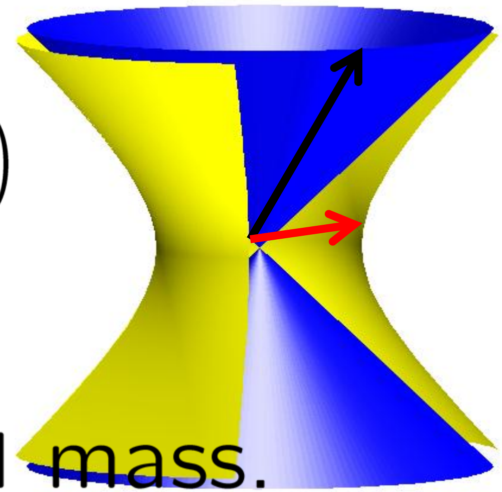
$$\bar\lambda = -\lambda - (d-1) = -\frac{d-1}{2} - \nu$$

$$\psi_{\bar\lambda}(X,\xi) = (X\cdot\xi)^{-\frac{d-1}{2}-\nu} \neq \overline{\psi_\lambda(X,\xi)} = \psi_\lambda(X,\xi)$$

$$m^2 = \lambda\bar\lambda = \left(\frac{d-1}{2}\right)^2 - \nu^2$$

$$-\left(\frac{d-1}{2}\right) < \nu < \left(\frac{d-1}{2}\right)$$

Discrete de Sitter waves



$$\lambda = -\tfrac{d-1}{2} + \nu\,,\ \ \nu \in \mathbf{R},\ \ |\nu| > \left(\tfrac{d-1}{2}\right)$$

$$\psi_\lambda(X,\xi) = (X\cdot\xi)^{-\frac{d-1}{2}+\nu}$$

have real but negative squared mass.

$$m^2 = \lambda\bar{\lambda} = \left(\tfrac{d-1}{2}\right)^2 - \nu^2 < 0$$

dS Tachyons?

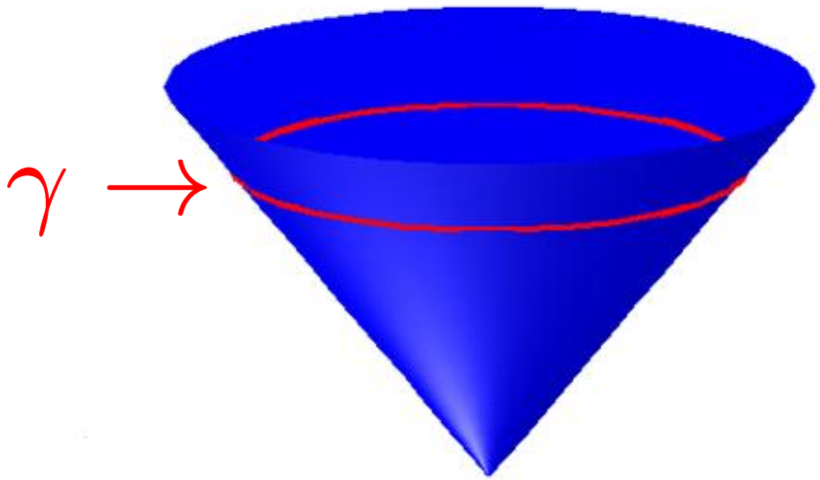
$(X\cdot\xi)^n$ and $(X\cdot\xi)^n \log(X\cdot\xi)$, n integer

$(X\cdot\xi)^{-n-d+1}$ and $(X\cdot\xi)^{-n-d+1} \log(X\cdot\xi)$

dS: construction of two-point functions

$$W(x_1, x_2) = \int e^{-ip\cdot x_1} e^{ip\cdot x_2} \theta(p^0)\delta(p^2 - m^2)dp$$

$$F_{\lambda,\gamma}(X_1, X_2) = \int (X_1 \cdot \xi)^{\lambda} (\xi \cdot X_2)^{-\lambda-d+1} d\mu_{\gamma}(\xi)$$



$$\left(\Box_{X_i} + \lambda\bar{\lambda}\right) F_{\lambda}(X_1, X_2) = 0,$$

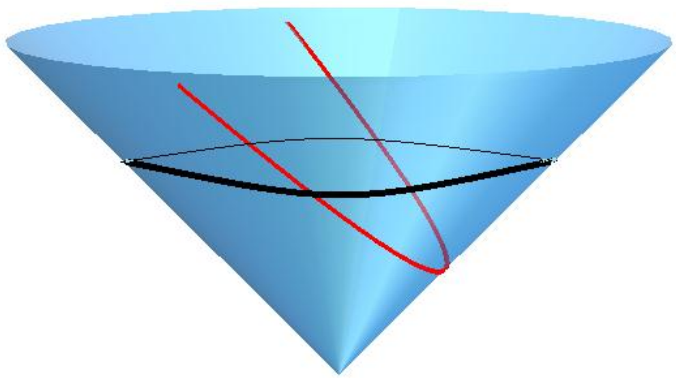
They are dS invariant

Plane waves are homogeneous functions of ξ

$$\psi(X, a\xi) = (X \cdot a\xi)^{\lambda} = a^{\lambda}\psi(X, \xi)$$

$$(X_1 \cdot a\xi)^{\lambda}(a\xi \cdot X_2)^{\bar{\lambda}} = a^{\lambda+\bar{\lambda}}(X_1 \cdot \xi)^{\lambda}(\xi \cdot X_2)^{\bar{\lambda}}$$

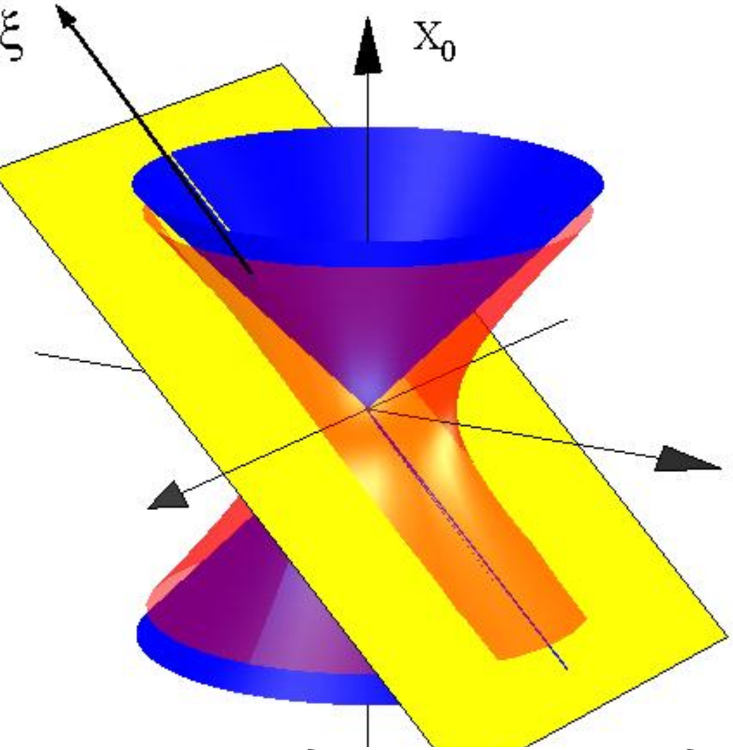
$$\lambda + \bar{\lambda} = -(d-1)$$



$$F_{\lambda,\gamma}(X_1, X_2) = \int (X_1 \cdot \xi)^{\lambda}(\xi \cdot X_2)^{-\lambda-d+1} d\mu_{\gamma}(\xi)$$

$$F_{\lambda}(gX_1, gX_2) = F_{\lambda}(X_1, X_2) = F_{\lambda}(X_1 \cdot X_2)$$

The plane waves are however irregular



$$\psi_\lambda(X,\xi) = (X\cdot\xi)^\lambda$$

$$X \in dS : (X\cdot\xi) = 0$$

$$(X\cdot\xi)^\lambda \to |X\cdot\xi|^\lambda \left(a(\lambda)\theta(X\cdot\xi) + b(\lambda)\theta(-X\cdot\xi)\right)$$

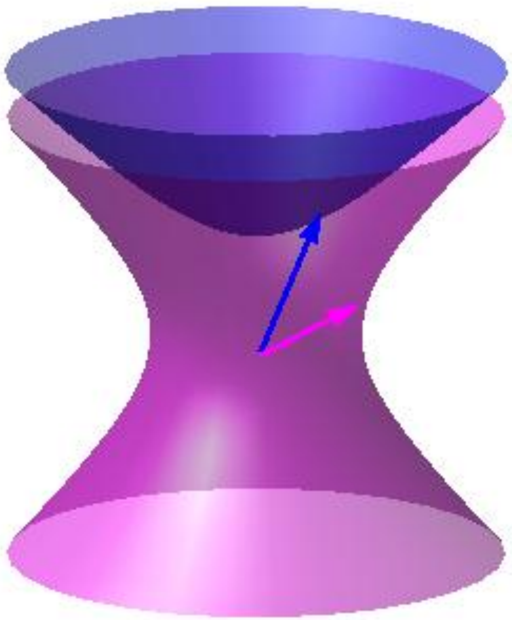
skip

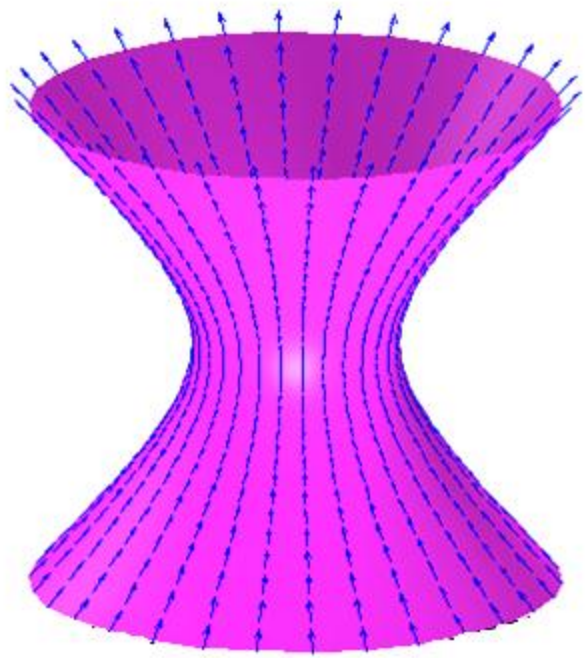
Spectral condition: de Sitter tubes

$$dS^c = Z_0^2 - Z_1^2 - \ldots Z_d^2 = -R^2$$

$$Z = X + iY, \quad X^2 - Y^2 = -R^2 \quad X \cdot Y = 0$$

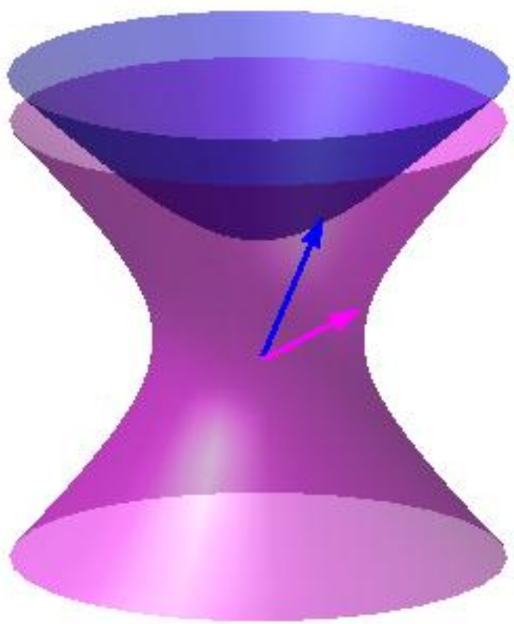
$\mathcal{T}^+ = Y$ in the forward cone.

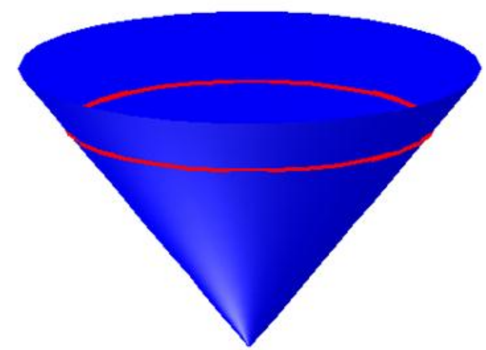




$\mathcal{T}^- = Y$ in the backward cone.

$\psi_\lambda(Z,\xi) = (Z\cdot\xi)^\lambda$ is globally well defined in both the past and future tubes because the imaginary part $Y\cdot\xi$ is always positive (negative) for $Z \in \mathcal{T}^+$ (alternatively $Z \in \mathcal{T}^+$)





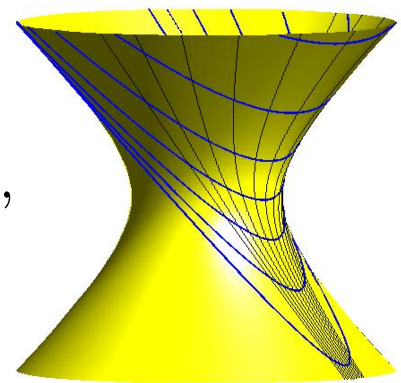
Boundary values on the reals:

$$(X\cdot\xi)^\lambda_\pm \to |X\cdot\xi|^\lambda\Big(\theta(X\cdot\xi) + e^{\pm i\pi\lambda}\theta(-X\cdot\xi)\Big)$$

$$\psi_{i\nu}^{\pm}(z,\xi) = (x \pm iy\cdot\xi)^{-\frac{d-1}{2}+i\nu},$$

are globally well-defined in the tubes.

$$\begin{cases} x^0 &= R\sinh\frac{t}{R} + \frac{1}{2R}\mathbf{x}^2 \exp\frac{t}{R} \\ x^i &= \mathbf{x}^\mathbf{i} \ \exp\frac{t}{R}, \\ x^d &= R\cosh\frac{t}{R} - \frac{1}{2R}\mathbf{x}^2 \ \exp\frac{t}{R}. \end{cases},$$



$$\begin{cases} \xi^0 &= \frac{1}{2}(1+\eta^2) \\ \xi^i &= \eta \\ \xi^d &= \frac{1}{2}(1-\eta^2) \end{cases},$$

$$\widetilde{\psi_{i\nu}^{\pm}}(t,\mathbf{k}) = \int [\mathbf{x}(\mathbf{t}\pm\mathbf{i}\epsilon,\mathbf{x})\cdot\xi(\eta)]^{-\frac{3}{2}+\mathbf{i}\nu}\mathbf{e}^{\mathbf{ikx}}\mathbf{dx}$$

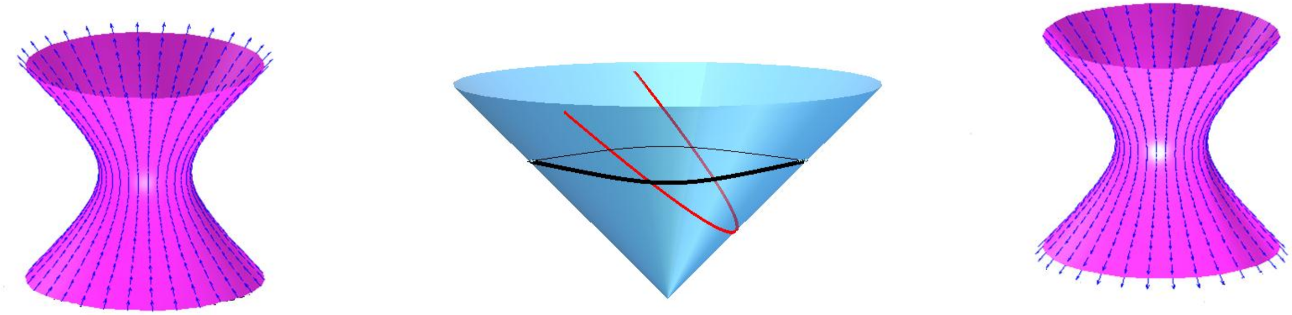
$$\widetilde{\psi_{i\nu}^{+}}(t,\mathbf{k}) = \frac{i\pi}{\Gamma\left(\frac{3}{2}-\nu\right)}(\mathbf{2}\pi\mathbf{e}^{-\mathbf{t}})^{\frac{3}{2}}\exp(\mathbf{ik}\eta)\,\mathbf{k}^{-\nu}\,\mathbf{H}_{\mathbf{i}\nu}^{(\mathbf{2})}(\mathbf{ke}^{-\mathbf{t}})$$

$$\widetilde{\psi_{i\nu}^{-}}(t,\mathbf{k}) = \frac{i\pi}{\Gamma\left(\frac{3}{2}-\nu\right)}(\mathbf{2}\pi\mathbf{e}^{-\mathbf{t}})^{\frac{3}{2}}\exp(\mathbf{ik}\eta)\,\mathbf{k}^{-\nu}\,\mathbf{H}_{\mathbf{i}\nu}^{(\mathbf{1})}(\mathbf{ke}^{-\mathbf{t}})$$

Fourier representation for BD 2-point functions

For $Z_1 \in \mathcal{T}^-$ e $Z_2 \in \mathcal{T}^+$

$$W_\lambda(Z_1, Z_2) = \int_\gamma (Z_1 \cdot \xi)^\lambda \ (\xi \cdot Z_2)^{-\lambda-(d-1)} d\mu(\xi)$$



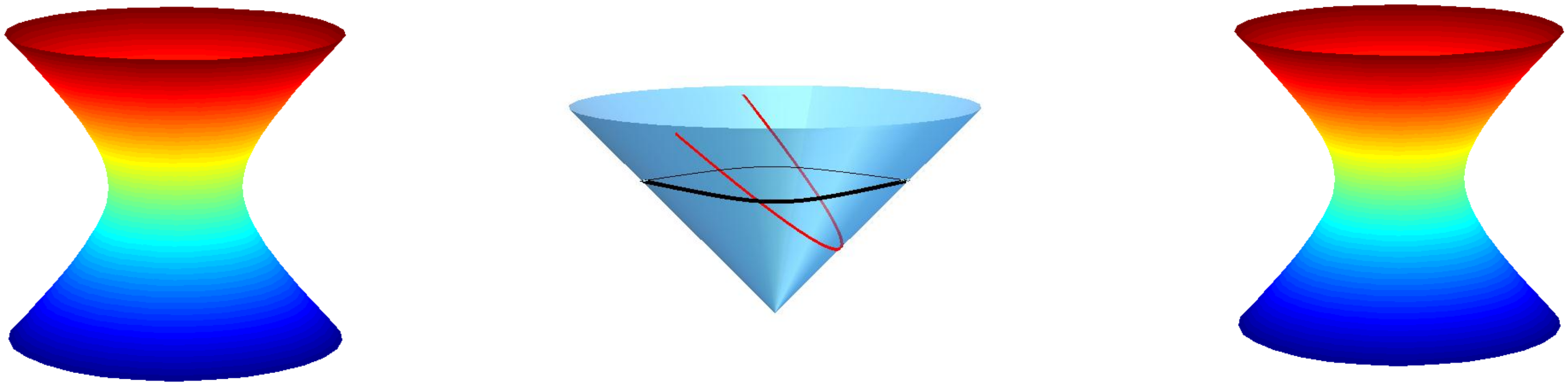
To be compared with the standard flat case:

$$W(z_1 - z_2) = \int e^{-ip\cdot z_1} e^{ip\cdot z_2} \theta(p^0)\delta(p^2 - m^2) d^4p$$

$z_1 \in T^-$ $z_2 \in T^+$

Fourier representation on the real manifold

$$W_\lambda(X_1, X_2) = \int_\gamma |X_1 \cdot \xi|^\lambda \, |X_2 \cdot \xi|^{\bar{\lambda}} \times$$
$$\times \left(\theta(X_1 \cdot \xi) + e^{-i\pi\lambda}\theta(-X_1 \cdot \xi)\right) \left(\theta(X_2 \cdot \xi) + e^{i\pi\bar{\lambda}}\theta(-X_2 \cdot \xi)\right) d\mu(\xi)$$



$$
\begin{aligned}
W_\lambda(Z_1,\ Z_2) &= \int_\gamma (Z_1\cdot\xi)^\lambda\ (\xi\cdot Z_2)^{-\lambda-(d-1)} d\mu(\xi) \\
&= \frac{\Gamma(-\lambda)\Gamma(\lambda+d-1)}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\ {}_2F_1\left(-\lambda,\ \lambda+d-1;\ \frac{d}{2};\ \frac{1-\zeta}{2}\right)\ . \\
&= \frac{\Gamma\,(-\lambda)\,\Gamma\,(\lambda+d-1)}{2(2\pi)^{\frac{d}{2}}}\ (\zeta^2-1)^{-\frac{d-2}{4}}\, P^{-\frac{d-2}{2}}_{-\lambda-\frac{d}{2}}(\zeta)
\end{aligned}
$$

a) $W(Z_2, Z_2)$ is invariant under the complex de Sitter group

$W(Z_1, Z_2)$ is maximally analytic.

The cut reflects causality

$$\zeta = Z_1\cdot Z_2$$

ζ = -1

b) $W(X_1, X_2)$ is b.v.of $W(Z_1, Z_2)$ from $\mathcal{T}^- \times \mathcal{T}^+$

The permuted function $W(X_2, X_1)$

is b.v.of the same $W(Z_1, Z_2)$ from $\mathcal{T}^+ \times \mathcal{T}^-$

Physical fields

$$\int W_\lambda(X_1, X_2)\bar{f}(X_1)f(X_2)dX_1dX_2 \geq 0$$

- For principal fields positivity is easy

$$W_\nu(Z_1, Z_2) = \int_\gamma (Z_1{\cdot}\xi)^{-\frac{d-1}{2}+i\nu}(\xi{\cdot}Z_2)^{-\frac{d-1}{2}-i\nu}d\mu(\xi)$$

- de Sitter Fourier transform(s)

$$\tilde{f}_\pm(\xi, -i\nu) = \int (\xi \cdot X)_\pm^{-\frac{d-1}{2}-i\nu} f(x)\, dx$$

$$\tilde{\bar{f}}_-(\xi, i\nu) = \bar{\tilde{f}}_+(\xi, -i\nu)$$

$$\int W_\nu(X_1, X_2)\bar{f}(X_1)f(X_2)dX_1dX_2 = \int_\gamma |\tilde{f}_+(\xi, -i\nu)|^2 d\mu(\xi) \geq 0$$

Complementary fields (inflation)

$$W_{\nu'}(Z_1, Z_2) = \int_\gamma (Z_2 \cdot \xi)^{-\frac{d-1}{2}+\nu'} (\xi \cdot Z_2)^{-\frac{d-1}{2}-\nu'} d\mu(\xi)$$

- Positivity is a little less direct. For ν>0 we have

$$[(X \pm iY) \cdot \xi]^{-\frac{d-1}{2}+\nu} = C_\nu \int (\xi \cdot \xi')^{-\frac{d-1}{2}+\nu} [(X \pm iY) \cdot \xi']^{-\frac{d-1}{2}-\nu} d\mu(\xi')$$

$$\overline{\widetilde{f}}_-(\xi, \nu) = C_\nu \int (\xi \cdot \xi')^{-\frac{d-1}{2}+\nu} \overline{\widetilde{f}}_+(\xi, -\nu) d\mu(\xi')$$

- The kernel in the following expression is positive definite for 0< ν<(d-1)/2

$$\langle W_\nu,\ \bar{f} \otimes f \rangle = C_\nu \int \overline{\widetilde{f}_+ (\xi, -\nu)} \, (\xi \cdot \xi')^{-\frac{d-1}{2}+\nu} \, \widetilde{f}_+ \left(\xi', -\nu\right) d\mu(\xi') d\mu(\xi)$$

De Sitter Tachyons

$$W_\lambda(Z_1,\ Z_2) = \Gamma(-\lambda)\, G_\lambda(\zeta)\ , \quad \zeta = Z_1 \cdot Z_2\,,$$

$$G_\lambda(\zeta) = \frac{\Gamma(\lambda + d - 1)}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\, {}_2F_1\left(-\lambda,\ \lambda + d - 1;\ \frac{d}{2};\ \frac{1-\zeta}{2}\right)\ .$$

$W_n(\zeta) = \infty$

$G_n(\zeta) = c_n {}_2F_1\left(-n,\ n + d - 1;\ \frac{d}{2};\ \frac{1-\zeta}{2}\right)$ is a polynomial.

$C_n(X_1, X_2) = \lim_{\lambda \to n} C_\lambda(X_1, X_2) =$

$= \lim_{\lambda \to n}[W_\lambda(X_1, X_2) - W_\lambda(X_2, X_1)]$ exists and is nontrivial.

$$\widehat{W}_n(Z_1, Z_2) = \lim_{\lambda \to n} \Gamma(-\lambda)\ \left[G_\lambda(\zeta) - G_n(\zeta)\right]$$

$\widehat{W}_n$ has the right commutator:

$\widehat{W}_n(X_1, X_2) - \widehat{W}_n(X_2, X_1) = C_n(X_1, X_2)$

The field equation gets an anomaly

$$\widehat{W}_n(Z_1, Z_2) = \lim_{\lambda \to n} \Gamma(-\lambda) \; [G_\lambda(\zeta) - G_n(\zeta)]$$

$$[\Box - n(n+d-1)] \, \widehat{W}_n(\zeta) = G_n(\zeta),$$

$$[\Box - n(n+d-1)] \, \phi(X) = Q_n(X)$$

$$Q_n^- \, |phys\rangle = 0$$

Euclidean approach (Folacci)

Euclidean action for a massive field

$$S(\phi) = \frac{1}{2}\int_{S^4} \sqrt{g}dx \left[(\nabla\phi)^2 + m^2\phi^2\right] = \frac{1}{2}\int_{S^4} dV \left(-\phi\Box\phi + m^2\phi^2\right)$$

The Laplace-Beltrami operator on S^4 possesses a discrete spectrum of eigenvalues λ_n.

$$\Box\phi_n^i = -\lambda_n\phi_n^i \qquad i = 1, ..., d_n$$
$$\lambda_n = n(n+3) \qquad n = 0, 1, 2, ...$$

The degeneracy is $d_n = \frac{1}{6}(n+1)(n+2)(2n+3)$. ϕ_n^i may be taken real and orthonormal:

$$\int_{S^4} dV \;\phi_n^i\phi_m^j = \delta_{nm}\delta_{ij}, \qquad \sum_n \sum_{i=1}^{d_n} \phi_n^i(x)\phi_n^i(x') = \delta^4(x, x').$$

Zero modes

The lowest eigenvalue of the laplacian $\Box$ is $\lambda_0 = 0$. The unique eigenfunction ϕ_0 is called the zero mode: it is a constant given by the normalization relation

$$\phi_0 = V^{-1/2} = \sqrt{\frac{3}{8\pi^2}}.$$

Orthonormality gives

$$\int_{S^4} dV \;\; \phi_0 \phi_n = \frac{1}{\sqrt{V}} \int_{S^4} dV \;\; \phi_n = 0 \quad \text{if } n \neq 0$$

Infrared divergence

$$\phi = \sum_n a_n \phi_n, \qquad S(\phi) = \frac{1}{2}\sum_n (\lambda_n + m^2) {a_n}^2.$$

$$G(x, x'; m^2) = \frac{\int d[\phi]\ \phi(x)\phi(x') \exp(-S)}{\int d[\phi]\ \exp(-S)} = \sum_n \frac{\phi_n(x)\phi_n(x')}{\lambda_n + m^2}$$

In the massless limit, $G(x, x'; m^2)$ is divergent.

$$G(x, x'; m^2) \simeq \frac{3}{8\pi^2 m^2} + \frac{4}{3\pi^2}\left[\frac{1/2}{(x - x')^2} - \ln\left(x - x')^2\right)\right] + \mathcal{O}(m^2).$$

$$\widehat{G}(x, x'; m^2) = \sum_{n \neq 0} \frac{\phi_n(x)\phi_n(x')}{\lambda_n + m^2} \neq \frac{\int d[\phi]\ \phi(x)\phi(x') \exp(-S)}{\int d[\phi]\ \exp(-S)}$$

BRS quantization

The Euclidean massless action

$$S(\phi) = \frac{1}{2}\int_{S^4} dV\ (\nabla\phi)^2 = -\frac{1}{2}\int_{S^4} dV\ \phi\Box\phi$$

is invariant under the gauge symmetry $\phi \to \phi$+constant

Introduce an anticommuting operator $\mathbf{s}$ so that $\mathbf{s}^2 = 0$:

$$\mathbf{s}\phi = c, \quad \mathbf{s}c = 0, \quad \mathbf{s}\overline{c} = b, \quad \mathbf{s}b = 0.$$

c is the ghost associated to the gauge symmetry. $\overline{c}$ is the antighost and b is the auxillary field.

Add the gauge fixing term to the massless action:

$$S^{GF} = \mathbf{s}\int_{S^4} dV\ \left(\overline{c}\phi - \frac{1}{2}\alpha\overline{c}b\right) = b\left(\int_{S^4} dV\ \phi\right) - V\overline{c}c - \frac{1}{2}\alpha V b^2$$

BRS quantization

The shift $b \to b + \frac{1}{\alpha V}\int_{S^4} dV \ \phi$ finally gives the total action

$$S^Q(\phi, c, \overline{c}, b) = \frac{1}{2}\int_{S^4} dV \ (\nabla\phi)^2 + \frac{1}{2\alpha V}\left(\int_{S^4} dV \ \phi\right)^2 - V\overline{c}c - \frac{1}{2}\alpha V b^2.$$

$$G(x, x') = \frac{\int d[\phi] \ d\overline{c} \ dc \ db \ \phi(x)\phi(x') \exp(-S^Q)}{\int d[\phi] \ d\overline{c} \ dc \ db \ \exp(-S^Q)}.$$

Inserting $\phi = \sum_n a_n \phi_n$ and Wick rotating b and c we get:

$$\begin{aligned} G(x, x') &= \sum_{n \neq 0} \frac{\phi_n(x)\phi_n(x')}{\lambda_n} + \alpha\phi_0(x)\phi_0(x') \\ &= \frac{4}{3\pi^2}\left[\frac{1/2}{(x - x')^2} - \ln\Big(x - x')^2\Big)\right] + \frac{3\alpha}{8\pi^2 m^2} \end{aligned}$$

Fourier representation

$$\widehat{W}_n(z_1, z_2) = W_n(z_1, z_2) - F_n^1(z_1, z_2) - F_n^2(z_1, z_2) + G_n(z_1,\ z_2).$$

$$\begin{aligned}
W_n(z_1, z_2) &= \int_\gamma \int_\gamma (z_1 \cdot \xi)^{1-d-n} (\xi \cdot \xi')^n \log(\xi \cdot \xi') (z_2 \cdot \xi')^{1-d-n} \, d\mu(\xi) d\mu(\xi') \\
F_n^1(z_1, z_2) &= \int_\gamma \int_\gamma \log(z_1 \cdot \xi)(z_1 \cdot \xi)^{1-d-n} (\xi \cdot \xi')^n (z_2 \cdot \xi')^{1-d-n} \, d\mu(\xi) d\mu(\xi') \\
F_n^2(z_1, z_2) &= \int_\gamma \int_\gamma (z_1 \cdot \xi)^{1-d-n} (\xi \cdot \xi')^n \log(z_2 \cdot \xi') (z_2 \cdot \xi')^{1-d-n} \, d\mu(\xi) d\mu(\xi')
\end{aligned}$$

$$\Psi \in E_n, \quad E_n = \left\{ \Psi \in C_0^\infty(X_d) \ : \int G_n(x_1 \cdot x_2) \Psi(x_2) dx_2 = 0 \right\}$$

$$\widehat{W}_n(z_1 \cdot z_2)|_{E_n \times E_n} = W_n(z_1, z_2)|_{E_n \times E_n}.$$

(With a little hard work one can show that) The theory is local, de Sitter invariant and positive definite

The equation of motion anomaly-free on the physical space

The positive physical space disappears in the flat limit

Sketch of the proof

$$\widehat{W}_n(z_1 \cdot z_2)|_{E_n \times E_n} = W_n(z_1, z_2)|_{E_n \times E_n}.$$

$$\int W_n(x_1, x_2)\bar{f}(x_1) f(x_2) dx_1 dx_2 =$$
$$\int \int_\gamma \int_\gamma \bar{f}(x_1)(x_1 \cdot \xi)_-^{1-d-n} (\xi \cdot \xi')^n \log(\xi \cdot \xi')\, (x_2 \cdot \xi')_+^{1-d-n} f(x_2)\, d\mu(\xi) d\mu(\xi') dx_1 dx_2$$

$$f \in E_n, \quad E_n = \left\{ \Psi \in C_0^\infty(X_d) \ : \int G_n(x_1 \cdot x_2) f(x_2) dx_2 = 0 \right\}$$

The proof makes use of the following

LEMMA: For any integer $n \geq 0$ the power series expansion $u_n(z) = (-1)^{n+1}(1-z)^n \log(1-z) = \sum_{m=0}^{\infty} u_{n,m} z^m$ converges for $|z| < 1$ and satisfies $u_{n,m} > 0$ for all $m > n$.

Remarks

The theory is local, de Sitter invariant and positive definite
The equation of motion anomaly-free on the physical space
The positive physical space disappears in the flat limit

For $d \geq 2$ is an even integer $z = -(z_1 - z_2)^2/4 = (1 + \zeta)/2$.

$$\widehat{w}_n = z^{1-\frac{d}{2}} A(z,\ n,\ d) - \log(z) B(z,\ n,\ d) + C(z,\ n,\ d),$$

A, B, C are polynomials in z. The most singular term is locally Hadamard(CCRs hold)

A fully positive de Sitter non-invariant Allen-Folacci type quantization does not exist for $m \neq 0$. Note that the Allen-Folacci two-point function does not coincide with $\widehat{w}_n$ on the physical space.

Conclusion and Outlook

- Physics of the de Sitter Tachyons remains to be understood
- This work was done in collaboration and friendship with Jacques Bros and Henri Epstein
- See you in Cargese again!