Radiation-Dominated Epoch in R^2 Gravity

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Outline

Why Modified Gravity?

Despite its many successes, the standard cosmological model does *not* successfully explain everything about the present Universe. Among the most interesting problems, we have

- **Horizon, Flatness** and **Monopole Problems** $\rightarrow$ **Inflation**!
- **Dark Matter** (anomalous galactic rotation curves, ...)
- **Dark Energy** (Supernovae Ia measurements, CMB observables, ...) $\rightarrow$ 2011 Nobel Prize in Physics!
- anomalous acceleration of Pioneer 10/11
- ^{7}Li problem in standard BBN
- ...

Can **Modified Gravity** be the solution?
Can it at least provide some hints to find such solution ?

Modified-Gravity Theories

The classical Einstein-Hilbert action for General Relativity is

$$A_{grav} = -\frac{m_{Pl}^2}{16\pi} \int d^4x \sqrt{-g}\, R \equiv \int d^4x \sqrt{-g}\, \mathcal{L}_{grav}$$

We can construct modified theories of gravity by:

- adding a new (scalar) field with non-minimal coupling to gravity

$$\mathcal{L}_{grav,\varphi} = R\, f(\varphi)$$

- adding new geometrical invariants:

$$\mathcal{L}_{grav} = f(R, R_{\mu\nu}R^{\mu\nu}, \Box R, \cdots)$$

- even more complicated theories are possible (our imagination is the limit!)

Quadratic Gravity

Probably, the most "natural" modified-gravity model is **quadratic gravity**:

$$\mathcal{L}_{grav} = -\frac{m_{Pl}^2}{16\pi}\left[R + a\,R^2 + b\,R_{\mu\nu}R^{\mu\nu} + c\,R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right]$$

- terms arising naturally from loop corrections to the r.h.s. of the Einstein equations (Starobinsky 1980)
- **inflation** without phase transitions in the (very) early Universe
- graceful exit to a **scalaron-dominated** (MD) regime
- **reheating** process, leading to the known early RD-Universe
- at large curvatures $|R| \gg |R_0|$, "saving" $f(R)$ models capable of generating the present cosmic acceleration

From Quadratic Gravity to R^2 Gravity

$$\mathcal{L}_{grav} = -\frac{m_{Pl}^2}{16\pi}\left[R + a\,R^2 + b\,R_{\mu\nu}R^{\mu\nu} + c\,R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right]$$

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- in FLRW spacetimes only the R^2 term is relevant

From Quadratic Gravity to R^2 Gravity

$$\mathcal{L}_{grav} = -\frac{m_{Pl}^2}{16\pi}\left[R - \frac{R^2}{6m^2} + b\, R_{\mu\nu}R^{\mu\nu} + c\, R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right]$$

- In the "standard" scenario, coupling constants are typically **very small**, so these terms are usually unimportant, except in the very early Universe
- in FLRW spacetimes only the R^2 term is relevant
- the only free parameter of the model is now m [mass]

R^2 Gravity

Action and Field Equations

The action is:

$$A_{grav} = -\frac{m_{Pl}^2}{16\pi}\int d^4x\,\sqrt{-g}\,\left(R - \frac{R^2}{6m^2}\right)$$

The new field equations are 4th order, and read

$$R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu} - \frac{1}{3m^2}\left(R_{\mu\nu} - \frac{1}{4}R\,g_{\mu\nu} + g_{\mu\nu}\mathcal{D}^2 - \mathcal{D}_\mu\mathcal{D}_\nu\right)R = \frac{8\pi}{m_{Pl}^2}\,T_{\mu\nu}$$

- Matter and gravity are now related **differentially**, so we expect more solutions than in GR
- the l.h.s. is conserved, so **energy-momentum conservation** still holds

Friedmann Universe

In a FLRW Universe, the only non-trivial equations (only **two** are independent!) are

- $\ddot{R} + 3H\dot{R} + m^2 \left[R + \frac{8\pi}{m_{Pl}^2}(1-3w)\varrho \right] = 0$
- $\ddot{H} + 3H\dot{H} - \frac{\dot{H}^2}{2H} + \frac{m^2}{2}H = \frac{4\pi m^2}{3m_{Pl}^2 H}\varrho$
- $\dot{\varrho} + 3H(1+w)\varrho = 0$

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- $$\dot{\varrho} + 3H(1+w)\varrho = 0 \quad \rightarrow \quad \dot{\varrho} + 4H\varrho = 0$$

During the **RD epoch**, $w = 1/3$. In usual GR, the RD solution is given by $H = 1/(2t)$ ($R = 0$), which also fulfills this system. But:

is this solution stable?

RD Epoch

Equations in dimensionless form

We start looking at the Universe when $H = H_0$, and define the new dimensionless variables

$$h \equiv \frac{H}{H_0}\,, \quad \tau \equiv H_0\, t\,, \quad r \equiv \frac{R}{H_0^2}\,, \quad \omega \equiv \frac{m}{H_0}\,, \quad y \equiv \frac{8\pi}{3m_{Pl}^2 H_0^2}\varrho$$

The system reads

$$\begin{cases} h'' + 3hh' - \dfrac{h'^2}{2h} + \dfrac{\omega^2}{2h}(h^2 - y) = 0 \\ \\ y' + 4hy = 0 \end{cases} \quad \text{or} \quad \begin{cases} r'' + 3hr' + \omega^2 r = 0 \\ \\ r + 6h' + 12h^2 = 0 \end{cases}$$

Initial Conditions
Avoiding Inflation

We assume that the Universe is deep in the radiation epoch, so some mechanism of **inflation** and **reheating** is needed.
However, we do **not** assume that the R^2-term is necessarily responsible for this, and treat **m** as a true free parameter.
In order to avoid inflation, we look for a sort of (very-)"**fast-roll**" condition.
Consider, for instance, the evolution equation for R:

$$\ddot{R} + 3H\dot{R} + m^2 R = 0 \quad \Rightarrow \quad r'' + 3hr' + \omega^2 r = 0$$

The fast-roll condition becomes

$$\frac{m^2}{H^2} \gg 1 \quad \Rightarrow \quad \omega^2 \gg 1$$

From R^2 to Scalar-Tensor Gravity

Jordan Frame → Einstein Frame

We start with the gravitational lagrangian

$$\mathcal{L}_{grav} = -\frac{m_{Pl}^2}{16\pi} f(R)$$

then we perform the conformal transformation

$$g_{\mu\nu} \to \widetilde{g}_{\mu\nu} = f'_R \, g_{\mu\nu}$$

and define a new scalar field

$$\varphi \equiv \sqrt{\frac{3m_{Pl}^2}{16\pi}} \ln f'_R$$

From R^2 to Scalar-Tensor Gravity

Jordan Frame → Einstein Frame

We start with the gravitational lagrangian

$$\mathcal{L}_{grav} = -\frac{m_{Pl}^2}{16\pi} f(R) \to -\frac{m_{Pl}^2}{16\pi}\left(R - \frac{R^2}{6m^2}\right)$$

then we perform the conformal transformation

$$g_{\mu\nu} \to \widetilde{g}_{\mu\nu} = f'_R \, g_{\mu\nu} \to \left(1 - \frac{R}{3m^2}\right) g_{\mu\nu}$$

and define a new scalar field

$$\varphi \equiv \sqrt{\frac{3m_{Pl}^2}{16\pi}} \ln f'_R \to \sqrt{\frac{3m_{Pl}^2}{16\pi}} \ln\left(1 - \frac{R}{3m^2}\right)$$

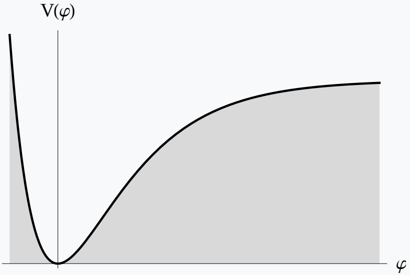
Scalar-Field Potential
Fast-Roll Region

The new lagrangian in the Einstein frame reads

$$\widetilde{\mathcal{L}}_{grav,\varphi} = -\frac{m_{Pl}^2}{16\pi}\widetilde{R} + \frac{1}{2}(\widetilde{\partial\varphi})^2 - V(\varphi)$$

where

$$V(\varphi) = \frac{3\, m^2 m_{Pl}^2}{32\pi}\, e^{-2\kappa\varphi}\,(1 - e^{\kappa\varphi})^2$$



$$\varphi \simeq \ln\left(1 - \frac{R}{3m^2}\right)$$

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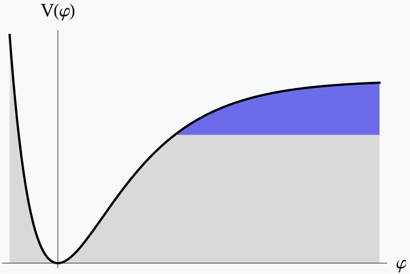

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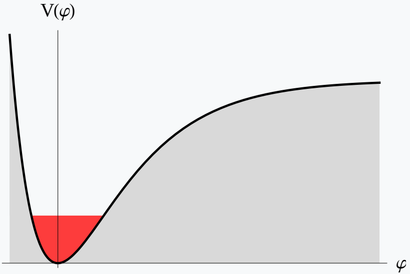
$$V(\varphi) = \frac{3\,m^2 m_{Pl}^2}{32\pi}\,e^{-2\kappa\varphi}\,(1 - e^{\kappa\varphi})^2 \simeq \frac{1}{2}m^2\varphi^2$$



$$\varphi \simeq \ln\left(1 - \frac{R}{3m^2}\right)$$

$$\varphi \simeq -\frac{R}{3m^2} = -\frac{r}{3\omega^2} \quad \text{small!}$$

Linear Regime

System of Equations

We linearize the system considering small perturbations from the GR behaviour

$$h = \frac{1}{2\tau} + h_1 \,, \qquad y = \frac{1}{4\tau^2} + y_1 \,, \qquad z_1 \equiv h_1'$$

obtaining

$$\begin{cases} z_1' = -\frac{5}{2\tau} z_1 + \left(\frac{1}{\tau^2} - \omega^2\right) h_1 + \tau\omega^2 y_1 \\ h_1' = \quad + z_1 \\ y_1' = \quad -\frac{1}{\tau^2} h_1 - \frac{2}{\tau} y_1 \end{cases} \Rightarrow \begin{pmatrix} z_1 \\ h_1 \\ y_1 \end{pmatrix}' = \mathbf{M}(\tau) \begin{pmatrix} z_1 \\ h_1 \\ y_1 \end{pmatrix}$$

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Linear Regime
Results

At large times, we can treat $\tau-$dependent coefficients as "constants", and find the eigenvalues/eigenvectors of the system. The only relevant solution is

$$h_1 \sim \frac{\cos(\omega\tau + \varphi)}{\tau^{3/4}}$$

so

$$h \simeq \frac{1}{2\tau} + \frac{b\cos(\omega\tau + \varphi)}{\tau^{3/4}}$$

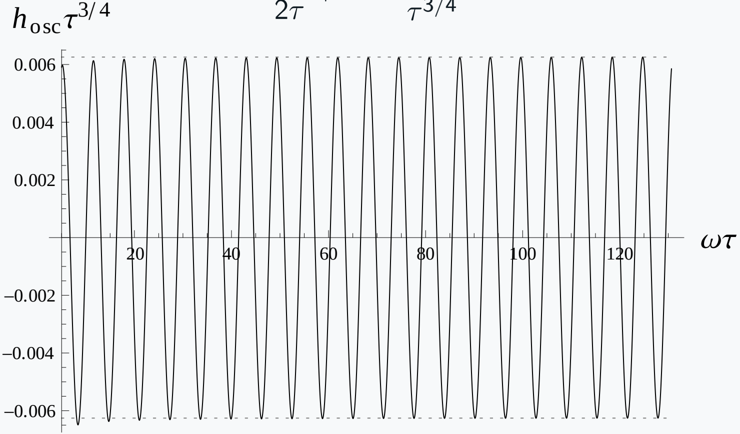
This corresponds to

$$r \simeq \frac{c}{\tau^{3/2}} + \frac{d\sin(\omega\tau + \phi)}{\tau^{3/4}}$$

Linear Regime

Numerical Results - h $(\omega = 10,\ h_0 = 1.01)$

$$h \simeq \frac{1}{2\tau} + \frac{b\cos(\omega\tau + \varphi)}{\tau^{3/4}}$$



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Numerical Results - r $(\omega = 10,\ h_0 = 1.01)$

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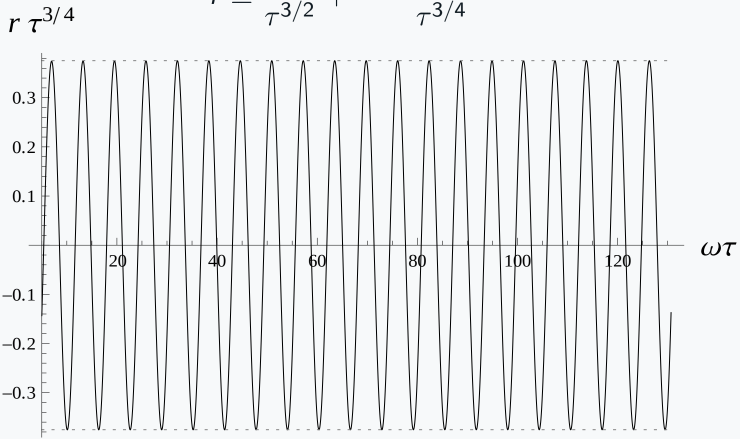

Asymptotic Regime

The oscillating part decreases more slowly than $1/\tau$, so we expect that at some point the linear regime will no longer hold.
However, for $\omega\tau \gg 1$, we can perform **analytical** estimates even when $\delta h/h \sim 1$.
We look for solutions of the form

$$h(\tau) = \frac{a}{\tau} + \frac{b}{\tau}\cos(\omega\tau + \varphi) + \cdots$$

$$r(\tau) \equiv -6H' + 12H^2 = \frac{c}{\tau^2} + \frac{d}{\tau}\sin(\omega\tau + \varphi) + \cdots$$

The original equations are transformed into equations for the coefficients of the expansion, e.g.

$$r'' + 3hr' + \omega^2 r = 0$$

$$\Rightarrow \quad f_{\text{power}} + f_{\sin}\sin(\omega\tau) + f_{\cos}\cos(\omega\tau) + \text{ higher freq.} = 0$$

where $f_i = f_i(a, b, c, d, \omega, \tau)$.

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Conditions on Coefficients

Using this procedure, we obtain some conditions on the coefficients of the expansion, in particular:

$$\begin{cases} b^2 = 2a(2a-1) \\ \\ b(2-3a) = 0 \end{cases} \quad \rightarrow \quad \begin{array}{lll} a = 1/2\,, & b = 0 & \text{GR, RD} \\ \\ a = 2/3\,, & b = 2/3 & \text{"Maximal" osc.} \end{array}$$

Note that the first condition gives $a \geq 1/2$
$\rightarrow$ "average" expansion at least as fast as in GR!

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Note that the first condition gives $a \geq 1/2$

$\rightarrow$ “average” expansion at least as fast as in GR!

The asymptotic behaviour is that of “maximal” oscillations around a MD Universe:

$$h \simeq \frac{2}{3\tau}\left[1 + \cos(\omega\tau + \varphi)\right]$$

Numerical Treatment

Eventually, radiation becomes negligible, and the Universe is dominated by the scalaron degree of freedom:

$$h \simeq \frac{2}{3\tau}\left[1 + \cos(\omega\tau + \varphi)\right]$$

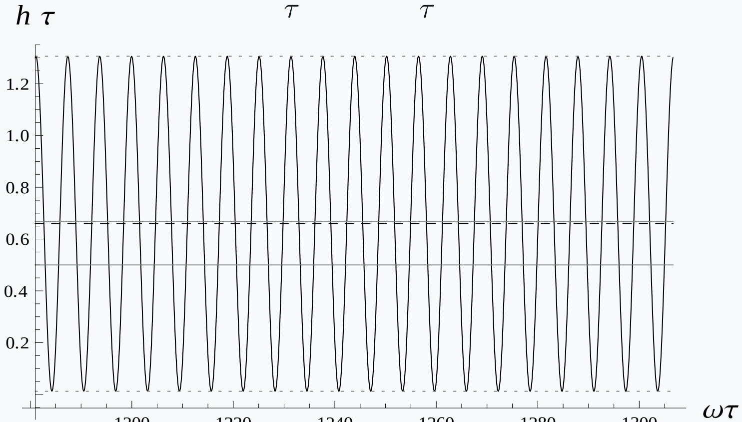
Numerically, we want to avoid lengthy calculations, so we explore this regime by **shifting the initial time**, or equivalently **changing** H_0, and **increasing the initial displacements from GR**.
Other parameters change accordingly, for instance

$$H_0 \to \frac{H_0}{K} \quad \Rightarrow \quad \omega \equiv \frac{m}{H_0} \to K\,\omega$$

Asymptotic Regime

Numerical Results - h $(\omega = 100,\ h_0 = 3.5)$

$$h \simeq \frac{a}{\tau} + \frac{b\cos(\omega\tau + \varphi)}{\tau}$$



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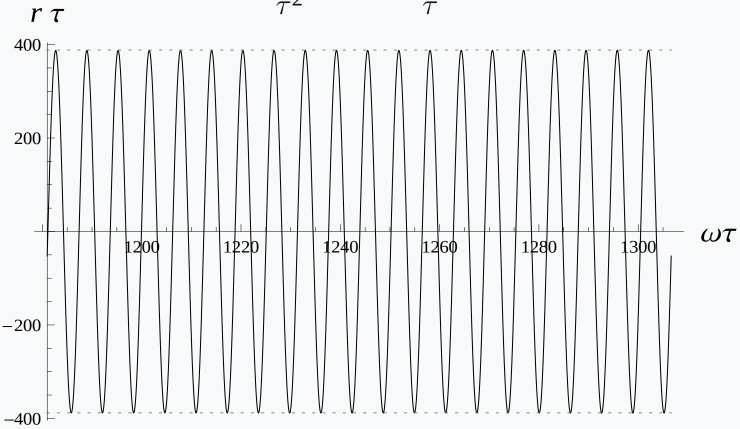
Numerical Results - r $(\omega = 100,\ h_0 = 3.5)$

$$r \simeq \frac{c}{\tau^2} + \frac{d \sin(\omega\tau + \varphi)}{\tau}$$

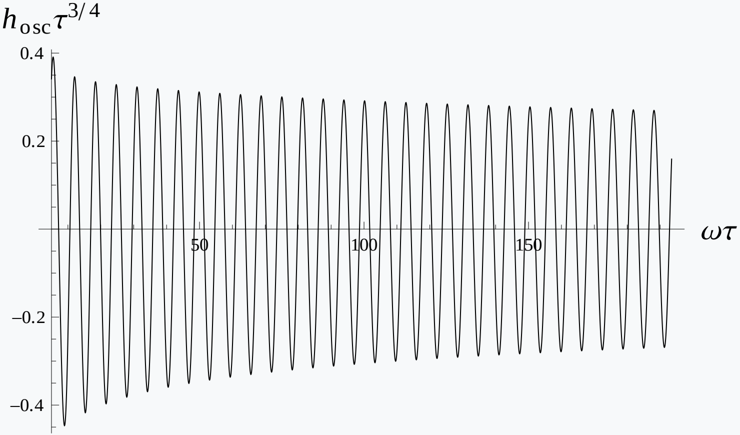


Intermediate Regime

We also expect some intermediate regime, in which we can only rely on numerical results. For instance, large h_1 but not-too-large ω's give results which are in between the two regimes discussed earlier.

Intermediate Regime

Numerical Results - h $(\omega = 10,\ h_0 = 1.8)$



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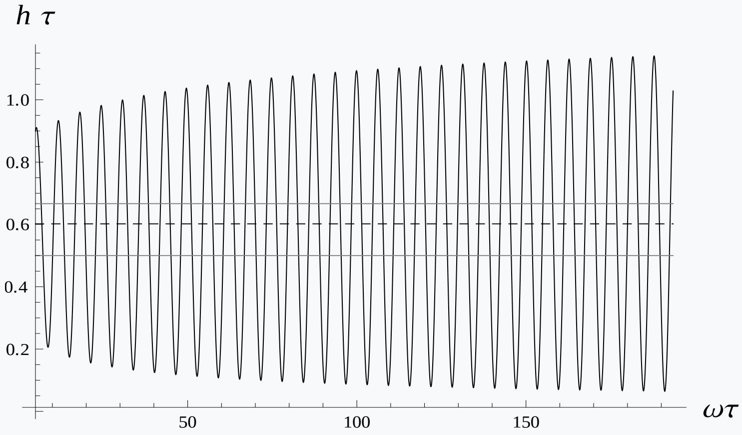
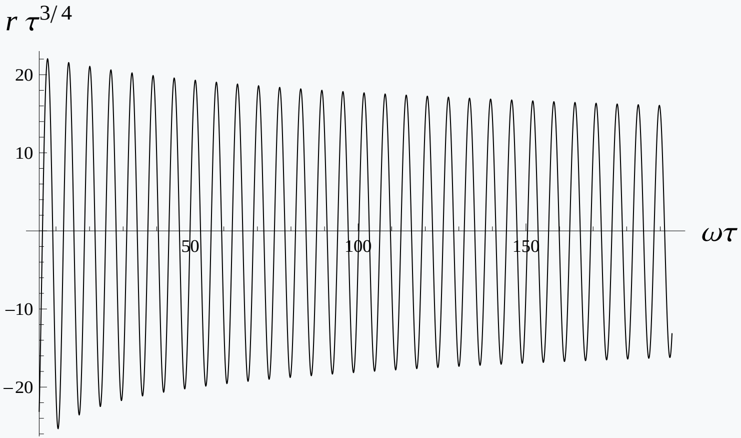


Intermediate Regime

Numerical Results - r $(\omega = 10,\ h_0 = 1.8)$



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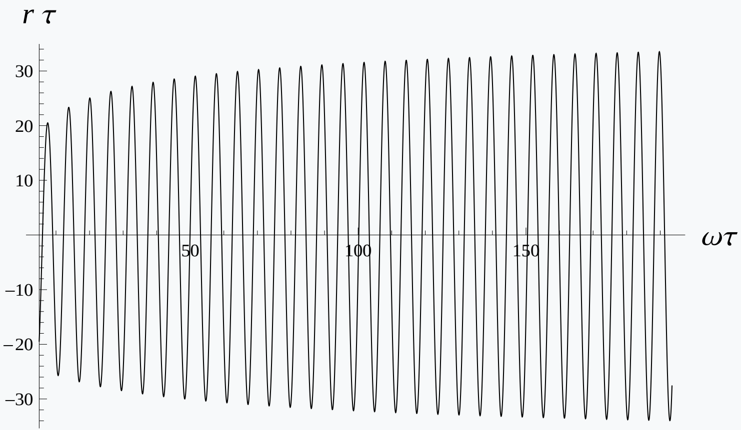

What Next?

- Include non-relativistic matter and/or trace anomaly
- oscillating gravity $\rightarrow$ **particle production!**
  (see talk by E. Arbuzova)
  - which fraction of total primordial energy density?
  - **non-thermal Universe?**
  - (heavy) **Dark Matter** production?
- study effects on BBN, both from particle production and deviation from GR expansion
- astrophysics: stability/evolution in pure R^2, $f(R) + R^2$, cosmic rays
- ...