Part I: A First Class Formulation of Massive Gravity

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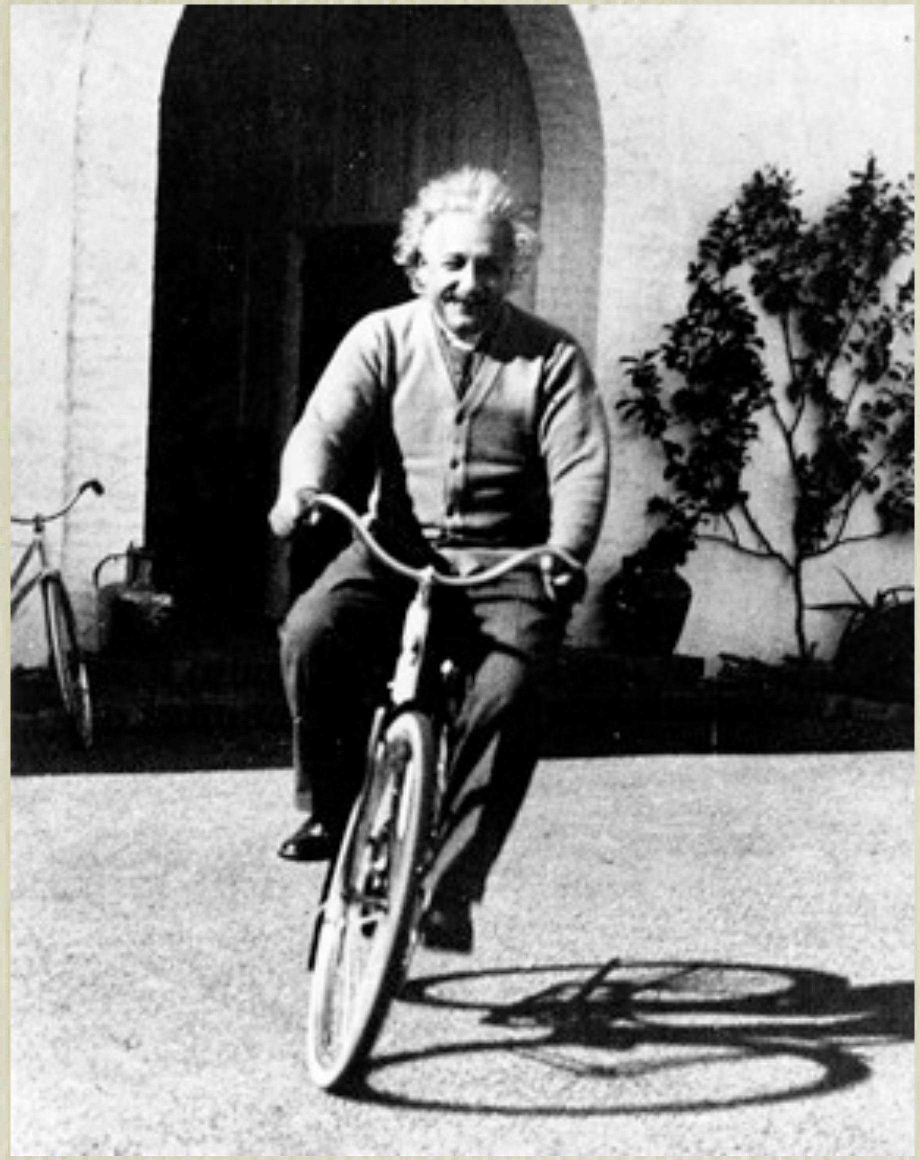
Based on work to appear

Overview

- Why is General Relativity so Special?
- Symmetries and First Class Constraints
- Vierbein Formalism
- `Hidden U(1) symmetry'

Why is General Relativity so special?



1. GR is Diffeomorphism Invariant

i.e. it exhibits 4 local symmetries -
General Coordinate Transformations

$$x^{\mu} \to x^{\mu}(x')$$

Every theory can be written in a coordinate invariant way, but there is usually a preferred system of coordinates/frame of reference

- in GR there is no preferred system in the absence of matter
  - in the presence of matter there is a preferred reference frame, e.g. the rest frame of the cosmic microwave background

2. In GR Gravity is described by the curvature of spacetime

Einsteins equations take the form:

Curvature of spacetime $\propto$ Energy Momentum Density

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

radius of curvature^2 $\propto$ 1/energy density

3. GR is locally Lorentz Invariant

Every geometry is locally Minkowski -

GR can be rewritten as spin-two perturbations around Minkowski

E.o.Ms for GR are Lorentz invariant to all orders

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

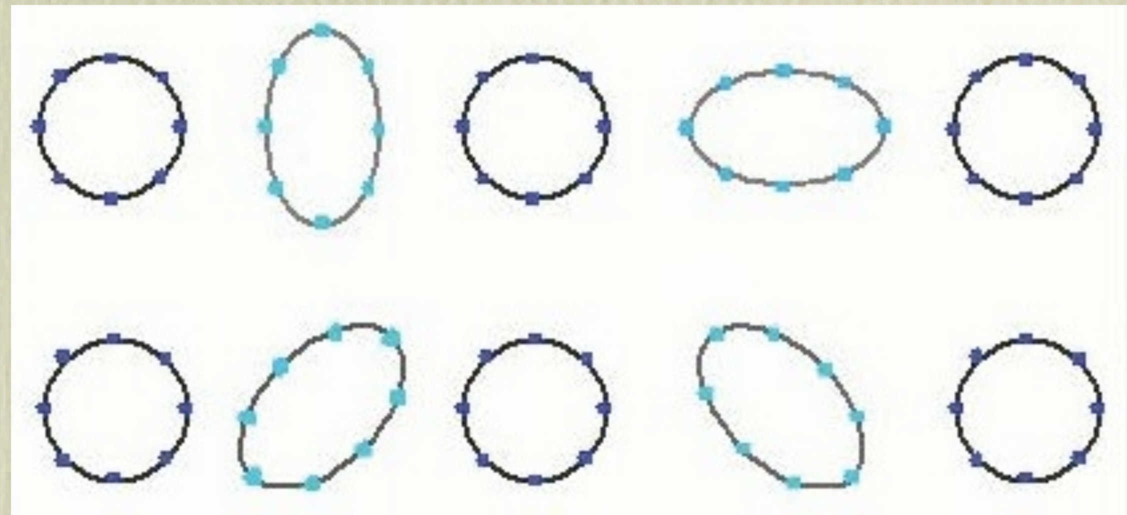
$$h_{\mu\nu} \sim R_{\mu\nu\alpha\beta}(x_P)(x^\alpha - x_P^\alpha)(x^\beta - x_P^\beta) + \mathcal{O}((x - x_P)^3)$$

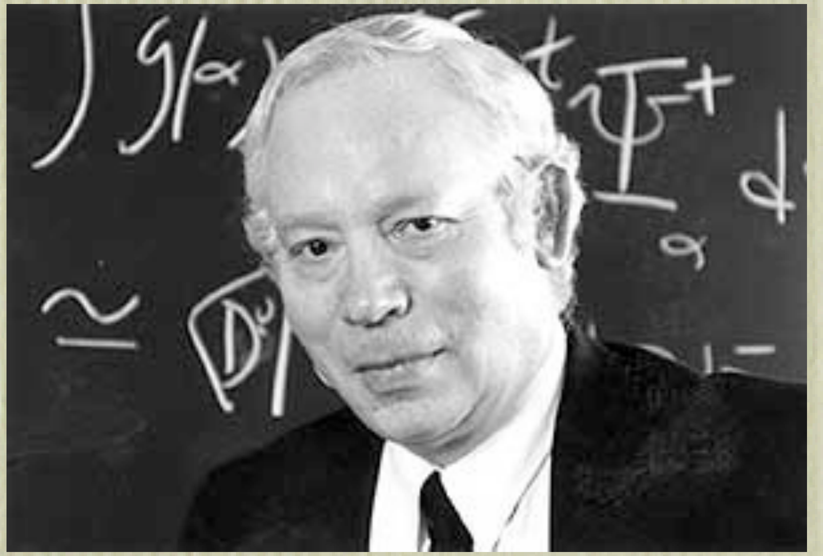
Essentially a different phrasing of the equivalence principle - ability to choice locally inertial frames

4. GR is unique theory of a massless spin-two field

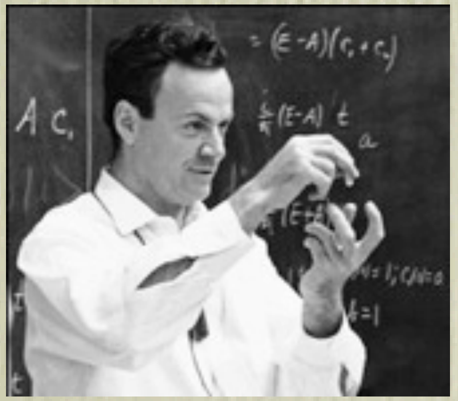
Metric perturbations transform as massless fields of spin 2!!

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$





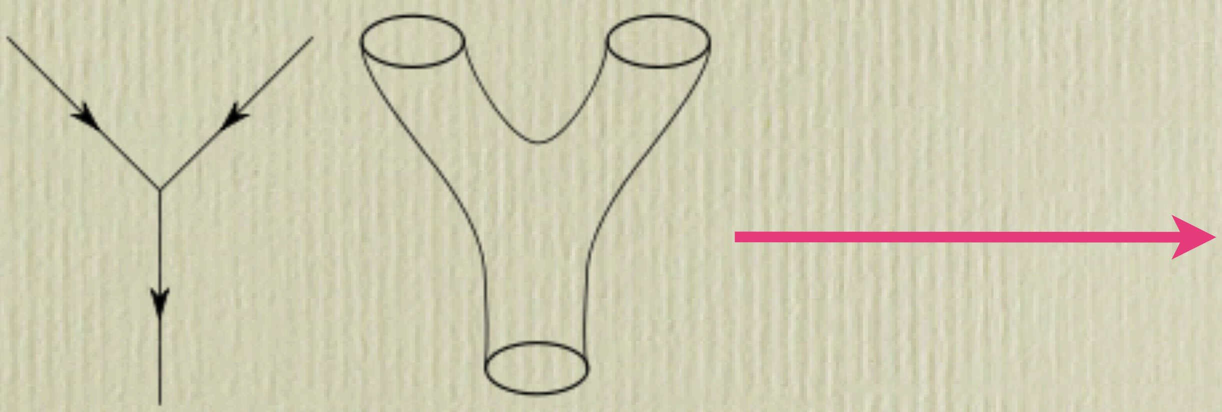
There are only two physical polarizations of gravitational waves!



4. GR is unique theory of a massless spin-two field

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

It is this argument that has played a more significant role in the particle physics community



String Theory is a theory of Quantum Gravity BECAUSE in the spectrum of oscillations of a quantized string is a m=0 s=2 state

Markus Fierz and Wolfgang Pauli, 1939

$$\Box h_{\mu\nu} + \cdots = m^2(h_{\mu\nu} - 1\,\eta_{\mu\nu} h)$$

$$5 = 2s + 1$$

Fierz-Pauli mass term

guarantees 5 rather than 6 propagating degrees of freedom

Massless spin-two in Minkowski makes sense!

Known problem ....

Interacting theories of massive gravitons pathological - extra ghostly non-pert. d.o.f., unbounded energy, causality violations

Boulware + Deser (1972)

Previously Two known solutions ....

Nappi + L. Witten (1989)

Argyres + Nappi (1989)

Extra dimensions

Strings

Price: Infinite tower of massive KK modes

Price: Infinite tower of higher spins

de Rham-Gabadadze-Tolley (dRGT) Massive Gravity

$$S = \frac{M_{\rm Pl}^2}{2} \int d^4x \sqrt{-g} \left(R - \frac{1}{4} m^2 U(g,f) \right)$$

Resummation of Massive Gravity

de Rham, Gabadadze, Tolley, PRL, 106, 231101 (2011)

Proven fully ghost free in ADM formalism:

Hassan and Rosen 2011

Result reconfirmed in Stueckelberg decomposition:

de Rham, Gabadadze, Tolley 2011

Hassan, Schmidt-May, von Strauss 2012

Result reconfirmed in helicity decomposition:

de Rham, Gabadadze, Tolley 1108.4521

Now several other proofs: Mehrdad Mirbaryi 2011...

dRGT model: allowed mass terms

de Rham, Gabadadze, Tolley, PRL, 106, 231101 (2011)

Build out of unique combination

$$K^{\mu}{}_{\nu} = \delta_{\mu\nu} - \sqrt{g^{\mu\alpha} f_{\alpha\nu}}$$

$$U(g,f) = \sum_i \beta_i U_i(K)$$

Mass terms are ***characteristic polynomials***

$$det(\delta^{\mu}{}_{\nu} + \lambda K^{\mu}{}_{\nu}) = \sum_{n=0}^{n=d} \lambda^n U_n(K)$$

Finite number of allowed interactions in any dimension

Interactions protected by a Nonrenormalization theorem

Second Class Constraints

But why does it work??????

Theory requires **two** second class constraints

The first was difficult to show - the second was extremely difficult!!!

(as explained in Luigi Pilo's talk)

Upgrading from Second to First Class

In many systems it is more natural to formulate a system with two second class constraints as a system with one first class constraint

$$2 \times 1 \text{ second class} = 1 \text{ local symmetry} \; = 1 \text{ first class constraint} + 1 \text{ gauge choice}$$

We are looking for an extra local/gauge symmetry!

Example: Extra Dimensions

Massless Graviton in 5D has 5 degrees of freedom
because of the existence of 5=4+1 gauge symmetries

15-5 (constraint)-5 (gauge choice) = 5

5d symmetric matrix

DGP model:

More irrelevant ⟷ More relevant

$$S = \int d^4x\sqrt{-g_4}\frac{M_4^2}{2}R_4 + \int d^4x\sqrt{-g_4}\mathcal{L}_M + \int d^5x\sqrt{-g_5}\frac{M_5^3}{2}R_5$$

Dominates in UV ⟷ Dominates in IR

Gravity in Higher Dimensions

In *4+n* dimensional spacetime, gravitational potential scales as

$$V(r) \sim \frac{1}{r^{1+n}}$$

weaker gravity

we want to achieve this in the IR

$$V(r) \sim \frac{1}{r} \longrightarrow V(r) \sim \frac{1}{r^{1+n}}$$

UV, small r

IR, large r

Gravity in Higher Dimensions

Form of potential

$$V(r) = \int_0^\infty ds^2 \rho(s^2) \frac{e^{-sr}}{r}$$

corresponds to propagator

$$G_F(k) = \int_0^\infty ds^2 \rho(s^2) \frac{1}{k^2 + s^2 - i\epsilon} = \frac{1}{k^2 + m^2(k) - i\epsilon}$$

for DGP

$$m^2(k) \propto \sqrt{-k^2}$$

Finding the Hidden Symmetry

GOAL: Can we reformulation the 4D ghost-free massive gravity theories as theories possessing the same number of gauge symmetries as 5D gravity

Advantage:

1. Local symmetries protect against quantum corrections
2. Starting point for a consistent quantization
3. Proper understanding of helicity zero mode in full nonlinear theory
4. Guarantees the correct number of degrees of freedom
5. Coupling to matter must respect symmetry
6. Useful implications of Ward identities etc.

Main point: How to introduce helicity zero mode

DGP

$$ds^2 = N^2 dy^2 + g_{\mu\nu}(dx^\mu + N^\mu dy)(dx^\nu + N^\nu dy)$$

4 Stueckelberg fields =
4 ADM Shifts

Massive gravity

$$g_{\mu\nu} \qquad\qquad f_{\mu\nu} = \partial_\mu \phi^A \partial_\nu \phi^B \eta_{AB}$$

No analogue of N
No analogue of y coordinate transformations

Main point: How to introduce helicity zero mode

Need to introduce a NEW Stueckelberg field for a broken U(1) symmetry

Arkani-Hamed, Georgi, Schwartz 2003
Creminelli, Nicolis, Papucci, Trincherini 2005

Previous attempts:

$$\phi^A = V^A + \partial^A \pi$$

Introduces a new symmetry

$$V^A \to V^A + \partial^A \chi$$
$$\pi \to \pi - \chi$$

Gets correct decoupling limit

de Rham, Gabadadze 2010

π is a Galileon field

$$\pi \to \pi + v_\mu x^\mu$$

Main point: How to introduce helicity zero mode

Arkani-Hamed, Georgi, Schwartz 2003
Creminelli, Nicolis, Papucci, Trincherini 2005

Previous attempts:

$$\phi^A = V^A + \partial^A \pi$$

But it fails nonlinearly !!!!!!!!

Alberte, Chamseddine, Mukhanov 2010

Superficially a problem but not really .....

de Rham, Gabadadze, Tolley 2011
Hassan, Schmidt-May, von Strauss 2012

It only indicates we have not **correctly** introduced the helicity zero mode

Massive Gravity in the Vierbein Formalism

Nibblelink Groot, Peloso, Sexton 2006
Hinterbichler and Rosen 2012

Write metric in terms of vierbeins (vielbeins)

Minkowski metric

$\eta = - + + +$

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$$

Lorentz Transformation

Introduce Local Lorentz symmetry

$$e^a_\mu \to e^b_\mu \Lambda^a_b \qquad \Lambda^a_c \Lambda^b_d \eta_{ab} = \eta_{cd} \qquad g_{\mu\nu} \to g_{\mu\nu}$$

Massive Gravity in the Vierbein Formalism

Nibblelink Groot, Peloso, Sexton 2006
Hinterbichler and Rosen 2012

Write reference (Minkowski) metric in terms of vierbeins

$$f_{\mu\nu} = b^a_\mu b^b_\nu \eta_{ab}$$

Most general reference vierbein is ...

$$b^a_\mu = \partial_\mu \phi^b \lambda^a_b \qquad \lambda^a_c \lambda^b_d \eta_{ab} = \eta_{cd}$$

Massive Gravity in the Vierbein Formalism

Under a diff (coordinate) transformation

ϕ^a and λ^a_b transform as scalars

Under a local Lorentz transformation

$$b^a_\mu = \partial_\mu \phi^b \lambda^a_b \qquad b^a_\mu \to b^b_\mu \Lambda^a_b \qquad \lambda^a_c \to \lambda^b_c \Lambda^a_b$$

The λ^a_b are the Stueckelberg fields for the broken Lorentz invariance

Allowed dRGT mass terms

All the mass terms arise as characteristic polynomials in the expansion of a determinant

$$\mathcal{L}_{\text{mass}} = m^2 M_{\text{pl}}^2 \det\left(e_\mu^a + \mu\, \partial_\mu \phi^b \lambda_b^a\right)$$

Proof (Nibblelink et al.)

Varying w.r.t. λ_b^a gives constraint

$$e_a^\mu \partial_\mu \phi^c \lambda_{cb} = e_b^\mu \partial_\mu \phi^c \lambda_{ca}$$

Allowed dRGT mass terms

more abstractly this says

$$e^{-1}\partial\phi\lambda = \left(e^{-1}\partial\phi\lambda\right)^T = \lambda^T(\partial\phi)^T e^{T^{-1}}$$

but $\lambda\eta\lambda^T = \eta$ since the λ^a_b are Lorentz Stueckelbergs

and so ....

$$\left(e^{-1}(\partial\phi)\lambda\eta\right)^2 = e^{-1}(\partial\phi)\lambda\eta\lambda^T(\partial\phi)^T e^{T^{-1}}\eta$$

$$= e^{-1}(\partial\phi)\eta(\partial\phi)^T e^{T^{-1}}\eta$$

which implies ....

$$\left(e^{-1}(\partial\phi)\lambda\eta\right) = \sqrt{e^{-1}(\partial\phi)\eta(\partial\phi)^T e^{T^{-1}}\eta}$$

The Famous Square Root of the dRGT model

Finally the answer ....

Perturbatively spotted by Mirbabayi, 1112.1435
Nonperturbatively (AJT to appear)

$$\mathcal{L}_{\mathrm{mass}} = m^2 M_{\mathrm{pl}}^2 \det\left(e^a_\mu + \mu\,\partial_\mu \phi^b \lambda^a_b\right)$$

$$\phi^a \to \phi^a + e^b_\mu {\lambda^{-1}}^a{}_b (V^\mu + \nabla^\mu \pi)$$

Local Symmetries:

4 coordinate transformations

6 local Lorentz transformations

1 `previously hidden' U(1) symmetry

$$V_\mu \to V_\mu + \partial_\mu \chi \qquad \pi \to \pi - \chi$$

The answer

$$\mathcal{L}_{\rm mass} = m^2 M_{\rm pl}^2 \det\left(e^a_\mu + \mu\,\partial_\mu\phi^b\lambda^a_b\right)$$

$$\phi^a \to \phi^a + e^b_\mu {\lambda^{-1}}^a_b\left(V^\mu + \nabla^\mu\pi\right)$$

This only works if we can show that the equations of motion for the helicity zero mode π

After some hard work .... this can be proven

(AJT to appear)

π looks not dissimilar although not the same as a gauged covariant Galileon

Summary (Part I)

Formulated dRGT massive gravity in D dimensions with the same number of first class constraints corresponding to the number of symmetries as in (D+1) dimensional General Relativity

In this form there are no second class constraints

The extra U(1) symmetry guarantees the correct number of degrees of freedom and absence of BD ghost

It provided a suitable starting point for a consistent quantization and coupling to matter

It allows us to define what we mean by the helicity zero mode about in an arbitrarily curved geometry

Part II: Connections between models of Massive Gravity

Work with Miguel F. Paulos - arXiv:1203.4268

Two roads to Massive Gravity

1. Problem of Dark Energy and the cosmological constant

2. Holographic Principle - Higher derivative corrections in AdS/CFT c-theorem

The first road:

1. Problem of Dark Energy and the cosmological constant

Infrared Modification of Gravity

Ghost-free massive gravity
de Rham, Gabadadze, Tolley

$M_{Pl} \to \infty$
$M_{Pl} m^2$ fixed

Ghost-free bigravity models
Hassan and Rosen

Galileon Theories
Nicolis, Ratazzi, Trincherini

The second road:

2. Holographic Principle - Higher derivative corrections in AdS/CFT c-theorem

New Massive Gravity (in 3D)
Bergshoeff, Hohm and Townsend

Counterterms in AdS_4/CFT_3
Jatkar and Sinha

Born-Infeld New Massive Gravity
Gullu, Sisman, Tekin

Cubic New Massive Gravity
Jatkar and Sinha

Unification of Massive Gravity

All these different theories are different ***scaling/ decoupling limit***s of the same family of bigravity models

A scaling/decoupling limit is a limit where one parameter e.g. Planck Mass is sent to infinity in such away that other parameters in the theory are kept finite

$$M_{\rm Planck} \to \infty$$

Famous Example: t'Hooft coupling - QCD for SU(N)

$$\lambda_{'\rm t\ Hooft} = g_{YM}^2 N$$

$$N \to \infty \quad g_{YM} \to 0$$

$$\lambda_{'\rm t\ Hooft} \text{ held fixed}$$

Why Massive Gravity? Part I

Adding a mass to gravity weakens the strength of gravity at large (cosmological) distances

$$V_{Yukawa} \sim \frac{e^{-mr}}{r}$$

Gravitons can condense to form a condensate whose energy density compensates the cosmological constant

Screening mechanism - The Cosmological Constant can be LARGE with the cosmic acceleration SMALL

In a Massive Theory - the c.c. is a `redundant' operator

Why Massive Gravity? Part I

mass term

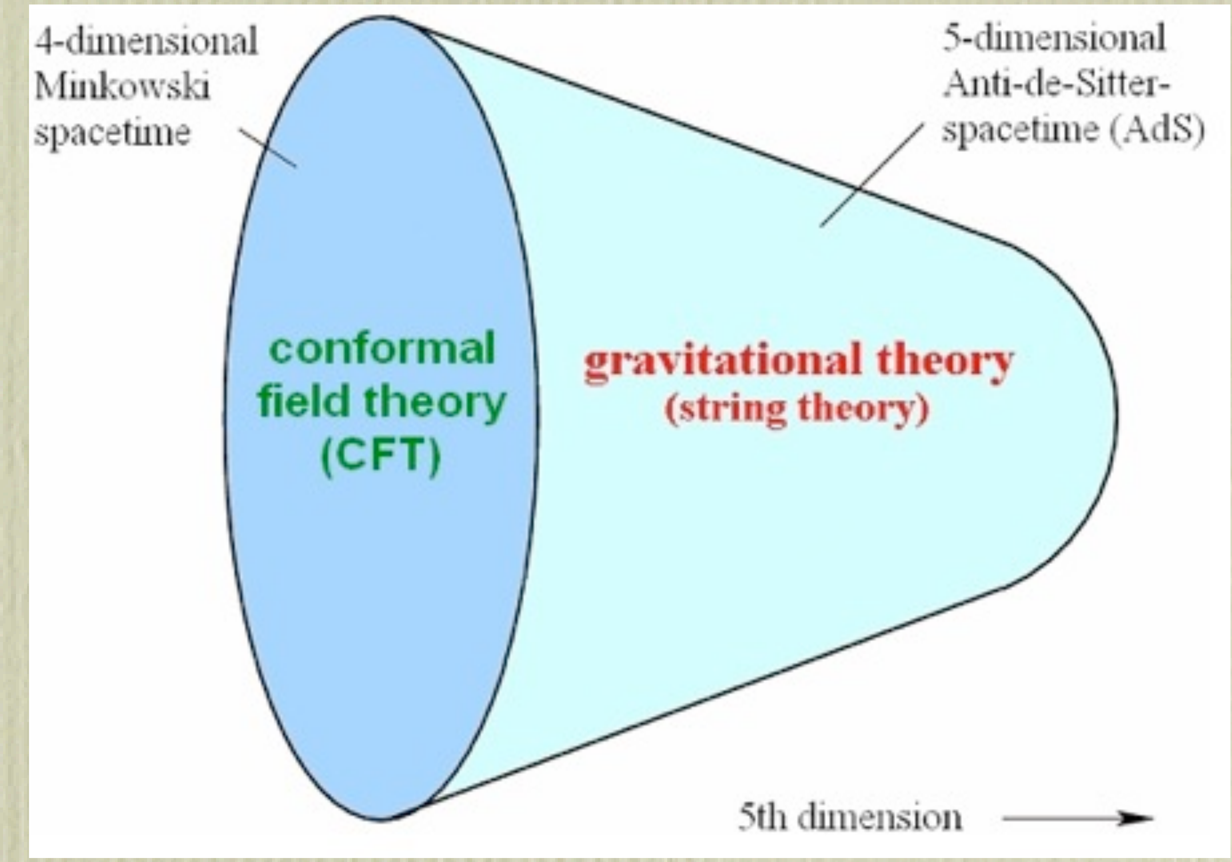
$$G_{\mu\nu} + m^2 \frac{\partial L_M}{\partial g_{\mu\nu}} = -\Lambda g_{\mu\nu}$$

Graviton condensate:
Spacetime is Minkowski in presence of an arbitrary large Λ

$$g_{\mu\nu} = \left(1 + f\left(\frac{\Lambda}{m^2}\right)\right)\eta_{\mu\nu} \qquad G_{\mu\nu} = 0 \qquad m^2 \frac{\partial L_M}{\partial g_{\mu\nu}} = -\Lambda g_{\mu\nu}$$

Equivalent Statement: The cosmological constant can be reabsorbed into a **redefinition** of the metric and coupling constants - and is hence a **redundant** operator

Why Massive Gravity? Part II



According to the AdS/CFT correspondence

2 dimensional CFTs are dual to 3 dimensional gravity theories

AdS3/CFT2

Why Massive Gravity? Part II

Generic 3D Gravity theory = Einstein Hilbert (no degrees of freedom) plus ***HIGHER CURVATURE INVARIANTS***

$$\mathcal{L} = \sqrt{-g}\left(\frac{1}{2}M_P R + R^2 + R^3 + \ldots\right)$$

Form of higher derivative corrections is fixed by the requirement that the dual QFT satisfy the requirements of the **c-theorem**

Zamolodchikov's c-theorem states that the number of degrees of freedom for a 2D field theory increases with energy

Why Massive Gravity? Part II

Form of higher derivative corrections is fixed by the requirement that the dual CFT satisfy the requirements of the **c-theorem**

To quadratic order the theory is unique:

$$\mathcal{L} = \sqrt{-g}\Big(\underbrace{\frac{1}{2}M_P R + c_1\left(S_{\mu\nu}^2 - S^2\right)} + c_3\left(2S_{\mu\nu}^2 - 3SS_{\mu\nu}^2 + S^3\right)\dots\Big)$$

New Massive Gravity — Bergshoeff, Hohm, Townsend

To cubic order the theory is unique:

$$\mathcal{L} = \sqrt{-g}\Big(\underbrace{\frac{1}{2}M_P R + c_1\left(S_{\mu\nu}^2 - S^2\right) + c_3\left(2S_{\mu\nu}^2 - 3SS_{\mu\nu}^2 + S^3\right)}\dots\Big)$$

Cubic Extension of New Massive Gravity — Sinha

$$S_{\mu\nu} = \text{Shouten tensor} = R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R$$

Why Massive Gravity? Part II

Form of higher derivative corrections is fixed by the requirement that the dual CFT satisfy the requirements of the **c-theorem**

To all orders the theory is not uniquely fixed by the c-theorem:

M. Paulos

$$\mathcal{L} = \sqrt{-g}\left(\frac{1}{2}M_P R + \sum_{n=2}^{\infty} c_n P^{(n)}(S_{\mu\nu})\right)$$

However adding the requirement that the result theory is free of the **Boulware-Deser** ghost fixed the all orders theory uniquely

M. Paulos, AJT

$$\mathcal{L} = \sqrt{-g}\left(\frac{1}{2}M_P(R-2\Lambda)\right) + m^2 M_P \sqrt{\det\left[g_{\mu\nu} + \alpha\, G_{\mu\nu}\right]}$$

c-theorem + no ghost implies Born-Infeld New Massive Gravity

Born-Infeld NMG is a scaling limit of Ghost-free Bigravity

M. Paulos, AJT

$$\mathcal{L} = \sqrt{-g}\frac{1}{2}M_a R[g] + \sqrt{-f}\frac{1}{2}M_b R[f] + \frac{m^2}{M_a^{-1} + M_b^{-1}} \det\left(g_{\mu\nu} + \lambda f_{\mu\nu}\right)$$

$$M_{\mathrm{P}} = M_a + M_b$$

$$M_a \to \infty$$

$$(g_{\mu\nu} - f_{\mu\nu}) \to 0$$

$$M_a(g_{\mu\nu} - f_{\mu\nu}) \text{ fixed}$$

$$\mathcal{L} = \sqrt{-g}\left(\frac{1}{2}M_P(R - 2\Lambda)\right) + m^2 M_P \sqrt{\det\left[g_{\mu\nu} + \alpha\, G_{\mu\nu}\right]}$$

Bigravity in 3D implies Born-Infeld New Massive Gravity

Summary

All known consistent ghost-free models of massive gravity and bigravity fall are limits of the same common set of bigravity theories which can be defined in any number of dimensions and have a finite number of operators

$$\mathcal{L} = \sqrt{-g}\frac{1}{2}M_a R[g] + \sqrt{-f}\frac{1}{2}M_b R[f] + \frac{m^2}{M_a^{-1} + M_b^{-1}} \det\left(g_{\mu\nu} + \lambda f_{\mu\nu}\right)$$

In AdS3/CFT2 absence of Boulware-Deser ghost is sufficient to guarantee holographic c-theorem for dual theory.

In 3 dimensions unique theory satisfying c-theorem and no ghost is Born-Infeld extension of New Massive Gravity

$$\mathcal{L} = \sqrt{-g}\left(\frac{1}{2}M_P(R - 2\Lambda)\right) + m^2 M_P \sqrt{\det\left[g_{\mu\nu} + \alpha\, G_{\mu\nu}\right]}$$