de Sitter Strings and more….

Ugo Moschella
Università dell'Insubria, Como, Italia
Roland's SW7
ugomoschella@gmail.com

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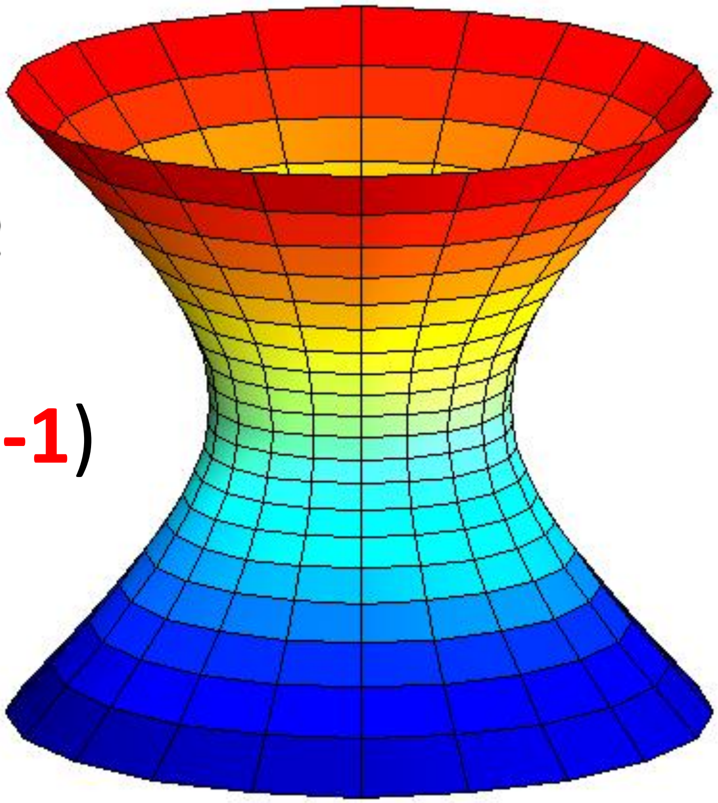
- A. G. Riess et al., “Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant”, Astronomical Journal 116, 1009 (1998).
- J. Maldacena, “The Large N Limit Of Superconformal Field Theories and Supergravity”, Adv. Theor. Math. Phys. 2 (1998) 231.

The shape of our universe

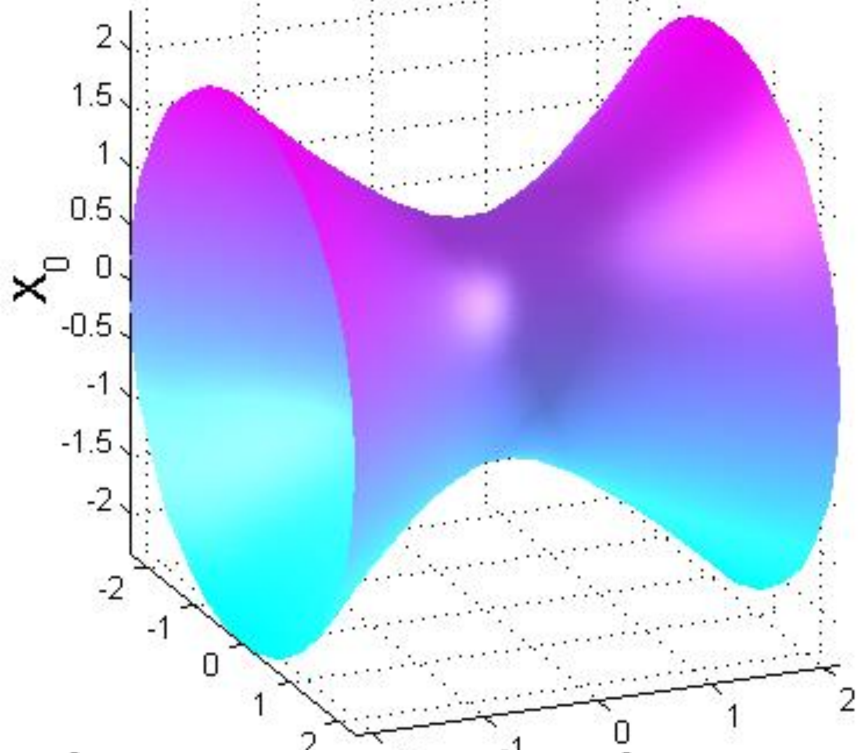
$$X_0^2 - X_1^2 - \ldots X_d^2 = -R^2$$

$M^{(d+1)}$: $\eta_{\mu\nu}$ = diag(**1,-1,……,-1**)

$$G = SO(1, d)$$



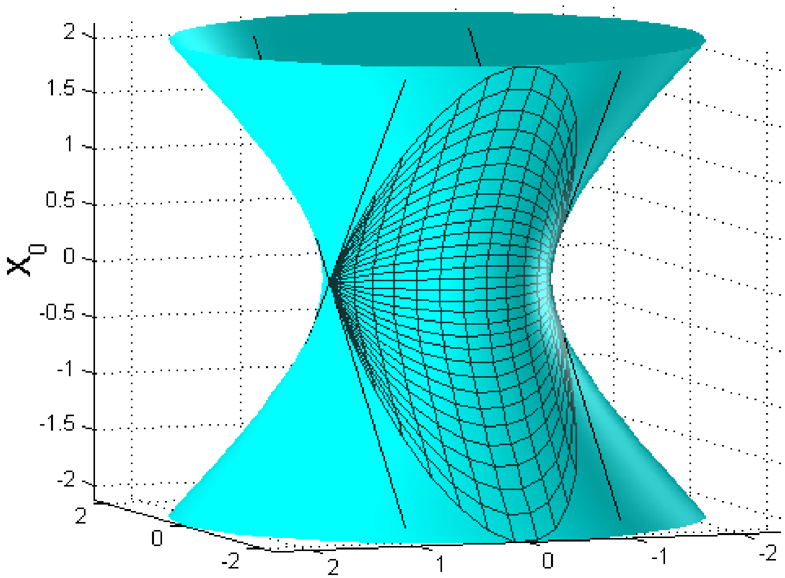
Its cousin: the anti de Sitter universe

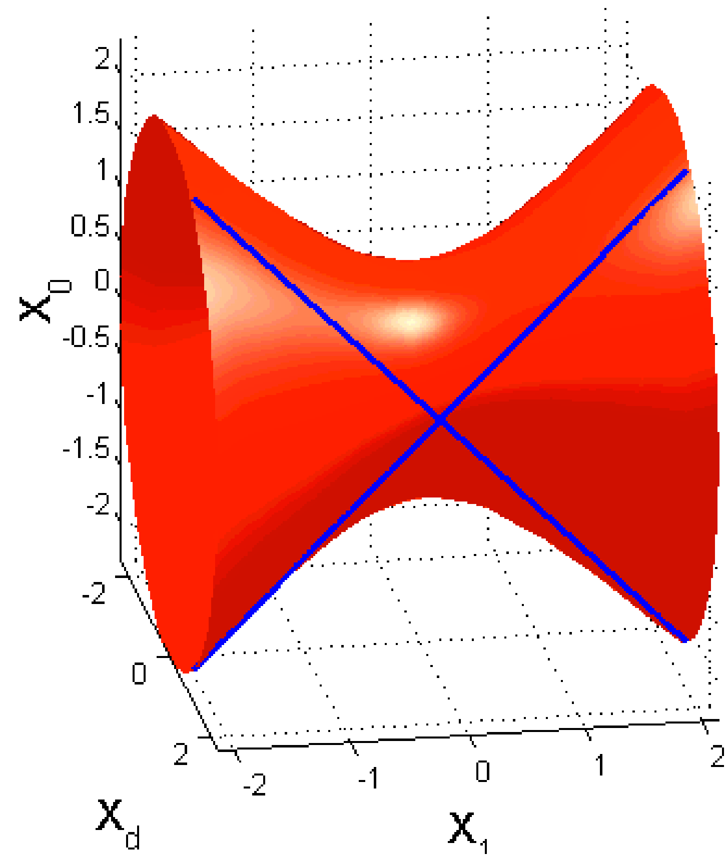


$$X_0^2 - X_1^2 - \ldots - X_{d-1}^2 + X_d^2 = R^2$$

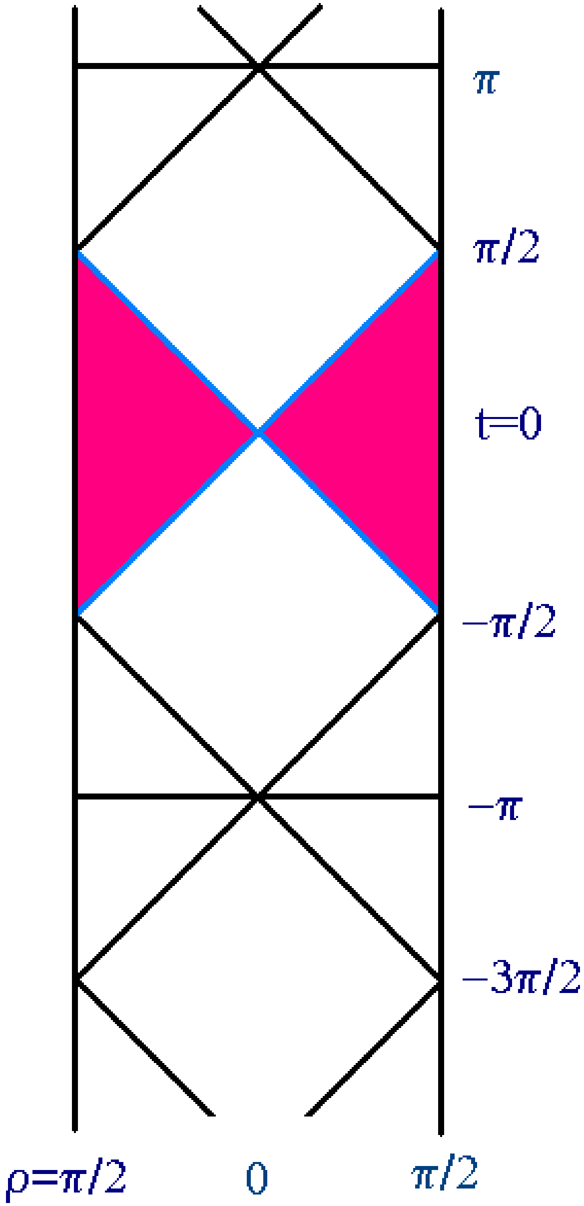
$E^{(2,d-1)}$: $\eta_{\mu\nu}$ = diag(**1**,**-1**,**……**,**-1**,**1**) $SO(2, d-1)$

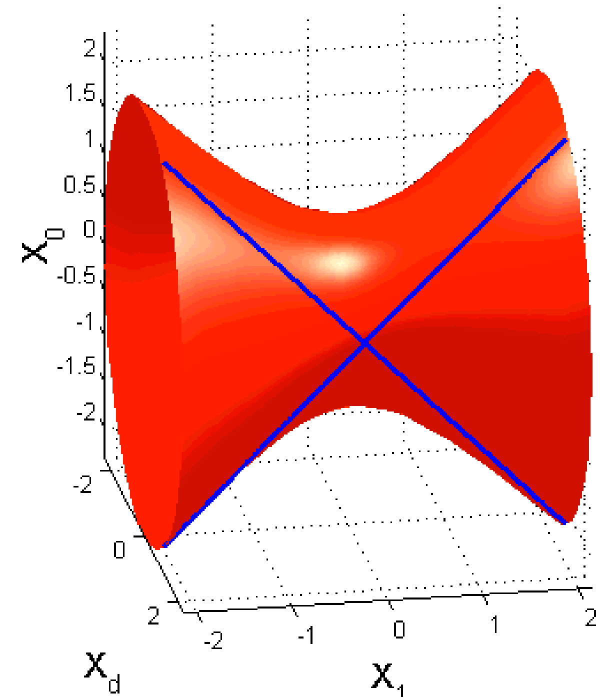
dS / AdS physics.
What Are the Problems?





Penrose Diagrams

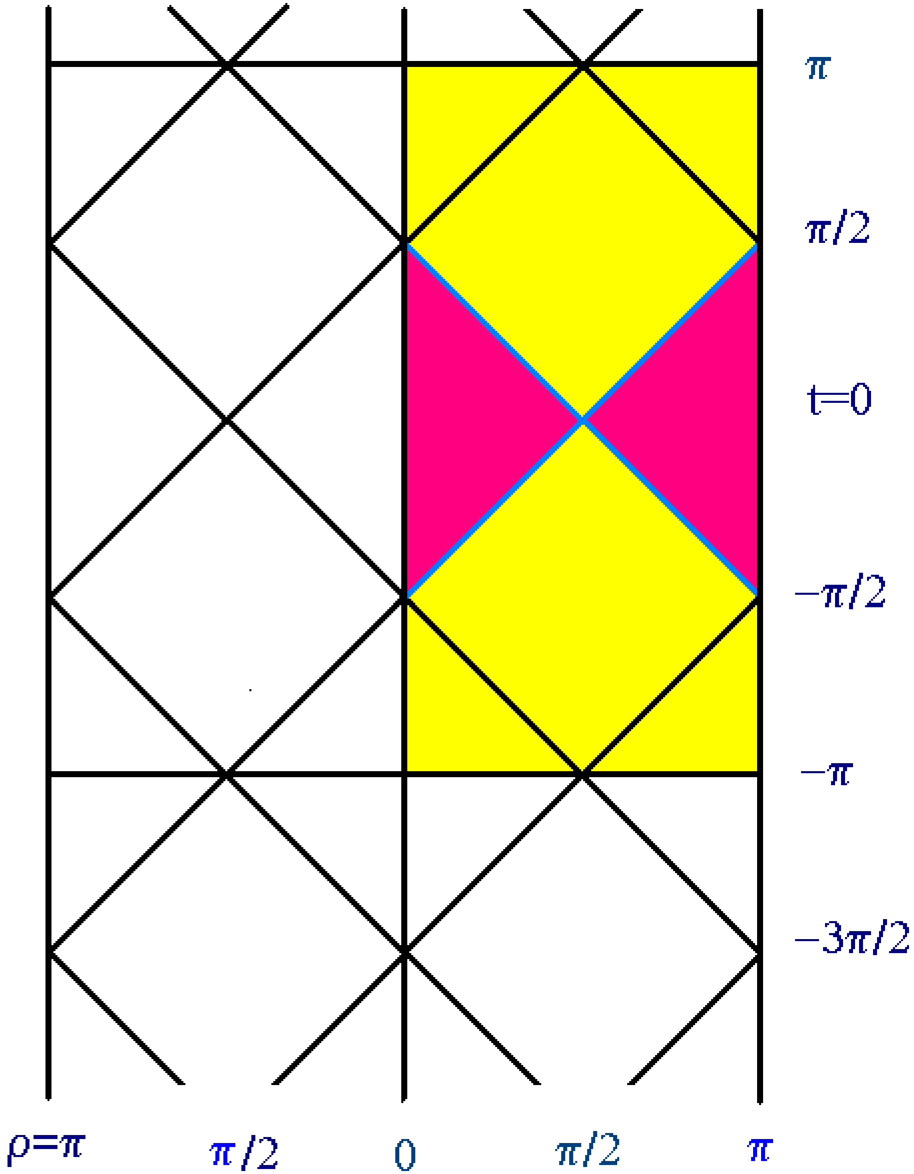


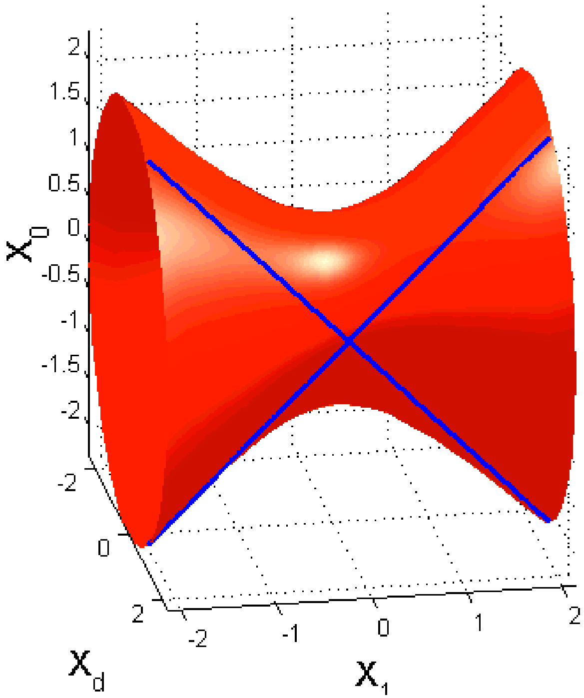


$t \in R$

$$\begin{cases} X_0 = \dfrac{\sin t}{\cos\rho} & 0 \le \rho < \frac{\pi}{2} \\ X_i = \tan\rho\,\omega_i & |\vec{\omega}|^2 = 1 \\ X_d = \dfrac{\cos t}{\cos\rho} & \end{cases}$$

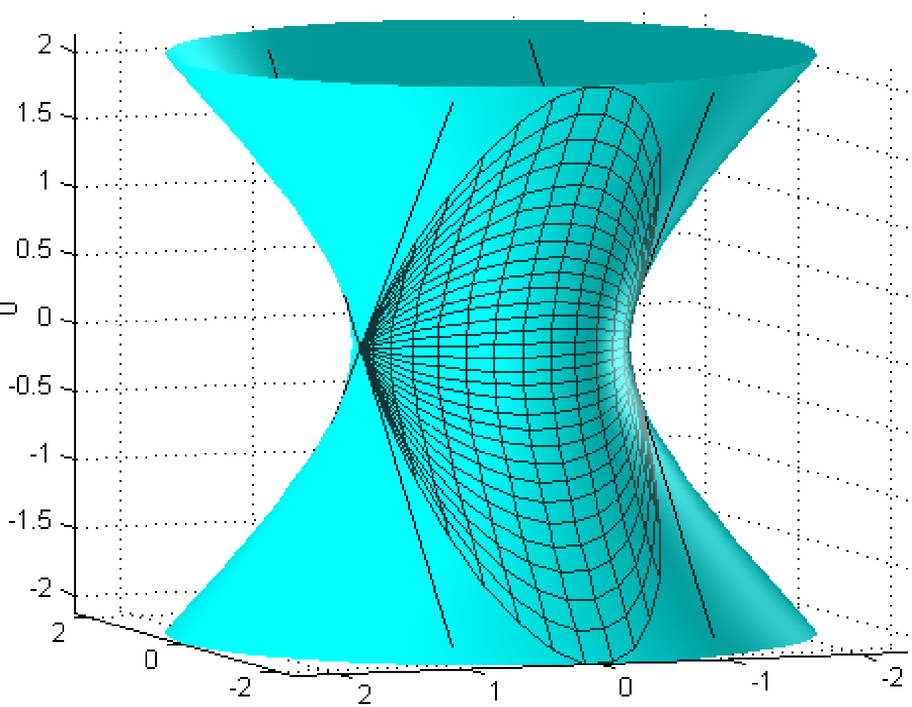
QFT: (Conformal) Embedding in the ESU

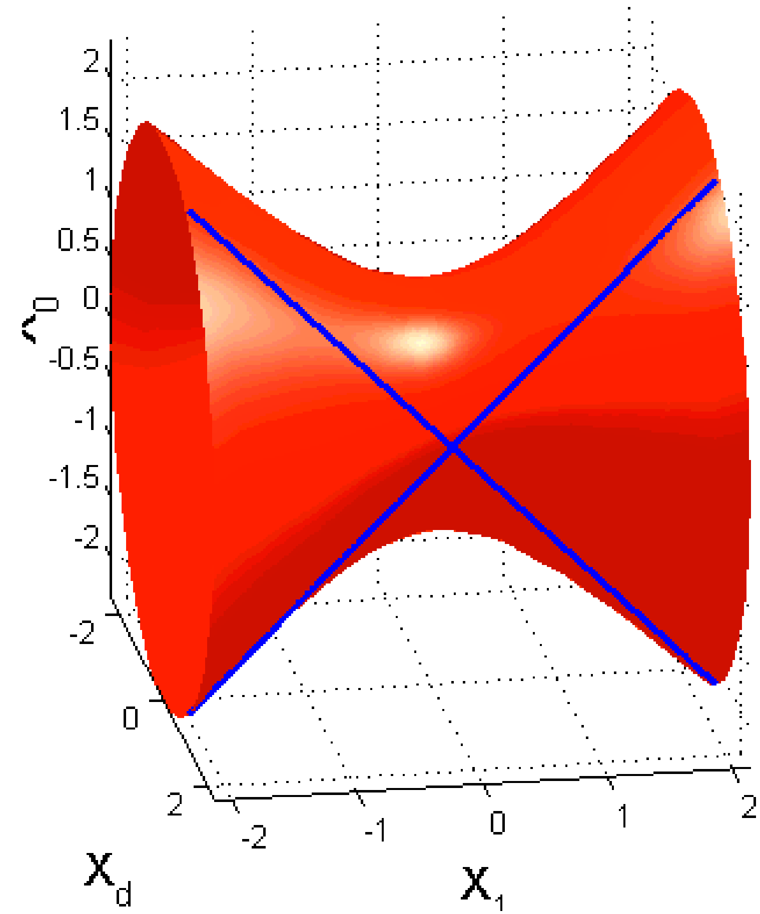




Avis,Isham,Storey (1978)

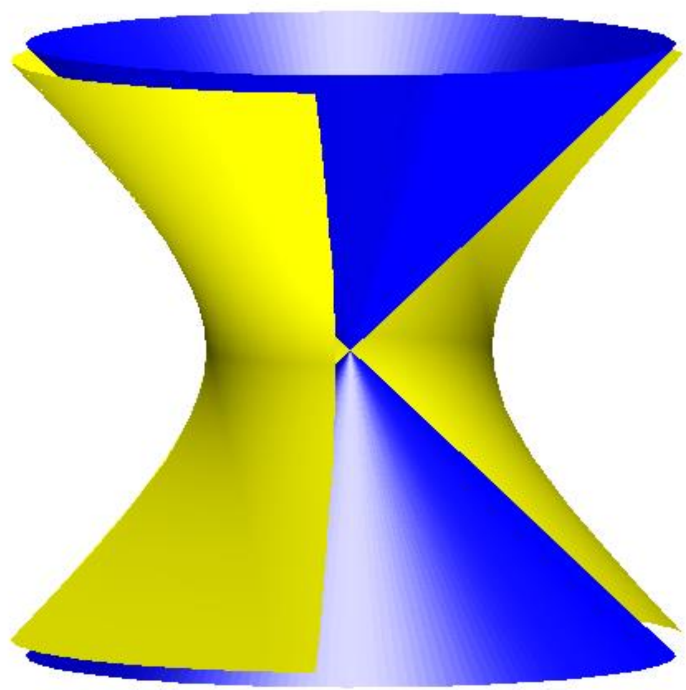
BTW: dS is more malicious (and misterious) than AdS





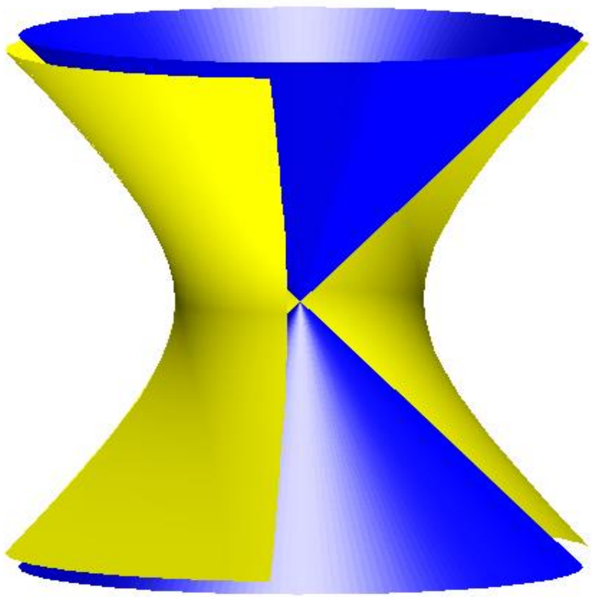
The asymptotic cone

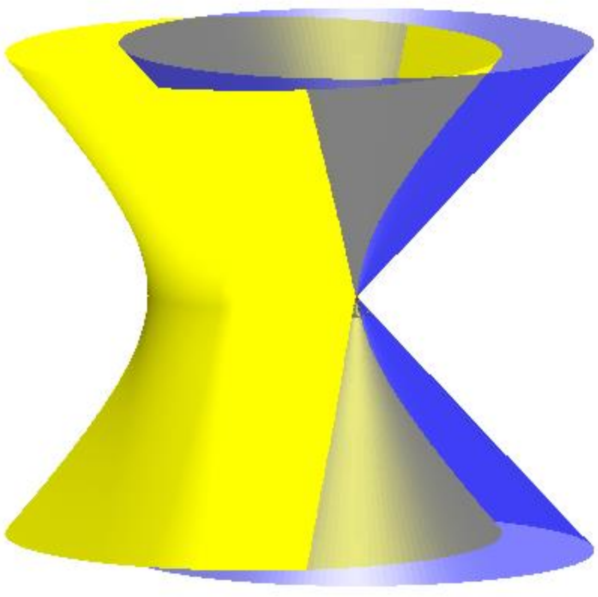
$$\{\xi_0^2 - \xi_1^2 - \ldots - \xi_d^2 = 0\}$$



$M^{(d+1)}$: $\eta_{\mu\nu}$ = diag(**1**,**-1**,……,**-1**)

The asymptotic cone: causal structure



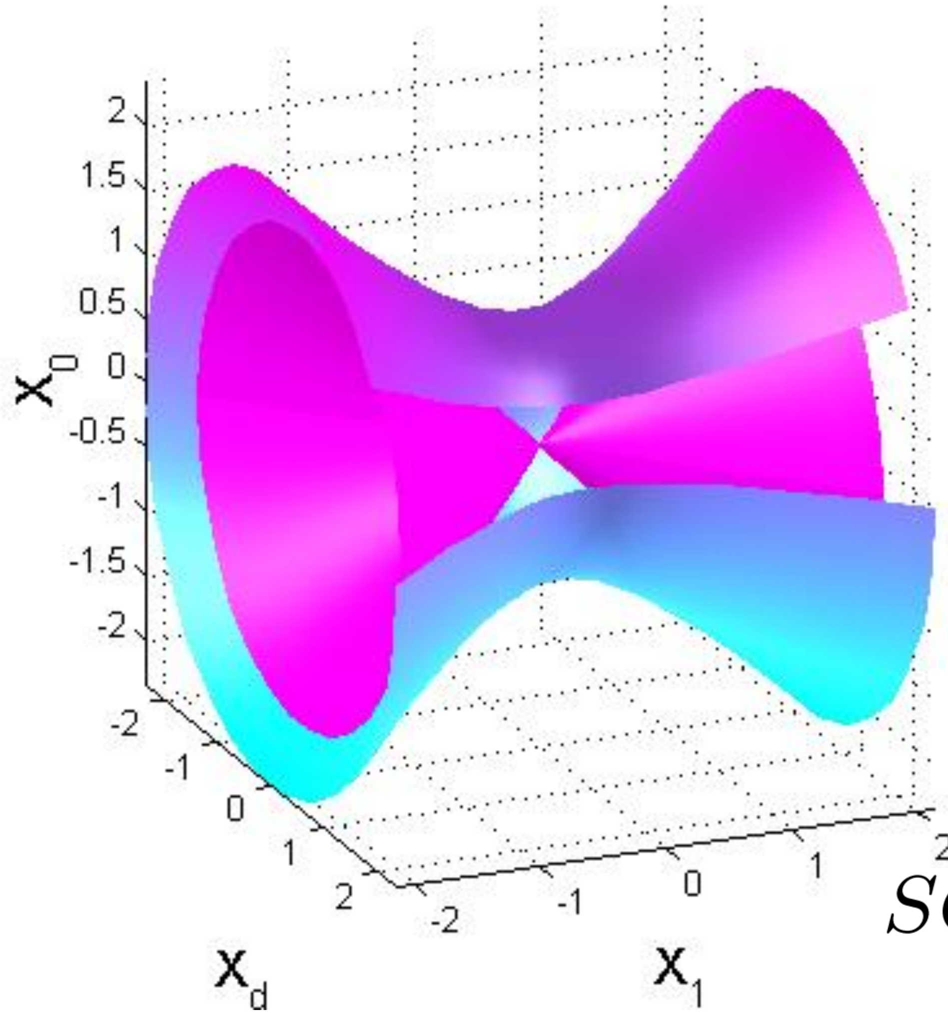


X, Y are spacelike separated iff $(X - Y)^2 < 0$ ($X - Y$ is outside the cone)

$$(X - Y)^2 = X^2 + Y^2 - 2X \cdot Y = -2R^2 - 2X \cdot Y$$

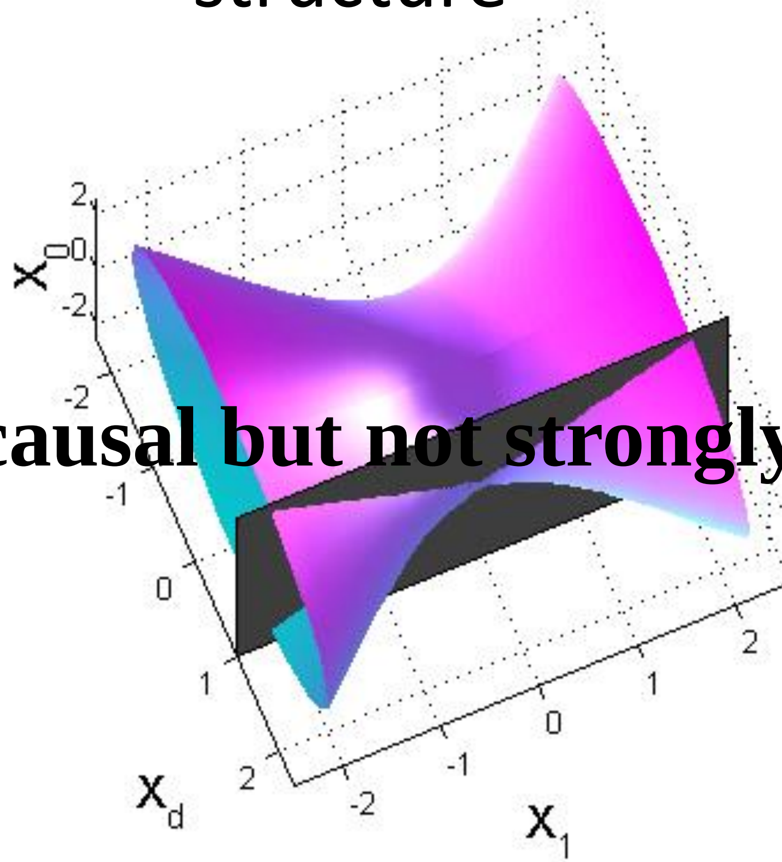
The asymptotic cone

$$\{\xi_0^2 - \xi_1^2 - \cdots - \xi_{d-1}^2 + \xi_d^2 = 0\}$$



$SO(2, d-1)$

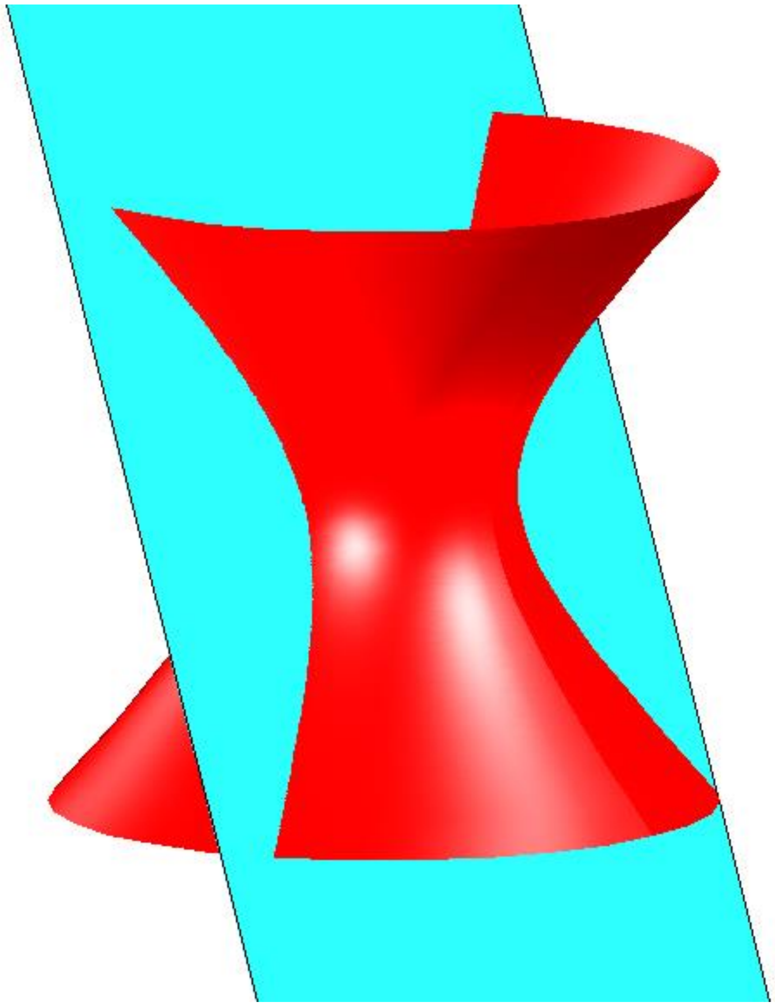
The asymptotic cone provides the causal structure



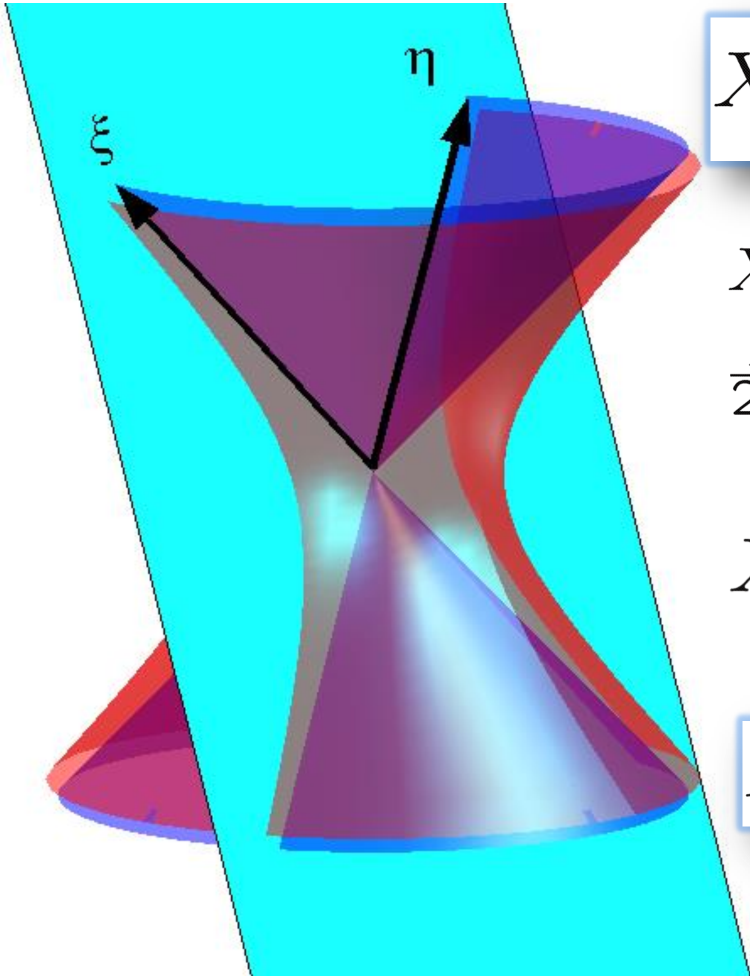
AdS: causal but not strongly causal

X e Y on AdS are spacelike separated iff $(X-Y)^2 < 0$
(in the ambient space sense)

Particles geodesics



The asymptotic cone as the de Sitter momentum space



$$X_\mu(\tau) = \frac{R}{\sqrt{2\xi\cdot\eta}}\left(\xi_\mu e^{\frac{c\tau}{R}} - \eta_\mu e^{-\frac{c\tau}{R}}\right)$$

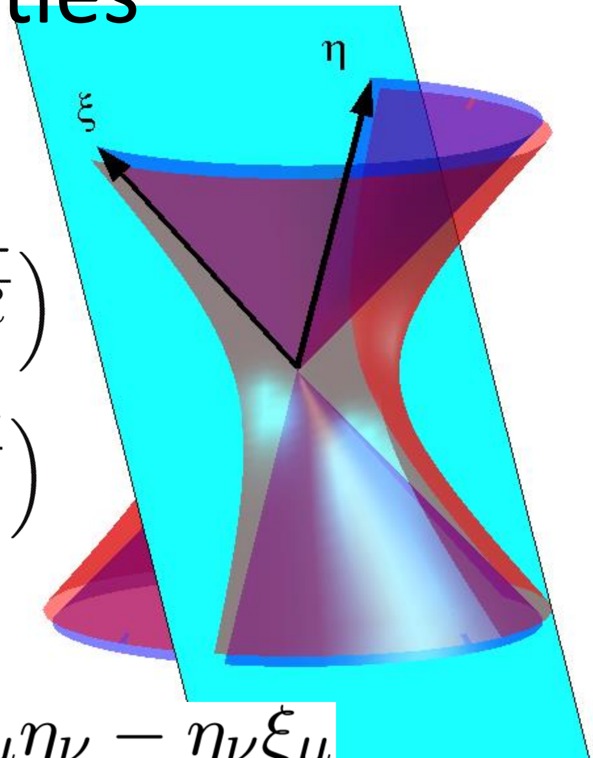
$$X^\mu(\tau)X_\mu(\tau) =$$
$$\frac{R^2}{2\xi\cdot\eta}\left(\xi^2 e^{\frac{2c\tau}{R}} + \eta^2 e^{-\frac{2c\tau}{R}} - 2\xi\cdot\eta\right) = -R^2$$

$$X_\mu(0) = \frac{R}{\sqrt{2\xi\cdot\eta}}(\xi_\mu - \eta_\mu)$$

$$X_\mu(\tau) = X_\mu(0)e^{-\frac{c\tau}{R}} + \frac{kR}{m}\xi_\mu \sinh\frac{c\tau}{R}.$$

Minkowski $x_\mu(\tau) = x_\mu(0) + \frac{p_\mu\tau}{mc}$

Conserved quantities



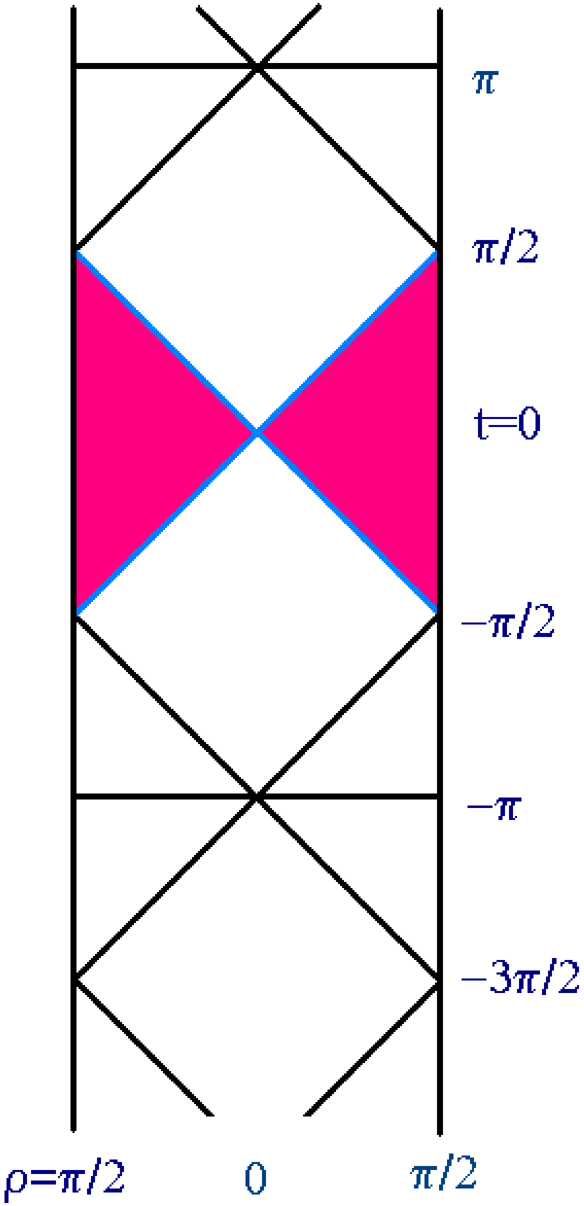
$$X_\mu(\tau) = \frac{R}{\sqrt{2\xi\cdot\eta}}\left(\xi_\mu e^{\frac{c\tau}{R}} - \eta_\mu e^{-\frac{c\tau}{R}}\right)$$

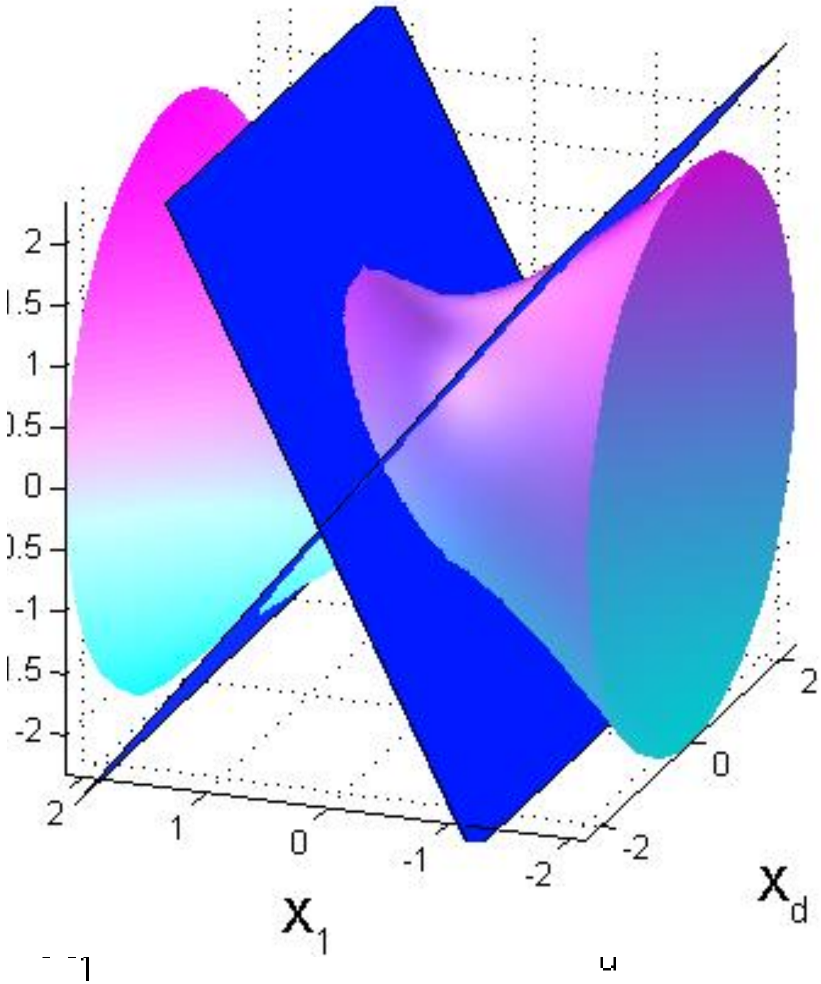
$$V_\mu(\tau) = \frac{c}{\sqrt{2\xi\cdot\eta}}\left(\xi_\mu e^{\frac{c\tau}{R}} + \eta_\mu e^{-\frac{c\tau}{R}}\right)$$

$$K_{\mu\nu} = \frac{m(X_\mu V_\nu - X_\nu V_\mu)}{R\sqrt{V\cdot V}} = mc\frac{\xi_\mu\eta_\nu - \eta_\nu\xi_\mu}{\xi\cdot\eta}$$

$$K_{\xi,\eta} = mc\frac{\xi\wedge\eta}{\xi\cdot\eta}$$

AdS: timelike geodesics



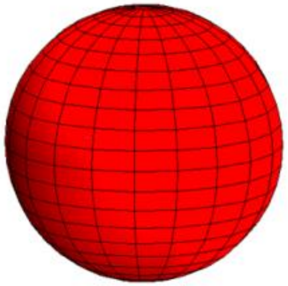


Coordinates: Vacuum Cosmological Equations

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right)$$

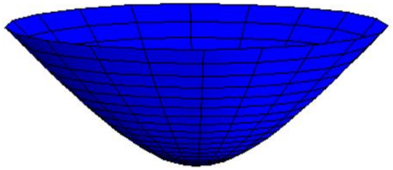
$$\ddot{a} = \frac{1}{3}\Lambda a , \qquad \dot{a}^2 = \frac{1}{3}\Lambda a^2 - K$$

$$K = 1 \quad \rightarrow \quad a(t) = \sqrt{\frac{3}{\Lambda}} \cosh \sqrt{\frac{\Lambda}{3}}\, t$$

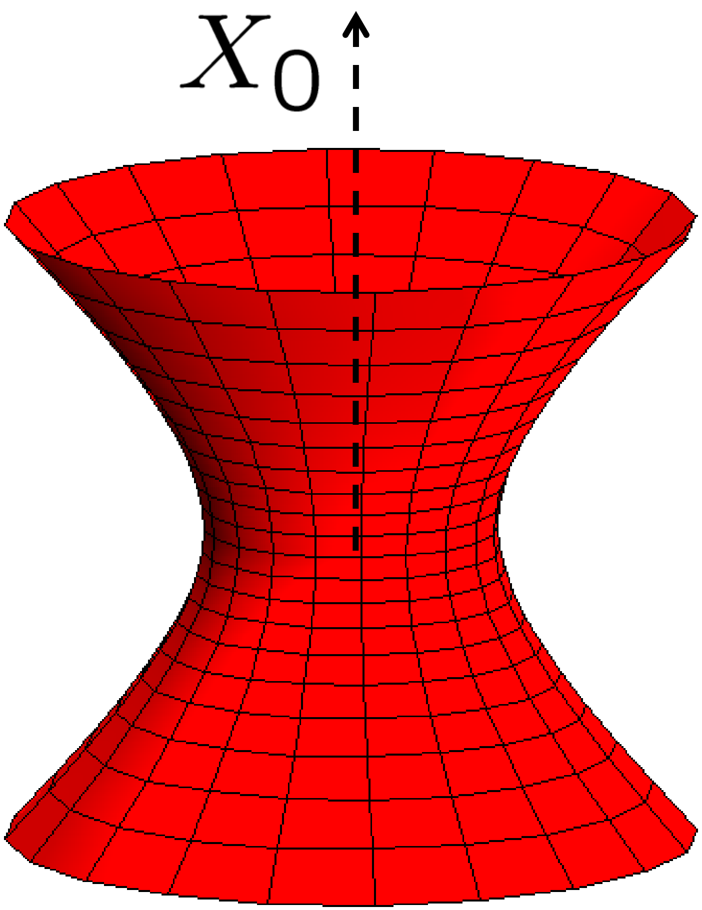


$$K = 0 \quad \rightarrow \quad a(t) = \exp \sqrt{\frac{\Lambda}{3}}\, t$$

$$K = -1 \quad \rightarrow \quad a(t) = \sqrt{\frac{3}{\Lambda}} \sinh \sqrt{\frac{\Lambda}{3}}\, t$$



Spherical de Sitter model



$$\begin{cases} X_0 &= R\sinh(t/R) \\ X_1 &= R\cosh(t/R)\sin\theta\sin\chi\sin\phi \\ X_2 &= R\cosh(t/R)\sin\theta\sin\chi\cos\phi \\ X_3 &= R\cosh(t/R)\sin\theta\cos\chi \\ X_4 &= R\cosh(t/R)\cos\theta \end{cases}$$

$$R = \sqrt{\frac{3}{\Lambda}}$$

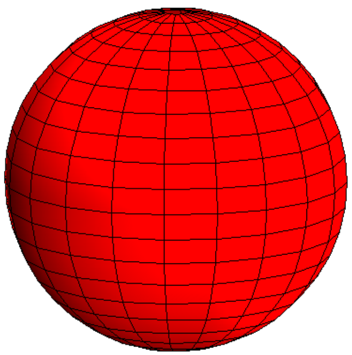
$$ds^2 = dX_0^2 - dX_1^2 - \ldots dX_4^2\Big|_{dS} =$$

$$= dt^2 - R^2\cosh^2\frac{t}{R}\left(d\theta^2 + \sin^2\theta(d\chi^2 + \sin^2\chi d\phi^2)\right)$$

Only Λ>0

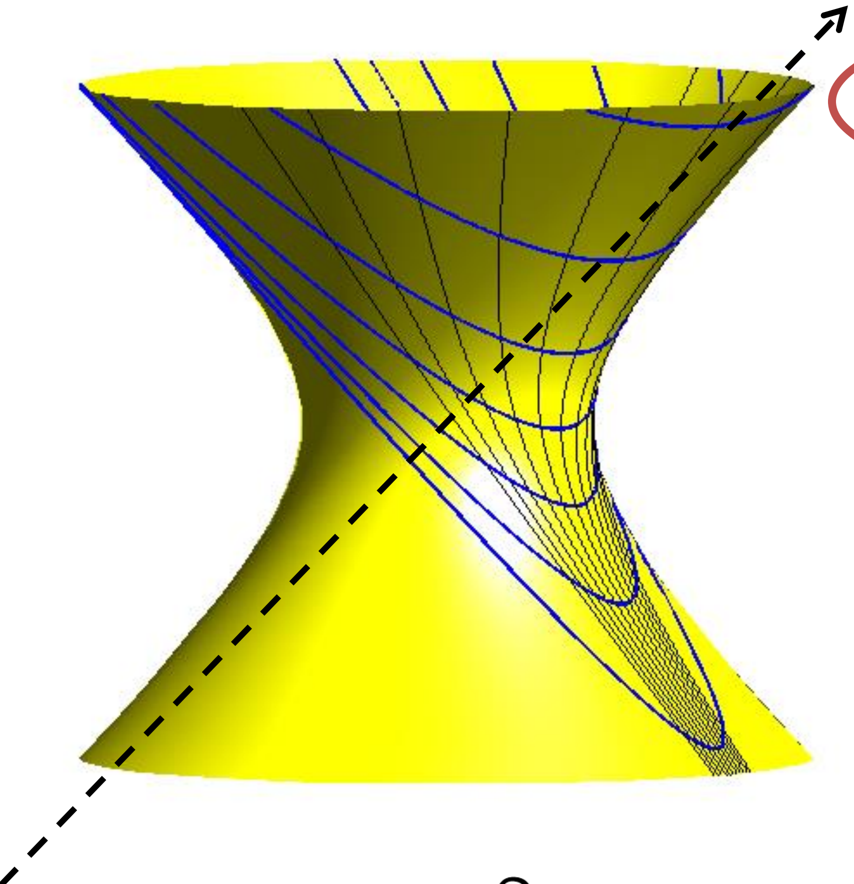
$$\ddot{a} = \frac{1}{3}\Lambda\, a \qquad \dot{a}^2 = \frac{1}{3}\Lambda\, a^2 - K$$

$$K = 1$$



$$a(t) = \sqrt{\frac{3}{\Lambda}} \cosh \sqrt{\frac{\Lambda}{3}}\, t$$

Flat de Sitter model (Lemaître, 1924)



$$X_0 + X_4 = R \exp \frac{t}{R}$$

$$\begin{cases} X_0 &= R \sinh \frac{t}{R} + \frac{1}{2R} e^{\frac{t}{R}} |\vec{x}|^2 \\ X_1 &= \exp\left(\frac{t}{R}\right) x_1 \\ X_2 &= \exp\left(\frac{t}{R}\right) x_2 \\ X_3 &= \exp\left(\frac{t}{R}\right) x_3 \\ X_4 &= R \cosh \frac{t}{R} - \frac{1}{2R} e^{\frac{t}{R}} |\vec{x}|^2 \end{cases}$$

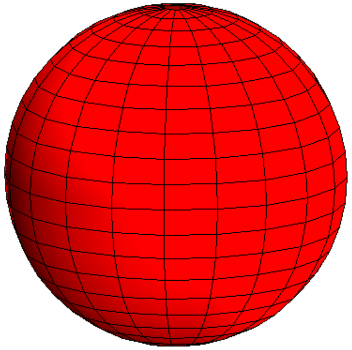
$$ds^2 = dX_0^2 - dX_1^2 - \dots dX_4^2 \Big|_{dS} = $$

$$= dt^2 - \exp \frac{2t}{R} \left(dx_1^2 + dx_2^2 + dx_3^2 \right)$$

Only Λ>0

$$\ddot{a} = \frac{1}{3}\Lambda\, a \qquad \dot{a}^2 = \frac{1}{3}\Lambda\, a^2 - K$$

$K = 1$

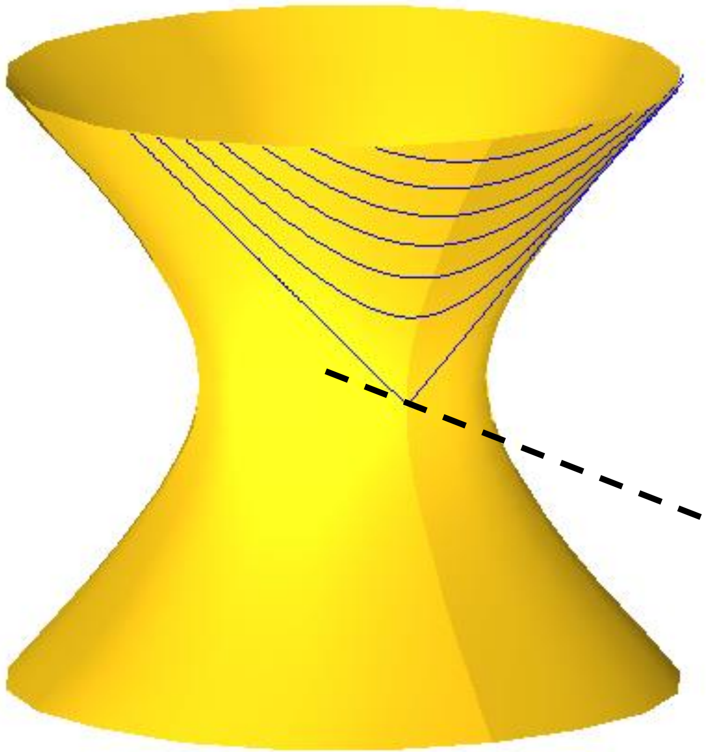


$$a(t) = \sqrt{\frac{3}{\Lambda}} \cosh \sqrt{\frac{\Lambda}{3}}\, t$$

$K = 0$

$$a(t) = \exp \sqrt{\frac{\Lambda}{3}}\, t$$

Open de Sitter model (de Sitter 1917)



$$\begin{cases} X_0 &= R\sinh\frac{t}{R}\cosh\chi \\ X_1 &= R\sinh\frac{t}{R}\sinh\chi\sin\theta\sin\phi \\ X_2 &= R\sinh\frac{t}{R}\sinh\chi\sin\theta\cos\phi \\ X_3 &= R\sinh\frac{t}{R}\sinh\chi\cos\theta \\ X_4 &= R\cosh\frac{t}{R} \end{cases}$$

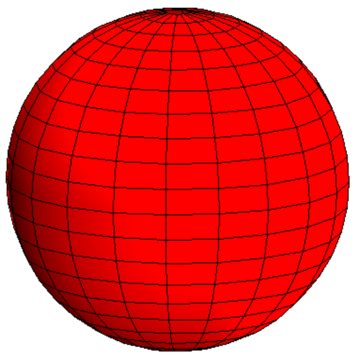
$$ds^2 = dX_0^2 - dX_1^2 - \ldots dX_4^2\Big|_{dS} =$$

$$= dt^2 - R^2\sinh^2\frac{t}{R}\left(d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right)$$

Only$\Lambda>0$

$$\ddot{a} = \frac{1}{3}\Lambda a \qquad \dot{a}^2 = \frac{1}{3}\Lambda a^2 - K$$

$K = 1$

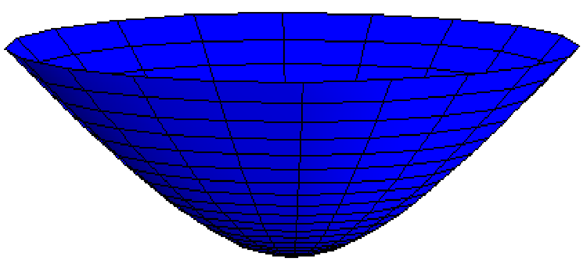


$$a(t) = \sqrt{\frac{3}{\Lambda}}\cosh\sqrt{\frac{\Lambda}{3}}t$$

$K = 0$

$$a(t) = \exp\sqrt{\frac{\Lambda}{3}}t$$

$K = -1$



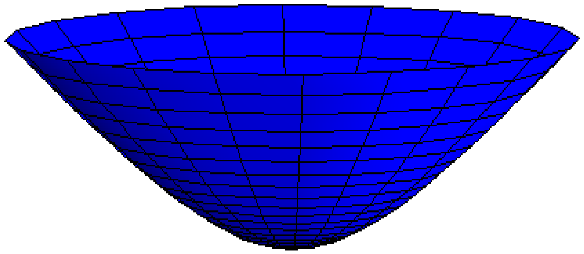
$$a(t) = \sqrt{\frac{3}{\Lambda}}\sinh\sqrt{\frac{\Lambda}{3}}t$$

Only Λ=0

$$\ddot{a} = 0 \qquad \dot{a}^2 = -K$$

$K = 0$

$a(t) = 1$

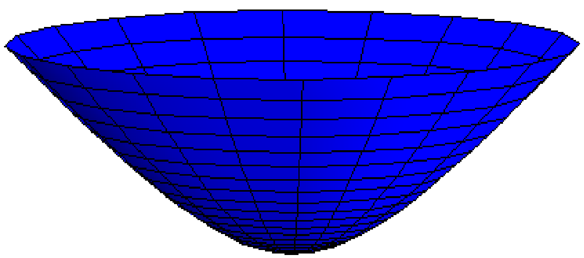


$K = -1$

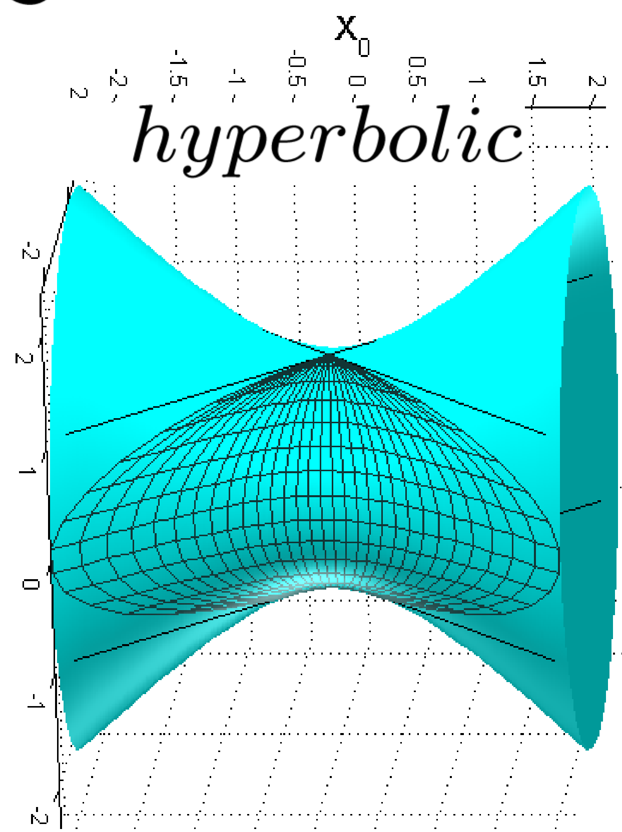
$a(t) = t$

Only Λ<0

$$\ddot{a} = -\frac{1}{3}|\Lambda|\, a \qquad \dot{a}^2 = -\frac{1}{3}|\Lambda|\, a^2 - K$$

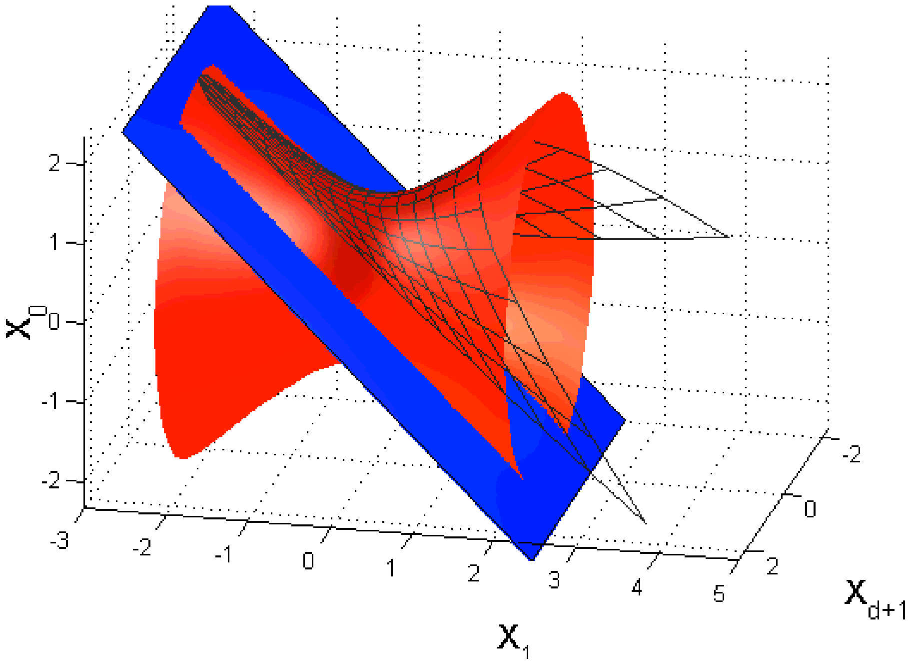


$$K = -1$$



$$a(t) = \sqrt{\frac{3}{|\Lambda|}} \sin \sqrt{\frac{|\Lambda|}{3}}\, t$$

AdS: Poincaré coordinates



$$X_d + X_{d+1} = \exp v$$

$$\begin{cases} X_\mu &= e^v x_\mu \\ X_d &= \sinh v + \frac{1}{2} e^v x^2 \\ X_{d+1} &= \cosh v - \frac{1}{2} e^v x^2 \end{cases}$$

$$x^2 = x_0^2 - x_1^2 - \ldots - x_{d-1}^2$$

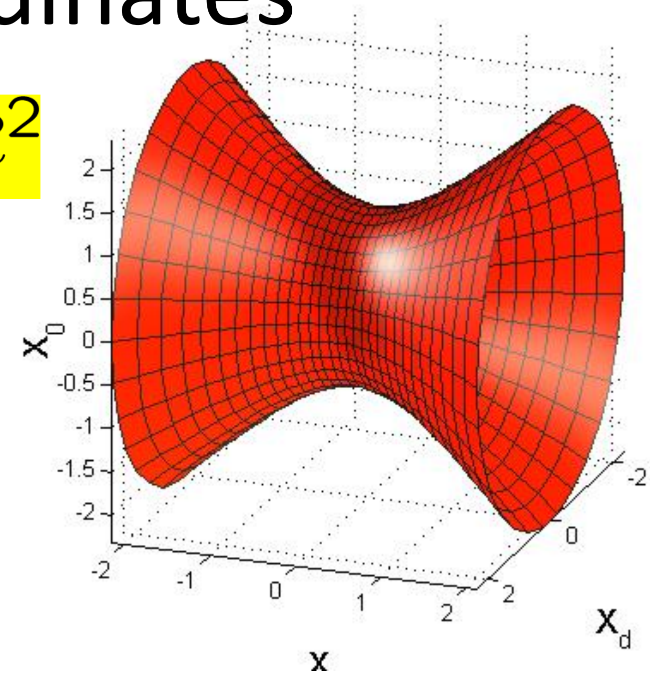
$$ds^2 = e^{2v} dx^\mu dx_\mu - dv^2$$

$$X_0^2 - X_1^2 - \ldots - X_d^2 + X_{d+1}^2 = 1$$

AdS: covering coordinates

$$X_0^2 - X_1^2 - \ldots - X_d^2 + X_{d+1}^2 = R^2$$

$$\begin{cases} X_0 = \frac{\sin t}{\cos \rho} & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ X_i = \tan \rho \, \omega_i & |\vec{\omega}|^2 = 1 \\ X_{d+1} = \frac{\cos t}{\cos \rho} & 0 \leq \rho < \frac{\pi}{2} \end{cases}$$



$$ds^2 = \frac{1}{\cos^2 \rho}(dt^2 - \sin^2 \rho \, d\omega^2 - d\rho^2)$$ $(r = \tan \rho)$

C(r) = AdS|$_{\{r=const\}}$ is a Lorentz manifold
(not strongly causal)

Jacobi Elliptic functions

$$\sin^{-1}(x) = \int_0^x \frac{dx}{\sqrt{1-x^2}},$$

$$u = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

$$\mathrm{snu} = \sin\phi, \qquad \mathrm{cnu} = \cos\phi,$$

$$\mathrm{dnu} = \sqrt{1-k^2\sin\phi}$$

Jacobi elliptic functions: a reminder

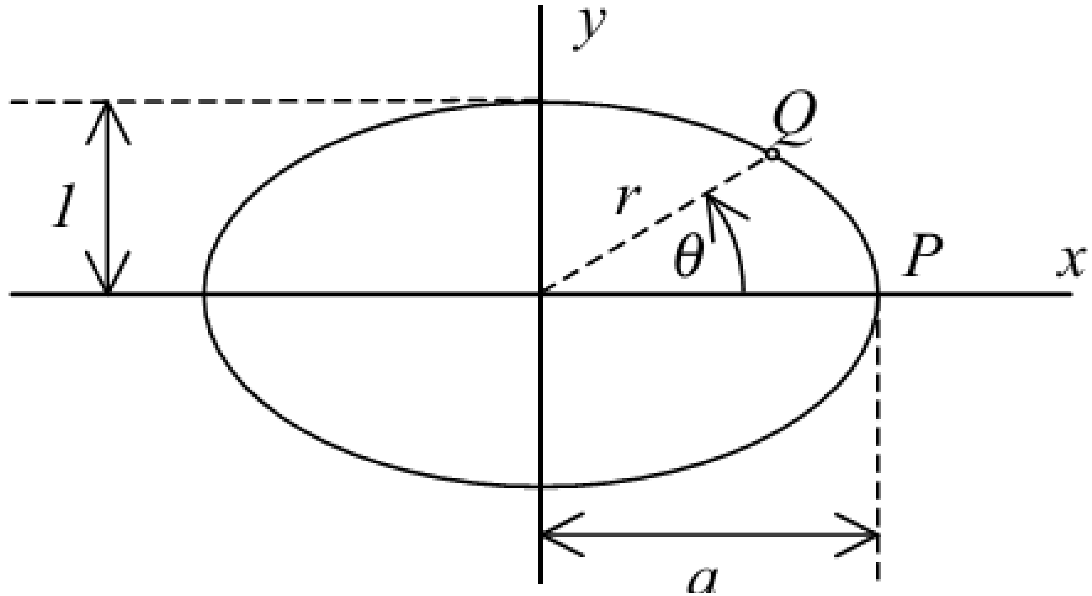
$$u = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}$$

$$\mathrm{sn} u = \sin\phi, \qquad \mathrm{cn} u = \cos\phi,$$

$$\mathrm{dn} u = \sqrt{1 - k^2 \sin\phi}$$

$$\frac{d}{du}\,\mathrm{sn}\, u \;=\; \mathrm{cn}\, u\, \mathrm{dn}\, u \;=\; \sqrt{1 - \mathrm{sn}^2\, u}\sqrt{1 - k^2\,\mathrm{sn}^2\, u},$$

$$\left(\frac{dy}{du}\right)^2 \;=\; (1 - y^2)(1 - k^2\, y^2).$$



$$\left(\frac{x}{a}\right)^2 + y^2 = 1. \qquad \epsilon \equiv k = \sqrt{1 - \frac{1}{a^2}},$$

$$x^2 + y^2 = r^2.$$

$$u \equiv \int_P^Q r\, d\theta,$$

$$\begin{aligned} \mathrm{sn}(u,k) &= y, \\ \mathrm{cn}(u,k) &= x/a, \\ \mathrm{dn}(u,k) &= r/a. \end{aligned}$$

Jacobi elliptic functions

$$\operatorname{ns} u = \frac{1}{\operatorname{sn} u} \quad \operatorname{nc} u = \frac{1}{\operatorname{cn} u} \quad \operatorname{nd} u = \frac{1}{\operatorname{dn} u}$$

$$\operatorname{sc} u = \frac{\operatorname{sn} u}{\operatorname{cn} u} \quad \operatorname{dc} u = \frac{\operatorname{dn} u}{\operatorname{cn} u} \quad \operatorname{cs} u = \frac{\operatorname{cn} u}{\operatorname{sn} u}$$

$$\operatorname{ds} u = \frac{\operatorname{dn} u}{\operatorname{sn} u} \quad \operatorname{sd} u = \frac{\operatorname{sn} u}{\operatorname{dn} u} \quad \operatorname{cd} u = \frac{\operatorname{cn} u}{\operatorname{dn} u}$$

Theta functions

$$\vartheta(z,q) = \vartheta(z|\tau) = \vartheta_4(z,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \exp(2inz)$$

$$q = \exp i\pi\tau$$

$$\vartheta(z+\pi,q) = \vartheta(z,q)$$

$$\vartheta(z+\pi\tau,q) = -q^{-1}\exp(-2iz)\,\vartheta(z,q)$$

The Jacobi theta function is the unique solution to the heat equation with periodic boundary conditions:

$$\frac{\partial}{\partial\tau}\vartheta(x|\tau) + \frac{1}{4\pi i}\frac{\partial^2}{\partial x^2}\vartheta(x|\tau) = 0$$

.

The four types of theta functions

$$\vartheta_4(z,q) = \vartheta(z,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \exp(2inz)$$

$$\vartheta_3(z,q) = \vartheta_4\left(z + \frac{1}{2}\pi, q\right)$$

$$\vartheta_1(z,q) = -i\exp(iz + \frac{1}{4}i\pi\tau)\vartheta\left(z + \frac{1}{2}\pi\tau, q\right)$$

$$\vartheta_2(z,q) = \vartheta_1\left(z + \frac{1}{2}\pi, q\right)$$

$$q = \exp i\pi\tau$$

Strings

$$t, s \to Y(t, s) \in dS(AdS)$$

String equations

$$\partial_t^2 Y_i - \partial_s^2 Y_i + [(\partial_t Y)^2 - (\partial_s Y)^2] Y_i = 0$$

Conformal gauge constraints:

$$(\partial_t Y \pm \partial_s Y)^2 = 0.$$

GKP's rotating folded string (2002)

$$AdS_3 = \{Y \in \mathbf{R}^4 : \; Y^2 = Y \cdot Y = Y_0^2 + Y_1^2 - Y_2^2 - Y_3^2 = 1\}.$$

$$\begin{cases} Y_0 = \cosh \rho(s) \cos(\omega_1 t) \\ Y_1 = \cosh \rho(s) \sin(\omega_1 t) \\ Y_2 = \sinh \rho(s) \cos(\omega_2 t) \\ Y_3 = \sinh \rho(s) \sin(\omega_2 t) \end{cases} \qquad \rho(s) = \rho(s + 2L)$$

Conformal gauge constraints:

$$(\partial_t Y \pm \partial_s Y)^2 = 0.$$

GKP's rotating string (2002)

$$(\partial_t Y \pm \partial_s Y)^2 = (\partial_t Y_0 \pm \partial_s Y_0)^2 + (\partial_t Y_1 \pm \partial_s Y_1)^2 +$$
$$-(\partial_t Y_2 \pm \partial_s Y_2)^2 - (\partial_t Y_3 \pm \partial_s Y_3)^2 = 0$$

$$\left(\frac{d\rho}{ds}\right)^2 = \omega_1^2 \cosh^2 \rho(s) - \omega_2^2 \sinh^2 \rho(s)$$

$$2L = \int_0^{2L} ds = 4 \int_0^{\rho_0} \frac{d\rho}{\sqrt{\omega_1^2 \cosh^2 \rho - \omega_2^2 \sinh^2 \rho}} = \frac{4K(k)}{\omega_2}$$

$$\tanh \rho_0 = \pm \frac{\omega_1}{\omega_2} = \pm k$$

GKP’s rotating string (2002)

$y = \cosh\rho$ transforms the constraint into a nonlinear Jacobian differential equation:

$$(y')^2 = \omega_2^2 \left[-1 + \left(2 - k^2\right) y^2 - \left(1 - k^2\right) y^4\right]$$

Initial condition $\rho(0) = 0$; solution

$$\cosh\rho = \mathrm{nd}\,(\omega s; k)\,, \qquad \sinh\rho = k \;\; \mathrm{sd}(\omega s; k),$$

$$\begin{cases} Y_0 = \mathrm{nd}\,(\omega s; k)\cos\,(k\omega t), & Y_1 = \mathrm{nd}\,(\omega s; k)\sin\,(k\omega t), \\ Y_2 = k\ \mathrm{sd}(\omega s; k)\cos\,(\omega t), & Y_3 = k\ \mathrm{sd}(\omega s; k)\sin\,(\omega t). \end{cases}$$

GKP's rotating string (2002)

$$\mathcal{E} = \int_0^{2L} (\dot{Y}_0 Y_1 - Y_0 \dot{Y}_1) d\sigma \;\; = \frac{4kE(k)}{1-k^2}$$

$$\mathcal{S} = \int_0^{2L} (\dot{Y}_2 Y_3 - Y_3 \dot{Y}_2) d\sigma \;\; = \frac{4E(k)}{1-k^2} - 4K(k)$$

$$\mathcal{E} = \mathcal{E}(\mathcal{S})$$

3D -> 2D

$$AdS_3 = \{Y \in \mathbf{R}^4 : \; Y^2 = Y \cdot Y = Y_0^2 + Y_1^2 - Y_2^2 - Y_3^2 = 1\}.$$

$$\begin{cases} Y_0 = \cosh\rho(s)\cos(\omega_1 t) \\ Y_1 = \cosh\rho(s)\sin(\omega_1 t) \\ Y_2 = \sinh\rho(s)\cos(\omega_2 t) \\ Y_3 = \sinh\rho(s)\sin(\omega_2 t) \end{cases}$$

$$\rho(s) = \rho(s + 2L)$$

$$AdS_3 = \{Y \in \mathbf{R}^4 : \; Y^2 = Y \cdot Y = Y_0^2 + Y_1^2 - Y_2^2 = 1\}.$$

$$\begin{cases} Y_0 = \cosh\rho(s)\cos(\omega_1 t) \\ Y_1 = \cosh\rho(s)\sin(\omega_1 t) \\ Y_2 = \sinh\rho(s) \end{cases}$$

2D GKP's string

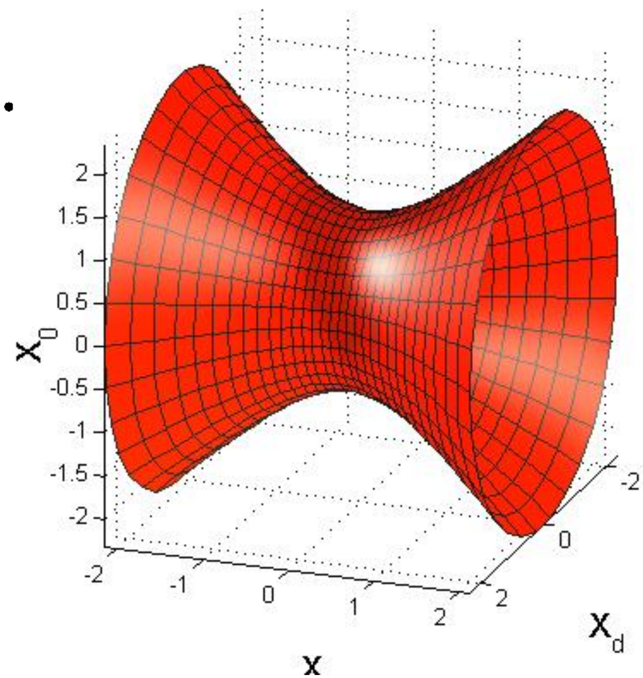
$$AdS_3 = \{Y \in \mathbf{R}^4 : \; Y^2 = Y \cdot Y = Y_0^2 + Y_1^2 - Y_2^2 = 1\}.$$

$$\begin{cases} Y_0 = \cosh\rho(s)\cos(\omega_1 t) \\ Y_1 = \cosh\rho(s)\sin(\omega_1 t) \\ Y_2 = \sinh\rho(s) \end{cases}$$

$$(\partial_t Y \pm \partial_s Y)^2 = 0.$$

$$\cosh\rho(s) \to \frac{1}{\cos(\omega_1 s)}$$

$$\sinh\rho(s) \to \tan(\omega_1 s)$$



Elliptic function and theta functions

$$\mathrm{sn}(z,k)=\frac{\vartheta_3\vartheta_1(z/\vartheta_3^2)}{\vartheta_2\vartheta_4(z/\vartheta_3^2)},\quad \mathrm{cn}(z,k)=\frac{\vartheta_4\vartheta_2(z/\vartheta_3^2)}{\vartheta_2\vartheta_4(z/\vartheta_3^2)},\quad \mathrm{dn}(z,k)=\frac{\vartheta_4\vartheta_3(z/\vartheta_3^2)}{\vartheta_3\vartheta_4(z/\vartheta_3^2)}.$$

$$k=\frac{\vartheta_2^2}{\vartheta_3^2}=\frac{\vartheta_2^2(0\,|\,\tau)}{\vartheta_3^2(0\,|\,\tau)},\qquad k'=\frac{\vartheta_4^2}{\vartheta_3^2}=\frac{\vartheta_4^2(0\,|\,\tau)}{\vartheta_3^2(0\,|\,\tau)}$$

$$\begin{cases} Y_0=\mathrm{nd}\,(\omega s;k)\cos\,(k\omega t), & Y_1=\mathrm{nd}\,(\omega s;k)\sin\,(k\omega t),\\ Y_2=k\;\mathrm{sd}(\omega s;k)\cos\,(\omega t), & Y_3=k\;\mathrm{sd}(\omega s;k)\sin\,(\omega t).\end{cases}$$

$$\begin{cases} Y_0=\frac{\vartheta_3\vartheta_4(\widehat{s})}{\vartheta_4\vartheta_3(\widehat{s})}\cos\,(kt), & Y_1=\frac{\vartheta_3\vartheta_4(\widehat{s})}{\vartheta_4\vartheta_3(\widehat{s})}\sin\,(kt),\\ Y_2=\frac{\vartheta_2\vartheta_1(\widehat{s})}{\vartheta_4\vartheta_3(\widehat{s})}\cos\,(t), & Y_3=\frac{\vartheta_2\vartheta_1(\widehat{s})}{\vartheta_4\vartheta_3(\widehat{s})}\sin\,(t),\end{cases}$$

$$\omega=1,\qquad \widehat{s}=s/\vartheta_3^2.$$

The trick of the tale: GKP's string in hom. coordinates

$$\begin{cases} Y_0 = \frac{\vartheta_3\vartheta_4(\hat{s})}{\vartheta_4\vartheta_3(\hat{s})}\cos(kt), & Y_1 = \frac{\vartheta_3\vartheta_4(\hat{s})}{\vartheta_4\vartheta_3(\hat{s})}\sin(kt), \\ Y_2 = \frac{\vartheta_2\vartheta_1(\hat{s})}{\vartheta_4\vartheta_3(\hat{s})}\cos(t), & Y_3 = \frac{\vartheta_2\vartheta_1(\hat{s})}{\vartheta_4\vartheta_3(\hat{s})}\sin(t), \end{cases}$$

$$C_{2,3} = \{\xi \in \mathbf{R}^{d+2} : \ \xi^2 = \xi\cdot\xi = \xi_0^2+\xi_1^2-\xi_2^2-\xi_3^2-\xi_4^2 = 0\}$$

String on the cone

$$(t,s) \to \xi(t,s) = \begin{cases} \xi_0 = \vartheta_3\vartheta_4(\hat{s})\cos(kt), \\ \xi_1 = \vartheta_3\vartheta_4(\hat{s})\sin(kt), \\ \xi_2 = \vartheta_2\vartheta_1(\hat{s})\cos(t), \\ \xi_3 = \vartheta_2\vartheta_1(\hat{s})\sin(t), \\ \xi_4 = \vartheta_4\vartheta_3(\hat{s}); \end{cases}$$

GKP's string on the cone

$$(t,s) \to \xi(t,s) = \begin{cases} \xi_0 = \vartheta_3\vartheta_4(\widehat{s})\cos(kt), \\ \xi_1 = \vartheta_3\vartheta_4(\widehat{s})\sin(kt), \\ \xi_2 = \vartheta_2\vartheta_1(\widehat{s})\cos(t), \\ \xi_3 = \vartheta_2\vartheta_1(\widehat{s})\sin(t), \\ \xi_4 = \vartheta_4\vartheta_3(\widehat{s}); \end{cases}$$

$\xi \in C_{2,3}$ is a well-known quadratic identity between theta functions (Whittaker, p. 466):

$$\xi^2 = \xi_0^2 + \xi_1^2 - \xi_2^2 - \xi_3^2 - \xi_4^2 =$$

$$= {\vartheta_3}^2\vartheta_4(\widehat{s})^2 - {\vartheta_2}^2\vartheta_1(\widehat{s})^2 - {\vartheta_4}^2\vartheta_3(\widehat{s})^2 = 0.$$

Constraints on the cone

AdS string in homogeneous coordinates

$$t, s \to Y_i(t,s) = \frac{\xi_i(t,s)}{\xi_{d+1}(t,s)}, \quad i = 0, 1, \ldots, d; \qquad (1)$$

$t, s \to \xi_\mu(t,s)$, $\mu = 0, 1, \ldots, d+1$, is a two-surface in $C_{2,d}$.

$$\xi^2 = 0 \quad \longrightarrow \quad \partial_z Y^i \partial_w Y_i = \frac{1}{\xi_{d+1}^2} \partial_z \xi^\mu \partial_w \xi_\mu, \qquad (2)$$

z, w can be either t or s.

If Y_i satisfy the constraints in AdS_d, the functions ξ_μ also do in $C(2,d)$ and viceversa. .

Doubly elliptic strings on the cone

A fundamental quadratic identity between theta functions:

$$\vartheta_1(\hat{t})^2\,\vartheta_1(\hat{s})^2 - \vartheta_2(\hat{t})^2\,\vartheta_2(\hat{s})^2 + \vartheta_3(\hat{t})^2\,\vartheta_3(\hat{s})^2 - \vartheta_4(\hat{t})^2\,\vartheta_4(\hat{s})^2 = 0$$

$$(t,s) \to \xi(t,s) = \begin{cases} \xi_0 = \vartheta_1(\hat{t})\,\vartheta_1(\hat{s}), & \xi_1 = \vartheta_3(\hat{t})\,\vartheta_3(\hat{s}), \\ \xi_2 = \vartheta_2(\hat{t})\,\vartheta_2(\hat{s}), & \xi_3 = \vartheta_4(\hat{t})\,\vartheta_4(\hat{s}). \end{cases}$$

$$\xi^2 \;= \xi_0^2 + \xi_1^2 - \xi_2^2 - \xi_3^2 = 0$$

$$(t,s) \to \xi(t,s) \in C_{2,2}$$

Constraints

Amounts to another (possibly unknown) identity among theta functions and their derivatives

$$\partial_t\xi \cdot \partial_t\xi + \partial_s\xi \cdot \partial_s\xi = -\theta_3^{-2} \sum_{i=1}^{4} (-1)^\alpha (\vartheta'_\alpha(\hat{t})^2\, \vartheta_\alpha(\hat{s})^2 + \vartheta_\alpha(\hat{t})^2\, \vartheta'_\alpha(\hat{s})^2) = 0.$$

Proof: apply the Laplace operator to the defining identity

$$0 = \frac{1}{2}(\partial_x^2 + \partial_y^2) \sum_{\alpha=1}^{4} (-1)^\alpha (\vartheta_\alpha(x)^2 \vartheta_\alpha(y)^2) =$$

$$= \sum_{\alpha=1}^{4} (-1)^\alpha (\vartheta'_\alpha(x)^2 \vartheta_\alpha(y)^2 + \vartheta_\alpha(x)^2 \vartheta'_\alpha(y)^2 + \vartheta_\alpha(x)\vartheta''_\alpha(x)\vartheta_\alpha(y)^2 + \vartheta_\alpha(x)^2\vartheta_\alpha(y)\vartheta''_\alpha(y)) =$$

$$= \sum_{\alpha=1}^{4} (-1)^\alpha (\vartheta'_\alpha(x)^2 \vartheta_\alpha(y)^2 + \vartheta_\alpha(x)^2 \vartheta'_\alpha(y)^2) + \frac{2i}{\pi}\frac{\partial}{\partial\tau} \sum (-1)^\alpha (\vartheta_\alpha(x)^2 \vartheta_\alpha(y)^2) =$$

$$= \sum_{\alpha=1}^{4} (-1)^\alpha (\vartheta'_\alpha(x)^2 \vartheta_\alpha(y)^2 + \vartheta_\alpha(x)^2 \vartheta'_\alpha(y)^2) = 0.$$

Finite open strings

Project back to AdS/dS

$$(t,s) \to \xi(t,s) = \begin{cases} \xi_0 = \vartheta_1(\hat{t})\,\vartheta_1(\hat{s}), & \xi_1 = \vartheta_3(\hat{t})\,\vartheta_3(\hat{s}), \\ \xi_2 = \vartheta_2(\hat{t})\,\vartheta_2(\hat{s}), & \xi_3 = \vartheta_4(\hat{t})\,\vartheta_4(\hat{s}). \end{cases}$$

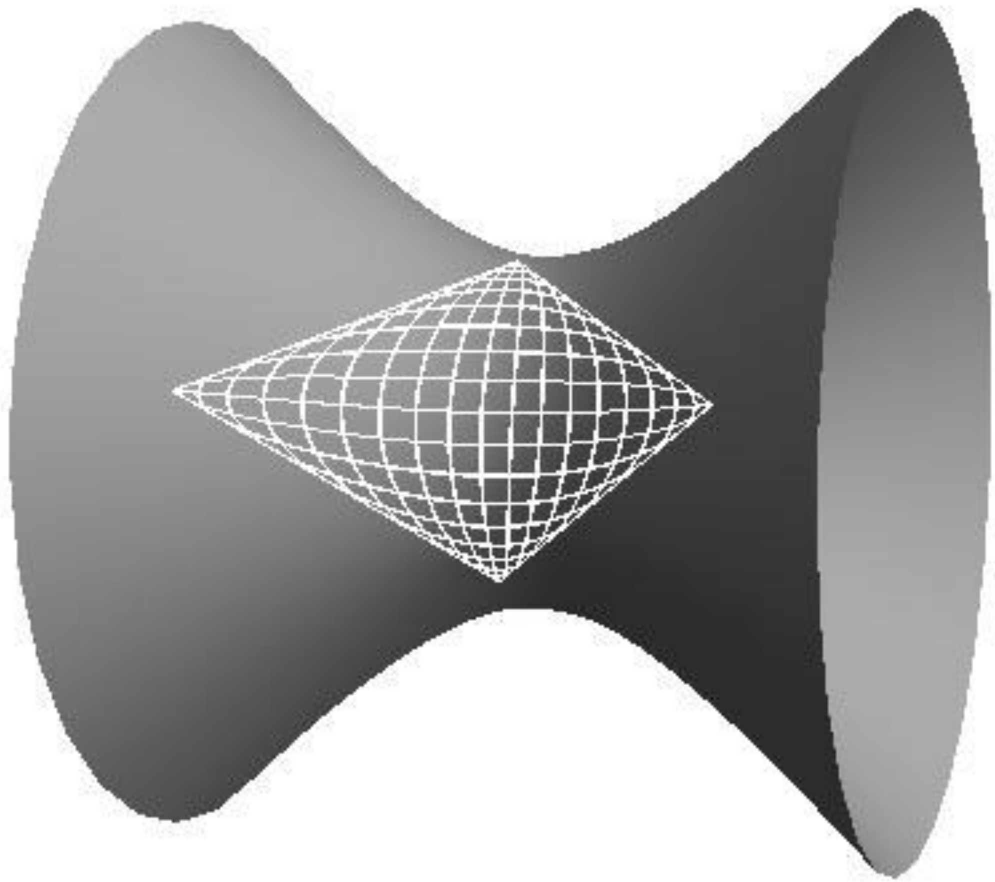
$$(t,s) \to Y^{(1)} = \begin{cases} Y_0(t,s) = \frac{\xi_0}{\xi_3} = \frac{\vartheta_1(\hat{t})\vartheta_1(\hat{s})}{\vartheta_4(\hat{t})\vartheta_4(\hat{s})} = k\ \mathrm{sn}(t,k)\,\mathrm{sn}(s,k), \\ Y_1(t,s) = \frac{\xi_1}{\xi_3} = \frac{\vartheta_3(\hat{t})\vartheta_3(\hat{s})}{\vartheta_4(\hat{t})\vartheta_4(\hat{s})} = \frac{1}{k'}\,\mathrm{dn}(t,k)\,\mathrm{dn}(s,k), \\ Y_2(t,s) = \frac{\xi_2}{\xi_3} = \frac{\vartheta_2(\hat{t})\vartheta_2(\hat{s})}{\vartheta_4(\hat{t})\vartheta_4(\hat{s})} = \frac{k}{k'}\,\mathrm{cn}(t,k)\,\mathrm{cn}(s,k), \end{cases}$$

Constraints are satisfied.

String equations also (I leave this as an exercise!)

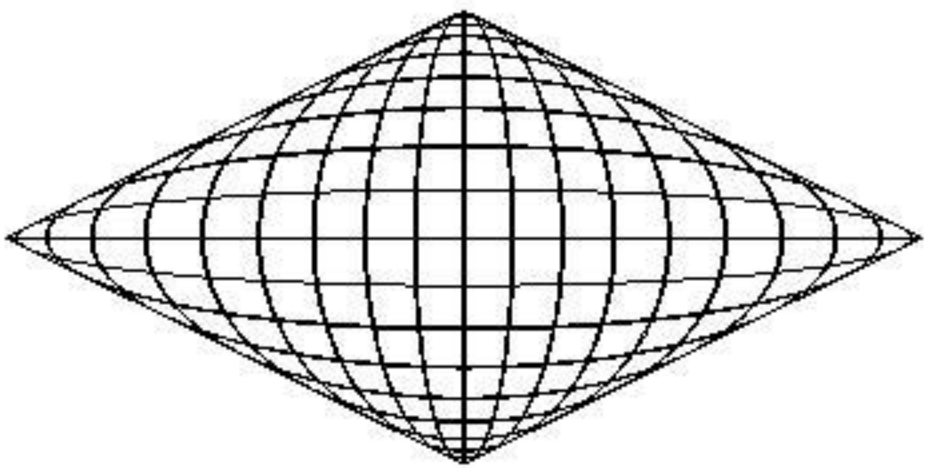
Finite open strings

$$(t,s) \to Y^{(1)} = \begin{cases} Y_0(t,s) &= k\ \mathrm{sn}(t,k)\,\mathrm{sn}(s,k), \\ Y_1(t,s) &= \frac{1}{k'}\,\mathrm{dn}(t,k)\,\mathrm{dn}(s,k), \\ Y_2(t,s) &= \frac{k}{k'}\,\mathrm{cn}(t,k)\,\mathrm{cn}(s,k), \end{cases}$$



Finite open strings

$$(t,s) \to Y^{(1)} = \begin{cases} Y_0(t,s) &=& k\ \mathrm{sn}(t,k)\,\mathrm{sn}(s,k), \\ Y_1(t,s) &=& \frac{1}{k'}\,\mathrm{dn}(t,k)\,\mathrm{dn}(s,k), \\ Y_2(t,s) &=& \frac{k}{k'}\,\mathrm{cn}(t,k)\,\mathrm{cn}(s,k), \end{cases}$$

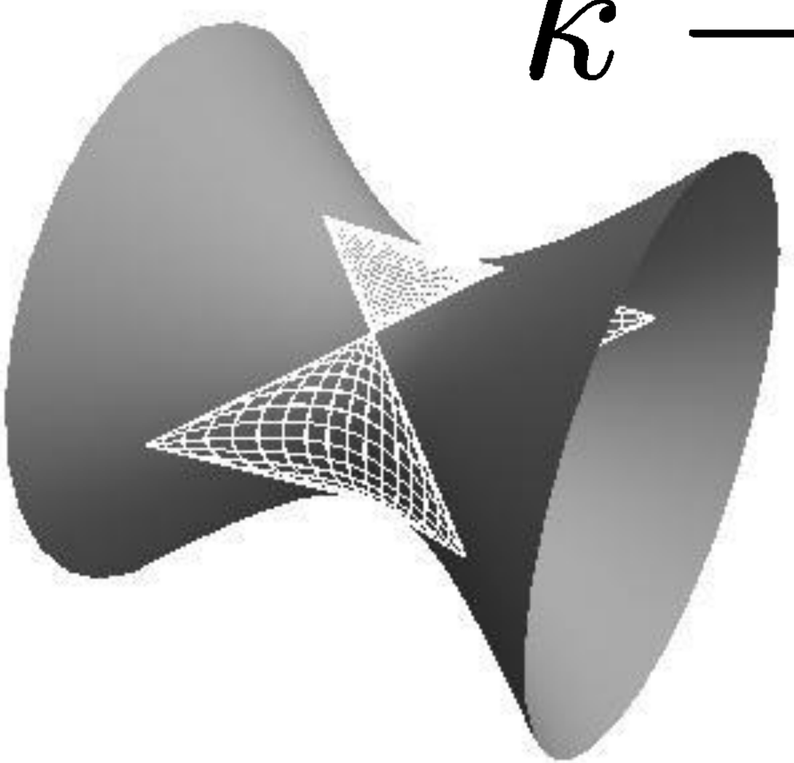


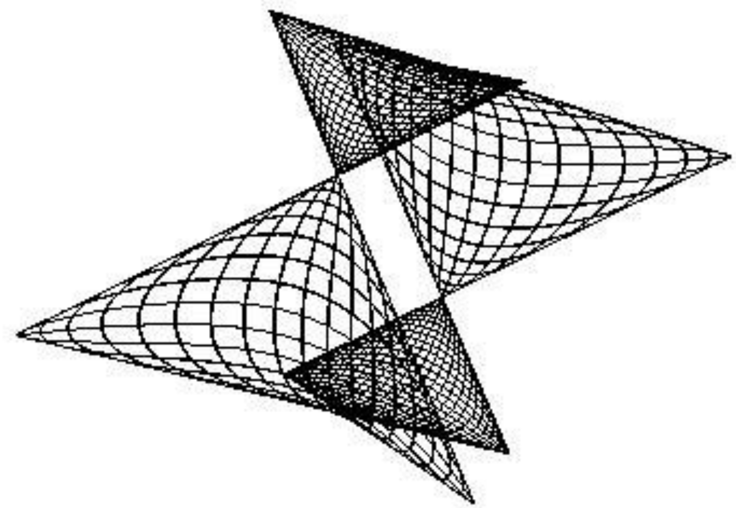
$$\mathcal{A} = \int \sqrt{h}dtds = k^2 \int \sqrt{\left(\mathrm{sn}\,(t,k)^2 - \mathrm{sn}\,(s,k)^2\right)^2} dtds =$$

$$= 8(K(k) - E(k))K(k).$$

Finite open strings

$$k \to k'$$





$$\mathcal{A} \;=\; \int \sqrt{h} dt ds = 8[K(k)^2 + K(k')^2 - E(k)K(k) - E(k')K(k')].$$

Semi infinite Strings

Project back to AdS/dS

$$(t,s) \to \xi(t,s) = \begin{cases} \xi_0 = \vartheta_1(\hat{t})\,\vartheta_1(\hat{s}), & \xi_1 = \vartheta_3(\hat{t})\,\vartheta_3(\hat{s}), \\ \xi_2 = \vartheta_2(\hat{t})\,\vartheta_2(\hat{s}), & \xi_3 = \vartheta_4(\hat{t})\,\vartheta_4(\hat{s}). \end{cases}$$

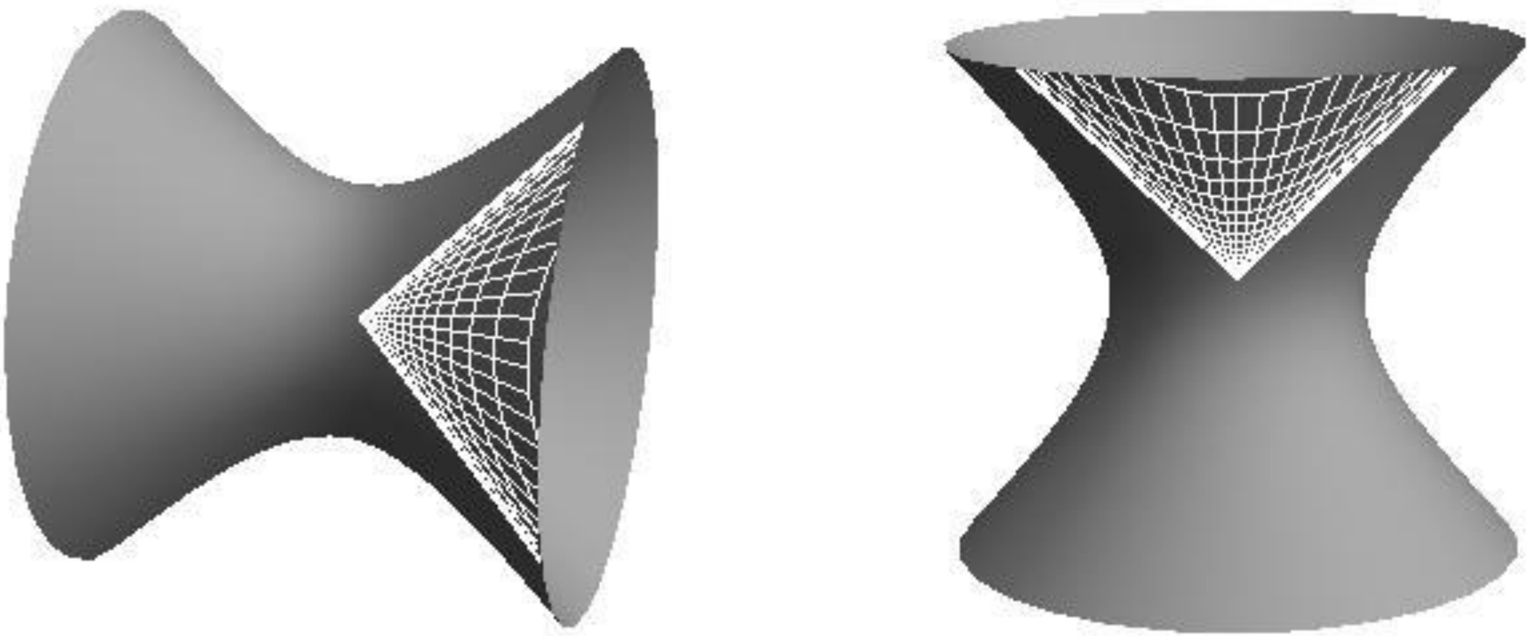
$$(t,s) \to Y^{(2)}(t,s;k) = \begin{cases} Y_0(t,s) & = & \frac{\xi_0}{\xi_2} = \frac{\vartheta_1(\hat{t})\vartheta_1(\hat{s})}{\vartheta_2(\hat{t})\vartheta_2(\hat{s})} = k' \ \mathrm{sc}(t,k)\,\mathrm{sc}(s,k), \\ Y_1(t,s) & = & \frac{\xi_1}{\xi_2} = \frac{\vartheta_3(\hat{t})\vartheta_3(\hat{s})}{\vartheta_2(\hat{t})\vartheta_2(\hat{s})} = \frac{1}{k}\,\mathrm{dc}(t,k)\,\mathrm{dc}(s,k), \\ Y_2(t,s) & = & \frac{\xi_3}{\xi_2} = \frac{\vartheta_4(\hat{t})\vartheta_4(\hat{s})}{\vartheta_2(\hat{t})\vartheta_2(\hat{s})} = \frac{k'}{k}\,\mathrm{nc}(t,k)\,\mathrm{nc}(s,k); \end{cases}$$

Constraints are satisfied.

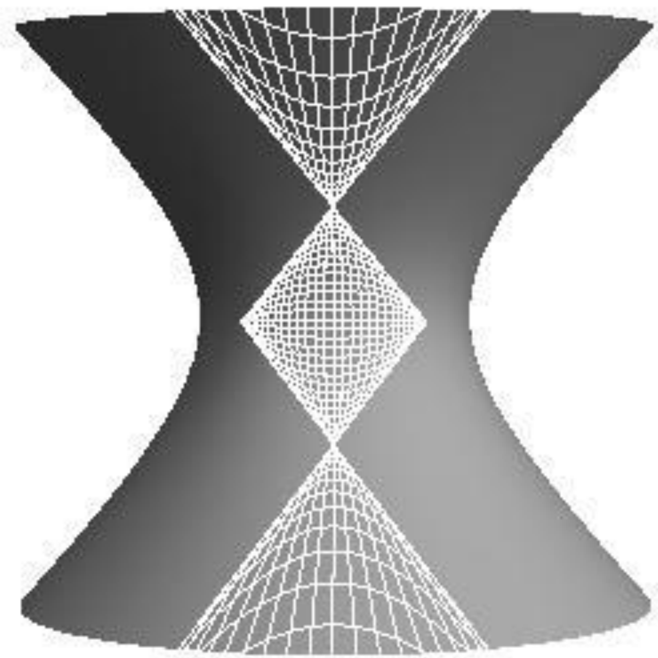
String equations also (I leave this as another exercise!)

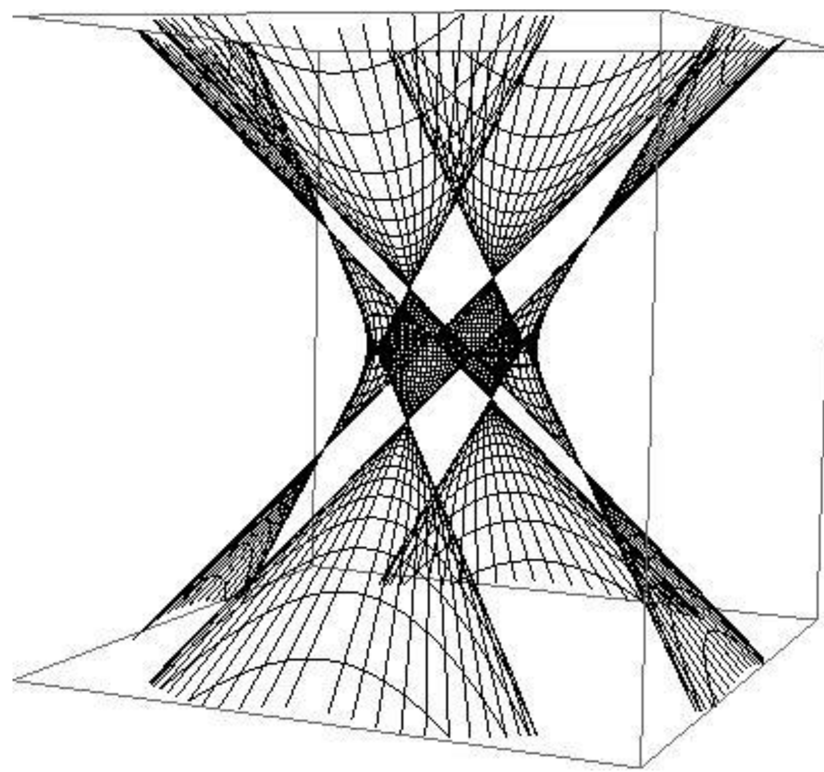
Finite open strings

$$(t,s) \to Y^{(2)}(t,s;k) = \begin{cases} Y_0(t,s) &= \frac{\xi_0}{\xi_2} = \frac{\vartheta_1(\hat{t})\vartheta_1(\hat{s})}{\vartheta_2(\hat{t})\vartheta_2(\hat{s})} = k' \,\mathsf{sc}(t,k)\,\mathsf{sc}(s,k), \\ Y_1(t,s) &= \frac{\xi_1}{\xi_2} = \frac{\vartheta_3(\hat{t})\vartheta_3(\hat{s})}{\vartheta_2(\hat{t})\vartheta_2(\hat{s})} = \frac{1}{k}\,\mathsf{dc}(t,k)\,\mathsf{dc}(s,k), \\ Y_2(t,s) &= \frac{\xi_3}{\xi_2} = \frac{\vartheta_4(\hat{t})\vartheta_4(\hat{s})}{\vartheta_2(\hat{t})\vartheta_2(\hat{s})} = \frac{k'}{k}\,\mathsf{nc}(t,k)\,\mathsf{nc}(s,k); \end{cases}$$



All in all





Infinite open (AdS)/ closed (dS) strings

A second well-known relation between theta functions

$$\xi^2 = \vartheta_1(\hat{t})^2\,\vartheta_3(\hat{s})^2 + \vartheta_2(\hat{t})^2\,\vartheta_4(\hat{s})^2 - \vartheta_3(\hat{t})^2\,\vartheta_1(\hat{s})^2 - \vartheta_4(\hat{t})^2\,\vartheta_2(\hat{s})^2 = 0.$$

$$(t,s) \to \xi(t,s) = \begin{cases} \xi_0 = \vartheta_1(\hat{t})\,\vartheta_3(\hat{s}), & \xi_1 = \vartheta_2(\hat{t})\,\vartheta_4(\hat{s}), \\ \xi_2 = \vartheta_3(\hat{t})\,\vartheta_1(\hat{s}), & \xi_3 = \vartheta_4(\hat{t})\,\vartheta_2(\hat{s}). \end{cases}$$

$$(t,s) \to \begin{cases} Y_0(t,s) &= \frac{\xi_0}{\xi_3} &= \mathrm{sn}(t,k)\ \mathrm{dc}(s,k), \\ Y_1(t,s) &= \frac{\xi_1}{\xi_3} &= \mathrm{cn}(t,k)\ \mathrm{nc}(s,k), \\ Y_2(t,s) &= \frac{\xi_2}{\xi_3} &= \mathrm{dn}(t,k)\ \mathrm{sc}(s,k) \end{cases}$$

Infinite open (AdS)/ closed (dS) strings

$$
(t,s) \to \begin{cases} Y_0(t,s) &= \frac{\xi_0}{\xi_3} = \mathsf{sn}(t,k)\ \mathsf{dc}(s,k), \\ Y_1(t,s) &= \frac{\xi_1}{\xi_3} = \mathsf{cn}(t,k)\ \mathsf{nc}(s,k), \\ Y_2(t,s) &= \frac{\xi_2}{\xi_3} = \mathsf{dn}(t,k)\ \mathsf{sc}(s,k) \end{cases}
$$