Stochastic Formalism Revisited and Stochastic Effects in Hybrid Inflation

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Based on arXiv:1304.6408
and ongoing work with
Robert Brandenberger (McGill) and Vincent Vennin (IAP)

Outline

Stochastic Inflation Formalism (mostly Starobinsky's work)

- Heuristically and some intuition;
- Motivating a recursive method;

Microphysics justification:

- CPT (in-in) formalism & rederivation of the Langevin eqns
- Perturbative expansion

Application: Hybrid Inflation

- What is hybrid inflation?
- Recursive strategy: how to implement it
- New results: noise amplitude, tilt, and dispersion

Setup: Inflation

We work in FLRW spacetime:

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2 = a^2(\tau)\left(-d\tau^2 + d\vec{x}^2\right) \; ;$$

$$H^2(t) \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_p^2}\rho \, , \quad q(t) \equiv -\frac{a\ddot{a}}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2} \, .$$

Inflation is defined by a phase of accelerated expansion of the scale factor $a(t)$, meaning:

$$H(t) > 0 \qquad \text{with} \; -1 \leq q(t) < 0$$

$\Rightarrow$ Physical lengths grow quasi-exponentially

(we allow $\dot{H} \neq 0$ but do not perturb the metric for now)

Slow-Roll Inflation

The quasi-exponential expansion of $a(t)$ is driven by the slow roll of a scalar field $\hat{\Phi}$ down the slope of a flat potential.

$$\ddot{\varphi} + 3H\dot{\varphi} + V_{,\varphi} = 0$$

$$H^2 = \frac{1}{3M_{pl}^2}\left[\frac{\dot{\varphi}^2}{2} + V\right]$$

Assume:

$$\left.\begin{array}{l} \hat{\Phi} \approx \varphi \\ \ddot{\varphi} \ll 3H\dot{\varphi} \\ \rho \approx V \sim cst \end{array}\right. \quad \Rightarrow \quad \left.\begin{array}{l} H \approx cst \\ a(t) \sim e^{Ht} \end{array}\right\} \text{quasi-de Sitter}$$

$\Rightarrow$ Physical lengths grow quasi-exponentially

Split: $\hat{\Phi} = \varphi + \delta\phi$

$\boxed{\text{Classical Background (fixed)}}$ $\boxed{\text{Small quantum perturbations}}$

homogeneous $\Rightarrow$ acceleration $\hookleftarrow$ $\hookrightarrow$ Structure

Quantum fluctuations: $\delta\phi$ are created on small scales, are stretched by the inflating space beyond the Hubble radius where they freeze out (when $k/aH \sim 1$), get squeezed and undergo classicalisation. (they later re-enter the Hubble, and seed the fluctuations of the CMB and the LLS of the Universe)

Is this split true/good?

Idea: we are interested in the *classical* theory, beyond the Hubble radius, since these are the range of scales that are observable in the CMB.

$\Rightarrow$ Write an effective classical theory for these modes, by coarse-graining, or averaging, over scales $\sim H^{-1}$

«Problem»: Modes smaller than the coarse-graining scale, that is quantum-fluctuating modes, are constantly escaping the coarse-grained region and sourcing the classical theory. From this perspective, they act as a *noise* for the classical theory

Stochastic inflation describes how to perform this averaging, and how quantum fluctuation give rise to a classical noise term in the effective coarse-grained classical equation.

Why does this even matter?

Shouldn't the constant contribution of incoming quantum modes into the coarse grained theory be negligible anyway?

- Matters a lot, *e.g*. when the classical trajectory in field space is constrained to small fields values, quantum dispersion may dominates

- Also, in eternal inflation, quantum corrections must dominate over the classical trajectory

In general,

allows to constantly «renormalise» the background trajectory, $i.e.$ re-sums the incoming quantum modes in the background.

so $e.g.\ H(t)$ assumes it physical values at all t

$\Rightarrow$ **Powerful non-perturbative method**

How does it work?
(Heuristically)

Consider a set of 2 quantum fields $\left\{\hat{\Phi}, \hat{\Psi}\right\}$ (generalisation to larger numbers easy)

Split each one into long and short wavelengths at a coarse graining scale using a window function

$$\boxed{\Phi = \varphi + \phi_> \,, \;\; \Psi = \chi + \psi_> \,,}$$

$$\phi_> \,, \;\; \psi_> \text{ correspond to } \;\; k > H(t)a(t) \,,$$
$$\varphi \,, \;\; \chi \text{ correspond to } \;\; H(t)a(t) > k > 0 \,,$$

Plugging this expansion in the KG equation:

$$-\Box\varphi + m_\Phi^2\varphi + V_{{\rm pert},\Phi}(\varphi,\chi) + \left[-\Box\phi_> + m_\Phi^2\phi_> + V^{\rm pert}_{,\Phi\Phi}(\varphi,\chi)\phi_> + V^{\rm pert}_{,\Phi\Psi}(\varphi,\chi)\psi_>\right]$$
$$= -V^{\rm pert}_{,\Phi\Phi\Psi}(\varphi,\chi)\phi_>\psi_> - \frac{1}{2}V^{\rm pert}_{,\Phi\Phi\Phi}(\varphi,\chi)\phi_>^2 - \frac{1}{2}V^{\rm pert}_{,\Phi\Psi\Psi}(\varphi,\chi)\psi_>^2 + \ldots \,,$$

How Does it Work

(Heuristically) continued...

To subtract the linearised equation for the small-scale $\phi_>, \psi_>$ from the φ, χ EoMs, expand $\phi_>, \psi_>$ in creation/annihilation opts on a time-dept background:

$$\boxed{\begin{aligned}\phi_>(\mathbf{x},t) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} W_H(k,t)\left[\phi_{\mathbf{k}}\hat{a}_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}} + \phi^*_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}\right] ,\\ \psi_>(\mathbf{x},t) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} W_H(k,t)\left[\psi_{\mathbf{k}}\hat{b}_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}} + \psi^*_{\mathbf{k}}\hat{b}^\dagger_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}\right] ,\end{aligned}}$$

$W_H(k,t)$ is the time-dependent window function filtering only the sub-Hubble modes.

Simplest choice: $W_H(k,t) = \theta(k/\epsilon a H - 1)$

Also, only choice to make $\phi_>, \psi_>$ appear as white noise to φ, χ

$\Rightarrow \varphi, \chi$ become Markovian processes (memoryless)

BUT: not very physical...

Winitzki & Vilenkin, 2000, Matarrese et al. 2004

How Does it Work

(Heuristically) continued...

Plug this expansion back in the KG equations, and subtract the linearised quantum fields EoM. Left with:

$$-\Box\varphi + m_\Phi^2\varphi + V_{\mathrm{pert},\Phi}(\varphi,\chi) = \delta S_{\phi_>}$$
$$-V^{\mathrm{pert}}_{,\Phi\Phi\Psi}(\varphi,\chi)\phi_>\psi_> - \frac{1}{2}V^{\mathrm{pert}}_{,\Phi\Phi\Phi}(\varphi,\chi)\phi_>^2 - \frac{1}{2}V^{\mathrm{pert}}_{,\Phi\Psi\Psi}(\varphi,\chi)\psi_>^2 + \dots .$$

Where:

$$\delta S_{\phi_>} = 3H\xi_1^\phi + \dot{\xi}_1^\phi - \xi_2^\phi$$

With:

$$\xi_1^\phi = -\int \frac{d^3\mathbf{k}}{(2\pi)^3}\dot{W}_H\left(\frac{k}{\epsilon a(t)H(t)}\right)\left[\phi_\mathbf{k}\hat{a}_\mathbf{k}e^{-i\mathbf{k}\cdot\mathbf{x}} + \phi_\mathbf{k}^*\hat{a}_\mathbf{k}^\dagger e^{i\mathbf{k}\cdot\mathbf{x}}\right],$$
$$\xi_2^\phi = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\dot{W}_H\left(\frac{k}{\epsilon a(t)H(t)}\right)\left[\dot{\phi}_\mathbf{k}\hat{a}_\mathbf{k}e^{-i\mathbf{k}\cdot\mathbf{x}} + \dot{\phi}_\mathbf{k}^*\hat{a}_\mathbf{k}^\dagger e^{i\mathbf{k}\cdot\mathbf{x}}\right].$$

Stochastic equations of motion:

$$\boxed{\Box\varphi + m_\Phi^2\varphi + V_{\mathrm{pert},\Phi}\,\varphi = 3H\xi_1^\phi + \underline{\dot{\xi}_1^\phi - \xi_2^\phi}}$$
$$- V_{\mathrm{pert},\Phi\Phi\Psi}(\varphi,\chi)\phi_>\psi_> - \frac{1}{2}V_{\mathrm{pert},\Phi\Phi\Phi}(\varphi,\chi)\phi_>^2 - \frac{1}{2}V_{\mathrm{pert},\Phi\Psi\Psi}(\varphi,\chi)\psi_>^2\,,$$

Mode coupling terms

Slow roll-suppressed

$$\boxed{\Box\chi + m_\psi^2\chi + V_{\mathrm{pert},\Psi}\,\chi = 3H\xi_1^\psi + \underline{\dot{\xi}_1^\psi - \xi_2^\psi}}$$
$$- V_{\mathrm{pert},\Psi\Psi\Phi}(\varphi,\chi)\psi_>\phi_> - \frac{1}{2}V_{\mathrm{pert},\Psi\Psi\Psi}(\varphi,\chi)\psi_>^2 - \frac{1}{2}V_{\mathrm{pert},\Psi\Phi\Phi}(\varphi,\chi)\phi_>^2\,.$$

These form a **system of *classical* Langevin equations**, sourced by random gaussian noise terms (which are completely determined by their 2-pt functions).
They describe a stochastic process.

BUT: To find the 2-pt functions, we still need the linearised mode functions:

$$\ddot{\phi}_k + 3H\dot{\phi}_k + \left(\frac{k^2}{a^2} + m_\Phi^2 + V_{\mathrm{pert},\Phi\Phi}\right)\phi_k = -V_{\mathrm{pert},\Phi\Psi}\,\psi_k$$
$$\ddot{\psi}_k + 3H\dot{\psi}_k + \left(\frac{k^2}{a^2} + m_\Psi^2 + V_{\mathrm{pert},\Psi\Psi}\right)\psi_k = -V_{\mathrm{pert},\Psi\Phi}\,\phi_k$$

We now have a ***stochastic process***.

$\Rightarrow$ we are NOT solving for one realisation of φ, χ, we must solve for their probability distribution over many realisations, $\rho(t, \varphi, \chi)$ through a Fokker-Planck equation:

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial \varphi}\left(\frac{V_{,\Phi}}{3H}\rho\right) + \frac{\partial}{\partial \chi}\left(\frac{V_{,\Psi}}{3H}\rho\right) + \frac{1}{2}\frac{\partial^2}{\partial \varphi^2}\left(\langle \xi_1^\phi \xi_1^\phi \rangle \rho\right) + \frac{1}{2}\frac{\partial^2}{\partial \chi^2}\left(\langle \xi_1^\psi \xi_1^\psi \rangle \rho\right)$$

This is the 1-pt PDF

$$\rho_1[t, \varphi(\mathbf{x_1}, t), \chi(\mathbf{x_1}, t)]$$

It's possible to find an expression (messier) for the 2-pt PDF

$$\rho_2[t, \varphi(\mathbf{x_1}, t), \chi(\mathbf{x_1}, t), \varphi(\mathbf{x_2}, t), \chi(\mathbf{x_2}, t)]$$

Expectation values of functionals of the stochastic fields, in particular their correlation functions, are calculated via:

$$\langle F[\varphi(t, \mathbf{x}), \chi(t, \mathbf{x})]\rangle = \int \mathcal{D}\varphi \mathcal{D}\chi \rho(t, \varphi, \chi) F[\varphi, \chi]\,,$$

$$e.g. \qquad \bar{\varphi}(t, \mathbf{x}) = \langle \varphi(t, \mathbf{x})\rangle = \int\int \mathcal{D}\varphi \mathcal{D}\chi \rho(t, \varphi, \chi)\varphi$$

$$\langle \varphi(t, \mathbf{x_1})\varphi(t, \mathbf{x_2})\rangle = \int \mathcal{D}\varphi_1 \mathcal{D}\chi_1 \mathcal{D}\varphi_2 \mathcal{D}\chi_2 \rho_2(t, \varphi(\mathbf{x_1}), \chi(\mathbf{x_1}), \varphi(\mathbf{x_2}), \chi(\mathbf{x_2}))\varphi_1\varphi_2$$

A Recursive Method?

We now have two coupled systems of 2 equations each:
the **classical stochastic system** and the **quantum system**

In order to solve it consistently, we must solve both *to the same order of accuracy in the slow-roll parameters*

$\Rightarrow$ We use a recursive approach!

Outline of the Recursive Approach

1. Solve for the quantum fields $\phi_>, \chi_>$ mode functions to zeroth order in slow-roll, that is, as if they were free, massless fields in dS space. Get the zeroth order noise:

$$\langle \xi_1^{\phi,\psi}(\mathbf{x},t), \xi_1^{\phi,\psi}(\mathbf{x},t)\rangle = \frac{H^3}{4\pi^2}\frac{\sin(\epsilon a H r)}{\epsilon a H r}\delta(t-t')$$

2. Use this noise to find the classical fields φ, χ to zeroth order in slow-roll and their corresponding PDFs. *i.e.* we need to solve:

$$3H^2\frac{\mathrm{d}\varphi}{\mathrm{d}N} = -V_{,\Phi} + \frac{3H^3}{2\pi}\xi_\phi(N) \ ,$$
$$3H^2\frac{\mathrm{d}\chi}{\mathrm{d}N} = -V_{,\Psi} + \frac{3H^3}{2\pi}\xi_\psi(N)$$

where we changed the time variable to the *e*-fold number: $N \equiv \ln(a/a_\mathrm{i})$

Recursive Approach

continued...

3. Go back to the linearised mode functions for the quantum fields and **replace all occurrences of the coarse-grained fields** by their average values, variances, and higher momenta:

$$\begin{array}{rcl} \varphi, \chi & \to & \langle\varphi\rangle, \langle\chi\rangle, \\ \varphi^2, \chi^2 & \to & \langle\varphi^2\rangle, \langle\chi^2\rangle, \\ \varphi^p \chi^q & \to & \langle\varphi^p \chi^q\rangle\,, \end{array}$$

solve the corrected linearised equations for $\phi_>, \chi_>$, this time expanding to next-to-leading order in slow-roll.

i.e. **solve for the full linearised mode functions**

4. Go back to the coarse-grained φ, χ system, and reevaluate the PDF with the corrected linearised noise. Keep up to $\mathcal{O}\left(\sqrt{\langle\xi_{1,2}^{\phi,\psi}\,\xi_{1,2}^{\phi,\psi}\rangle}\right)$ in slow-roll. Can find the spectrum, including the tilt, in a consistent manner.

etc... until the process converges

How does it work?
(Actually)

To understand why this is a sensible thing to do (in particular step 3) and in general why the stochastic approach of coarse-graining the full quantum EoM makes sense to derive a classical theory, look at the microphysics of the process.

This is very similar to quantum Brownian motion

$\Rightarrow$ use similar techniques, *i.e.*

the *in-in* (or CPT) Schwinger-Keldysh formalism

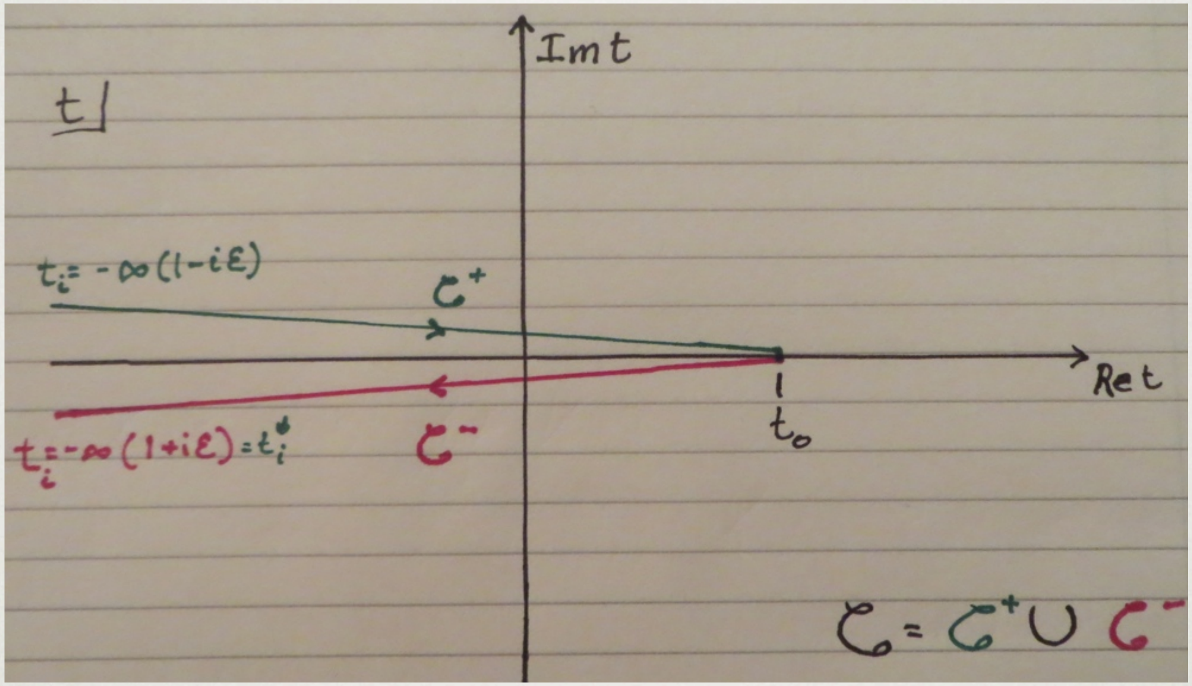
As opposed to the *in-out* formalism where:

- one calculates *S*-matrix elements,
- for transition amplitudes between *in* and *out* asymptotic states,
- with one-particles states defined in the ∞-distant past and future.

In the *in-in* formalism:

- one calculates expectation values of operators at a fixed time t_0, (EEVs for quantum statistical mechanics)
- with one-particles states defined in the ∞-distant past only.

In-In Formalism



- Split the fields into: - a bath $\phi_>, \psi_>$ (same *k*-mode exp. as before)
  - a system $\varphi = \Phi = \phi_>, \chi = \Psi - \psi_>$
- Split each of the bath & system fields into: part ϵ C^+ & part ϵ C^-
  get: $\varphi^+, \varphi^-, \phi_>^+, \phi_>^-$ & similarly for Ψ
- The *in* state, at $-\infty$, is taken to be the Bunch-Davis vacuum,
- Evaluate operators at fixed t_0

Goal: Integrate out the bath degrees of freedom.

- In the same spirit as Wilsonian renormalisation, we want to get a V_{eff} for the system fields once the bath has been integrated out.

-> Similar to quantum Brownian motion!

- Because assume Bunch-Davis vacuum, the initial density matrix factorizes:

$$\hat{\rho}(t=t_i) = \hat{\rho}_{sys}(t_i) \times \hat{\rho}_{bath}(t_i)$$

can write the reduced evolution operator for the system fields as a functional representation, so the effective action can be written as:

$$\int_{\varphi_i^\pm}^{\varphi_f^\pm} \mathcal{D}\varphi^\pm \int_{\chi_i^\pm}^{\chi_f^\pm} \mathcal{D}\chi^\pm \exp\left\{\frac{i}{\hbar} S_{eff}[\varphi^\pm, \chi^\pm]\right\}$$

$$\equiv \int_{\varphi_i^\pm}^{\varphi_f^\pm} \mathcal{D}\varphi^\pm \int_{\chi_i^\pm}^{\chi_f^\pm} \mathcal{D}\chi^\pm \exp\left(\frac{i}{\hbar}\left\{S_{sys}[\varphi^+, \chi^+] - S_{sys}[\varphi^-, \chi^-]\right\}\right) F[\varphi^\pm, \chi^\pm]\,,$$

$F[\varphi^\pm, \chi^\pm]$ is known as the *influence functional*. In general, it is a non-local, non-trivial object: -depends on the time history, mixes the forward and backward histories along the CTP in an irreducible manner.

Influence Functional

It can be written as:

$$F[\varphi^{\pm},\chi^{\pm}] = \int_{-\infty}^{\infty} d\phi^{+}_{>_f} d\psi^{+}_{>_f} \int_{-\infty}^{\infty} d\phi^{\pm}_{>_i} d\psi^{\pm}_{>_i} \int_{\phi^{\pm}_{>_i}}^{\phi^{+}_{>_f}} \mathcal{D}\phi^{\pm}_{>} \int_{\psi^{\pm}_{>_i}}^{\psi^{+}_{>_f}} \mathcal{D}\psi^{\pm}_{>}$$
$$\exp\left(\frac{i}{\hbar}\left\{(S^0_{bath})^+ - (S^0_{bath})^- + S^+_{pert} - S^-_{pert}\right\}\right)\hat{\rho}_{bath}(\phi^{\pm}_{>_i},\psi^{\pm}_{>_i})$$
$$\equiv \exp\left[\frac{i}{\hbar}S_{IA}[\varphi^{\pm},\chi^{\pm}]\right].$$

where S_{IA} in the influence action. Using the vector notation:

$$\tilde{\phi}_> = \begin{pmatrix} \phi^+_> \\ \phi^-_> \end{pmatrix} \quad \tilde{\psi}_> = \begin{pmatrix} \psi^+_> \\ \psi^-_> \end{pmatrix} \quad \tilde{\varphi} = \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix} \quad \tilde{\chi} = \begin{pmatrix} \chi^+ \\ \chi^-_> \end{pmatrix} \quad \tilde{\Lambda}_\phi = \begin{pmatrix} \Lambda_\phi & 0 \\ 0 & -\Lambda_\phi \end{pmatrix} \quad \tilde{\Lambda}_\psi = \begin{pmatrix} \Lambda_\psi & 0 \\ 0 & -\Lambda_\psi \end{pmatrix},$$

$$\Lambda_\phi = -a^3(t)\left[\partial_t^2 + 3H\partial_t - \frac{\nabla^2}{a^2(t)} + m_\Phi^2\right]; \qquad \Lambda_\psi = -a^3(t)\left[\partial_t^2 + 3H\partial_t - \frac{\nabla^2}{a^2(t)} + m_\Psi^2\right].$$

it can be written explicitly in the bilinear form when $V_{pert} = 0$

$$\int_{-\infty}^{\infty} d\phi^{+}_{>_f} d\psi^{+}_{>_f} \int^{\phi^{+}_{>_f}} \mathcal{D}\phi^{\pm}_{>} \int^{\psi^{+}_{>_f}} \mathcal{D}\psi^{\pm}_{>} e^{\frac{i}{\hbar}\int d^4x\left[\left(\frac{1}{2}\tilde{\phi}^T_>\tilde{\Lambda}_\phi\tilde{\phi}_> + \tilde{\varphi}^T\tilde{\Lambda}_\phi\tilde{\phi}_>\right) + \left(\frac{1}{2}\tilde{\psi}^T_>\tilde{\Lambda}_\psi\tilde{\psi}_> + \tilde{\chi}^T\tilde{\Lambda}_\psi\tilde{\psi}_>\right)\right]}$$

Integrating Out the Bath DOFs

- In flat space, the term linear in $\tilde{\phi}_>$ or $\tilde{\psi}_>$ would be set to zero to ensure that ϕ_k and ψ_k are indeed solutions to the linearised mode eqn. (*c.f.* the tadpole method Weinberg 94, Boyanovsky et al. 94 - to ensure we are expanding around the right background)

- However, because of the time-dependence of $W_H(k/\epsilon a H)$, the time derivative in the $\Lambda_{\phi,\psi}$ operators act on the window function, giving a non-zero result. This is *precisely* the effect of the modes leaving the quantum theory and joining the coarse-grained theory.
(else, system & bath are orthogonal in *k*-space in $W_H \to \theta(\frac{k}{\epsilon a H} - 1)$ limit)

- We can perform this Gaussian integral over $\tilde{\phi}_>, \tilde{\psi}_>$

Integrating Out the Bath DOFs (continued)

- Performing the path integral over the bath fields, we obtain:

$$S^{(1)}_{IA} = \underbrace{\frac{i}{2\hbar}\int d^4x d^4x' \varphi_q(x) \mathrm{Re}\left[\Pi_\phi(x,x')\right]\varphi_q(x')}_{\text{Imaginary term}} - \underbrace{\frac{2}{\hbar}\int d^4x d^4x' \theta(t-t')\varphi_q \mathrm{Im}\left[\Pi_\phi(x,x')\right]\varphi_c}_{\text{Real term}} + (\chi \leftrightarrow \varphi)\ ,$$

Morikawa 1990,
Matarrese et al. 2004

- Where:

$$\Pi_\phi(x,x') = \int \frac{d^3\mathbf{k}}{(2\pi)^3} a^3(t)\left[P_t \phi_\mathbf{k}(t)\right] e^{-i\mathbf{k}\cdot\mathbf{x}} a^3(t')\left[P_{t'}\phi^*_\mathbf{k}(t')\right] e^{i\mathbf{k}\cdot\mathbf{x}'}\ ,$$

$$P_t = \left[\ddot{W}_H(t) + 3H\dot{W}_H(t) + 2\dot{W}_H(t)\partial_t\right]\ ;$$

& defined the *quantum* and *classical* fields, rotating to the Keldysh basis:

$$\begin{pmatrix} \varphi_c \\ \varphi_q \end{pmatrix} \equiv \begin{pmatrix} \frac{\varphi^+ + \varphi^-}{2} \\ \varphi^+ - \varphi^- \end{pmatrix}, \qquad \begin{pmatrix} \chi_c \\ \chi_q \end{pmatrix} \equiv \begin{pmatrix} \frac{\chi^+ + \chi^-}{2} \\ \chi^+ - \chi^- \end{pmatrix}.$$

Fluctuation-Dissipation Theorem

- Leading order influence action splits into a real and an imaginary part -> they represent dissipation and noise, respectively.

- The kernels $\mathrm{Im}\left[\Pi_{\phi,\psi}(x,x')\right]$ are the dissipation kernels. *i.e.* their non-symmetric part add a non-local extra term in the classical fields EoM, proportional to $\dot{\varphi}_c$ and $\dot{\chi}_c \;\Rightarrow$ friction, or dissipation.

  Slow-roll $\Rightarrow$ negligible compared to *H*-friction

  Not negligible $\Rightarrow$ *e.g.* warm inflation Berera et al. 2009

- The kernels $\mathrm{Re}\left[\Pi_{\phi,\psi}(x,x')\right]$ each give an imaginary part to the effective action. Interpret them as a result of a weighted average over configurations of stochastic noise terms, representing the «coupling» btw φ and $\phi_>$ and χ and $\psi_>$ $\Rightarrow$ reintroduce these noises w/ right PDF

  Morikawa 1986

- Fluctuation-dissipation thm: they are linked since they come from the same underlying dofs $\Rightarrow$ real and imaginary part of same kernel!

- To interpret the imaginary part as noise, introduce two real classical random fields per field in the system ξ_1^ϕ, ξ_2^ϕ and ξ_1^ψ, ξ_2^ψ, each obeying the Gaussian pdf: Stratonovich 1957, Hubbard 1959

$$\mathcal{P}\left[\xi_1^{\phi,\psi}, \xi_2^{\phi,\psi}\right] = \exp\left\{-\frac{1}{2}\int d^4x d^4x' [\xi_1^{\phi,\psi}(x), \xi_2^{\phi,\psi}(x)] \mathbf{A}^{-1}(x,x') \left[\begin{array}{c} \xi_1^{\phi,\psi}(x') \\ \xi_2^{\phi,\psi}(x') \end{array}\right]\right\},$$

With:

$$\mathbf{A}_\phi^{i,j}(x,x') = \frac{H^2}{2\pi^2}\int \frac{dk}{k}\frac{\sin(kr)}{kr} k^5 \tau\tau' \partial_{k\tau} W_H(k\tau)\, \partial_{k\tau'} W_H(k\tau') \left[1 + \frac{\dot{H}(\tau)}{H^2(\tau)}\right]\left[1 + \frac{\dot{H}(\tau')}{H^2(\tau')}\right] \mathrm{Re}\Big[\mathbf{M}_\phi^{i,j}(k\tau, k\tau')\Big],$$

$$\mathbf{M}_\phi^{i,j}(k\tau,k\tau') = \begin{pmatrix} \phi_{\mathbf{k}}(t)\phi_{\mathbf{k}}^*(t') & q(-k\tau)\,\phi_{\mathbf{k}}(t)\phi_{\mathbf{k}}^*(t') \\ \phi_{\mathbf{k}}(t)\phi_{\mathbf{k}}^*(t')q(-k\tau') & q(-k\tau)\,\phi_{\mathbf{k}}(t)\phi_{\mathbf{k}}^*(t')q(-k\tau') \end{pmatrix},$$

$$\int_{\varphi_i^\pm}^{\varphi_f^\pm}\mathcal{D}\varphi^\pm \int_{\chi_i^\pm}^{\chi_f^\pm}\mathcal{D}\chi^\pm \exp\left[\frac{i}{\hbar}S_{eff}^{(1)}\right]$$
$$= \int \mathcal{D}\varphi^{q,c}\mathcal{D}\chi^{q,c}\int \mathcal{D}\xi_1\mathcal{D}\xi_2\mathcal{P}[\xi_1,\xi_2]\exp\Big(i\int d^4x a^3(t)\,\{\varphi_q\left[\left(\Box - m_\Phi^2\right)\varphi_c - V_{\mathrm{pert},\Phi}(\varphi_c,\chi_c)\right]$$
$$+\chi_q\left[\left(\Box - m_\Psi^2\right)\chi_c - V_{\mathrm{pert},\Psi}(\varphi_c,\chi_c)\right] + \varphi_q\left[p_\phi(t)\xi_1^\phi + \xi_2^\phi\right] - \dot{\varphi}_q\xi_1^\phi + \chi_q\left[p_\psi(t)\xi_1^\psi + \xi_2^\psi\right] - \dot{\chi}_q\xi_1^\psi\}\Big),$$

- To take the classical limit of the action: rescale $\varphi_q, \chi_q \to \hbar\varphi_q, \hbar\chi_q$ and expand in powers of $\hbar$. EoM in the classical limit are given by:

$$\left.\frac{\delta S_{eff}^{(1)}}{\delta\varphi_q}\right|_{\varphi_q=0} = 0 \quad ; \quad \left.\frac{\delta S_{eff}^{(1)}}{\delta\chi_q}\right|_{\chi_q=0} = 0.$$

We obtain:

$$\boxed{\begin{aligned}(-\Box + m_\Phi^2)\varphi_c + \tilde{V}_{,\Phi}(\varphi_c, \chi_c) &= p_\phi(t)\xi_1^\phi + \xi_2^\phi + \dot{\xi}_1^\phi + 3H\xi_1^\phi, \\ (-\Box + m_\Psi^2)\chi_c + \tilde{V}_{,\Psi}(\varphi_c, \chi_c) &= p_\psi(t)\xi_1^\psi + \xi_2^\psi + \dot{\xi}_1^\psi + 3H\xi_1^\psi.\end{aligned}}$$

The noise correlations are found by solving the linearised mode functions:

$$\boxed{\begin{aligned}\Lambda_\phi \phi_k &= 0 \\ \Lambda_\psi \psi_k &= 0\end{aligned}}$$

These are indeed two coupled systems

These are the same as in the heuristic approach, provided we perform a simple redefinition of $\xi_2^{\phi,\psi}$

$$\xi_2^\phi \to -p_\phi(t)\xi_1^\phi - \xi_2^\phi$$

Perturbative Expansion

- Easy to extend this formalism to include non-trivial interacting potential;

- Introduce a current per branch of the CPT contour, J^+ and J^- and define a diagrammatic expansion of *V*, integrate order by order the bath fields, and derive a similar influence action;

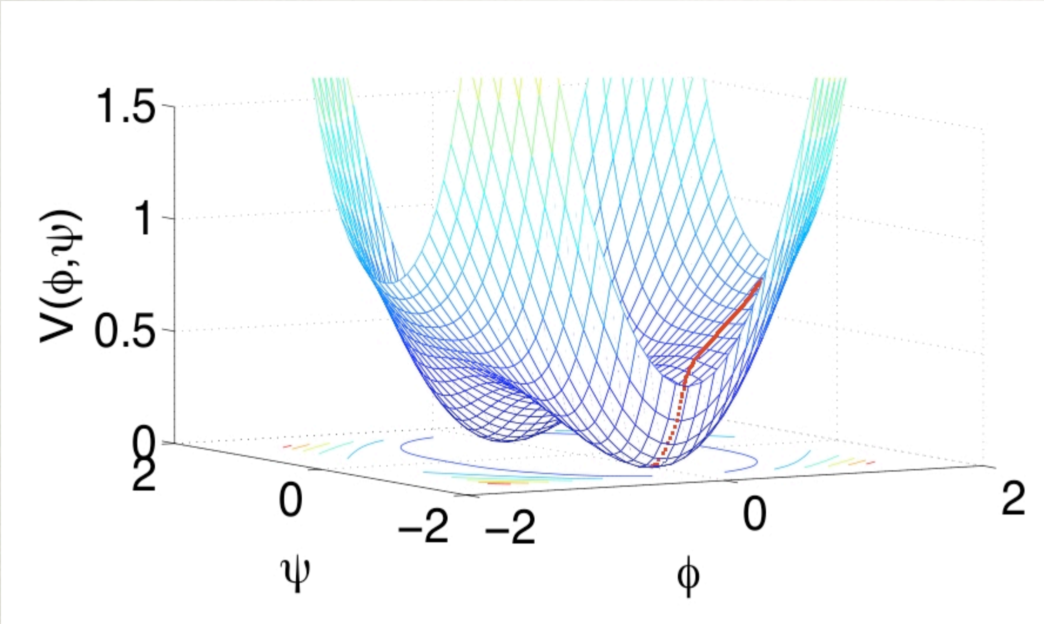
Morikawa 1986, Hu et al. 1993, Boyanovsky 1995

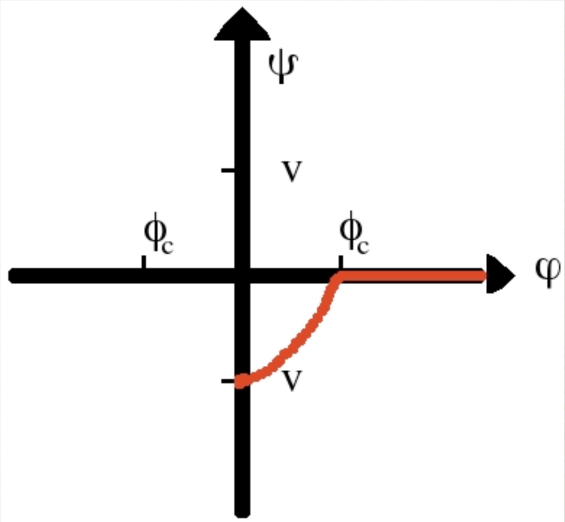
- Quadratic terms in the bath fields are considered as part of the free bath propagator, $e.g.\ \phi_>^2\varphi^2$ coming from $V_{pert} \supset \Phi^4$

$\Rightarrow$ To solve for the noise variance using the full linearised mode function EoM, we obtain 2 coupled system, which justifies a recursive approach

- For every loop correction, we obtain an extra noise term, dissipation term (real and imaginary part of the same kernel), and mass-renormalisation term

Hybrid Inflation





- Two scalar fields inflation, the inflaton ϕ and the waterfall field ψ
- Inflation takes place when ϕ is slowly rolling for $\phi > \phi_c$
- The energy density is dominated by the mass of ψ
- For $\phi < \phi_c$, the $\psi \rightarrow -\psi$ symmetry is broken and ψ develop a tachyonic instability, which trigger its rapid rolling toward a true ground state

Dynamics and Steps 1-2

- Potential:

$$V(\Phi,\Psi)=\frac{1}{2}m^2\Phi^2+\frac{\lambda}{4}(\Psi^2-v^2)^2+\frac{g^2}{2}\Phi^2\Psi^2$$

Recursive Solution:

- Step 1: Free, massless dS noise:

$$\langle\xi_1^{\phi,\psi}(\mathbf{x},t),\xi_1^{\phi,\psi}(\mathbf{x},t)\rangle=\frac{H^3}{4\pi^2}\frac{\sin(\epsilon aHr)}{\epsilon aHr}\delta(t-t')$$

- Step 2: Zeroth order stochastic equations:

$$3H^2\frac{\mathrm{d}\varphi}{\mathrm{d}N}=-m^2\varphi\left(1+\frac{g^2\chi^2}{m^2}\right)+3H\xi_\phi(N)\ ,$$

$$3H^2\frac{\mathrm{d}\chi}{\mathrm{d}N}=-\lambda v^2\chi\left(\frac{\varphi^2-\Phi_{\mathrm{c}}^2}{\Phi_{\mathrm{c}}^2}+\frac{\chi^2}{v^2}\right)+3H\xi_\psi(N)$$

- Step 2 (continued): Zeroth order stochastic solutions: Martin & Vennin 2011

$$\left\langle \chi^2 \right\rangle = \frac{1}{384\pi^2}\frac{\lambda^2 v^8}{m^2 M_{pl}^4}\left(\frac{m^2 \mathrm{e}^x}{\lambda v^2 x}\right)^{\frac{\lambda v^2}{m^2}}\Gamma\left(\frac{\lambda v^2}{m^2}, \frac{\lambda v^2}{m^2}x\right),$$

$$\varphi = \exp\left[-4\frac{m^2 M_{pl}^2}{\lambda v^4}\left(N - N_{in}\right)\right]\left[\varphi_{in} + 2\sqrt{\frac{3}{\lambda}}\frac{M_{pl}}{v^2}\int_{N_{in}}^{N}\exp\left(4\frac{m^2 M_{pl}^2}{\lambda v^4}n\right)\xi_\phi\left(n\right)dn\right],$$

Zeroth order dispersions:

$$\sigma_\varphi = \frac{\lambda v^4}{8\sqrt{6}\pi m M_{pl}^2}.$$

$$\sigma_\chi \equiv \sqrt{\left\langle \chi^2 \right\rangle - \left\langle \chi \right\rangle^2} = \frac{\lambda v^4}{8\sqrt{6}\pi m M_{pl}^2}\left(\frac{m^2 \mathrm{e}^x}{\lambda v^2 x}\right)^{\frac{\lambda v^2}{2m^2}}\Gamma^{\frac{1}{2}}\left(\frac{\lambda v^2}{m^2}, \frac{\lambda v^2}{m^2}x\right),$$

$$\sigma_{\chi_c} \simeq \left(\frac{\lambda}{2\pi}\right)^{3/4}\left(\frac{v}{3m}\right)^{1/2}\frac{v^3}{8M_{pl}^2}.$$

Step 3

Linearised quantum perturbations on a stochastically shifted background:
- Replace coarse-grained quantities with their stochastic mean:

$$F\left[\varphi^{(0)}, \chi^{(0)}\right] \to \left\langle F\left[\varphi^{(0)}, \chi^{(0)}\right]\right\rangle .$$

- We work in the spatially flat gauge;
- The EoM & solution for the canonically normalised field, $\delta\phi_k^{(1)} = a^{-1} v_{\mathbf{k}}$

$$v_{\mathbf{k}}'' + \left[k^2 - \frac{2 - m^2/H^2 - g^2\sigma_\chi^2/H^2 + 9\varepsilon_1}{\tau^2}\right] v_{\mathbf{k}} = 0$$

$$v_{\mathbf{k}} \to \begin{cases} -e^{i(\nu+\frac{1}{2})\frac{\pi}{2}} \frac{2^{\nu-1}}{\sqrt{\pi}} \Gamma(\nu) \frac{(-\tau)^{-\nu+1/2}}{k^\nu} & 0 < \nu \leq 3/2 \\ e^{i\frac{\pi}{2}} (-\tau)^{1/2} \ln(-k\tau) & \nu = 0 \end{cases}$$

$$\nu^2 = 9/4 - (m^2 + g\sigma_\chi^2)/H^2 + 9\varepsilon_1$$

STEP 3 (CONTINUED...)

Similarly, for $\delta\psi_k^{(1)} = a^{-1} u_{\mathbf{k}}$

$$u_{\mathbf{k}}'' + \left[k^2 - m_u^2\left(\tau\right)\right] u_{\mathbf{k}} = 0\,, \quad m_u^2(\tau) \equiv \frac{2 - m_\psi^2/H^2}{\tau^2}$$

$$= \frac{1}{\tau^2}\left[2 + 15\varepsilon_1 - 3\frac{\lambda\sigma_\chi^2}{H^2} - \frac{12M_{pl}^2}{v^2}\left(\frac{{\varphi^{(0)}}^2}{\varphi_c^2} - 1\right)\right]$$

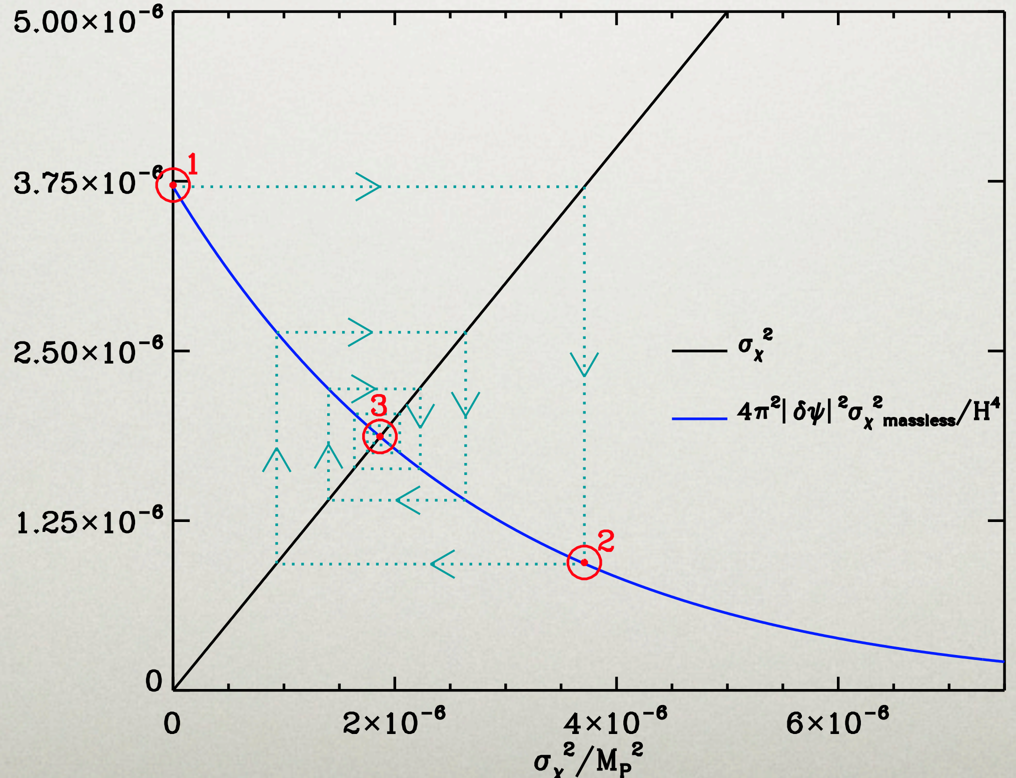
Solution in terms of Airy functions, but not very enlightening to write down...

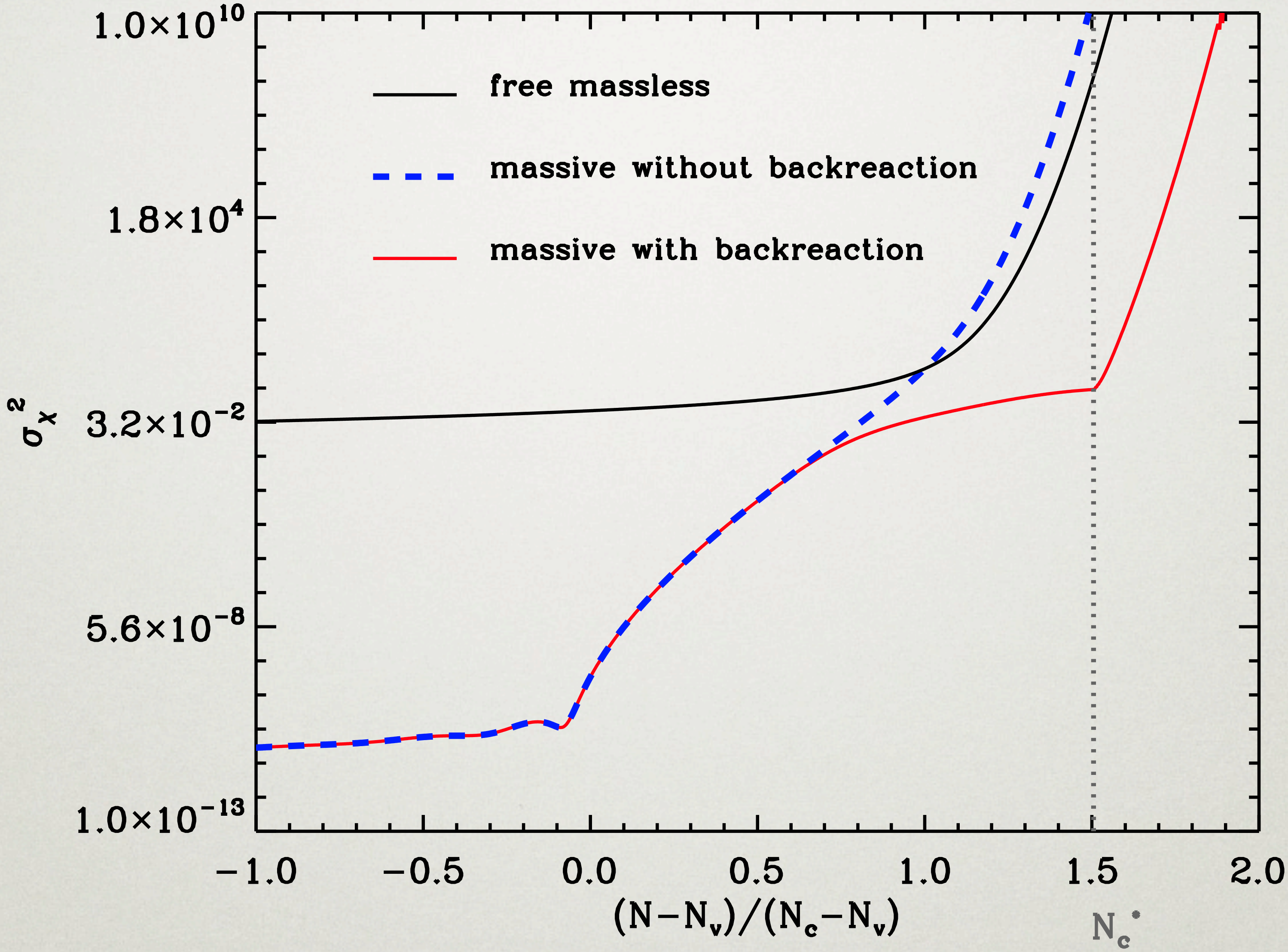
Step 4: Results

Under the quasi-static approximation, i.e. the relaxation time for the χ distribution is very small, and it swiftly acquires its "stationary" local dispersion.

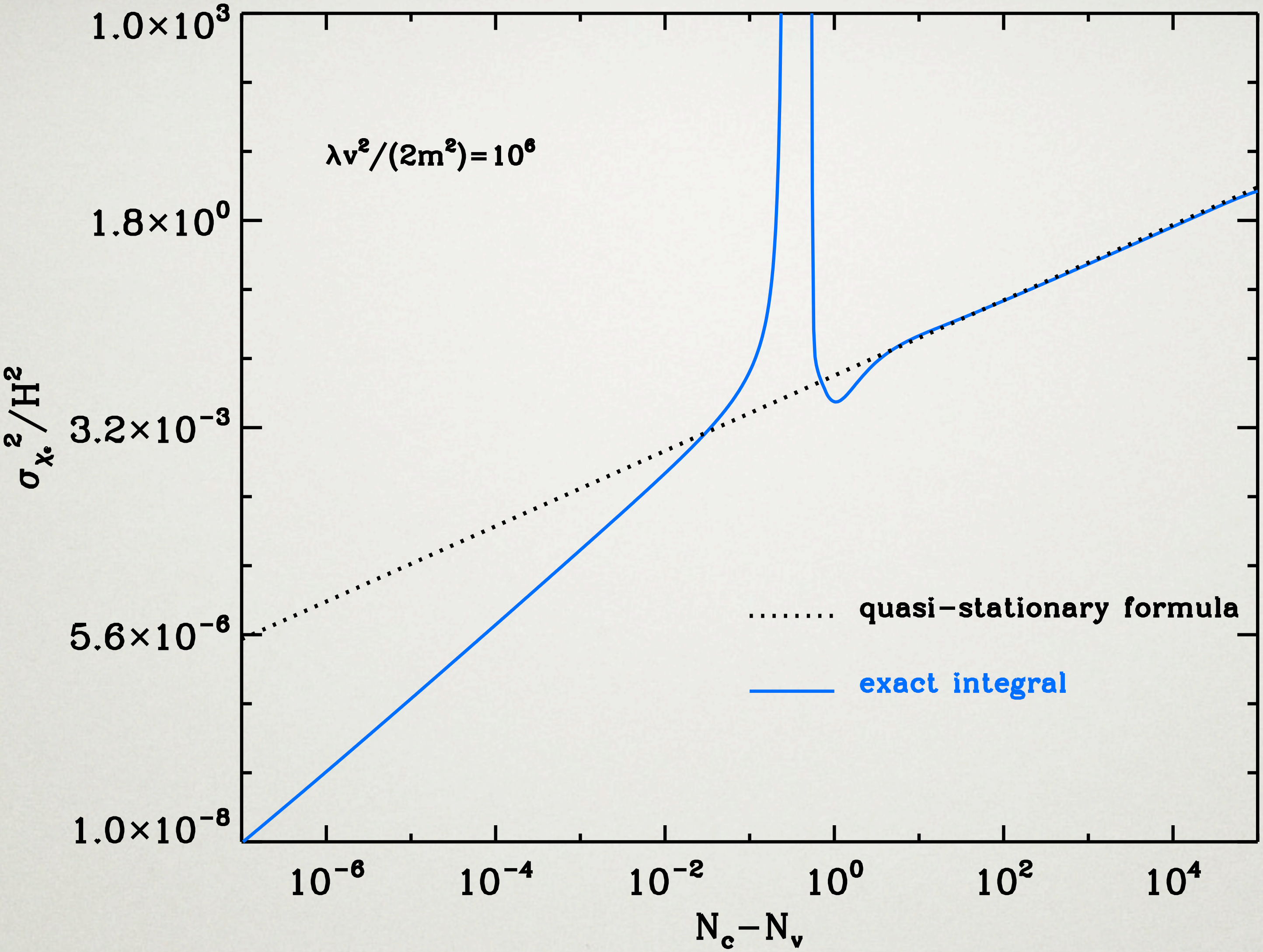
$$\sigma_\chi^2 / \left.\sigma_\chi^2\right|_{\rm massless} \simeq \langle \xi_\psi^2 \rangle / \langle \xi_\psi^2 \rangle_{\rm massless} = |\delta\psi^{(1)}|^2 / |\delta\psi^{(1)}|^2_{\rm massless}$$

$$\Rightarrow \sigma_\chi^2 \simeq \frac{\left|\delta\psi^{(1)}\right|^2}{H^4/(4\pi^2)} \left.\sigma_\chi^2\right|_{\rm massless}$$





free massless
massive without backreaction
massive with backreaction
1.0×10^10
1.8×10^4
3.2×10^-2
5.6×10^-8
1.0×10^-13
σ_χ^2
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
(N−N_v)/(N_c−N_v)
N_c*



$$\sigma^2_{\chi_c} = \frac{9}{8\pi^2}\frac{H^4}{m^2}e^{\frac{\lambda v^2}{2m^2}}\int_1^\infty e^{-\frac{\lambda v^2}{2m^2}(x-\ln x)}\,(\epsilon a H)^3\,\frac{\left|\delta\psi^{(1)}_{\mathbf{k}}\right|^2_{k=\epsilon a H}}{H^2}\frac{dx}{x}$$

For the inflaton:

- Noise amplitude

$$\langle \xi_\phi(N)\,\xi_\phi(N')\rangle = \frac{H^4}{4\pi^2}\delta(N-N')\left[1+\frac{2}{3}\frac{m^2+g\sigma_\chi^2}{H^2}(\ln 2\epsilon+\gamma-2)\right],$$

- Classical perturbations

$$\langle(\delta\varphi^{(1)})^2\rangle \approx \frac{3H^4\varphi_0^2}{8\pi^2\tilde{m}^2}\left(1-\frac{\varphi_0^2}{(\varphi_0)^2_{in}}\right)\left(1+\frac{2}{3}\frac{A}{H^2}\right)$$

$$\boxed{\Rightarrow \left|\delta\varphi_k^{(1)}\right|^2 \approx \left(\frac{k}{aH}\right)^{\frac{2\tilde{m}^2}{3H^2}-\frac{4}{9}\frac{\tilde{m}^4}{H^4}(\ln 2\epsilon+\gamma-2)}}$$

with:

$$A = \tilde{m}^2(\ln 2\epsilon+\gamma-2) \qquad \tilde{m}^2 = (m^2+g^2\sigma_\chi^2)$$

Conclusion

- Reviewed stochastic inflation starting from the EoM
- Proposed a recursive approach
- Showed how this is motivated from the microphysics of stochastic inflation
- Applied the recursive to derive new results in Hybrid inflation (tilt, dispersion of the waterfall field...)