Curvature Singularities from Gravitational Contraction in $f(R)$ Gravity

Lorenzo Reverberi

PhD Student

Università di Ferrara and INFN Sezione di Ferrara (Italy)

SW7 – HOT TOPICS IN MODERN COSMOLOGY
Cargèse, Corsica (France)
6-10 May 2013

- LR, *Phys. Rev.* **D** 87, 084005 (2013) [arXiv:1212.2870]

Dark Energy

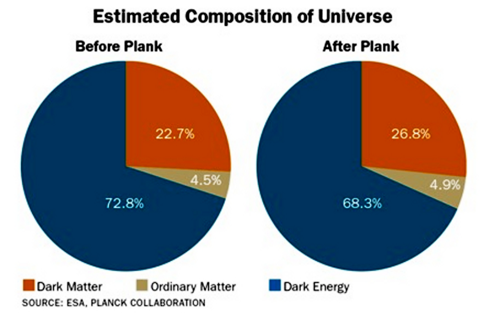
As is widely known, the Universe is almost completely dominated by unknown forms of matter/energy: **Dark Matter** and **Dark Energy**.
Theoretically, Dark Energy is quite natural both in QFT ($\sim$ vacuum energy) and in GR (Λ-term). However,

Coincidence Problem

why $\Omega_\Lambda \sim \Omega_m \sim \mathcal{O}(1)$?
$(d\Omega_\Lambda/dN)_{today} \simeq$ max?

Smallness Problem

$$\Lambda_{obs} \sim 10^{-123}\, m_{Pl}^4 \sim 10^{-44}\, \Lambda_{QCD}^4$$



Alternative: Dark Energy is gravitational but **dynamical**, originated by modifications of the Einstein-Hilbert action:

$$A_{grav} \sim -\int d^4x \sqrt{-g}\,(R + 2\Lambda) \to -\int d^4x \sqrt{-g}\, f(R)$$

The modified theory has an additional **massive scalar** degree of freedom (**scalaron**).

$f(R)$ models of Dark Energy

Action $A_{grav} \sim -\int d^4x \sqrt{-g}\, f(R) \equiv -\int d^4x \sqrt{-g}\, [R + F(R)]$

Field Eqs $(1 + F'_R)\, R_{\mu\nu} - \frac{1}{2}[R + F(R)]\, g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu)\, F'_R = T_{\mu\nu}$

Trace $3\Box F' + RF' - 2F - R = T$

In order to generate an effective Λ, one needs constant (non-zero) curvature solutions, i.e. roots of

$$RF' - 2F - R = 0$$

First models proposed:

- $F \sim \frac{1}{R}$ (Capozziello et al. 2003, Carroll et al. 2004)

  Due to the negative power of R, corrections start dominating when $R \to 0$. However: **strong instabilities** in the presence of gravitating bodies.

Near $R = 0$, corrections must be at most $\sim \Lambda + R^2$ (Olmo, *PRL* 2005)

Recent Proposals

Stability conditions (Amendola et al. 2007, Sawicki and Hu 2007)

- $f' > 0$ **(graviton $\neq$ ghost)**
- $f'' < 0$ **(scalaron $\neq$ tachyon)**

Hu-Sawicki 2007

$$F_{HS} = -\lambda R_c \left[1 + \left(\frac{R_c}{R} \right)^{2n} \right]^{-1}$$

Starobinsky 2007

$$F_S = -\lambda R_c \left[1 - \left(1 + \frac{R^2}{R_c^2} \right)^{-n} \right]$$

- evade Solar System and cosmological tests
- $F(0) = 0 \quad \Rightarrow \quad$ vacuum = Minkowski ("disappearing cosmological constant")
- at large $|R|$, $|F'| \ll 1$ and $F \simeq -\lambda R_c$ so

$$\Lambda_{\text{eff}} \simeq -\frac{\lambda R_c}{2}$$

(from now on $\lambda = 1$, $R_c = -\Lambda$)

Contracting Astronomical Objects

Let us consider a cloud under the following conditions:

- **"high" density**: $\rho_m \gg \rho_c \sim 10^{-29}$ g cm^{-3} $(|R/R_c| \gg 1)$
- **low gravity**: $|g_{\mu\nu} - \eta_{\mu\nu}| \ll 1 \quad \Rightarrow \quad \nabla_\mu \to \partial_\mu$
- **spherical symmetry + homogeneity** $\quad \Rightarrow \quad$ no space derivatives
- **pressureless dust**: $T = 8\pi\rho_m/m_{Pl}^2$ (not necessary but reasonable)

For simplicity, we will assume the contraction law

$$T(t) = T_0 \left(1 + \frac{t}{t_{contr}}\right)$$

$$\rho_{29} \equiv \frac{\rho_0}{10^{-29}\ \text{g cm}^{-3}} \gg 1$$

$$t_{10} \equiv \frac{t_{contr}}{10^{10}\ \text{years}}$$

This should be more or less reliable at least up to $t \sim t_{contr}$.

Negligible Pressure

$$\frac{p}{\rho} \sim \frac{v^2}{3c^2} \simeq 10^{-9}\, M_{11}\, t_{10}^{-2}\, \rho_{29}^{-2/3} \ll 1$$

$$M_{11} \equiv \frac{M_{cloud}}{10^{11} M_\odot}$$

Low Gravity

Taking for definiteness $g_{\mu\nu} - \eta_{\mu\nu} = \text{diag}(0, -\psi)$, we have $R \simeq -3\ddot{\psi}$. It can be explicitly proved that $|\psi| < 1$ in all cases considered (see later).

$f(R)$ Dynamics

For $|R/R_c| \gg 1$, $F \approx 2\Lambda$ and $|F'| \ll 1$ so we can recast the trace equation as a simple **time-dependent oscillator** equation:

$$\ddot{\xi} + R + T = 0 \qquad \Leftrightarrow \qquad \ddot{\xi} + \frac{\partial U}{\partial \xi} = 0$$

- **Scalaron**: $\boldsymbol{\xi} \equiv -3F' = 6n\lambda \left(\frac{R_c}{R}\right)^{2n+1}$
- reabsorb Λ in the definition of T: $T \to T + 4\Lambda$
- solutions oscillate around the GR solution $R = -T$, with frequency:

$$\omega_\xi^2 = \frac{\partial^2 U}{\partial \xi} \simeq -\frac{R_c}{6n(2n+1)\lambda}\left(-\frac{T}{R_c}\right)^{2n+2} > 0$$

- energy conservation, modified by the explicit time dependence of U:

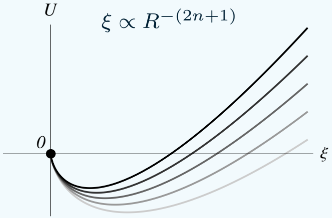
$$\frac{1}{2}\dot{\xi}^2 + U(\xi) - \int^t dt' \, \frac{\partial T}{\partial t'} \, \xi(t') = \text{ const}$$

SINGULARITY

$$R \to \infty \qquad \text{for} \qquad \xi = 0$$

Along the GR solution we have $\xi \propto T^{-(2n+1)} \neq 0$ but oscillations may allow $\xi = 0$!

Scalaron Potential



$$U(\xi) = T\,\xi - 3(2n+1)|R_c| \left(\frac{\xi}{6n}\right)^{\frac{2n}{2n+1}}$$

- bottom corresponds to the GR solution $R = -T$
- not symmetric around the position of the bottom
- for increasing T, the bottom rises: $U_0 = -3n\lambda\, R_c\, |R_c/T|^{2n}$
- potential is finite for $\xi = 0 \;\Leftrightarrow\; R \to \infty$
- ξ oscillates with frequency ω; the potential changes on a timescale t_{contr}:

$$\omega_0\, t_{contr} \simeq 0.5\, \frac{\rho_{29}^{n+1}\, t_{10}}{\sqrt{(2n+1)n\lambda}}$$

"Slow-Roll" Regime: $\omega\, t_{contr} \ll 1$

Oscillations are slow w.r.t. changes of the potential, so the motion of ξ is mainly driven by changes of U (and initial conditions if $\dot{\xi}_0 \neq 0$)

"Fast-Roll" Regime: $\omega\, t_{contr} \gg 1$

Oscillations are fast, so they are practically **adiabatic**. Near a given time t, ξ oscillates between two values ξ_{min} and ξ_{max} with roughly $U(\xi_{min}) = U(\xi_{max})$.

Slow-Roll Regime

Let us first consider the slow-roll regime, that is $\omega_0 \, t_{contr} \ll 1$. The initial "velocity" of the field dominates over the acceleration due to the potential, so in first approximation

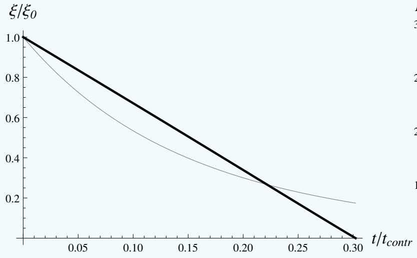
$$\xi(t) = \xi_0 + \dot{\xi}_0 t$$

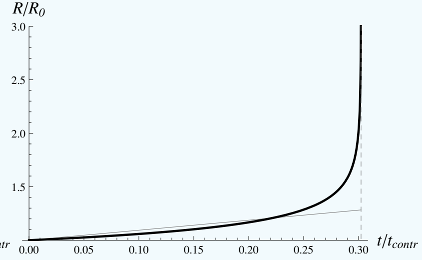
This can also be understood as follows:

$$\ddot{\xi} \sim \frac{\xi}{t_{contr}^2}\,, \qquad R + T = \frac{\partial U}{\partial \xi} \sim \omega^2 \xi \quad \Rightarrow \quad \frac{\ddot{\xi}}{R+T} \sim \frac{1}{\omega^2\, t_{contr}^2} \gg 1$$

Therefore the trace equation reduces to

$$\ddot{\xi} + R + T \simeq \ddot{\xi} = 0$$





Initial Conditions

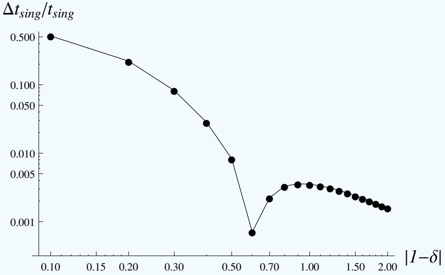
$$R_0 = -T_0 \qquad \dot{R}_0 = -(1-\boldsymbol{\delta})\dot{T}_0$$

The singularity appears when $\xi = 0$, that is at

Singularity – Critical T and t

$$\frac{t_{sing}}{t_{contr}} = -\frac{\xi_0}{\dot{\xi}_0} \simeq \frac{1}{(2n+1)|1-\delta|} \qquad \frac{T_{sing}}{T_0} = 1 + \frac{1}{(2n+1)|1-\delta|}$$

$$\begin{array}{l} n = 3 \\ \rho_{29} = 30 \\ t_{10} = 10^{-5} \end{array} \quad \Rightarrow \quad \frac{t_{sing}}{t_{contr}} \sim \mathcal{O}(1-0.1)$$



"Cusp" due to change in sign of $\Delta t/t$. Precision is outstanding, given the relatively large $\omega_0\, t_{contr} \simeq 1$. Taking

$$\begin{cases} n = 3 \\ \rho_{29} = 100 \\ t_{10} = 10^{-9} \\ \delta = 0 \end{cases} \quad \Rightarrow \quad \omega_0\, t_{contr} \simeq 10^{-2}$$

yields $\Delta t_{sing}/t_{sing} \simeq 10^{-7}$.

But: **short contraction timescales** $\Rightarrow$ (maybe) **unphysical**.

We expand ξ around its "GR value" ξ_a, defined by $R(\xi_a) = -T$, assuming small perturbations so that the potential $\sim$ harmonic

$$\xi = \xi_a + \xi_1 \equiv \xi_a + \alpha \sin \int^t \omega \, dt' \qquad R = -T + R_1 \equiv -T + \beta \sin \int^t \omega \, dt'$$

Expanding the trace equation at first order in ξ_1 and using the fast-roll condition, we obtain

$$\ddot{\xi}_1 + \omega^2 \xi_1 \simeq 0 \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad \alpha \sim \omega^{-1/2}$$

Using initial conditions specifying R_0 and $\dot{R}_0$, and therefore $\xi_0(R_0, \dot{R}_0)$ and $\dot{\xi}_0(R_0, \dot{R}_0)$, we find

$$|\alpha| = \frac{|\dot{\xi}_0 - \dot{\xi}_{a,0}|}{\omega_0} \left(\frac{\omega}{\omega_0} \right)^{-1/2}$$

Explicit Solutions

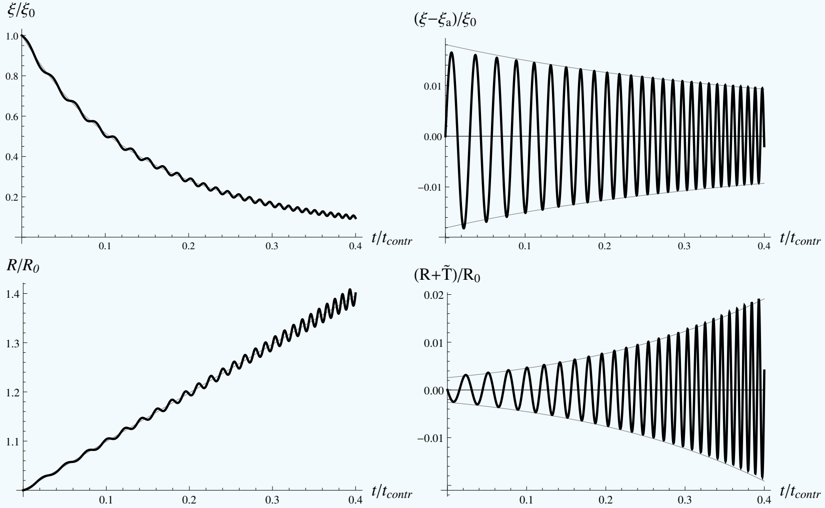
$$\alpha \simeq \frac{[6(2n+1)n]^{3/2} |\delta| |R_c|^{3n+\frac{3}{2}}}{T_0^{\frac{5n+3}{2}} \, t_{contr}} \, T(t)^{-\frac{n+1}{2}}$$

$$\beta = \omega^2 \alpha \simeq \frac{\sqrt{6n(2n+1)} |\delta| |R_c|^{n+\frac{1}{2}}}{T_0^{\frac{5n+3}{2}} \, t_{contr}} \, T(t)^{\frac{3n+3}{2}}$$

Problem: $\delta = 0$ gives $\alpha = \beta = 0$.
But: we are neglecting terms of order $\sim (\omega \, t_{contr})^{-1}$!
Oscillations are **always** excited if $\dot{T} \neq 0$.

Fast-Roll Regime

Harmonic Oscillations – Solutions ($n = 3$, $\delta = 0.5$, $\rho_{29} = 200$, $t_{10} = 10^{-6}$)



$$(\, \omega_0 \, t_{contr} \simeq 180 \gg 1 \,)$$

The singularity should be reached when

Singularity Condition I

amplitude of oscillations $=$ $\alpha(t) = \xi_a(t)$ $=$ distance from singular point

However, when α becomes of the order of ξ_a, the field starts "feeling" the anharmonicity of the potential, which results in an asymmetry of the oscillations around $\xi = \xi_a$. We can define, at each oscillation,

$$\xi_{min} \equiv \xi_a - \alpha_- \qquad \text{with} \qquad \alpha_- \neq \alpha_+ \qquad (\neq \text{ harm. case})$$

$$\xi_{max} \equiv \xi_a + \alpha_+ \qquad\qquad U(\xi_{min}) \simeq U(\xi_{min}) \qquad (\text{still} \sim \text{adiabatic!})$$

Singularity Condition II

$$U(\xi_a) + \Delta U = U(\xi_{sing}) = 0$$

Comparing the system to a classical oscillator, we have

$$\Delta U = \frac{1}{2}\,\dot{\xi}^2\Big|_{max} \simeq \frac{1}{2}\,\alpha^2\omega^2 = \begin{array}{l}\text{max. kinetic energy near given } t\\ \text{(within one oscillation)}\end{array}$$

Here, α, ω = as in the harmonic region! ΔU comes into play in energy conservation, which does not care about harmonic/anharmonic oscillations!

Expanding U near the singularity $\xi = 0$ and imposing the condition

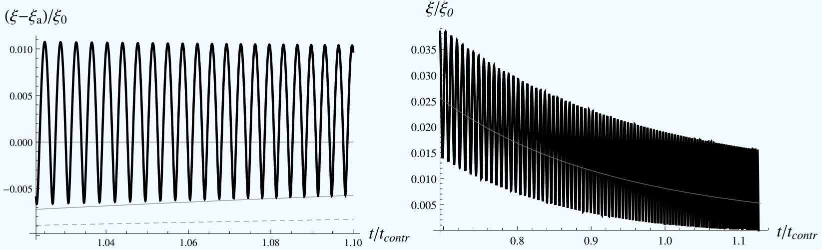
$$U(\xi_a - \boldsymbol{\alpha_-}) = U(\xi_a) + \frac{1}{2}\,(\alpha^2\omega^2)_{\textsf{harm}}$$

yields the solution

$$\boldsymbol{\alpha_- = \xi_a - 6n\left[\frac{U(\xi_a) - \Delta U}{3(2n+1)R_c}\right]^{\frac{2n+1}{2n}}}$$

$$U(\xi_a) = 3R_c\left(-\frac{R_c}{T}\right)^{2n}$$

$$\Delta U = \frac{18[(2n+1)n]^2\delta^2|R_c|^{4n+2}\,T^{n+1}}{T_0^{5n+3}\,t_{contr}^2}$$



Anharmonic Features – Solutions ($n = 3$, $\delta = 0.5$, $\rho_{29} = 200$, $t_{10} = 5 \cdot 10^{-7}$)

Expanding U near the singularity $\xi = 0$ and imposing the condition

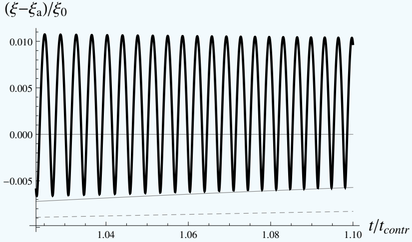
$$U(\xi_a - \boldsymbol{\alpha}_-) = U(\xi_a) + \frac{1}{2}\,(\alpha^2\omega^2)_{\text{harm}}$$

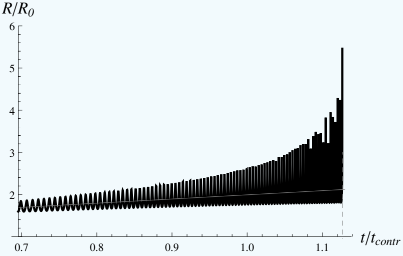
yields the solution

$$\boldsymbol{\alpha_- = \xi_a - 6n\left[\frac{U(\xi_a) - \Delta U}{3(2n+1)R_c}\right]^{\frac{2n+1}{2n}}}$$

$$U(\xi_a) = 3R_c\left(-\frac{R_c}{T}\right)^{2n}$$

$$\Delta U = \frac{18[(2n+1)n]^2\delta^2|R_c|^{4n+2}\,T^{n+1}}{T_0^{5n+3}\,t_{contr}^2}$$





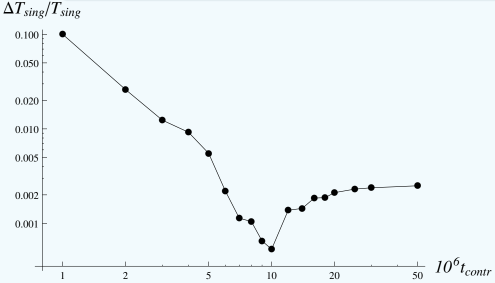
Fast-Roll Regime – Singularity

Collecting all results, the singularity condition $U(\xi_a) + \Delta U = 0$ (or equivalently $\alpha_- = \xi_a$) gives the critical T at which the singularity appears:

Singularity – Critical T and t

$$\frac{T_{sing}}{T_0} = \left[\frac{T_0^{2n+2}\, t_{contr}^2}{6n^2(2n+1)^2\,\delta^2 |R_c|^{2n+1}}\right]^{\frac{1}{3n+1}} \simeq \left[0.28\,\frac{\rho_{29}^{2n+2}\, t_{10}^2}{n^2(2n+1)^2\,\delta^2}\right]^{\frac{1}{3n+1}}$$

$$\frac{t_{sing}}{t_{contr}} = \frac{T_{sing}}{T_0} - 1$$



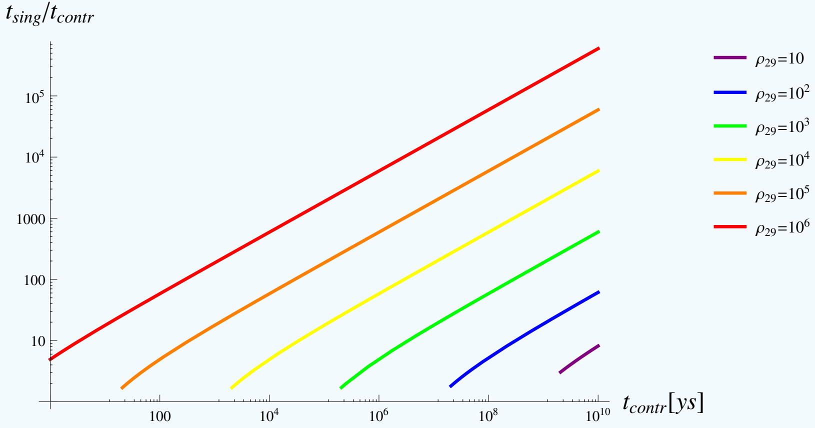
$n = 3$, $\rho_{29} = 10^2$, $\delta = 0.5$ $\Rightarrow$ $\frac{t_{sing}}{t_{contr}} \sim \mathcal{O}(1)$

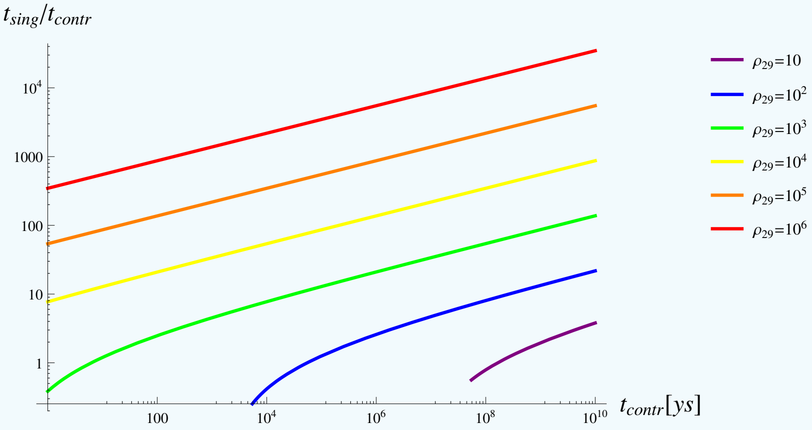
Relative errors tend to constant value $\sim 2 \cdot 10^{-3}$ (maybe numerical feature?). "Cusp" due to change in sign of $\Delta T/T$. Precision is nevertheless satisfactory.
Computational time proportional to total number of oscillations:

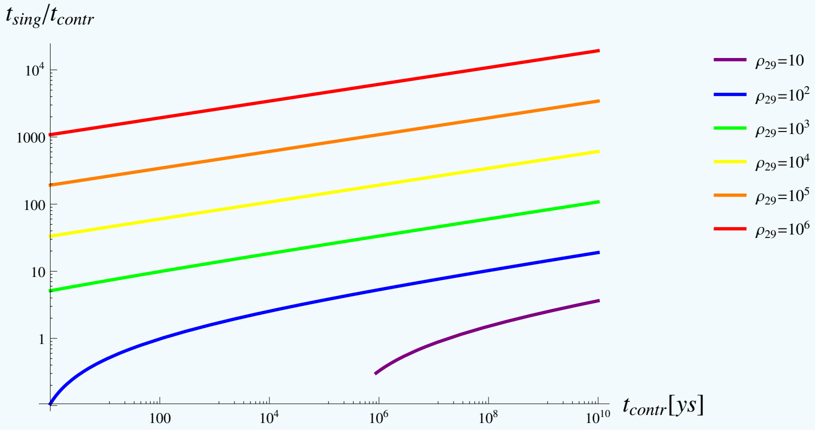
$$N_{osc} \sim \int^{t_{sing}} \omega\, dt \propto \left(\rho_{29}^{n+1}\, t_{10}\right)^{\frac{5n+5}{3n+1}}$$

Large ρ_{29} = expect better agreement, but difficult to test numerically.

Fast-Roll Regime – Results ($n = 1$)




t_{sing}/t_{contr}
10^4
1000
100
10
1
100
10^4
10^6
10^8
10^{10}
$t_{contr}[ys]$
ρ_{29}=10
ρ_{29}=10^2
ρ_{29}=10^3
ρ_{29}=10^4
ρ_{29}=10^5
ρ_{29}=10^6



t_{sing}/t_{contr}
10^4
1000
100
10
1
100
10^4
10^6
10^8
10^{10}
$t_{contr}[ys]$
$\rho_{29}=10$
$\rho_{29}=10^2$
$\rho_{29}=10^3$
$\rho_{29}=10^4$
$\rho_{29}=10^5$
$\rho_{29}=10^6$

About the Low-Gravity Approximation

Let us take for definiteness the line element

$$ds^2 = dt^2 - (1+\psi)\, d\mathbf{x}^2 \qquad |\psi| \ll 1$$

so that

$$R \simeq -3\ddot{\psi}$$

Therefore, using $R = -T + R_{osc}$,

$$\psi_{max} \simeq \frac{1}{3}\int^{t_{sing}} dt \int^{t} dt'\, T \simeq 0.28\, \rho_{29}\, t_{10}^2\, x^2 \left(1 + \frac{x}{3}\right) \qquad x \equiv \frac{t_{sing}}{t_{contr}}$$

Slow-Roll

We always have $x < 1$, so

$$\psi \sim \rho_{29}\, t_{10}^2\, x \sim (\omega_0\, t_{contr})^2 \rho_{29}^{-(2n+1)}\, x \ll 1$$

Fast-Roll

- $x \lesssim 1$: the condition $x < 1$ gives

$$\rho_{29}^{2n+2}\, t_{10}^2 \lesssim \mathcal{O}(1) \quad \Rightarrow \quad \psi \sim \rho_{29}\, t_{10}^2 \ll 1$$

- $x \gtrsim 1$: imposing $\psi \lesssim 1$ yields $\rho_{29}\, t_{10}^2\, x^3 \lesssim 10$, or

$$\rho_{29}^{9n+7}\, t_{10}^{6n+8} \lesssim 4 \cdot 10^{3n+1}\, [n(2n+1)\delta]^6$$

One can check that this is fulfilled for all presented results.

Summary and Conclusions

- **Dark Energy** is a central problem in modern physics. Answer: particle physics? gravitation?
- there exist $f(R)$ models capable of generating an effective DE which survive **cosmological** and **Solar System** tests
- in such models, the additional **scalaron** field ξ moves in a potential in which the singular point $\xi(R \to \infty)$ is in principle accessible
- in contracting systems, the increasing energy/mass density excites oscillations of ξ which may push the field towards the singularity
- the interplay between the oscillation frequency ω and the typical contraction time of the system t_{contr} determines two regimes:
  - **Slow Roll**: the singularity is reached rapidly, with t_{sing}/t_{contr} at most of order unity
  - **Fast Roll**: depending on parameters, the singularity can be reached with $t_{sing}/t_{contr} \lesssim 1$ or $\gg 1$
- **"Naive" approach**: constrain models using real observational data
- **large R** $\Rightarrow$ high-curvature corrections: evaluate their contribution (particle production – see talk by E. Arbuzova, etc.)