From Brans-Dicke gravity to geometric scalar-tensor theory

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Basic features of Einstein's gravity that make it a geometrical theory

- The assumption that gravity is not an ordinary force, but rather a property of space-time geometry.

This is followed by two postulates:

- The principle of equivalence, which in mathematical terms corresponds to the so-called **geodesic postulate**.
- The **field equations** that determine how matter curves space-time.

In the words of American physicist John Wheeler, **"space tells matter how to move, and matter tells space how to curve"**

However, as we know, these requirements do not determine a unique theory of gravity. Many other theories share the same features.

- One example is **Brans-Dicke scalar-tensor theory**.

Brans-Dicke action

The action in the so-called Jordan frame is

$$S^{(BD)} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[\phi R + \frac{\omega}{\phi} \phi_{,\nu} \phi^{,\nu} - V(\phi) \right] + S_m, \quad (1)$$

where

$$S_m = \int d^4x \sqrt{-g} \mathcal{L}_m. \quad (2)$$

BD scalar field is interpreted as

$$G_{eff}(\phi) = \frac{1}{\phi}. \quad (3)$$

Palatini formalism

- Consider the metric and the connection as independent variables.
  The Riemann and the Ricci tensor are constructed with the independent connnection.
- Assume that matter action does not depend on the connection.

Field equations

BD action [Brans-Dicke,1961]

$$S_G = \int d^4x\sqrt{-g}(\Phi R + \frac{\omega}{\Phi}\Phi^{,\alpha}\Phi_{,\alpha}). \tag{4}$$

Performing the variable transformation

$$\Phi = e^{-\phi}, \tag{5}$$

we have

$$S_G = \int d^4x\sqrt{-g}e^{-\phi}(R + \omega\phi^{,\alpha}\phi_{,\alpha}). \tag{6}$$

Variation with respect to the independent connection

Compatibility condition between the metric and the connection

$$\nabla_\alpha g^{\mu\nu} = -\phi_{,\alpha} g^{\mu\nu}. \tag{7}$$

It is associated to **Weyl integrable geometry**.

- **Extended Palatini variational method**: consider variation of the action also with respect to the scalar field.

Weyl's integrable geometry

- The geometry conceived by Weyl can be considered as a simple generalization of the Riemann geometry.
- Instead Riemannian compatibility condition

$$\nabla_\alpha g_{\mu\nu} = 0, \tag{8}$$

we consider

$$\nabla_\alpha g_{\mu\nu} = \sigma_\alpha g_{\mu\nu}, \tag{9}$$

with σ_α a one-form field.

- If $\sigma = d\phi$, ϕ scalar field, we have an integrable Weyl geometry.
- From (9), the components of the affine conection

$$\Gamma^\alpha_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\} - \frac{1}{2} g^{\alpha\beta} \left(g_{\beta\mu}\phi_{,\mu} + g_{\beta\nu}\phi_\mu - g_{\mu\nu}\phi_\beta \right) \tag{10}$$

- The set (M, g, ϕ) is called *Weyl frame*.
- The particular case $(M, \bar{g}, 0)$ is named *Riemann frame*.
- Performing the transformations

$$\bar{g}_{\mu\nu} = e^{-f} g_{\mu\nu}, \qquad (11)$$
$$\bar{\phi} = \phi - f,$$

  the new frame $(M, \bar{g}, \bar{\phi})$ is reached.
- The geodesic equations remain invariant under Weyl transformations.
- It is always possible to get a Riemannian frame by choosing $f = \phi$.

Matter action

Brans-Dicke matter action in Palatini formalism

$$S_m = \int d^4x\sqrt{-g}\mathcal{L}_m, \qquad (12)$$

- does not couple to the scalar field,
- does not satisfy the Weak Principle of Equivalence.

Rewriting the compatibility condition

$$\nabla_\alpha(e^\phi g^{\mu\nu}) = 0. \tag{13}$$

Effective metric is defined

$$\gamma^{\mu\nu} = e^\phi g^{\mu\nu}. \tag{14}$$

We assume minimal coupling to the effective metric,

$$\begin{aligned} S_m &= \kappa \int d^4x\sqrt{-\gamma}L_m(\gamma_{\mu\nu}, \Psi, \nabla^{(\gamma)}\Psi) \\ &= \kappa \int d^4x\sqrt{-g}e^{-2\phi}L_m(e^{-\phi}g_{\mu\nu}, \Psi, \nabla^{(e^{-\phi}g)}\Psi). \end{aligned} \tag{15}$$

Field equations for the metric g

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa T_{\mu\nu} - \omega(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi^{,\alpha}\phi_{,\alpha}), \qquad (16)$$

where

$$T_{\mu\nu} = \frac{1}{\sqrt{-\gamma}}\frac{\delta(\sqrt{-\gamma}L_m)}{\delta\gamma^{\mu\nu}}, \qquad (17)$$

and

$$\frac{\delta S_m}{\delta g^{\mu\nu}}\delta g^{\mu\nu} = \int d^4x\sqrt{-g}e^{-\phi}T_{\mu\nu}\delta g^{\mu\nu}, \qquad (18)$$

Field equation for ϕ

$$R + 3\omega\phi^{,\alpha}\phi_{,\alpha} + 2\omega\Box\phi = \kappa T, \tag{19}$$

where $T = g^{\mu\nu}T_{\mu\nu}$ and

$$\frac{\delta S_m}{\delta\phi}\delta\phi = -\int d^4x\sqrt{-g}e^{-\phi}T\delta\phi. \tag{20}$$

The trace of (16) with respect to $g_{\mu\nu}$

$$R + \omega\phi^{,\alpha}\phi_{,\alpha} = \kappa T, \tag{21}$$

in (19) leads to

$$\Box\phi + \phi^{,\alpha}\phi_{,\alpha} = 0. \tag{22}$$

Riemann frame picture

Weyl transformation with $f = \phi$ leads to $(M, \gamma = e^{\phi} g, 0)$.
The transformed BD action

$$S = \int d^4 x \sqrt{-\gamma} (\tilde{R} + \omega \phi^{,\alpha} \phi_{,\alpha}) + S_m(\gamma, \Psi, \nabla^{\gamma} \Psi), \qquad (23)$$

and field equations

$$\tilde{R}_{\mu\nu} - \frac{1}{2} \gamma_{\mu\nu} \tilde{R} = -\kappa T_{\mu\nu}(\gamma) - \omega(\phi_{,\mu} \phi_{,\nu} - \frac{1}{2} \gamma_{\mu\nu} \phi^{,\alpha} \phi_{,\alpha}), \qquad (24)$$

$$\tilde{\Box} \phi = 0. \qquad (25)$$

With $\varphi = \sqrt{2\omega}\phi$, field equations for Einstein's theory of gravity minimally coupled with a massless scalar field are obtained.

Field equations solution

The static, spherically symmetric and asymptotically flat solution (Fisher, 1948)

$$ds^2 = W(r)^S dt^2 - W(r)^{-S} dr^2 - r^2 W(r)^{1-S} d\Omega, \qquad (26)$$

$$\phi = \frac{1}{\sqrt{2\omega}} \frac{\Sigma}{\eta} \ln |W(r)| \qquad (27)$$

$$W(r) = 1 - \frac{r_0}{r} \qquad (28)$$

where $S = \frac{M}{\eta}$, $r_0 = 2\eta$, $\eta = \sqrt{M^2 + \Sigma^2}$ and $M > 0$ is the body's mass in the center of coordinates.

The post-Newtonian approximation

- Fisher space-time predicts the same effects on solar-system experiments as the Schwarzschild one does.[J. B. Formiga, 2011]
- Invariance of geodesic equations assure the same results in Weyl frame.

Simple cosmological model

Robertson-Walker metric

$$ds^2 = -dt^2 + A^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\psi^2\right). \quad (29)$$

Fields equations in the Riemann frame

$$3\frac{\dot{A}^2 + k}{A^2} = \kappa T_{tt}(\gamma) + \frac{\omega}{2}\dot{\phi}^2, \quad (30)$$

$$\frac{2A\ddot{A} + \dot{A}^2 + k}{A^2}\gamma_{jj} = -\kappa T_{jj}(\gamma) - \frac{\omega}{2}\dot{\phi}^2\gamma_{jj}, \quad (31)$$

,

$$\ddot{\phi} = -3\frac{\dot{A}}{A}\dot{\phi}. \quad (32)$$

For vacumm and $k = 0$,

$$2\frac{\dot{A}^2}{A^2} = -\frac{\ddot{A}}{A}, \tag{33}$$

with solution

$$A(t) = A_0(3H_0 t + 1)^{\frac{1}{3}}, \tag{34}$$

$$\phi_{\pm}(t) = \phi_0 \pm \sqrt{\frac{2}{3\omega}} \ln(3H_0 t + 1), \tag{35}$$

where $\omega > 0$, A_0 and ϕ_0 are integration constants and H_0 is the Hubble's parameter.

Applying the Weyl transformations, the solutions in the Weyl frame are obtained.

$$\tilde{A}_{\pm}(\tau) = A_0 e^{\phi_0/2} \left\{ 3H_0 e^{-\phi_0/2} \frac{1 \pm \sqrt{6\omega}}{\sqrt{6\omega}} \tau + 1 \right\}^{\frac{(6\omega,+3) \pm 2\sqrt{6\omega}}{6\sqrt{6\omega} \pm 3(6\omega+1)}} \tag{36}$$

$$\phi_{\pm}(\tau) = \phi_0 + \frac{2}{\sqrt{1 \pm 6\omega}} \ln \left\{ 3H_0 e^{-\phi_0/2} \frac{1 \pm \sqrt{6\omega}}{\sqrt{6\omega}} \tau + 1 \right\}, \tag{37}$$

where τ is the proper time in the Weyl frame.

Field equations for the RW metric in Weyl frame

$$3\frac{\tilde{A}'^2}{\tilde{A}^2} - 3\frac{\tilde{A}'}{\tilde{A}}\phi' + \frac{1}{2}\left(\frac{3}{2} - \omega\right)\phi'^2 + 3\frac{k}{\tilde{A}^2} = \kappa T_{\tau\tau}, \tag{38}$$

$$\left[2\frac{\tilde{A}''}{\tilde{A}} + \frac{\tilde{A}'^2}{\tilde{A}^2} + \frac{k}{\tilde{A}^2} - 2\frac{\tilde{A}'}{\tilde{A}}\phi' + \frac{1}{2}\left(\omega + \frac{1}{2}\right)\phi'^2 - \phi''\right] g_{jj} = -\kappa T_{jj}, \tag{39}$$

$$-\phi'' - 3\frac{\tilde{A}'}{\tilde{A}}\phi' + \phi'^2 = 0, \tag{40}$$

For vacuum and $k = 0$:

$$\frac{3}{2}\frac{\tilde{A}'^2}{\tilde{A}^2} - 3\frac{\tilde{A}''}{\tilde{A}} = \frac{1}{4}\left(5\omega - \frac{3}{2}\right)\phi'^2 - \frac{3}{2}\phi''. \tag{41}$$

Comparing with the solutions of Brans-Dicke theory in the metric formalism

Vacuum solution, with $V(\phi) = 0$ is the O'Hanlon-Tupper solution

$$A(t) = A_0 \left(\frac{t}{t_0}\right)^{q\pm}, \tag{42}$$

$$\phi(t) = \phi_0 \left(\frac{t}{t_0}\right)^{s\mp}, \tag{43}$$

where

$$q\pm = \frac{1}{3\omega + 4}\left[\omega + 1 \pm \sqrt{\frac{2\omega + 3}{3}}\right], \tag{44}$$

$$s\mp = \frac{1 \mp \sqrt{3(2\omega + 3)}}{3\omega + 4}, \tag{45}$$

with $\omega > -3/2$, $\omega \neq 0, -4/3$ and $3q + s = 1$.

- Solution has as a valid range $\omega > 0$.
- $\tilde{A}_{\pm}(\tau)$ and $\Phi_{\pm} = e^{-\phi}$ tend to a constant value as $\tau \to 0$.
- When $\omega \to +\infty$, $\tilde{A}_{\pm}(\tau) \propto \tau^{\pm 1/3}$ and $\Phi_{\pm} = const$. This solution does not reproduce the corresponding general relativistic solution which is Minkowsky space.
- At early times, Φ decreases as τ increases for $\omega > 0$, which implies an increasing G_{eff}.

- O'Hanlon-Tupper solution for $\omega = -4/3$ approaches the de Sitter space

$$A(t) = A_0 e^{Ht}, \tag{46}$$

$$\Phi(t) = \Phi_0 e^{-3Ht}. \tag{47}$$

The present theory reproduces this result for $\tilde{A}_-$ in the limit $\omega = 1/6$.

References

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