Spherically symmetric solutions on geometric scalar theory of gravity

Mario Novello, **Josephine Rua**, Eduardo Bittencourt

Instituto de Cosmologia Relatividade e Astrofisica - ICRA/CBPF
Rio de Janeiro, Brasil

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Outline

Introduction

What is the GSG?

Paper by M. Novello, E. Bittencourt, U. Moschella, et al. (2013).

- The gravitational interaction is described by a scalar field Φ;
- The field Φ satisfies a nonlinear dynamics;
- The theory satisfies the principle of general covariance. In other words this is not a theory restricted to the realm of special relativity;
- All kind of matter and energy interact with Φ only through the pseudo-Riemannian metric $q_{\mu\nu} = a\,\eta_{\mu\nu} + b\,\partial_\mu\Phi\,\partial_\nu\Phi$;

- Test particles follow geodesics relative to the gravitational metric $q_{\mu\nu}$;
- Φ is related in a nontrivial way with the Newtonian potential Φ_N;
- Electromagnetic waves propagate along null geodesics relative to the metric $q^{\mu\nu}$.

They have introduced the contravariant metric tensor $q_{\mu\nu}$ by the binomial formula

$$q^{\mu\nu} = \alpha\,\eta^{\mu\nu} + \frac{\beta}{w}\,\eta^{\mu\rho}\eta^{\nu\sigma}\partial_\rho\Phi\,\partial_\sigma\Phi, \tag{1}$$

where parameters α and β are dimensionless functions of Φ and $w = \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. The corresponding covariant expression, defined as the inverse $q_{\mu\nu}\,q^{\nu\lambda} = \delta_\mu^\lambda$, is also a binomial expression:

$$q_{\mu\nu} = \frac{1}{\alpha}\,\eta_{\mu\nu} - \frac{\beta}{\alpha\,(\alpha+\beta)\,w}\,\partial_\mu\Phi\,\partial_\nu\Phi. \tag{2}$$

The relation between the parameters α and β is given by the following:

Theorem

Given the Lagrangian $L = V(\Phi)w$ with an arbitrary potential $V(\Phi)$, the field theory satisfying equation

$$\frac{1}{\sqrt{-\eta}}\partial_\mu(\sqrt{-\eta}\,\eta^{\mu\nu}\partial_\nu\Phi) + \frac{1}{2}\frac{V'}{V}w = 0 \tag{3}$$

in Minkowski spacetime is equivalent to a massless Klein-Gordon field $\Box\Phi = 0$ in the metric $q^{\mu\nu}$ provided that the functions $\alpha(\Phi)$ and $\beta(\Phi)$ satisfy the condition

$$\alpha + \beta = \alpha^3\,V. \tag{4}$$

Thereby one might choose to work either with this correspondence or directly consider a field theory describe by the action

$$S = \int (\partial\phi)^2 \sqrt{-q}\, dx. \tag{5}$$

At last the potential $V(\Phi)$ is obtained when analysing the planetary orbits from GSG, given

$$V(\Phi) = \frac{1}{4}\frac{(\alpha - 3)^2}{\alpha^3}, \tag{6}$$

where $\alpha(\Phi) = e^{-2\Phi}$.

The constrains are then reviewed to match with the observed regime for the solar tests and astonishingly reveal the same line element as described by the Schwarzschild metric.
Following the steps of general relativity, the dynamic equation of GSG could be written as

$$\sqrt{V}\,\Box\Phi = -\kappa\,\chi, \tag{7}$$

where

$$\chi = \frac{1}{2}\left(\frac{\alpha'}{2\alpha}(T-E) + \frac{(\alpha+\beta)'}{2(\alpha+\beta)}E - \nabla_\lambda C^\lambda\right),$$

and we have defined

$$T = T^{\mu\nu}\,q_{\mu\nu}, \quad E = \frac{T^{\mu\nu}\,\partial_\mu\Phi\,\partial_\nu\Phi}{\Omega}, \quad \Omega = \partial_\mu\partial_\nu\Phi q^{\mu\nu}, \quad X' = \frac{dX}{d\Phi},$$

and

$$C^\lambda = \frac{\beta}{\alpha\,\Omega}\left(E\,q^{\lambda\mu} - T^{\lambda\mu}\right)\partial_\mu\Phi.$$

Spherically symmetric metric of GSG

Let us define the auxiliary Minkowski background metric in spherical coordinates

$$ds_M^2 = dt^2 - dR^2 - R^2\, d\Omega^2. \tag{8}$$

Changing the radial coordinate to $R = \sqrt{\alpha}\, r$, where $\alpha = \alpha(r)$ we get

$$ds_M^2 = dt^2 - \alpha \left(\frac{1}{2\alpha} \frac{d\alpha}{dr}\, r + 1 \right)^2 dr^2 - \alpha r^2\, d\Omega^2. \tag{9}$$

The gravitational metric (2) takes the form

$$ds^2 = \frac{1}{\alpha}\, dt^2 - B\, dr^2 - r^2\, d\Omega^2, \tag{10}$$

where we have defined

$$B \equiv \frac{\alpha}{\alpha + \beta} \left(\frac{1}{2\alpha} \frac{d\alpha}{dr}\, r + 1 \right)^2,$$

which can be rewritten accordingly with the form for the potential $V(\Phi)$ in equation (6) as

$$B = \frac{4\alpha}{(\alpha - 3)^2} \left(\frac{1}{2\alpha} \frac{d\alpha}{dr}\, r + 1 \right)^2.$$

If we choose a class of comoving observers as $V_\mu = 1/\sqrt{\alpha}\,\delta^0_\mu$ the equation of motion (7) gives

$$\sqrt{V}\,\Box\Phi = -\kappa\left[\frac{3-2\alpha}{3-\alpha}p - \frac{\rho}{2}\right] \tag{11}$$

The projector tensor upon the three-space is

$$h_{\mu\nu} = q_{\mu\nu} - V_\mu V_\nu. \tag{12}$$

Then the projection on the three-space of the conservation equation of the energy-momentum tensor $T^{\mu\nu}$ for a perfect fluid, gives

$$T^{\mu\nu}_{;\nu} h_{\mu\alpha} = 0$$

$$\Rightarrow \frac{dp}{dr} = -(\rho + p)\frac{d\phi}{dr}. \tag{13}$$

Equations (11), (13) and equation of state $p = p(\rho)$ complete the set necessary to find general spherically symmetric solutions on GSG. It would be usefull to rewritten these equations in another form. Let us define the parameter α as function of $\mu(r)$:

$$\alpha(r) = \left(1 - \frac{2\mu(r)}{r}\right)^{-1}, \tag{14}$$

which gives us the following

$$B = \alpha\Sigma^2, \tag{15}$$

$$\Sigma = \left(1 - \frac{3\mu}{r}\right)^{-1}\left(\mu' - \frac{3\mu}{r} + 1\right), \tag{16}$$

$$V = \frac{(1 - r\Phi')^2}{\alpha^3\,\Sigma^2}, \tag{17}$$

$$\Phi = \frac{1}{2}\ln\left(1 - \frac{2\mu}{r}\right) \Rightarrow \Phi' = \frac{\alpha}{r^2}\Big(\mu - r\mu'\Big). \tag{18}$$

Then we have for the equation of motion (11)

$$\sqrt{\frac{(1 - r\Phi')^2}{\alpha^3\,\Sigma^2}}\frac{1}{\Sigma r^2}\left(\frac{r^2\Phi'}{\alpha\Sigma}\right)' = k\left(\frac{\rho}{2} - \frac{3 - 2\alpha}{3 - \alpha}p\right). \tag{19}$$

Outline

Newtonian stars

A Newtonian star is characterized by negligible isotropic pressure p in comparison with the energy density ρ and the surround geometry represented by a Minkowski's metric, means

$$p << \rho;$$

$$\alpha \rightarrow 1 \text{ and } B \rightarrow 1;$$

$$\Rightarrow \Sigma^2 \rightarrow 1;$$

$$\Rightarrow V \rightarrow 1.$$

At this point equation (11) establishs

$$\frac{df(r)}{dr} = \frac{k}{2}\rho r^2, \tag{20}$$

where the function $f(r)$ is define as $f(r) \equiv r^2 \frac{d\Phi}{dr}$.
Differently from general relativity equation (20) is valid only for *some* solutions of GSG. A distribution of matter which obey this equation is a particular case or, perhaps, a class of solutions, therefore there is not generality on it.

Using this auxiliary function equation (13) sets

$$\frac{dp}{dr} = -\rho \frac{f(r)}{r^2}, \tag{21}$$

which is kind the Chandrasekhar equation for hydrostatic. Note that f(r) is not the gravitational mass unless we are in the low energy regime.

Later we will explore a little more of this constraint and prove that is not imposed by hand but a natural consequence of GSG.

Outline

The Tolman-Oppenheimer-Volkoff equation on GSG

Once again we will rewrite equation (19) now using the auxiliary function $f(r)$

$$f\frac{d}{dr}\ln\left[f\left(1-\frac{f}{r}\right)\left(1-\frac{3\mu}{r}\right)\right]=$$
$$=-(-1)^n\,k\,\frac{r^2}{2}\left(1-\frac{3\mu}{r}\right)^{-4}\left(1-\frac{2\mu}{r}\right)^{1/2}\Bigg[$$
$$\left[\left(1-\frac{6\mu}{r}\right)p-\left(1-\frac{3\mu}{r}\right)\rho\right], \tag{22}$$

where $n=1$ if $2\mu<r<3\mu$ and $n=2$ if $r>3\mu$.

We can write the equations (22) and (13) in even more convenient way. In do it so we have the three equations of the dynamical system responsible for describing the spherically symmetric solution of a perfect fluid on GSG:

$$\left(1-\frac{3}{2}\sigma\right)\left\{\left[\frac{1}{2}\sigma''r^2+\sigma'r+\frac{1}{2}\sigma'^2r^2\left(1-\sigma\right)^{-1}\right]-\right.$$
$$\left.-\frac{1}{4}\sigma'^2r^2\left(1-\sigma+\frac{1}{2}\sigma'r\right)\right\}+\frac{3}{4}\sigma'^2r^2=$$
$$-(-1)^n k\frac{r^2}{2}\left(1-\frac{3}{2}\sigma\right)^{-3}\left(1-\sigma\right)^{3/2}\Bigg[$$
$$\left[\left(1-3\sigma\right)p-\left(1-\frac{3}{2}\sigma\right)\rho\right], \tag{23}$$

$$\frac{dp}{dr} = \frac{1}{2}(\rho + p)(1 - \sigma)^{-1}\frac{d\sigma}{dr}\,, \tag{24}$$

$$p = p(\rho), \tag{25}$$

where σ is defined as $\sigma(r) \equiv 2\mu/r$.

On the other hand, we have the set of equations from general relativity:

$$\frac{d\sigma}{dr} = r\rho - \frac{\sigma}{r}, \tag{26}$$

$$\frac{dp}{dr} = -(\rho + p)(1-\sigma)^{-1}\left(pr + \frac{\sigma}{2r}\right), \tag{27}$$

$$p = p(\rho). \tag{28}$$

Where the equation (27) is the original TOV equation.

Taking equation (24) in order to write σ' and σ'' as functions of p and ρ we obtain a new formulation for eq. (23)

$$\frac{2}{r}\frac{1}{\rho+p}\frac{dp}{dr}+(1-\sigma)\left(\frac{1}{\rho+p}\frac{dp}{dr}\right)^2\Bigg[$$
$$\left[2(1-\sigma)^{-1}-(1-\sigma)\left(1+\frac{r}{\rho+p}\frac{dp}{dr}\right)+3\left(1-\frac{3}{2}\sigma\right)^{-1}\right]+$$
$$+\left[\left(\frac{1}{\rho+p}\frac{dp}{dr}\right)'-2\left(\frac{1}{\rho+p}\frac{dp}{dr}\right)^2\right]=$$
$$-(-1)^n\frac{k}{2}\left(1-\frac{3}{2}\sigma\right)^{-4}(1-\sigma)^{1/2}\Bigg[$$
$$\left[(1-3\sigma)\,p-\left(1-\frac{3}{2}\sigma\right)\rho\right], \tag{29}$$

Outline

The empty space

Let us go back to the equation (11):

$$\sqrt{V}\,\Box\Phi = -\kappa\left[\frac{3-2\alpha}{3-\alpha}p - \frac{\rho}{2}\right].$$

If we want to calculate the solution for empty space, we must have $p = \rho = 0$, which implies $\Box\Phi = 0$. The solution is

$$\epsilon\sqrt{\alpha} + \frac{1}{\alpha} + \frac{a}{r} = 1, \tag{30}$$

where ϵ and a are constants.
Defining $x \equiv \sqrt{\alpha}$ we have

$$x^2\left[\epsilon x + \frac{a}{r} - 1\right] = -1. \tag{31}$$

Since x must be positive otherwise the metric will change signature, we work with two inequalities

$$\frac{1}{\epsilon}\left(\frac{a}{r}-1\right)<0; \tag{32}$$

$$x<\frac{1}{\epsilon}\left(1-\frac{a}{r}\right). \tag{33}$$

After exhaustive analysis, we find two possible solutions

(I) If $a < 0$ and $\epsilon = 0 \Rightarrow 0 < r < +\infty \;\Rightarrow\; \alpha(r) = \left(1 - \frac{a}{r}\right)^{-1}$;

(II) If $a > 0$ and $\epsilon = 0 \Rightarrow a < r < +\infty \;\Rightarrow\; \alpha(r) = \left(1 - \frac{a}{r}\right)^{-1}$.

Imposing the Newtonian regime when $r \to +\infty$, we have $a = 2MG$.

Then the (II) solution is the only one possible and corresponds to the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2MG}{r}\right) dt^2 - \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Birkhoff's theorem holds for GSG.

At last the interpretation for the function $f(r) = r^2 d\Phi/dr$.
In the empty space $f(r)$ is written as

$$f(r) = GM\left(1 - \frac{2GM}{r}\right)^{-1}. \tag{34}$$

For the Newtonian limit $r \to +\infty$, $f \to GM$.

We have then the completeness of Chandrasekhar's equation (35)

$$\frac{dp}{dr} = -\rho\frac{GM}{r^2}, \tag{35}$$

as expectated.

Conclusions

What is next?

Merci!