Gravitational Repulsion in Modified Gravity

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based on common works with A.D. Dolgov and L. Reverberi

Hot Topics in Modern Cosmology
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Outline

- Dark Energy or Modifed Gravity **?**
- Curvature Oscillations in $F(R)$ Theories in Systems with Rising Energy Density
- Spherically Symmetric Solutions in $F(R)$ Gravity and Gravitational Repulsion
- Jeans Instability in Modified Gravity
- Conclusions

Cosmological Acceleration

The data in favor of accelerated expansion:

- observation of the large scale structure of the universe
- measurements of the angular fluctuations of the CMBR
- determination of the universe age
- discovery of the dimming of distant Supernovae

With cosmological inflation, at the very beginning, the picture would be:

- first acceleration (initial push)
- then normal deceleration
- and lastly (today) surprising acceleration again

Cosmological Equations

Universe expansion is described by scale factor $\mathbf{a(t)}$, which satisfies the Friedmann equations. In particular, cosmological acceleration is given by:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\varrho + 3P)$$

NB: Pressure gravitates!

- Source of gravitational force $\varrho + 3P$, not only ϱ.
- Negative pressure is a source of the cosmological expansion, cosmic antigravity.

Equation of State

- There are two independent cosmological equations for three functions $\mathbf{a(t)}$, $\varrho(\mathbf{t})$, $\mathbf{P(t)}$.
- These equations should be supplemented by the equation of state $\mathbf{P} = \mathbf{P}(\varrho)$, which is determined by physical properties of matter.

Usually matter is described by linear equation of state:

$$\mathbf{P} = \mathbf{w}\varrho$$

- Non-relativistic matter: $\mathbf{w_{nr}} = \mathbf{0}$
- Relativistic matter: $\mathbf{w_{rel}} = \mathbf{1/3}$
- Dark energy: $\mathbf{w_{DE}} = -\mathbf{1.13^{+0.13}_{-0.10}}$

Possible source of the cosmic acceleration?

Dark Energy: P $< -\varrho/3$

- small vacuum energy, which is identical to cosmological constant
- energy density associated with an unknown, presumably scalar field, which slowly varies in the course of the cosmological evolution

Modification of Gravity, which is considered below.

Gravity Modification

Action in $\mathbf{f(R)}$ theories:

$$\mathbf{S} = \frac{\mathbf{m_{Pl}^2}}{\mathbf{16\pi}} \int \mathbf{d^4x\sqrt{-g}f(R)} + \mathbf{S_m}$$
$$= \frac{\mathbf{m_{Pl}^2}}{\mathbf{16\pi}} \int \mathbf{d^4x\sqrt{-g}[R + F(R)]} + \mathbf{S_m}$$

Here $\mathbf{m_{Pl} = 1.22 \cdot 10^{19} GeV}$ is the Planck mass and $\mathbf{S_m}$ is the matter action.

- Usual Einstein gravity: $\mathbf{f(R) = R}$
- Modified gravity: $\mathbf{f(R) = R + F(R)}$

Pioneering Works

The pioneering suggestion:

- *S. Capozziello, S. Carloni, A. Troisi,* Recent Res. Develop. Astron. Astrophys.1(2003)625; astro-ph/0303041.
- *S.M. Carroll, V. Duvvuri, M. Trodden, M.S. Turner,* Phys. Rev. D70 (2004) 043528, astro-ph/0306438.

$$\mathbf{F(R) = -\mu^4/R}$$

$\mu^2 \sim R_c \sim 1/t_u^2$ is a small parameter with dimension of mass squared.

- Agreement with Newtonian limit for sufficiently small μ.
- Strong instability in presence of matter

Can modified gravity explain accelerated cosmic expansion?
A.D. Dolgov, M. Kawasaki, Phys.Lett. B573 (2003) 1.

Modified modified gravity: free from exponential instability

W.Hu, I. Sawicki, Phys. Rev. D **76**, 064004 (2007).

$$\mathbf{F_{HS}(R) = -\frac{R_{vac}}{2}\frac{c\left(\frac{R}{R_{vac}}\right)^{2n}}{1+c\left(\frac{R}{R_{vac}}\right)^{2n}}},$$

A.Appleby, R. Battye,Phys. Lett. B **654**, 7 (2007).

$$\mathbf{F_{AB}(R) = \frac{\epsilon}{2}\log\left[\frac{\cosh\left(\frac{R}{\epsilon}-b\right)}{\cosh b}\right] - \frac{R}{2}},$$

A.A. Starobinsky, JETP Lett. **86**, 157 (2007).

$$\mathbf{F_S(R) = \lambda R_0\left[\left(1+\frac{R^2}{R_0^2}\right)^{-n} - 1\right]}.$$

Curvature Oscillations in Modified Gravity

- *E. V. Arbuzova, A. D. Dolgov, L. Reverberi*, Curvature Oscillations in Modified Gravity and High Energy Cosmic Rays, Eur.Phys.J.**C**(2012) 72:2247, arXiv:1211.5011; Particle Production in $f(R)$ Gravity during Structure Formation, Phys.Rev.**D** 88, 024035 (2013), arXiv:1305.5668.

Starobinsky model with $\mathbf{R^2}$ term:

$$\mathbf{F(R)} = -\lambda \mathbf{R_0} \left[1 - \left(1 + \frac{\mathbf{R^2}}{\mathbf{R_0^2}} \right)^{-\mathbf{n}} \right] - \frac{\mathbf{R^2}}{\mathbf{6m^2}}$$

- Parameter $\mathbf{m}$ is bounded by $\mathbf{m} \gtrsim \mathbf{10^5}$ GeV to preserve successful predictions of BBN.
- $\mathbf{R^2}$-term is included to prevent curvature singularities in the presence of contracting bodies.
- *E.V. Arbuzova, A.D. Dolgov,* Explosive phenomena in modified gravity. Phys.Lett.**B**700(2011)289: $\mathbf{R^2}$ prevents from hitting infinity but still the maximum amplitude of $\mathbf{R}$ reaches a large value much larger than in GR

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Basic Equations

The evolution of **R** is determined from the equation:

$$3\mathcal{D}^2 F'_R - R + RF'_R - 2F = T$$

- $\mathcal{D}^2$ is the covariant D'Alambertian operator, $F'_R \equiv dF/dR$
- $T \equiv 8\pi T^\mu_\mu / m^2_{Pl}$ and $T_{\mu\nu}$ is energy-momentum tensor of matter.

We are interested in the regime $|R_0| \ll |R| \ll m^2$, in which:

$$F(R) \simeq -\lambda R_0 \left[1 - \left(\frac{R_0}{R}\right)^{2n}\right] - \frac{R^2}{6m^2}\,.$$

We study the evolution of R in a contracting astrophysical system with rising energy density:

$$\varrho = \varrho_0 (1 + t/t_{contr})$$

We assume that the gravity of matter is not strong and thus the background metric is flat: $3\partial_t^2 F'_R - R - T = 0.$ Hot Topics in Modern

New Notations: Oscillator Equation

With the dimensionless quantities:

$$z \equiv \frac{T(t)}{T(t_{in})} \equiv \frac{T}{T_0} = \frac{\varrho_m(t)}{\varrho_{m0}}\,, \qquad y \equiv -\frac{R}{T_0}\,,$$

$$g = \frac{1}{6\lambda n (m t_U)^2} \left(\frac{\varrho_{m0}}{\varrho_c}\right)^{2n+2}\,, \quad \tau \equiv m\sqrt{g}\,t$$

and new function, proportional to $F'(R)$:

$$\xi \equiv \frac{1}{2\lambda n}\left(\frac{T_0}{R_0}\right)^{2n+1} F'_R = \frac{1}{y^{2n+1}} - gy$$

The equation of motion for ξ takes the simple oscillator form:

$$\xi'' + dU/d\xi = 0\,, \quad \text{where} \quad dU/d\xi = z - y(\xi).$$

y cannot be expressed through ξ analytically so we have to use different approximate expressions in different ranges of ξ.

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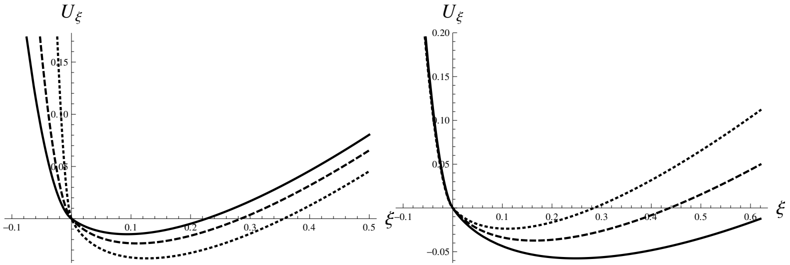
Potential: $\mathbf{U}(\xi) = \mathbf{U}_+(\xi)\Theta(\xi) + \mathbf{U}_-(\xi)\Theta(-\xi)$



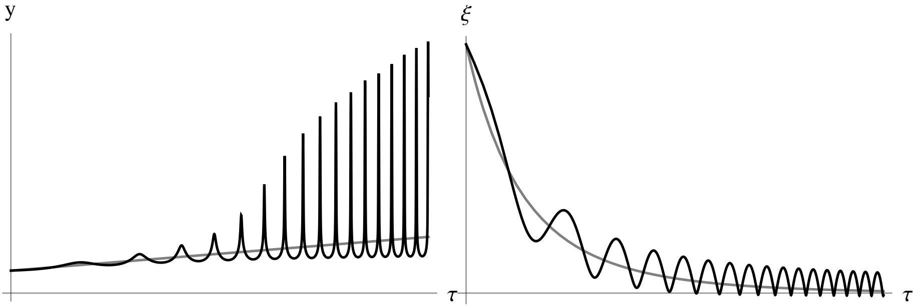
- *Left panel* ($\mathbf{n = 2, z = 1.5}$): solid line: $\mathbf{g = 0.02}$, dashed line: $\mathbf{g = 0.01}$, dotted line: $\mathbf{g = 0.002}$. *Right panel* ($\mathbf{n = 2, g = 0.01}$): solid line: $\mathbf{z = 1.3}$, dashed line: $\mathbf{z = 1.4}$, dotted line: $\mathbf{z = 1.5}$.

$$\mathbf{U}_+(\xi) = \mathbf{z}\xi - \frac{\mathbf{2n+1}}{\mathbf{2n}}\left[\left(\xi + \mathbf{g}^{(\mathbf{2n+1})/(\mathbf{2n+2})}\right)^{\mathbf{2n}/(\mathbf{2n+1})} - \mathbf{g}^{\mathbf{2n}/(\mathbf{2n+2})}\right],$$

$$\mathbf{U}_-(\xi) = \left(\mathbf{z} - \mathbf{g}^{-1/(\mathbf{2n+2})}\right)\xi + \frac{\xi^2}{\mathbf{2g}}.$$

Oscillations of Curvature. "Spike-like" Solutions

In contrast to ξ the oscillations of $\mathbf{y}$ are strongly unharmonic.
For negative and even very small $|\xi|$ the amplitude of $\mathbf{y}$ may be very large because $\mathbf{y} \approx -\xi/\mathbf{g}$, according to $\xi = \mathbf{1}/\mathbf{y}^{\mathbf{2n+1}} - \mathbf{g}\mathbf{y}$.



"Spikes" in the solutions. $\mathbf{n = 2}$, $\mathbf{g = 0.001}$, and $\mathbf{y'_0 = 0.2}$.
Note the strong anharmonicity of the oscillations of $\mathbf{y = -R/T_0}$ and the asymmetry of the oscillations of ξ around minimum of potential.

Amplitude and Frequency of Oscillations

- If the energy density rises with time, fast oscillations of the scalar curvature are induced, with an amplitude possibly much larger the the usual GR value $\mathbf{R = -\tilde{T}}$.

The amplitude of the spikes:

$$\mathbf{y_{max} \simeq \frac{1}{\sqrt{ng}} \left[|y'_0 - \kappa| (2n+1)^{3/2} \right]^{2n/(3n+1)}}$$

The calculated amplitude of $\mathbf{y_{max}}$ becomes noticeably larger with rising

$$\mathbf{z = \varrho(t)/\varrho_0 = 1 + \kappa\tau, \;\; \kappa = (m t_{contr} \sqrt{g})^{-1}}.$$

For negative ξ potential behaves as $\mathbf{U \approx \xi^2/(2g)}$, so the characteristic frequency of oscillations in the region of negative ξ:

$\mathbf{\Omega \sim 1/\sqrt{g}}$ in dimensionless time or $\omega \approx \mathbf{m}$ in physical time.

Evidently frequency of oscillations of $\mathbf{y}$ in this region is the same.

Spike-like Solutions

Solution of modified gravity equations for finite-size astronomical objects with rising energy density:

$$\mathbf{R} = \mathbf{R_{GR}(r)y(t)}, \quad \mathbf{R_{GR}} = -\tilde{\mathbf{T}}(\mathbf{r}) = -\frac{8\pi \mathbf{T}^{\mu}_{\mu}(\mathbf{r})}{\mathbf{m^2_{Pl}}}$$

The maximum value of $\mathbf{y}$ in the spike region is:

$$\mathbf{y_{max}(t)} \sim \mathbf{6n(2n+1)mt_u} \left(\frac{\mathbf{t_u}}{\mathbf{t_{contr}}}\right) \left[\frac{\varrho_{\mathbf{m}}(\mathbf{t})}{\varrho_{\mathbf{m0}}}\right]^{\mathbf{(n+1)/2}} \left(\frac{\varrho_{\mathbf{c}}}{\varrho_{\mathbf{m0}}}\right)^{\mathbf{2n+2}}$$

The energy density of the contracting cloud behaves as

$$\varrho_{\mathbf{m}}(\mathbf{t}) = \varrho_{\mathbf{m0}}(\mathbf{1} + \mathbf{t}/\mathbf{t_{contr}})$$

- The mass $\mathbf{m} \geq \mathbf{10^5}$ GeV to avoid a conflict with BBN.
- $\mathbf{mt_u} \geq \mathbf{10^{47}}$ and $\mathbf{y}$ can reach a very high value

Spike Region

Spikes of high amplitude are formed if:

$$6n^2(2n+1)^2\left(\frac{t_u}{t_{contr}}\right)^2\left[\frac{\varrho_m(t)}{\varrho_{m0}}\right]^{3n+1}\left(\frac{\varrho_c}{\varrho_{m0}}\right)^{2n+2} > 1$$

- Formation of galaxies or their clusters: $\varrho_{m0}/\varrho_c = 1 - 10^3$ and $\varrho_m(t)/\varrho_{m0}$ varying in the range $1 - 10^5$.
- Formation of stellar or planetary type objects from the intergalactic gas with the initial density 10^{-24} g/cm^3: $\varrho_{m0}/\varrho_c = 10^5$ and $\varrho_m(t)/\varrho_{m0}$ can vary in the range $1 - 10^{24}$ or more.

The oscillations of curvature in such systems are excited if their mass density started to rise with time.

Spherically Symmetric Solutions in $F(R)$ Gravity

Assumption: the background space-time is nearly flat and so the background metric is almost Minkowsky.

Is this approximation valid for large deviation of curvature from its GR value?

- *E.V. Arbuzova, A.D. Dolgov, L. Reverberi*, Spherically Symmetric Solutions in F(R) Gravity and Gravitational Repulsion. Astropart.Phys.54(2014)44-47.

We consider a spherically symmetric bubble of matter of radius $\mathbf{r_m}$, and study spherically symmetric solution of modified EoM

$$\left(1+F'_R\right)R_{\mu\nu}-\frac{1}{2}\left(R+F\right)g_{\mu\nu}+\left(g_{\mu\nu}D_\alpha D^\alpha-D_\mu D_\nu\right)F'_R=\tilde{T}_{\mu\nu}$$

$$3D^2F'_R-R+RF'_R-2F=\tilde{T}$$

We use the Schwarzschild metric and assume that the metric is close to the flat one:

$$ds^2=A(r,t)dt^2-B(r,t)dr^2-r^2(d\theta^2+\sin^2\theta\,d\phi^2)$$

$$A_1=A-1\ll 1 \text{ and } B_1=B-1\ll 1$$

Equations of Motion

It is convenient to use EoM in the following form:

$$\begin{aligned} R_{00} - R/2 &= \frac{\tilde{T}_{00} + \Delta F'_R + F/2 - RF'_R/2}{1 + F'_R} \\ R_{rr} + R/2 &= \frac{\tilde{T}_{rr} + (\partial_t^2 + \partial_r^2 - \Delta)F'_R - F/2 + RF'_R/2}{1 + F'_R} \end{aligned}$$

In the weak field limit:

$$\begin{aligned} R_{00} &\approx \frac{A'' - \ddot{B}}{2} + \frac{A'}{r}, \quad R_{rr} \approx \frac{\ddot{B} - A''}{2} + \frac{B'}{r} \\ R &\approx A'' - \ddot{B} + \frac{2A'}{r} - \frac{2B'}{r} + \frac{2(1 - B)}{r^2} \end{aligned}$$

If the energy density of matter inside the the cloud, i.e. for $r < r_m$, is much larger than the cosmological energy density, then:

$$F'_R \ll 1 \quad \text{and} \quad F \ll R$$

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General Solutions

We assume that spatial derivatives of F'_R are small and we find:

$$B_1' + \frac{B_1}{r} = r\tilde{T}_{00}$$

$$A_1'' - \frac{A_1'}{r} = -\frac{3B_1}{r^2} + \ddot{B}_1 + \tilde{T}_{00} - 2\tilde{T}_{rr} + \frac{\tilde{T}_{\theta\theta}}{r^2} + \frac{\tilde{T}_{\varphi\varphi}}{r^2 \sin^2\theta} \equiv S_A$$

Equation for B_1 has the solution:

$$B_1(r,t) = \frac{C_B(t)}{r} + \frac{1}{r}\int_0^r dr' r'^2 \tilde{T}_{00}(r',t)$$

To avoid a singularity at $r = 0$ we have to assume that $C_B(t) \equiv 0$.

$$A_1(r,t) = C_{1A}(t) r^2 + C_{2A}(t) + \int_r^{r_m} dr_1\, r_1 \int_{r_1}^{r_m} \frac{dr_2}{r_2} S_A(r_2,t)$$

The Schwarzschild Limit

The mass of matter inside a radius $\mathbf{r}$ is defined in the usual way:

$$\mathbf{M(r,t)} = \int_0^r \mathbf{d^3r\, T_{00}(r,t)} = 4\pi \int_0^r \mathbf{dr\, r^2\, T_{00}(r,t)}$$

If all matter is confined inside a radius $\mathbf{r_m}$, the total mass is $\mathbf{M \equiv M(r_m)}$ and it does not depend on time.

Since $\mathbf{\tilde{T}_{00} = 8\pi T_{00}/m_{Pl}^2}$, we obtain for $\mathbf{r > r_m}$:

$$\mathbf{B_1 = r_g/r}, \text{ where } \mathbf{r_g = 2M/m_{Pl}^2}$$

The metric coefficient $\mathbf{A_1}$ outside the source is:

$$\mathbf{A_1} = -\frac{\mathbf{r_g}}{\mathbf{r}} + \left[\mathbf{C_{1A}(t)} - \frac{\mathbf{r_g}}{\mathbf{2r_m^3}}\right]\mathbf{r^2} + \left[\mathbf{C_{2A}(t)} + \frac{\mathbf{3r_g}}{\mathbf{2r_m}}\right]$$

Modified Gravity Solutions

The coefficient $\mathbf{C_{1A}(t)}$ can be found from equation for $\mathbf{R}$:

$$\mathbf{R} \approx \mathbf{A''} + \frac{\mathbf{2A'}}{\mathbf{r}} - \ddot{\mathbf{B}} - \frac{\mathbf{2B'}}{\mathbf{r}} + \frac{\mathbf{2(1-B)}}{\mathbf{r^2}}$$

- In systems with rising energy density the curvature scalar may be much larger than its value in GR.

Using eqs. for $\mathbf{A_1}$ and $\mathbf{B_1}$ and comparing them to expression for $\mathbf{R}$, we get:

$$\mathbf{C_{1A}(t) = R(t)/6}$$

$$\mathbf{R_{max}(t) \sim -6n(2n+1)mt_u \left(\frac{t_u}{t_{contr}}\right) \left[\frac{\varrho_m(t)}{\varrho_{m0}}\right]^{(n+1)/2} \left(\frac{\varrho_c}{\varrho_{m0}}\right)^{2n+2} \tilde{T}}$$

The difference between modified and standard solutions in vacuum:

- In the standard case the term proportional to $\mathbf{r^2}$ appears both at $\mathbf{r < r_m}$ and $\mathbf{r > r_m}$ with the same coefficient and we have to choose the arbitrary constant so that this term vanishes.
- For modified gravity such condition is not applicable and the $\mathbf{C_{1A}r^2}$-term may be present at $\mathbf{r < r_m}$ and absent at $\mathbf{r \gg r_m}$.

Hot Topics in Modern

Metric Functions inside a Cloud

The metric functions inside the cloud are equal to:

$$\mathbf{B(r,t)} = 1 + \frac{2M(r,t)}{m_{Pl}^2 r} \equiv 1 + B_1^{(Sch)}$$

$$\mathbf{A(r,t)} = 1 + \frac{R(t)\, r^2}{6} + A_1^{(Sch)}(r,t)$$

- The matter is nonrelativistic, so the space components of $\mathbf{T}_{\mu\nu}$ are negligible in comparison to $\mathbf{T_{00}}$.
- $\mathbf{T_{00}} \equiv \varrho_m(\mathbf{t})$ is spatially constant but may depend on time.

For the Schwarzschild part of the solution we find:

$$A_1^{(Sch)}(r,t) = \frac{r_g r^2}{2r_m^3} - \frac{3r_g}{2r_m} + \frac{\pi \ddot{\varrho}_m}{3m_{Pl}^2}(r_m^2 - r^2)^2$$

The oscillating part $\mathbf{R(t)r^2/6}$ gives the dominant contribution into $\mathbf{A_1}$:

- $r^2 R(t) \sim r^2 y(t)\, R_{GR}$ with $y > 1$, $|R_{GR}| = 8\pi\varrho_m/m_{Pl}^2$.
- the canonical Schwarzschild terms: $r_g/r_m \sim \varrho_m r_m^2/m_{Pl}^2 \sim r_m^2 R_{GR}$.

Applicability of the Approximation

- Assumption: the background metric weakly deviates from the flat Minkowsky one.
- Though it is certainly true for the Schwarzschild part of solution, this may be questioned for the $\mathbf{r^2R(t)/6}$ - term.

The flat background metric is not noticeably distorted if

$$\mathbf{r^2 < 6/R(t)}$$

If the initial energy density of the cloud is of the order of the cosmological energy density

$$\mathbf{R_{GR} \sim 1/t_u^2},$$

the metric would deviate from the Minkowsky one for clouds with:

$$\mathbf{r_m > t_u/\sqrt{y}}.$$

At the stage of the rising $\mathbf{R(t)}$, when $\mathbf{y > 1}$ but not huge, the flat space approximation would be valid over all the volume of the collapsing cloud.

Anti-gravity inside a Cloud

In the lowest order in the gravitational interaction the motion of a non-relativistic test particle is governed by the equation:

$$\ddot{r} = -\frac{A'}{2} = -\frac{1}{2}\left[\frac{R(t)r}{3} + \frac{r_g r}{r_m^3}\right]$$

- Since $R(t)$ is always negative and large, the modifications of GR considered here lead to anti-gravity inside a cloud with energy density exceeding the cosmological one.

Gravitational repulsion dominates over the usual attraction if:

$$\frac{|R| r_m^3}{3 r_g} = \frac{|R| r_m^3 m_{Pl}^2}{6M} = \frac{|R| r_m^3 m_{Pl}^2}{8\pi \varrho\, r_m^3} = \frac{|R|}{\tilde{T}_{00}} > 1$$

so basically whenever **oscillations of R start rising**, regardless of the initial value of ϱ and to some extent of the specific $F(R)$ considered.

Jeans Instability in Modified Gravity

- Jeans instability in Newtonian theory with time dependent background
- Evolution of fluctuations in General Relativity
- Gravitational instability in $\mathbf{F(R)}$-modified theories

Non-relativistic Jeans problem in Newtonian Gravity

Spherically symmetric cloud of particles with initially zero pressure and velocities

$$\Delta\Phi = 4\pi G\varrho$$
$$\partial_t(\varrho \mathbf{v}) + \varrho(\mathbf{v}\nabla)\mathbf{v} + \nabla P + \varrho\nabla\Phi = 0$$
$$\partial_t\varrho + \nabla(\varrho \mathbf{v}) = 0$$

The problem: time independent ϱ is not a solution of these equations.

- *Ya.B.Zeldovich, I.D.Novikov*: solutions in the cosmological background
- *V. Mukhanov*: some repulsive force
- *ADR*: the Poisson equation is valid in particular for zero order terms, so the solution of equations of motion leads to time dependent background energy density and gravitational potential

Development of Jeans instability goes faster than in the standard theory

Time Dependent Problem

Initial values:

- homogeneous distribution $\varrho_0 = \mathbf{const}$ inside a sphere $\mathbf{r} < \mathbf{r_m}$
- particle velocities $\mathbf{v_0} = \mathbf{0}$ and pressure $\mathbf{P_0} = \mathbf{0}$

The potential Φ is a solution of the Poisson equation:

$$\Phi_0(\mathbf{r} > \mathbf{r_m}) = -\mathbf{MG/r}, \;\; \Phi_0(\mathbf{r} < \mathbf{r_m}) = 2\pi \mathbf{G} \varrho_0 \mathbf{r}^2/3 + \mathbf{C_0}$$

where $\mathbf{C_0} = -2\pi \mathbf{G} \varrho_0 \mathbf{r_m^2}$ and $\mathbf{M} = 4\pi \varrho_0 \mathbf{r_m^3}/3$.
From Euler equation:

$$\mathbf{v_1(r,t)} = -\nabla \Phi_0 \mathbf{t} = -4\pi \mathbf{G} \varrho_0 \mathbf{rt}/3$$

From the continuity equation:

$$\varrho_1 = \frac{2\pi}{3} \mathbf{G} \varrho_0^2 \mathbf{t}^2 \;\; \text{or} \;\; \varrho_\mathbf{b}(\mathbf{t},\mathbf{r}) = \varrho_0 \left(1 + \frac{2\pi}{3} \mathbf{G} \varrho_0 \mathbf{t}^2\right)$$

The time variation of the potential:

$$\Phi_\mathbf{b}(\mathbf{r},\mathbf{t}) = \Phi_0 + \Phi_1 = \frac{2\pi}{3} \mathbf{G} \mathbf{r}^2 \varrho_0 \left(1 + \frac{2\pi}{3} \mathbf{G} \varrho_0 \mathbf{t}^2\right)$$

Evolution of Perturbations over Time-dependent Background

$\varrho = \varrho_b(\mathbf{r},t) + \delta\varrho$, $\Phi = \Phi_b(\mathbf{r},t) + \delta\Phi$, $\mathbf{v} = \mathbf{v}_1(\mathbf{r},t) + \delta\mathbf{v}$, $\delta P = c_s^2 \delta\varrho$,

- c_s is the speed of sound

The usual first order expansion:

$$\Delta(\delta\Phi) = 4\pi G \delta\varrho$$
$$\partial_t \delta\mathbf{v} + \nabla\delta\Phi + \delta\varrho/\varrho_0 \nabla\Phi_b + \nabla\delta P/\varrho_0 = 0$$
$$\partial_t \delta\varrho + \varrho_0 \nabla(\delta\mathbf{v}) = 0$$

Making Fourier transformation $\sim \exp[-i\lambda t + i k_j x^j]$ and neglecting **r**-dependent term $\delta\varrho/\varrho_0 \nabla\Phi_b$ we obtain the eigenvalue equation:

$$k^2(\lambda^2 - c_s^2 k^2 + 4\pi G \varrho_0) = 0$$

For small **k** we find the usual exponential Jeans instability:

$$\delta\varrho/\varrho \sim \exp\left[t(4\pi G\varrho_0 - k^2 c_s^2)^{1/2}\right]$$

Jeans Instability in General Relativity

Basic equations are the usual GR equations:

$$\mathbf{G}_{\mu\nu} \equiv \mathbf{R}_{\mu\nu} - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathbf{R} = 8\pi\mathbf{G}\mathbf{T}_{\mu\nu} \equiv \tilde{\mathbf{T}}_{\mu\nu}$$

- Equations of motion of matter (the continuity and Euler equations) are automatically included
- Equations of motion of matter can be obtained from the covariant conservation condition of the energy-momentum tensor

$$\mathbf{D}_{\mu}\mathbf{T}^{\mu}_{\nu} = 0$$

NB: In the first case one has to include the terms proportional to the square of Christoffel symbols in the expression for the Ricci tensor.

If we confine ourselves to the first order in Γ in $\mathbf{R}_{\mu\nu}$ we do not obtain self-consistent equations.

Metric and Christoffel symbols

We take the Newtonian gauge, in which the metric has the form

$$ds^2 = Adt^2 - B\delta_{ij}dx^i dx^j$$

The corresponding Christoffel symbols are:

$$\Gamma^t_{tt} = \frac{\dot{A}}{2A}, \quad \Gamma^t_{jt} = \frac{\partial_j A}{2A}, \quad \Gamma^j_{tt} = \frac{\delta^{jk}\partial_k A}{2B}, \quad \Gamma^t_{jk} = \frac{\delta_{jk}\dot{B}}{2A},$$
$$\Gamma^k_{jt} = \frac{\delta^k_j \dot{B}}{2B}, \quad \Gamma^k_{lj} = \frac{1}{2B}(\delta^k_l \partial_j B + \delta^k_j \partial_l B - \delta_{lj}\delta^{kn}\partial_n B)$$

Ricci Tensor and Curvature Scalar

For the Ricci tensor with an account of the quadratic in Γ terms we obtain:

$$\begin{aligned}
R_{tt} &= \frac{\Delta A}{2B} - \frac{3\ddot{B}}{2B} + \frac{3\dot{B}^2}{4B^2} + \frac{3\dot{A}\dot{B}}{4AB} + \frac{\partial^j A \partial_j B}{4B^2} - \frac{\partial^j A \partial_j A}{4AB} \\
R_{tj} &= -\frac{\partial_j \dot{B}}{B} + \frac{\dot{B}\partial_j B}{B^2} + \frac{\dot{B}\partial_j A}{2AB} \\
R_{ij} &= \delta_{ij}\left(\frac{\ddot{B}}{2A} - \frac{\Delta B}{2B} + \frac{\dot{B}^2}{4AB} - \frac{\dot{A}\dot{B}}{4A^2} - \frac{\partial^k A \partial_k B}{4AB} + \frac{\partial^k B \partial_k B}{4B^2}\right) \\
&- \frac{\partial_i \partial_j A}{2A} - \frac{\partial_i \partial_j B}{2B} + \frac{\partial_i A \partial_j A}{4A^2} + \frac{3\partial_i B \partial_j B}{4B^2} + \frac{\partial_i A \partial_j B + \partial_j A \partial_i B}{4AB}
\end{aligned}$$

The corresponding curvature scalar is:

$$R = \frac{\Delta A}{AB} - \frac{3\ddot{B}}{AB} + \frac{2\Delta B}{B^2} + \frac{3\dot{A}\dot{B}}{2A^2 B} - \frac{\partial^j A \partial_j A}{2A^2 B} - \frac{3\partial^j B \partial_j B}{2B^3} + \frac{\partial^j A \partial_j B}{2AB^2}$$

Einstein Tensor

Expressions for the Einstein tensor $\mathbf{G_{\mu\nu} = R_{\mu\nu} - 1/2\, g_{\mu\nu} R}$:

$$
\begin{aligned}
\mathbf{G_{tt}} &= -\frac{\mathbf{A\Delta B}}{\mathbf{B^2}} + \frac{\mathbf{3\dot{B}^2}}{\mathbf{4B^2}} + \frac{\mathbf{3A\partial^j B\partial_j B}}{\mathbf{4B^3}} \\
\mathbf{G_{tj}} &= \mathbf{R_{tj}} \\
\mathbf{G_{ij}} &= \delta_{ij}\left(\frac{\mathbf{\Delta A}}{\mathbf{2A}} + \frac{\mathbf{\Delta B}}{\mathbf{2B}} - \frac{\mathbf{\ddot{B}}}{\mathbf{A}} + \frac{\mathbf{\dot{B}^2}}{\mathbf{4AB}} + \frac{\mathbf{\dot{A}\dot{B}}}{\mathbf{2A^2}} - \frac{\mathbf{\partial^k A\partial_k A}}{\mathbf{4A^2}} - \frac{\mathbf{\partial^k B\partial_k B}}{\mathbf{2B^2}}\right) \\
&- \frac{\mathbf{\partial_i\partial_j A}}{\mathbf{2A}} - \frac{\mathbf{\partial_i\partial_j B}}{\mathbf{2B}} + \frac{\mathbf{\partial_i A\partial_j A}}{\mathbf{4A^2}} + \frac{\mathbf{3\partial_i B\partial_j B}}{\mathbf{4B^2}} + \frac{\mathbf{\partial_i A\partial_j B + \partial_j A\partial_i B}}{\mathbf{4AB}}
\end{aligned}
$$

Energy-momentum Tensor

The energy-momentum tensor is taken in the ideal liquid form without dissipative corrections:

$$T_{\mu\nu} = (\varrho + P)U_\mu U_\nu - P g_{\mu\nu}$$

- ϱ and **P** are respectively the energy and pressure densities of the liquid
- the four-velocity is: $U^\mu = dx^\mu/ds$ and $U_\mu = g_{\mu\alpha}U^\alpha$

We assume that three-velocity $v^j = dx^j/dt$ is small and neglect quadratic in **v** terms. Correspondingly:

$$U_j = -\frac{Bv_j}{\sqrt{A}\sqrt{1-(B/A)v_j v^j}} \approx -\frac{Bv_j}{\sqrt{A}}$$

Now we can write:

$$\begin{aligned} T_{tt} &= (\varrho + P)U_t^2 - PA \approx \varrho A \\ T_{jt} &= (\varrho + P)U_t U_j \approx -(\varrho + P)Bv_j \\ T_{ij} &= (\varrho + P)U_i U_j - P g_{ij} \approx PB\delta_{ij} \end{aligned}$$

Equations of Motion

The equations for $\mathbf{G_{tt}}$ and for $\partial_i\partial_j$-component of equation for $\mathbf{G_{ij}}$ are:

$$-\Delta B = \tilde{\varrho}$$
$$\partial_i\partial_j(A + B) = 0$$

The continuity and Euler equations are respectively:

$$\dot{\varrho} + \partial_j[(\varrho + P)v^j] + \frac{3}{2}\varrho\dot{B} = 0$$
$$\varrho\dot{v}_j + \partial_j P + \frac{1}{2}\varrho\,\partial_j A = 0$$

We assume that the background metric slowly changes as a function of space and time and study small fluctuations around background quantities:

$$\varrho = \varrho_b + \delta\varrho,\ \delta P = c_s^2\delta\varrho,\ v = \delta v,\ A = A_b + \delta A,\ B = B_b + \delta B$$

Evolution of Fluctuations in GR

The corresponding linear equations for infinitesimal quantities:

$$-\Delta\delta B = \delta\tilde{\varrho}$$
$$\partial_i\partial_j(\delta A + \delta B) = 0$$
$$\dot{\delta}\varrho + \varrho\partial_j\delta v^j + \frac{3}{2}\varrho\delta\dot{B} = 0$$
$$\varrho\,\delta\dot{v}_j + \partial_j\delta P + \frac{1}{2}\varrho\,\partial_j\delta A = 0$$

We look for the solution in the form $\sim \exp[-i\lambda t + ik_j x^j]$ and obtain:

$$\lambda^2 = \frac{c_s^2 k^2 - \tilde{\varrho}/2}{1 + 3\tilde{\varrho}/(2k^2)}$$

This result almost coincides with the Newtonian one. An extra term in the denominator is small when $k \sim k_J = \sqrt{4\pi G\varrho_0}/c_s$

Instability in Modified Gravity

GR equations are modified as:

$$\left(1+F_{,R}\right)R_{\mu\nu}-\frac{1}{2}\left(R+F\right)g_{\mu\nu}+\left(g_{\mu\nu}D_\alpha D^\alpha-D_\mu D_\nu\right)F_{,R}=\frac{8\pi T_{\mu\nu}}{m_{Pl}^2}$$

where $F_{,R}=dF/dR$.

- *A.Zhuk, S.Capozzillo et al., J. Matsumoto*: Perturbative expansion of $F(R)$ was performed either around $R=0$ or $R=R_c$, where R_c is the cosmological curvature scalar.
- *ADR*: we expand $F(R)$ around curvature of the background metric R_m, which is typically much larger than R_c.

$F(R)$-function has very different values for $R \ll R_c$, $R \sim R_c$, $R \gg R_c$

Evolution of Fluctuations in $F(R)$ - theories

- We consider the case of $\mathbf{R \gg R_c}$ which is realized in astronomical systems with energy density grossly exceeding the cosmological one.
- We assume: $|\mathbf{F_{,R}}| \ll \mathbf{1}, |\mathbf{F}| \ll |\mathbf{R}|, |\partial^2 \mathbf{F_{,R}}| \gg (\partial \mathbf{F_{,R}})^2$

In this case the equations of motion are much simplified:

$$-\Delta \delta \mathbf{B} + \omega^{-2} \Delta \delta \mathbf{R} = \delta \tilde{\varrho}$$
$$\partial_i \partial_j (\delta \mathbf{A} + \delta \mathbf{B} - 2\omega^{-2} \delta \mathbf{R}) = 0$$

where $\omega^{-2} = -3\mathbf{F_{,RR}}$.
Continuity and Euler equations remain untouched:

$$\dot{\delta} \varrho + \varrho \partial_j \delta \mathbf{v}^j + \frac{3}{2} \varrho \delta \dot{B} = 0$$
$$\varrho\, \delta \dot{\mathbf{v}}_j + \partial_j \delta \mathbf{P} + \frac{1}{2} \varrho\, \partial_j \delta \mathbf{A} = 0$$

System for Fluctuations

After the Fourier transformation $\sim \mathbf{exp}\,[-\mathbf{i}\lambda\mathbf{t}+\mathbf{ik_j x^j}]$ we obtain four equations for four unknowns, $\delta\mathbf{A}$, $\delta\mathbf{B}$, $\delta\tilde{\varrho}$, and longitudinal component of velocity $\delta\mathbf{v_j}$:

$$\begin{aligned}
&\delta\mathbf{A}+\delta\mathbf{B}=\mathbf{2}\delta\mathbf{R}\omega^{-2}\\
&\mathbf{k^2}(\delta\mathbf{B}-\delta\mathbf{A})=\mathbf{2}\delta\tilde{\varrho}\\
&\mathbf{k_j}\lambda\tilde{\varrho}\delta\mathbf{v_j}-\mathbf{k^2c_s^2}\delta\tilde{\varrho}-\mathbf{k^2}\tilde{\varrho}\delta\mathbf{A}/\mathbf{2}=\mathbf{0}\\
&\mathbf{k_j}\lambda\tilde{\varrho}\delta\mathbf{v_j}-\lambda^2\delta\tilde{\varrho}-\mathbf{3}\lambda^2\tilde{\varrho}\delta\mathbf{B}=\mathbf{0}
\end{aligned}$$

where $\delta\mathbf{R}=\mathbf{3}\lambda^2\delta\mathbf{B}-\mathbf{k^2}\delta\mathbf{A}-\mathbf{2k^2}\delta\mathbf{B}$

Eigenvalues of Frequency

Equation for the eigenvalues of the frequency $\boldsymbol{\lambda}$:

$$\frac{3k^2}{\omega^2}\lambda^4 - \lambda^2\left[k^2 + \frac{3\tilde{\varrho}}{2} + \frac{3k^4}{\omega^2}\left(1 + c_s^2\right)\right] - \frac{\tilde{\varrho}k^2}{2} + k^4 c_s^2 + \frac{k^4}{\omega^2}\left(3c_s^2 k^2 - 2\tilde{\varrho}\right) = 0$$

In the limit of $\mathbf{k}^2 \ll \boldsymbol{\omega}^2$ we obtain the following eigenvalues:

$$\lambda_1^2 = \frac{c_s^2 k^2 - \tilde{\varrho}/2}{1 + 3\tilde{\varrho}/(2k^2)}$$

$$\lambda_2^2 = \omega^2\left(\frac{1}{3} + \frac{\tilde{\varrho}}{2k^2}\right)$$

The first root coincides with the usual result of GR.

Conclusions

We have shown:

- In contracting astrophysical systems with rising energy density high amplitude oscillations of curvature scalar, **R**, are induced.
- Initially harmonic, these oscillations evolve to strongly unharmonic ones with high frequency and large amplitude, which could be much larger than the value of curvature in the standard **GR.**
- These spikes are damped due to gravitational particle production but the corresponding life-time could be comparable or even larger than the cosmological time.
- Structure formation in modified gravity would be very much different from that in the standard GR.
- Sufficiently large primordial clouds could not shrink down to smaller and smaller bodies with more or less uniform density but form thin shells empty (or almost empty) inside.
- This anti-gravitating behavior may also be a possible driving force for the creation of cosmic voids.