Renormalized effective action and approximate renormalized stress-energy tensor

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Introduction

- Quantum field theory in curved spacetime is a semiclassical approximation of quantum gravity.

- In quantum field theory in curved spacetime :

  first,

  - we treat classically the spacetime metric $g_{\mu\nu}$

  and, on the other hand,

  - we consider from a quantum point of view all the other fields including the graviton field to at least one-loop order for reasons of consistency.

- This approach

  - avoids the difficulties due to the nonrenormalizability of quantum gravity

  and

  - provides a framework which permits us to study the low-energy consequences of a hypothetical "theory of everything".

Semiclassical Einstein equations of quantum field theory in curved spacetime

- In this presentation we consider a four-dimensional curved spacetime $(\mathcal{M}, g)$ without boundary.
- In quantum field theory in curved spacetime, it is conjectured that the back reaction of a quantum field on the spacetime geometry is governed by the semi-classical Einstein equations

$$G_{\mu\nu} = 8\pi \langle T_{\mu\nu} \rangle_{\rm ren}$$

where

- $G_{\mu\nu}$ is the Einstein tensor $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu}$ or some higher-order generalization of this geometrical tensor,

and

- $\langle T_{\mu\nu} \rangle_{\rm ren}$ is the renormalized stress-energy tensor of the quantum field or, more precisely, the renormalized expectation value of the stress-energy tensor operator associated with the quantum field.

Successes of quantum field theory in curved spacetime

- This approach allowed theoretical physicists to obtain fascinating results concerning early universe cosmology and quantum black hole physics. In particular, it permits to discover
  - particle creation in expanding universes (Parker, 1969),
  - the black hole radiance (Hawking, 1975).
- Moreover, it provides the natural framework to analyze the cosmic microwave background observations made in recent years.
- Furthermore, the semiclassical Einstein equations have been used
  - by Starobinsky (1980) to show that, after the Planck era, quantum effects lead to an inflationary universe, i.e., a universe with an exponentially expanding de Sitter phase,
  - by several authors to analyze the dynamics of evaporating black holes due to Hawking radiation [see, e.g., Bardeen (1981), Hiscock (1981), etc.],
  - to explain the acceleration of the expansion of the universe [see, e.g., Parker and Vanzella (2004)].

Difficulties with the semiclassical Einstein equations

- However, the back reaction problem is in general difficult to tackle :
  - the semiclassical Einstein equations are a set of coupled nonlinear hyperbolic partial differential equations
  - it is in general difficult to define the right-hand side of these equations, i.e., to construct the renormalized stress-energy tensor.
- Indeed, the expectation value of the stress-energy tensor operator is formally infinite and it is necessary to regularize it, i.e.,
  - to extract from this formally infinite quantity a meaningful finite part
  - to renormalize all the coupling constants appearing in the problem in order to remove the remaining infinite part.
- Currently, there are some powerful procedures permitting us to construct it :
  - the adiabatic regularization method,
  - the dimensional regularization method,
  - the ζ-function approach,
  - the point-splitting method ...

Difficulties with the semiclassical Einstein equations (2)

- It is however important to note that, in order to obtain an *analytical* expression for the renormalized stress-energy tensor in a four-dimensional gravitational background, we have to work under very strong hypotheses, e.g.
  - we have to consider field theories in maximally symmetric spacetimes,
  - we have to study massless or conformally invariant field theories on very particular spacetimes.
- In most cases, it is even impossible to construct, from a practical point of view, the renormalized stress-energy tensor and, when this is possible, it is necessary to perform a numerical analysis in order to extract its physical content.
- So, it is interesting to note that various approaches have been developed which permit us to deal with situations presenting a "lower degree of symmetry" and to construct, in this context, accurate analytical approximations of the renormalized stress-energy tensor.
- In this presentation, we shall focus on the approximation which is based on the DeWitt-Schwinger expansion of the effective action associated with a massive quantum field.

Effective action

- The mean value of the stress-energy tensor $\langle T_{\mu\nu}\rangle$ is derived functionally with respect to the metric tensor $g_{\mu\nu}$ from the effective action W associated with a quantum field, i.e.

$$\langle T_{\mu\nu}\rangle = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} W.$$

- At one-loop order, the effective action W takes the following form

$$W = -\frac{i}{2} \mathrm{Tr} \ln(-G^F)$$

where G^F is the Feynman propagator verifying the equation

$$\hat{D}_x G^F(x, x') = -\delta(x, x')$$

with $\delta(x, x') = [-g(x)]^{-1/2} \, \delta(x - x')$ and $\hat{D}_x$ a minimal differential operator of second order.

DeWitt-Schwinger representation of the Feynman propagator

- The DeWitt-Schwinger representation of the Feynman propagator G^F is given by

$$G^F_{DS}(x,x') = i\int_0^\infty \mathrm{d}s\, H(s,x,x').$$

- The function $H(s,x,x')$ is called the heat kernel and satisfies the "heat equation"

$$\left(i\,\frac{\partial}{\partial s} + \hat{D}_x\right) H(s,x,x') = 0 \quad \text{for } s > 0$$

with the boundary condition

$$H(s,x,x') \to \delta(x,x') \quad \text{for } s \to 0.$$

- $H(s,x,x')$ can be expanded for $s \to 0$ and x' near x and then has the form

$$H(s,x,x') = \frac{i}{(4\pi i s)^{d/2}} \exp\left(\frac{i}{2s}\left[\sigma(x,x') + i\epsilon\right] - is\, m^2\right) \sum_{n=0}^{\infty} A_n(x,x')(is)^n.$$

DeWitt-Schwinger representation of the Feynman propagator (2)

$$H(s,x,x') = \frac{i}{(4\pi i s)^{d/2}} \exp\left(\frac{i}{2s}\left[\sigma(x,x') + i\epsilon\right] - is\, m^2\right) \sum_{n=0}^{\infty} A_n(x,x')(is)^n$$

- Here, d denotes the dimension of spacetime and we shall take $d = 4$ at the end of the calculations.
- The so-called geodetic interval $\sigma(x,x')$ is a biscalar function which is defined as half of the square of the geodesic distance between x and x' and which satisfies $2\sigma(x,x') = \sigma^{;\mu}\sigma_{;\mu}$. We recall also that
  - $\sigma(x,x') < 0$ if x and x' are timelike related,
  - $\sigma(x,x') = 0$ if x and x' are null related,
  - $\sigma(x,x') > 0$ if x and x' are spacelike related.
- The DeWitt coefficients $A_n(x,x')$ are biscalar functions, symmetric in the exchange of x and x' and regular for $x' \to x$. The heat equation and the boundary condition satisfied by $H(s,x,x')$ permit us to find the recursion relations for these coefficients.

DeWitt coefficients

- The DeWitt coefficients $A_n(x,x')$ are purely geometrical two-point functions formally independent of the dimension d of spacetime.
- The coefficients of lowest orders encode the short-distance singular behavior of the Feynman propagator. As consequence, their determination is an important problem.
- Unfortunately, in general, these coefficients cannot be determined exactly. It is however possible to look for them in the form of a covariant Taylor series expansion for x' in the neighborhood of x

$$A_n(x,x') = a_n(x) - a_{n\mu_1}(x)\,\sigma^{;\mu_1}(x,x') + \frac{1}{2!}\,a_{n\mu_1\mu_2}(x)\,\sigma^{;\mu_1}(x,x')\,\sigma^{;\mu_2}(x,x') - \frac{1}{3!}\,a_{n\mu_1\mu_2\mu_3}(x)\,\sigma^{;\mu_1}(x,x')\,\sigma^{;\mu_2}(x,x')\,\sigma^{;\mu_3}(x,x') + \dots$$

- The construction of these coefficients can be done with the covariant recursive method of DeWitt and with the modern covariant nonrecursive approach of Avramidi.

DeWitt coefficients (2)

- The diagonal DeWitt coefficients $a_n(x) = A_n(x, x)$ appear in connection with
  - the renormalization in the effective action for quantum field theories and quantum gravity,
  - the determination of gravitational anomalies for conformal theories.

- It must be noted that the off-diagonal DeWitt coefficients is of fundamental importance in other different contexts :
  - in stochastic semiclassical gravity,
  - in connection with the self-force problem of gravitational wave theory,
  - etc.

Renormalized effective action

- The DeWitt-Schwinger representation of the Feynman propagator G^F permits us to rewrite the effective action W at one-loop order in the form

$$W = -\frac{i}{2}\,\mathrm{Tr}\ln(-G^F_{DS}) = \int_{\mathcal{M}} \mathrm{d}x\,\sqrt{-g}\,\left\{\frac{1}{2(4\pi)^{d/2}}\sum_{n=0}^{\infty}\frac{\Gamma\left(n-d/2\right)}{\left(m^2\right)^{n-d/2}}\,a_n(x)\right\}$$

  where $a_n(x) = A_n(x,x)$ are the diagonal DeWitt coefficients.

- The expression between the braces corresponds to the Lagrangian density.

- If d is even, it is easy to see that in the sum the first $d/2+1$ terms are divergent and we have $W = W_{\mathrm{div}} + W_{\mathrm{ren}}$.

- In a four-dimensional spacetime
  - The three terms involving the diagonal DeWitt coefficients $a_0(x)$, $a_1(x)$ and $a_2(x)$ correspond to the divergent part of the effective action W_{div}. In order to remove this infinite part all the coupling constants must be redefined.
  - All the other terms in W contribute to the renormalized effective action W_{ren}.

Approximate renormalized stress-energy tensor

- In a four-dimensional curved spacetime without boundary, by functional derivation respect to the metric tensor $g_{\mu\nu}$ of the renormalized effective action $W_{\rm ren}$ we obtain the renormalized stress-energy tensor

$$\begin{aligned}\langle T_{\mu\nu}\rangle_{\rm ren} &= \frac{2}{\sqrt{-g}}\,\frac{\delta}{\delta g^{\mu\nu}}\,W_{\rm ren}\\ &= \frac{2}{\sqrt{-g}}\,\frac{\delta}{\delta g^{\mu\nu}}\int_{\mathcal{M}}\mathrm{d}x\,\sqrt{-g}\,\left\{\frac{1}{32\pi^2}\sum_{n=3}^{\infty}\frac{\Gamma\left(n-2\right)}{\left(m^2\right)^{n-2}}\,a_n(x)\right\}.\end{aligned}$$

- Its approximation in the large mass limit is given by

$$\langle T_{\mu\nu}\rangle_{\rm ren} = \frac{2}{\sqrt{-g}}\,\frac{\delta}{\delta g^{\mu\nu}}\int_{\mathcal{M}}\mathrm{d}x\,\sqrt{-g}\,\left\{\frac{1}{32\pi^2 m^2}\,a_3(x)\right\}+\mathcal{O}\left(\frac{1}{m^4}\right).$$

- To obtain the $a_3(x)$ diagonal DeWitt coefficient, it is necessary to solve the recursion relations satisfied by the DeWitt coefficients $A_0(x,x')$, $A_1(x,x')$, $A_2(x,x')$ and $A_3(x,x')$ which must be expanded up to orders σ^3, σ^2, σ^1 and σ^0 respectively.

Semiclassical Einstein equations

- In a four-dimensional curved spacetime without boundary, the semiclassical Einstein equations take the following form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} + \alpha H^{(1)}_{\mu\nu} + \beta H^{(2)}_{\mu\nu} + \gamma H^{(3)}_{\mu\nu} = 8\pi \langle T_{\mu\nu} \rangle_{\mathrm{ren}}$$

  where all the coupling constants are finite (renormalized).
- The $H^{(*)}_{\mu\nu}$ have the following expressions

$$\begin{aligned} H^{(1)}_{\mu\nu} &\equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int_{\mathcal{M}} \mathrm{d}x \sqrt{-g}\, R^2 \\ &= 2\, R_{;\mu\nu} - 2\, R R_{\mu\nu} + g_{\mu\nu}(-2\,\Box R + 1/2\, R^2), \\ H^{(2)}_{\mu\nu} &\equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int_{\mathcal{M}} \mathrm{d}x \sqrt{-g}\, R_{\alpha\beta} R^{\alpha\beta} \\ &= R_{;\mu\nu} - \Box R_{\mu\nu} - 2\, R^{\alpha\beta} R_{\alpha\mu\beta\nu} + g_{\mu\nu}(-1/2\,\Box R + 1/2\, R_{\alpha\beta} R^{\alpha\beta}), \\ H^{(3)}_{\mu\nu} &= -H^{(1)}_{\mu\nu} + 4 H^{(2)}_{\mu\nu}. \end{aligned}$$

  The last relation can be derived from the topological invariant Euler number.

Massive field theories

- We now consider three massive field theories :
  - the massive scalar field ϕ solution of the Klein-Gordon equation

$$(\Box - m^2 - \xi R)\phi = 0,$$

  - the massive spinor field ψ solution of the Dirac equation

$$(\gamma^\mu \nabla_\mu + m)\psi = 0,$$

  - the massive vector field A^μ solution of the Proca equation

$$(g_{\mu\nu}\Box - m^2 g_{\mu\nu} - \nabla_\mu \nabla_\nu - R_{\mu\nu})A^\nu = 0.$$

- In these wave equations,
  - m denotes the mass of the fields,
  - ξ is a dimensionless factor which accounts for the possible coupling between the scalar field and the gravitational background,
  - γ^μ denote the usual Dirac matrices.

Approximate renormalized stress-energy tensor for massive fields

- It should be noted that, formally, the DeWitt-Schwinger approximation can be used only when the Compton length associated with the massive field is much less than a characteristic length constructed from the curvature of spacetime.

- From the expression of the diagonal DeWitt coefficient $a_3(x)$ [see e.g. Avramidi (2000)], we have

$$(96\pi^2 m^2)\langle T_{\mu\nu}\rangle_{\rm ren} = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int_{\mathcal{M}} \mathrm{d}x \, \Big(c_1\, R\Box R + c_2\, R_{pq}\Box R^{pq} + c_3\, R^3 + c_4\, R R_{pq}R^{pq} + c_5\, R_{pq}R^{p}{}_{r}R^{qr} + c_6\, R_{pq}R_{rs}R^{prqs} + c_7\, R R_{pqrs}R^{pqrs} + c_8\, R_{pq}R^{p}{}_{rst}R^{qrst} + c_9\, R_{pqrs}R^{pquv}R^{rs}{}_{uv} + c_{10}\, R_{prqs}R^{p}{}_{u}{}^{q}{}_{v}R^{rusv}\Big) .$$

- The coefficients c_i which depend on the field are given in following table :

	Scalar field	Dirac field	Proca field		Scalar field	Dirac field	Proca field
c_1	$(1/2)\xi^2 - (1/5)\xi + 1/56$	$-3/280$	$-27/280$	c_6	$2/315$	$47/1260$	$-19/105$
c_2	$1/140$	$1/28$	$9/28$	c_7	$-(1/30)(\xi - 1/6)$	$-7/1440$	$-1/10$
c_3	$-(\xi - 1/6)^3$	$1/864$	$-5/72$	c_8	$1/1260$	$19/1260$	$61/140$
c_4	$(1/30)(\xi - 1/6)$	$-1/180$	$31/60$	c_9	$17/7560$	$29/7560$	$-67/2520$
c_5	$-8/945$	$-25/756$	$-52/63$	c_{10}	$-1/270$	$-1/108$	$1/18$

Approximate renormalized stress-energy tensor for massive fields (2)

- After the functional derivation [see Décanini and Folacci (2007)] we obtain the approximate renormalized stress-energy tensor for massive fields written in an irreducible form

$$
\begin{aligned}
(96\pi^2 m^2)\langle T_{\mu\nu}\rangle_{\rm ren} = {} & d_1\,(\Box R)_{;\mu\nu} + d_2\,\Box\Box R_{\mu\nu} + d_3\,RR_{;\mu\nu} + d_4\,(\Box R)R_{\mu\nu} + d_5\,R_{;p(\mu}R^p{}_{\nu)} + d_6\,R\Box R_{\mu\nu} \\
& + d_7\,R_{p(\mu}\Box R^p{}_{\nu)} + d_8\,R^{pq}R_{pq;(\mu\nu)} + d_9\,R^{pq}R_{p(\mu;\nu)q} + d_{10}\,R^{pq}R_{\mu\nu;pq} + d_{11}\,R^{;pq}R_{p\mu q\nu} + d_{12}\,(\Box R^{pq})R_{p\mu q\nu} \\
& + d_{13}\,R^{pq;r}{}_{(\mu}R_{|rqp|\nu)} + d_{14}\,R^p{}_{(\mu}{}^{;qr}R_{|pqr|\nu)} + d_{15}\,R^{pqrs}R_{pqrs;(\mu\nu)} + d_{16}\,R_{;\mu}R_{;\nu} + d_{17}\,R_{;p}R^p{}_{(\mu;\nu)} \\
& + d_{18}\,R_{;p}R_{\mu\nu}{}^{;p} + d_{19}\,R^{pq}{}_{;\mu}R_{pq;\nu} + d_{20}\,R^{pq}{}_{;(\mu}R_{\nu)p;q} + d_{21}\,R^p{}_{\mu;q}R_{p\nu}{}^{;q} + d_{22}\,R^p{}_{\mu;q}R^q{}_{\nu;p} + d_{23}\,R^{pq;r}R_{rqp(\mu;\nu)} \\
& + d_{24}\,R^{pq;r}R_{p\mu q\nu;r} + d_{25}\,R^{pqrs}{}_{;\mu}R_{pqrs;\nu} + d_{26}\,R^{pqr}{}_{\mu;s}R_{pqr\nu}{}^{;s} + d_{27}\,R^2R_{\mu\nu} + d_{28}\,RR_{p\mu}R^p{}_\nu + d_{29}\,R^{pq}R_{pq}R_{\mu\nu} \\
& + d_{30}\,R^{pq}R_{p\mu}R_{q\nu} + d_{31}\,RR^{pq}R_{p\mu q\nu} + d_{32}\,R^{pr}R^q{}_rR_{p\mu q\nu} + d_{33}\,R^{pq}R^r{}_{(\mu}R_{|rqp|\nu)} + d_{34}\,RR^{pqr}{}_\mu R_{pqr\nu} \\
& + d_{35}\,R_{\mu\nu}R^{pqrs}R_{pqrs} + d_{36}\,R^p{}_{(\mu}R^{qrs}{}_{|p|}R_{|qrs|\nu)} + d_{37}\,R^{pq}R^{rs}{}_{p\mu}R_{rsq\nu} + d_{38}\,R_{pq}R^{prqs}R_{r\mu s\nu} + d_{39}\,R_{pq}R^{prs}{}_\mu R^q{}_{rs\nu} \\
& + d_{40}\,R^{pqrs}R_{pqt\mu}R_{rs}{}^t{}_\nu + d_{41}\,R^{prqs}R^t{}_{pq\mu}R_{trs\nu} + d_{42}\,R^{pqr}{}_sR_{pqrt}R^s{}_\mu{}^t{}_\nu \\
& + g_{\mu\nu}\,[d_{43}\,\Box\Box R + d_{44}\,R\Box R + d_{45}\,R_{;pq}R^{pq} + d_{46}\,R_{pq}\Box R^{pq} + d_{47}\,R_{pq;rs}R^{prqs} + d_{48}\,R_{;p}R^{;p} + d_{49}\,R_{pq;r}R^{pq;r} \\
& + d_{50}\,R_{pq;r}R^{pr;q} + d_{51}\,R_{pqrs;t}R^{pqrs;t} + d_{52}\,R^3 + d_{53}\,RR_{pq}R^{pq} + d_{54}\,R_{pq}R^p{}_rR^{qr} + d_{55}\,R_{pq}R_{rs}R^{prqs} \\
& + d_{56}\,RR_{pqrs}R^{pqrs} + d_{57}\,R_{pq}R^p{}_{rst}R^{qrst} + d_{58}\,R_{pqrs}R^{pquv}R^{rs}{}_{uv} + d_{59}\,R_{prqs}R^p{}_u{}^q{}_vR^{rusv}]
\end{aligned}
$$

where the coefficients d_i can be expressed in function of the coefficients c_j.

Mathematica packages

- This expression involves Riemann polynomials of order six in the derivatives of the metric tensor. As a consequence, for spacetimes presenting a low degree of symmetry the calculation of this approximate renormalized stress-energy tensor cannot be done by hand.
- For this reason we have written some *Mathematica* packages in order to perform this computation on various spacetimes of astrophysical or cosmological interest.
  - the package *SETSphericallySymmetricST* that could be used to obtain the approximate renormalized stress-energy tensor for massive field theories on any arbitrary spherically symmetric spacetime with line element

$$ds^2 = -M_{00}(t,r)\,dt^2 + M_{11}(t,r)\,dr^2 + M_{22}(t,r)\,d\sigma_2^2$$

    where $d\sigma_2^2 = d\theta^2 + \sin^2\theta\,d\varphi^2$ denotes the metric on the unit 2-sphere S^2 and $M_{00}(t,r)$, $M_{11}(t,r)$ and $M_{22}(t,r)$ are three arbitrary functions,
  - the package *SETArbitraryST* based on the suite of *Mathematica* packages *xAct* which permits us to perform tensor algebra very efficiently and, therefore, to obtain the approximate renormalized stress-energy tensor for massive field theories on any arbitrary spacetime.

Friedmann-Lemaître-Robertson-Walker universes

- Due to the fundamental importance of the renormalized stress-energy tensor associated with quantum fields in early universe cosmology, its construction in Friedmann-Lemaître-Robertson-Walker spacetimes has often been discussed in the past forty years [see, e.g., Parker and Fulling (1974), Davies, Fulling, Christensen and Bunch (1977), etc. and, more recently, Matyjasek and al. (2014)].

- We work with coordinates such that the spacetime metric takes the form

$$ds^2 = -dt^2 + R^2(t)\left(\frac{1}{1-\kappa r^2}\, dr^2 + r^2\, d\sigma_2^2\right)$$

where

  - $R(t)$ denotes the scale factor,
  - $\kappa = -1, 0, 1$ corresponds to the three possible spatial geometries (open, flat, closed).

Friedmann-Lemaître-Robertson-Walker universes (2)

- For the massive fields, we have four non-vanishing components of the approximate renormalized stress-energy tensor :

$$\langle T^{t}{}_{t}\rangle_{\rm ren} = \frac{1}{40320\pi^2 m^2 [R(t)]^6}\Big\{v_1\kappa^3 + v_2\kappa^2[R'(t)]^2 + v_3\kappa[R'(t)]^4 + v_4[R'(t)]^6 + v_5\kappa R(t)[R'(t)]^2R''(t)$$
$$+v_6 R(t)[R'(t)]^4R''(t) + v_7\kappa[R(t)]^2[R''(t)]^2 + v_8\kappa[R(t)]^2R'(t)R^{(3)}(t) + v_9[R(t)]^2[R'(t)]^2[R''(t)]^2$$
$$+v_{10}[R(t)]^2[R'(t)]^3R^{(3)}(t) + v_{11}[R(t)]^3[R''(t)]^3 + v_{12}[R(t)]^3R'(t)R''(t)R^{(3)}(t) + v_{13}[R(t)]^3[R'(t)]^2R^{(4)}(t)$$
$$+v_{14}[R(t)]^4[R^{(3)}(t)]^2 + v_{15}[R(t)]^4R''(t)R^{(4)}(t) + v_{16}[R(t)]^4R'(t)R^{(5)}(t)\Big\},$$

$$\langle T^{r}{}_{r}\rangle_{\rm ren} = \langle T^{\theta}{}_{\theta}\rangle_{\rm ren} = \langle T^{\varphi}{}_{\varphi}\rangle_{\rm ren} = \frac{1}{40320\pi^2 m^2[R(t)]^6}\Big\{w_1\kappa^3 + w_2\kappa^2[R'(t)]^2 + w_3\kappa[R'(t)]^4 + w_4[R'(t)]^6$$
$$+w_5\kappa^2R(t)R''(t) + w_6\kappa R(t)[R'(t)]^2R''(t) + w_7R(t)[R'(t)]^4R''(t) + w_8\kappa[R(t)]^2[R''(t)]^2$$
$$+w_9\kappa[R(t)]^2R'(t)R^{(3)}(t) + w_{10}[R(t)]^2[R'(t)]^2[R''(t)]^2 + w_{11}[R(t)]^2[R'(t)]^3R^{(3)}(t) + w_{12}\kappa[R(t)]^3R^{(4)}(t)$$
$$+w_{13}[R(t)]^3[R''(t)]^3 + w_{14}[R(t)]^3R'(t)R''(t)R^{(3)}(t) + w_{15}[R(t)]^3[R'(t)]^2R^{(4)}(t) + w_{16}[R(t)]^4[R^{(3)}(t)]^2$$
$$+w_{17}[R(t)]^4R''(t)R^{(4)}(t) + w_{18}[R(t)]^4R'(t)R^{(5)}(t) + w_{19}[R(t)]^5R^{(6)}(t)\Big\}.$$

- The coefficients v_i and w_i can be expressed in function of the coefficients c_j which depend on the field.

Kerr-Newman spacetime

- The renormalization of the stress-energy tensor in Kerr-Newman spacetime has never been considered, despite of the physical importance of this black hole. This is due to the complexity of the calculations involved. In arXiv:1404.7422 (Belokogne and Folacci, 2014) we have addressed this subject for the first time.

- A charged and rotating black hole is described by the Kerr-Newman metric which takes the following form in Boyer-Lindquist coordinates

$$ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right) dt^2 - 2\,\frac{a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}\,dt d\varphi + \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta\, d\varphi^2 + \frac{\Sigma}{\Delta}\,dr^2 + \Sigma\, d\theta^2$$

  where
  - $\Delta = r^2 - 2Mr + a^2 + Q^2$ and $\Sigma = r^2 + a^2\cos^2\theta$,
  - M, Q and $J = aM$ are the mass, the charge and the angular momentum of the black hole while a is the so-called rotation parameter.

- Here, we assume $M^2 \geq a^2 + Q^2$ and the outer horizon is located at $r_+ = M + \sqrt{M^2 - (a^2+Q^2)}$, the largest root of Δ.

Kerr-Newman spacetime (2)

- For massive fields propagating on Kerr-Newman spacetime, the expressions of the approximate renormalized stress-energy tensor are complicated and, for this reason, we give here only their structure. We have
  - for the four diagonal components :

$$\langle T^{t}{}_{t}\rangle_{\mathrm{ren}} = \frac{M^2 r^{10}}{10080\pi^2 m^2 \Sigma^9} \sum_{p=0}^{10} \left\{ \sum_{q=0}^{3} A^{tt}{}_{p,q}\left[\theta, \frac{M}{r}\right] \left(\frac{Q^2}{M^2}\right)^q \right\} \left(\frac{a}{r}\right)^p ,$$

    and similar expressions for $\langle T^{r}{}_{r}\rangle_{\mathrm{ren}}$, $\langle T^{\theta}{}_{\theta}\rangle_{\mathrm{ren}}$ and $\langle T^{\varphi}{}_{\varphi}\rangle_{\mathrm{ren}}$,
  - for the four off-diagonal non-vanishing components :

$$\langle T^{\varphi}{}_{t}\rangle_{\mathrm{ren}} = \frac{M^2 r^{9}}{5040\pi^2 m^2 \Sigma^9} \sum_{p=1}^{9} \left\{ \sum_{q=0}^{3} A^{\varphi t}{}_{p,q}\left[\theta, \frac{M}{r}\right] \left(\frac{Q^2}{M^2}\right)^q \right\} \left(\frac{a}{r}\right)^p ,$$

$$\langle T^{t}{}_{\varphi}\rangle_{\mathrm{ren}} = \frac{M^2 r^{11}\sin^2\theta}{5040\pi^2 m^2 \Sigma^9} \sum_{p=1}^{11} \left\{ \sum_{q=0}^{3} A^{t\varphi}{}_{p,q}\left[\theta, \frac{M}{r}\right] \left(\frac{Q^2}{M^2}\right)^q \right\} \left(\frac{a}{r}\right)^p ,$$

$$\langle T^{\theta}{}_{r}\rangle_{\mathrm{ren}} = \frac{M^2 r^{9}\sin\theta\cos\theta}{180\pi^2 m^2 \Sigma^9} \sum_{p=2}^{8} \left\{ \sum_{q=0}^{2} A^{\theta r}{}_{p,q}\left[\theta, \frac{M}{r}\right] \left(\frac{Q^2}{M^2}\right)^q \right\} \left(\frac{a}{r}\right)^p ,$$

$$\langle T^{r}{}_{\theta}\rangle_{\mathrm{ren}} = \frac{M^2 r^{11}\sin\theta\cos\theta}{180\pi^2 m^2 \Sigma^9} \sum_{p=2}^{10} \left\{ \sum_{q=0}^{3} A^{r\theta}{}_{p,q}\left[\theta, \frac{M}{r}\right] \left(\frac{Q^2}{M^2}\right)^q \right\} \left(\frac{a}{r}\right)^p .$$

Kerr-Newman spacetime (3)

- We cannot present all the coefficients $A^{\mu\nu}{}_{p,q}\left[\theta, M/r\right]$. We only give some coefficients $A^{tt}{}_{p,q}\left[\theta, M/r\right]$ for the Proca field. They vanish if p is odd and we have

$$A^{tt}{}_{0,0}\left[\theta, M/r\right] = 1665 - 3666\,(M/r)$$
$$A^{tt}{}_{0,1}\left[\theta, M/r\right] = 12150 - 69024\,(M/r) + 93537\,(M/r)^2$$
$$A^{tt}{}_{0,2}\left[\theta, M/r\right] = 41854\,(M/r)^2 - 107516\,(M/r)^3$$
$$A^{tt}{}_{0,3}\left[\theta, M/r\right] = 31057\,(M/r)^4$$
$$A^{tt}{}_{2,0}\left[\theta, M/r\right] = -44955\cos^2\theta + 72\left(1528\cos^2\theta + 5\right)(M/r)$$
$$A^{tt}{}_{2,1}\left[\theta, M/r\right] = -12150\left(7\cos^2\theta - 12\right) + 36\left(10435\cos^2\theta - 13313\right)(M/r) - 12\left(36771\cos^2\theta - 23602\right)(M/r)^2$$
$$A^{tt}{}_{2,2}\left[\theta, M/r\right] = -4\left(47215\cos^2\theta - 69264\right)(M/r)^2 + 4\left(91476\cos^2\theta - 81229\right)(M/r)^3$$
$$A^{tt}{}_{2,3}\left[\theta, M/r\right] = -(86153\cos^2\theta - 92104)\,(M/r)^4$$
$$\ldots$$

- In view of the complexity of these results, the exact expression for any value of the mass (and not only in the large mass limit) of the renormalized stress-energy tensor in Kerr-Newman spacetime is completely out of reach.

Kerr-Newman spacetime (4)

- By putting $Q = 0$ into the expression of the renormalized stress-energy tensor of a massive field in Kerr-Newman spacetime, we can recover the results obtained in Kerr spacetime.

- The same can be done for Schwarzschild and Reissner-Nordström spacetimes.

Shift in mass and angular momentum of the black hole

- As an application, we can determine the shift in mass and angular momentum of the black hole measured by a distant observer due to the non-vanishing renormalized stress-energy tensor.
- For a stationary axisymmetric black hole, we recall that the mass M_D and the angular momentum J_D of the black hole dressed with a quantum field can be expressed in terms of its mass M and its angular momentum J by

$$M_D - M = 2\int_{\mathcal{S}} \left(\langle T^{\mu}{}_{\nu} \rangle_{\mathrm{ren}} - \frac{1}{2} g^{\mu}{}_{\nu} \langle T^{\rho}{}_{\rho} \rangle_{\mathrm{ren}} \right) \xi^{\nu} \, d\mathcal{S}_{\mu},$$

$$J_D - J = -\int_{\mathcal{S}} \langle T^{\mu}{}_{\nu} \rangle_{\mathrm{ren}} \psi^{\nu} \, d\mathcal{S}_{\mu}$$

where

  - $\langle T_{\mu\nu} \rangle_{\mathrm{ren}}$ is the renormalized stress-energy tensors of the quantum field,
  - $\xi^{\mu} = (\partial_t)^{\mu}$ and $\psi^{\mu} = (\partial_{\varphi})^{\mu}$ denote the two Killing vectors of the Kerr-Newman black hole,
  - $\mathcal{S}$ is any spacelike hypersurface that extends from the outer horizon at r_+ to spatial infinity and $d\mathcal{S}_{\mu} = -\Sigma \sin\theta \, dr \, d\theta \, d\varphi \, (dt)_{\mu}$ is the associated surface element.

Shift in mass and angular momentum of the black hole (2)

- In order to simplify the expression for the shift in mass and angular momentum we assume $a \ll M$ and $Q \ll M$ and then we obtain

$$M_D - M = \frac{M}{336 \times 7! \pi m^2 M^4} \left\{ \alpha_0 + \alpha_1 \left(\frac{a}{M}\right)^2 + \alpha_2 \left(\frac{Q}{M}\right)^2 + \dots \right\},$$

$$J_D - J = \frac{1}{480 \times 7! \pi m^2 M^2} \left(\frac{a}{M}\right) \left\{ \beta_0 + \beta_1 \left(\frac{a}{M}\right)^2 + \beta_2 \left(\frac{Q}{M}\right)^2 + \dots \right\}$$

where

- the dots denote terms of fourth order (i.e., in $(a/M)^4$ or in $(Q/M)^4$ or in $a^2 Q^2/M^4$),
- the coefficients α_i and β_i are given in two tables :

	Scalar field	Dirac field	Proca field
α_0	$-392(9\xi - 2)$	196	-1176
α_1	$14(63\xi - 16)$	7	210
α_2	-66	-162	978

	Scalar field	Dirac field	Proca field
β_0	$-60(84\xi - 17)$	480	-1980
β_1	$-9(252\xi - 53)$	180	-837
β_2	$20(714\xi - 145)$	-640	19580

Conclusion

- The approximate expressions of the renormalized stress-energy tensor obtained in various spacetimes are based on the DeWitt-Schwinger expansion of the effective action associated with a massive quantum field. As a consequence, they do not take into account the quantum state of the field and are only valid in the large mass limit.
- For Friedmann-Lemaître-Robertson-Walker spacetimes
  - An analytical approximation can be used to simplify the back reaction problem or to obtain an analytical expression for the density and the pressure associated with the quantum field.
- For Kerr-Newman spacetime
  - These expressions neglect the existence of superradiance instabilities for massive fields in rotating black holes. This seems quite reasonable in the large mass limit.
  - Our results could be very helpful to study the back reaction of massive quantum fields on this gravitational background. Here, we have limited this problem to the determination of the shift in mass and angular momentum of the black hole (measured by a distant observer) due to the renormalized stress-energy tensor.
  - Our results could be also used to study the quasinormal modes of the Kerr-Newman black hole dressed by a massive quantum field.