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Dynamical Bridges

Eduardo Bittencourt

Sapienza Università di Roma and ICRANet

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Collaborators: M. Novello, U. Moschella, E. Goulart, J.M. Salim, J.D. Toniato and S. Faci

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Outline on DB

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Concluding Remarks

- New way of describing dynamical equations for fundamental fields.
- Mathematical map between dynamics acting on the space-time metrics.
- For free fields, the solutions of a given dynamic in a specific metric are mapped in solutions of another dynamic in another metric.

Motivation

Analogously to Gordon's paper[1], we study specific geometries such that the metric and its inverse assume a binomial form

$$\widehat{q}_{\mu\nu} = A\,\eta_{\mu\nu} + B\,\Phi_{\mu\nu}, \qquad \widehat{q}^{\mu\nu} = \alpha\,\eta^{\mu\nu} + \beta\,\Phi^{\mu\nu}, \tag{1}$$

where A, B, α and β are functions, $\eta^{\mu\nu}$ is the Minkowski metric and $\Phi^{\mu\nu}$ is a symmetric tensor satisfying

$$\Phi_{\mu\nu}\,\Phi^{\nu\lambda} = m\,\delta^{\lambda}_{\mu} + n\,\Phi^{\lambda}_{\mu}. \tag{2}$$

Note that m and n are not completely arbitrary functions.

[1] W. Gordon, *Ann. Phys. (Leipzig)* **72** 421 (1923).

Examples:

- Scalar fields: $\Phi_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi$;
- Electromagnetic fields: $\Phi_{\mu\nu} = F_\mu{}^\alpha F_{\alpha\nu}$;
- Spinor fields: $\Phi_{\mu\nu} = (J_\mu + \epsilon I_\mu)(J_\nu + \epsilon I_\nu)$.

We analyze each case in details afterwards.

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Geodesic motion condition

The Dynamical Bridges also present interesting results in the kinematics, as follows:

Theorem

Let Γ be a congruence of curves represented by the vector field V_μ in a given space-time. This vector follows along a geodesic in the metric $\hat{q}_{\mu\nu}$, if the condition below is satisfied[2]

$$\frac{1}{2}\hat{N}_{,\mu} + V_{[\mu,\nu]}\hat{q}^{\nu\alpha}V_\alpha - p(\lambda)V_\mu = 0,$$

where $\hat{N}$ is the norm of V_μ calculated with $\hat{q}_{\mu\nu}$ and $p(\lambda)$ is a function of the affine parameter λ.

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[2] M. Novello and EB, PRD **86** 124024 (2012); M. Novello and EB, GERG **45** 1005 (2013).

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Polynomial Metrics

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Applying the theorem for a special set of metrics, called *dragged metrics*, which depend on the background and the observer field under consideration

(a) $\hat{q}_{\mu\nu} = A\eta_{\mu\nu} + BV_\mu V_\nu, \qquad \hat{q}^{\mu\nu} = \alpha\eta^{\mu\nu} + \beta V^\mu V^\nu.$

(b) $\hat{q}_{\mu\nu} = A\eta_{\mu\nu} + BV_\mu V_\nu + \Delta a_\mu a_\nu + \Lambda a_{(\mu} V_{\nu)},$

$$\hat{q}^{\mu\nu} = \alpha\eta^{\mu\nu} + \beta V^\mu V^\nu + \delta a^\mu a^\nu + \lambda a^{(\mu} V^{\nu)}.$$

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Ex. 1: Normalized vector fields

Consider V_μ normalized with acceleration a_μ in the Minkowski space. The condition for V_μ follows along a geodesic in $\hat{q}_{\mu\nu}$ (given by (a)) is

$$a_\mu = -\frac{1}{2}\frac{(\alpha+\beta)_{,\mu}}{\alpha+\beta}. \qquad (3)$$

It means that

$$\alpha+\beta = e^{-2\Psi},$$

where $a_\mu \equiv \partial_\mu \Psi$ and Ψ is a scalar potential.

Ex. 2: Gradient vector fields

Consider $k_\mu \equiv \partial_\mu \Sigma$ with norm N. In N is not a constant, then k_μ is an accelerated vector in the background geometry. The condition for k_μ follows along a geodesic in $\hat{q}_{\mu\nu}$ (given by (b)) is

$$\hat{N} \equiv const. = (\alpha + \beta N + \lambda \dot{N})N + \frac{\delta}{4}\dot{N}^2, \tag{4}$$

where $\hat{N}$ is the norm calculated with $\hat{q}_{\mu\nu}$ and $\dot{N} \equiv N_{,\mu}\hat{q}^{\mu\nu}k_\nu$.

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Electromagnetic DB: Born-Infeld[3]

Consider the Born-Infeld Lagrangian

$$L = \beta^2 \left(1 - \sqrt{\hat{U}}\right), \tag{5}$$

where $\hat{U} \equiv 1 + \hat{F}/(2\beta^2) - \hat{G}^2/(16\beta^4)$, defined in

$$\hat{e}^{\mu\nu} \equiv a\,\eta^{\mu\nu} + b\,\Phi^{\mu\nu} \quad \text{and} \quad \Phi_{\mu\nu} \equiv F_{\mu\alpha}\,F^{\alpha}{}_{\nu} \tag{6}$$

where $\hat{F} \equiv F_{\mu\nu}\,F_{\alpha\beta}\hat{e}^{\mu\alpha}\hat{e}^{\nu\beta}$ and $G \equiv F_{\mu\nu}\,F^{*}_{\alpha\beta}\hat{e}^{\mu\alpha}\hat{e}^{\nu\beta}$.
The dynamical equations are

$$\frac{1}{\sqrt{-\hat{e}}}\partial_\nu \left[\frac{\sqrt{-\hat{e}}}{\hat{U}}\left(\hat{F}^{\mu\nu} - \frac{1}{4\,\beta^2}\,\hat{G}\,\hat{F}^{*\mu\nu}\right)\right] = 0. \tag{7}$$

[3] M. Novello and EB, *IJMPA* **29** 1450075 (2014)

Electromagnetic DB: Maxwell

Consider the Maxwell Lagrangian

$$L = -\frac{1}{4}F \tag{8}$$

where $F \equiv F_{\mu\nu}\, F_{\alpha\beta}\eta^{\mu\alpha}\eta^{\nu\beta}$ defined in Minkowski space $\eta_{\mu\nu}$. The dynamical equations are

$$\frac{1}{\sqrt{-\eta}}\partial_\nu \left(\sqrt{-\eta}F^{\mu\nu}\right) = 0. \tag{9}$$

To demonstrate this equivalence is necessary to know how $\hat{F}^{\mu\nu}$ and its dual are related to the Maxwell's fields. In this case, the correspondence is

$$\left\{ \begin{array}{rcl} n - \epsilon F m + \dfrac{\hat{G}}{2\beta^2}\epsilon m & = & -\dfrac{Q}{4}, \\ -\epsilon G m + \dfrac{\hat{G} n}{2\beta^2} & = & 0, \end{array} \right. \tag{10}$$

where $\epsilon \equiv b/a$ and

$$n = 1 + \frac{\epsilon^2 G^2}{16}, \quad m = 1 - \frac{\epsilon F}{4}, \quad Q = 1 - \frac{\epsilon F}{2} - \frac{\epsilon^2 G^2}{16}.$$

Special case

- First order corrections in $F \ll \beta^2$.
- Suppose $G = 0$ [second Eq. of (10) is trivial] and then choose $a = 1$.
- In this case, the coefficient ϵ can only be

$$\epsilon = \frac{2}{F}\left(1 - \frac{1}{\sqrt{1 - F/2\beta^2}}\right).$$

- Finally, the EM metric is

$$\hat{e}_{\mu\nu} \approx \eta_{\mu\nu} + \frac{1}{2\,\beta^2}\,\Phi_{\mu\nu}. \tag{11}$$

Interaction procedure for leptons

- *Universality*: all particles (charged or not) interact with $\hat{e}_{\mu\nu}$.
- The interaction is done through the *minimal coupling principle*.
- The Clifford algebra should be preserved in both space-times. Then, from Eq. (11) we obtain

$$\hat{\gamma}^{\mu} = \gamma^{\mu} - \frac{1}{4\,\beta^{2}}\Phi^{\mu}{}_{\alpha}\gamma^{\alpha}. \tag{12}$$

Coupling uncharged particles to the EM field

In the curved space, the dynamics of Ψ is

$$i\hbar c\hat{\gamma}^{\mu}\hat{\nabla}_{\mu}\Psi - mc^2\Psi = 0, \tag{13}$$

and $\hat{\nabla}_{\mu} \equiv \partial_{\mu} - \hat{\Gamma}^{FI}_{\mu} - \hat{V}_{\mu}$, where $\hat{V}_{\mu}$ is an arbitrary element of the algebra given by

$$\hat{V}_{\mu} = i\frac{mc}{\hbar}\frac{F_{\mu\nu}}{\beta}\hat{\gamma}^{\nu}\gamma_5. \tag{14}$$

This gives $\hat{\nabla}_{\mu}\hat{\gamma}^{\nu} = [\hat{V}_{\mu}, \hat{\gamma}^{\nu}]$, but maintains $\hat{\nabla}_{\alpha}\hat{e}^{\mu\nu} = 0$ and preserves the current $\hat{\nabla}_{\mu}\hat{J}^{\mu} = 0$.

Using Eqs. for $\hat{e}_{\mu\nu}$, $\hat{\gamma}^{\mu}$ and $\hat{V}_{\mu}$, we obtain the following equation for Ψ in Minkowski space

$$i\,\hbar c\,\gamma^{\mu}\partial_{\mu}\,\Psi + \frac{mc^2}{2\beta}\,F_{\mu\nu}\,[\gamma^{\mu},\gamma^{\nu}]\gamma_5\,\Psi - m\,c^2\,\Psi = 0. \qquad (15)$$

This equation provides a magnetic moment for Ψ, whose the intensity is

$$\mu^G = \frac{m\,c^2}{\beta}. \qquad (16)$$

We named this contribution as *geometric magnetic moment*.

Geometric magnetic moment for charged particles

Charged particles have a classical source for μ. For electrons, it is the Bohr magneton $\mu_B = e\,\hbar/2m_e$. Then, the total value of the electron magnetic moment should be read as

$$\mu_e = \mu_B + \frac{m_e\,c^2}{\beta} + \text{quantum corrections.}$$

The first term is classical, the second one is the geometric magnetic moment and then we have quantum corrections (due to the charge).

Comparison with experiments

Anomalous magnetic moment: the value a_l of the anomaly of the magnetic moment μ_l of the particle l is defined as

$$a_l = \frac{g_l - 2}{2} = \frac{m_l}{m_e}\frac{\mu_l}{\mu_B} - 1, \qquad (17)$$

where $l = (e, \mu, \tau)$, g_l is the Landé factor and m_l is the mass[4].

Data: there are measurements of the anomaly for e^- (the most precise) and for μ (test the entire SM). For τ, it is unobservable due to its short mean-life ($\sim 2.9 \times 10^{-13}$ s).

[4] Data from the *Particle Data Group* (2013).

μ-case

The difference Δa between the experimental and theoretical value of the anomaly for muons gives the upper limit in which μ^G can contribute. So, if the effects of $\hat{e}^{\mu\nu}$ appear at this order, we have

$$\mu_\mu^G \doteq \Delta a_\mu \frac{e\hbar}{2m_\mu}. \tag{18}$$

On the other hand, the formula is given by

$$\mu_\mu^G = \frac{m_\mu c^2}{\beta}. \tag{19}$$

Assuming that μ_μ^G correspond to the remaining anomaly of muon, then we can estimate the critical field:

$$\beta \approx 1.31 \times 10^{23}\ T.$$

e^--case

Assuming this value for β, the geometric corrections for the electron are

$$\mu_e^G = 6.74 \times 10^{-14} \mu_B. \tag{20}$$

The difference between theory and experiments is

$$\Delta a_e = -0.40 \times 10^{-12}. \tag{21}$$

We can see that they are compatible with the effects of $\hat{e}^{\mu\nu}$.

ν-case

It is expected that μ_ν and its quantum corrections are zero. However, the extended SM suggests $\mu_\nu \neq 0$ and $\mu_\nu \propto m_\nu$. Then, it seems natural to propose that μ_ν has only the geometrical origin. In this case,

$$\mu_\nu = \frac{m_\nu c^2}{\beta} = 1.32 \times 10^{-19} \mu_B. \tag{22}$$

This value is in agreement with[5] phenomenology and close to the ESM value $(3,2 \times 10^{-19} \mu_B)$.

[5] A.V. Kuznetsov e N.V. Mikheev, *JCAP* **11** 031 (2007); A.B. Balantekin, *Proc. OMEG05* (2006); O. Lychkovskiy e S. Blinnikov, *Phys. Atom. Nucl.* **73** 614 (2010).

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Chiral symmetry breaking without mass[6]

- Due to symmetry properties, it is not possible to give mass for ν in the SM.
- Several distinct experiments detect neutrino oscillation.
- They are compatible with lightly massive neutrinos, breaking the chiral symmetry.

Then we face the question: where are the right-handed neutrinos?

[6] EB et al., in preparation.

Spinor DB: Dirac

Consider the Dirac equation

$$i\hat{\gamma}^{\mu}\hat{\nabla}\Psi = 0, \tag{23}$$

in the metric

$$\hat{g}^{\mu\nu} = \eta^{\mu\nu} + 2\beta H^{\mu}H^{\nu} \tag{24}$$

and tetrad basis

$$\hat{e}^{\mu}{}_{A} = e^{\mu}{}_{A} + \beta H^{\mu}H_{A},$$

where $H^{\mu} = J^{\mu} - I^{\mu}$, $J^{\mu} = \bar{\Psi}\gamma^{\mu}\Psi$ and $I^{\mu} = \bar{\Psi}\gamma^{\mu}\gamma_5\Psi$.

Spinor DB: NJL

The corresponding dynamics in Minkowski space is a generalization of a Nambu-Jona-Lasinio (NJL) model

$$\begin{aligned} i\gamma^{\mu}\nabla\Psi_R + i\dot{\beta}(A + iB\gamma_5)\Phi_L = 0, \\ i\gamma^{\mu}\nabla\Psi_L = 0, \end{aligned} \tag{25}$$

where $A \equiv \bar{\Psi}\Psi$ and $B \equiv i\bar{\Psi}\gamma_5\Psi$.

Further comments:

- There is an exact solution given by the Inomata condition $\partial_\mu \Psi = -(1/2)\dot{\beta} H_\mu \Psi$.
- For massive fermions, we expected that the self-interacting term should be negligible compared to the mass.
- Main drawback: J^μ is not conserved in flat space for the right-handed component. Possibly, a conformal transformation in Eq. (24) can solve the problem.

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Formulation of the GSG

Let us summarize the main properties of the GSG[7]:

- The gravity is described by Φ which satisfies a nonlinear dynamics;
- The theory satisfies the principle of general covariance and minimal coupling;
- All kinds of matter and energy interact with Φ only through

$$q^{\mu\nu} = \alpha\eta^{\mu\nu} + \beta\frac{\partial^\mu\Phi\partial^\nu\Phi}{w},$$

  i.e., gravity is a **geometrical phenomenon**;
- Test particles follow geodesics relative to $q^{\mu\nu}$;
- Φ is nontrivially related to Φ_N;

[7] M. Novello, EB *et al.*, *JCAP* **06** 014 (2013).

Scalar DB or the birth of the geometry[8]

Theorem

Consider a lagrangian $L = V(\Phi) w$, whose the dynamics is

$$\frac{1}{\sqrt{-\eta}} \partial_\mu \left(\sqrt{-\eta}\, \eta^{\mu\nu}\, \partial_\nu \Phi \right) + \frac{1}{2} \frac{V'}{V} w = 0. \tag{26}$$

This equation is equivalent to the Klein-Gordon equation

$$\Box \Phi = 0$$

in the metric $q^{\mu\nu}$, where $\alpha(\Phi)$ and $\beta(\Phi)$ satisfy $\alpha + \beta = \alpha^3 V$.

[8] M. Novello and E. Goulart, *Class. Quantum Grav.* **28** 145022 (2011); E. Goulart *et al.*, *Class. Quantum Grav.* **28** 245008 (2011).

Static and spherically symmetric solution

Let us start with flat background

$$ds_M^2 = dt^2 - \alpha \left(\frac{1}{2\alpha} \frac{d\alpha}{dr} r + 1 \right)^2 dr^2 - \alpha r^2 \, d\Omega^2, \tag{27}$$

where r is related to the radial coordinate R through $R = \sqrt{\alpha(r)}\, r$. The static and spherically symmetry suggests $\Phi = \Phi(r)$. Therefore, the gravitational metric becomes

$$ds^2 = \frac{1}{\alpha} dt^2 - B\, dr^2 - r^2 \, d\Omega^2, \tag{28}$$

where $B \equiv \frac{\alpha}{\alpha+\beta} \left(\frac{1}{2\alpha} \frac{d\alpha}{dr} r + 1 \right)^2$ and $\alpha = e^{-2\,\Phi}$.

By successive approximations, we obtained an *Ansatz* for V, as follows:

$$V(\Phi) = \frac{(e^{\Phi} - 3e^{3\Phi})^2}{4} \tag{29}$$

and then, the solution of the field equation yields

$$\Phi = \frac{1}{2} \ln \left(1 - \frac{r_H}{r}\right), \tag{30}$$

where $r_H \equiv 2MG/c^2$. Now, the line element can be written down as

$$ds^2 = \left(1 - \frac{r_H}{r}\right) dt^2 - \left(1 - \frac{r_H}{r}\right)^{-1} dr^2 - r^2 d\Omega^2. \tag{31}$$

Note that GSG reproduces solar system tests with a good approximation.

Dynamics in the presence of matter

We start with the action for Φ written in the Minkowski metric. Then, we use the DB to rewrite it in $q^{\mu\nu}$ as

$$\delta S_M = -2 \int \sqrt{-q}\, d^4x \sqrt{V}\, \Box\Phi\, \delta\Phi. \tag{32}$$

Then, we add the matter content L_m and make the variation wrt $q^{\mu\nu}$ and obtain the usual conserved $T_{\mu\nu}$. As $q_{\mu\nu}$ is not the fundamental quantity and we must rewrite $\delta q_{\mu\nu}$ as function of $\delta\Phi$.

After some calculation, we get

$$\sqrt{V}\,\Box\Phi = \kappa\chi, \tag{33}$$

where the quantity χ involves a nontrivial coupling between $\nabla_\mu\Phi$ and $T_{\mu\nu}$. The Newtonian limit of the theory is obtained by fixing $\kappa \equiv 8\pi G/c^4$.

Concluding Remarks

- Class of geometries that describe accelerated motion in a given space as geodetic one in another space.
- Vector case: a new description of the anomalous magnetic moment, through a minimal coupling principle.
- Spinor case: an alternative explanation of the chiral symmetry breaking.
- Scalar case: local gravitational tests verified by a scalar theory of gravity.
- A universality *a priori* of the space-time geometry is left aside; changing the metric we were able to describe the physical world in accordance with the postulate we choose.

Thanks for your attention!