AN EXACTLY SOLVABLE INFLATIONARY MODEL

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• Inflationary models are the most successful attempt to describe the early stages of development of the universe, and in particular are at the basis of the current explanations of the formation of structures.

• Usually, their dynamics is derived from models of gravity minimally coupled to a scalar field with suitable potential. The scalar field starts its evolution from a value that does not minimize the potential and then rolls toward the minimum, inducing the inflation of the scale factor of the universe.

• In most cases, these models can only be treated in an approximate way (slow-roll approximation). The approximation breaks down when the field is close to the minimum of the potential, where the inflation ends. Later evolution of the universe is assumed to obey the standard Friedmann-Lemaitre equations and is characterized by a power-law expansion.

• It would be interesting to find exact solutions that describe the transition from an exponential expansion to a later Friedmann-Lemaitre behavior in a smooth way.

• For this purpose, it may be useful to exploit the observation (Skenderis, Townsend) that cosmological solutions of models of gravity coupled to a scalar can be obtained by analytic continuation to imaginary values of time and radial coordinates of domain-wall solutions of the same model with opposite scalar potential.

• I present an application of this observation, leading to an exact solution in which the scale factor expands exponentially at early times and exhibits to a power-law behavior for late times.

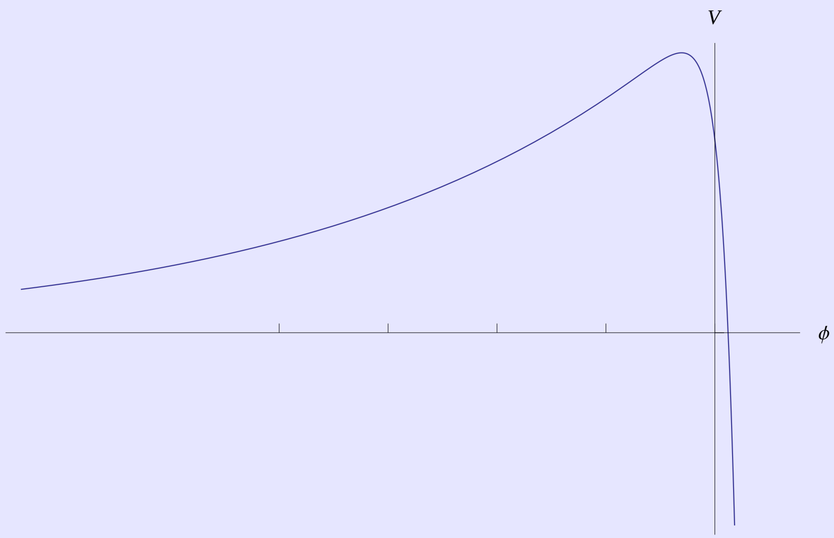
The model is based on gravity minimally coupled to a scalar field ϕ, with action

$$I = \int \sqrt{-g}\,\left[R - 2(\partial\phi)^2 - V(\phi)\right] d^4x$$

where the scalar potential

$$V(\phi) = \frac{2\lambda^2}{3\gamma}\left(e^{2\sqrt{3}\beta\phi} - \beta^2 e^{2\sqrt{3}\phi/\beta}\right)$$

depends on two parameters λ and β, with $\gamma = 1 - \beta^2$.



• The potential vanishes for $\phi \to -\infty$, has a maximum for $\phi = 0$, where it takes the value $V_0 = 2\lambda^2/3$, and goes to $-\infty$ for $\phi \to \infty$.

• $V(\phi)$ admits a duality for $\beta \to 1/\beta$. In the following we shall consider only the case $0 < \beta^2 < 1$.

• In the limit $\beta \to 0$, it reduces to a cosmological constant, while for $\beta^2 = 1$ it vanishes.

• A solution with $\phi = 0$ exists, that coincides with that of pure gravity with cosmological constant $\Lambda = 2\lambda^2/3$. This is of course the de Sitter solutions with cosmological constant Λ.

• The model with $V \to -V$ was considered in the context of AdS/CFT correspondence (Cadoni, S.M., Serra) and it was shown to admit solitonic domain-wall solutions of the form

$$\begin{aligned} ds^2 &= \hat{R}^{-\frac{2}{1+3\beta^2}} \left(1 + \mu \hat{R}^{-\frac{3\gamma}{1+3\beta^2}}\right)^{\frac{2\beta^2}{\gamma}} d\hat{R}^2 \\ &+ \hat{R}^{\frac{2}{1+3\beta^2}} \left(1 + \mu \hat{R}^{-\frac{3\gamma}{1+3\beta^2}}\right)^{\frac{2\beta^2}{3\gamma}} (-d\hat{T}^2 + d\bar{s}_2^2) \end{aligned}$$

with μ a free parameter.

• This metric interpolates between anti-de Sitter for $\hat{R} \to 0$ and domain-wall behavior for $\hat{R} \to \infty$.

• Analytically continuing this solution for $\hat{R} \to iT$, $\hat{T} \to iR$, one can obtain a cosmological solution that behaves as a de Sitter universe for $T \to 0$ and as a Friedmann universe with power-law expansion for $T \to \infty$. It is then a promising candidate to describe the evolution of an inflationary universe.

- However, this is not the most general cosmological solution of the model.
- It is therefore important to investigate the general solutions in order to see if this behavior is generic and to understand if it can describe a viable cosmological model.
- The system is exactly integrable if we choose a suitable parametrization of the metric.

• We are interested in the general isotropic and homogeneous cosmological solutions, with flat spatial sections, that we parametrize as

$$ds^2 = -e^{2a(\tau)}d\tau^2 + e^{2b(\tau)}d\bar{s}_3^2, \qquad \phi = \phi(\tau)$$

with τ a time variable.

• With this parametrization, the vacuum Einstein equations read

$$3\dot{b}^2 = \dot{\phi}^2 + \frac{V}{2}\,e^{2a},$$
$$2\ddot{b} + \dot{b}(3\dot{b} - 2\dot{a}) = -\dot{\phi}^2 + \frac{V}{2}\,e^{2a}$$

where a dot denotes a derivative with respect to τ.

• The scalar field obeys the equation

$$\ddot{\phi} + (3\dot{b} - \dot{a})\dot{\phi} = -\frac{1}{4}\frac{dV}{d\phi}\,e^{2a}$$

- In the gauge $b = a/3$ the previous equations simplify:

$$\frac{\dot{a}^2}{3} = \dot{\phi}^2 + \frac{1}{2} V e^{2a},$$
$$\ddot{a} = \frac{3}{2} V e^{2a},$$
$$\ddot{\phi} = -\frac{1}{4}\frac{dV}{d\phi}\, e^{2a}$$

- Defining new variables

$$\psi = a + \sqrt{3}\beta\phi, \qquad \chi = a + \frac{\sqrt{3}}{\beta}\phi$$

the field equations take the very simple form

$$\ddot{\psi} = \lambda^2 e^{2\psi}, \qquad \ddot{\chi} = \lambda^2 e^{2\chi},$$
$$\dot{\psi}^2 - \beta^2\dot{\chi}^2 = \lambda^2(e^{2\psi} - \beta^2 e^{2\chi})$$

- Note: they are invariant under time reversal, $\tau \to -\tau$.

• One can immediately write down the first integrals

$$\dot{\psi}^2 = \lambda^2 e^{2\psi} + Q_1, \qquad \dot{\chi}^2 = \lambda^2 e^{2\chi} + Q_2$$

with Q_1 and Q_2 integration constants, which satisfy

$$Q_1 = \beta^2 Q_2$$

• A further integration gives the solutions of the system, that depends on the sign of the integration constants.

- If $Q_i = q_i^2 > 0$ $(i = 1, 2)$,

$$\lambda^2 e^{2\psi} = \frac{q_1^2}{\sinh^2[q_1(\tau - \tau_1)]}, \qquad \lambda^2 e^{2\chi} = \frac{q_2^2}{\sinh^2[q_2(\tau - \tau_2)]}$$

with τ_1, τ_2, integration constants and $q_1^2 = \beta^2 q_2^2$.

- If $Q_1 = Q_2 = 0$,

$$\lambda^2 e^{2\psi} = \frac{1}{(\tau - \tau_1)^2}, \qquad \lambda^2 e^{2\chi} = \frac{1}{(\tau - \tau_2)^2}$$

- If $Q_i = -q_i^2 < 0$,

$$\lambda^2 e^{2\psi} = \frac{q_1^2}{\sin^2[q_1(\tau - \tau_1)]}, \qquad \lambda^2 e^{2\chi} = \frac{q_2^2}{\sin^2[q_2(\tau - \tau_2)]}$$

with $q_1^2 = \beta^2 q_2^2$.

- In the following we define $q = q_2 = q_1/\beta$.

The general solutions of the Friedman equations are therefore ($\varphi \equiv \sqrt{3}\phi/\beta$):

- If $Q_i > 0$,

$$e^{2a} = \frac{q^2}{\lambda^2}\, \frac{\left|\,\sinh[q(\tau-\tau_2)]\,\right|^{2\beta^2/\gamma}}{\left|\,\beta^{-1}\sinh[\beta q(\tau-\tau_1)]\,\right|^{2/\gamma}}, \quad e^{2\varphi} = \left|\frac{\sinh[\beta q(\tau-\tau_1)]}{\beta\sinh[q(\tau-\tau_2)]}\right|^{2/\gamma}$$

- If $Q_i = 0$,

$$e^{2a} = \frac{1}{\lambda^2}\, \frac{\left|\tau-\tau_2\right|^{2\beta^2/\gamma}}{\left|\tau-\tau_1\right|^{2/\gamma}}, \quad e^{2\varphi} = \left|\frac{\tau-\tau_1}{\tau-\tau_2}\right|^{2/\gamma}$$

- If $Q_i < 0$,

$$e^{2a} = \frac{q^2}{\lambda^2}\, \frac{\left|\,\sin[q(\tau-\tau_2)]\,\right|^{2\beta^2/\gamma}}{\left|\,\beta^{-1}\sin[\beta q(\tau-\tau_1)]\,\right|^{2/\gamma}}, \quad e^{2\varphi} = \left|\frac{\sin[\beta q(\tau-\tau_1)]}{\beta\sin[q(\tau-\tau_2)]}\right|^{2/\gamma}$$

• The actual value of q is not important, it only sets the scale of the unphysical time variable τ.

• To interpret the solutions, it is useful to define the cosmic time t such that $dt = \pm e^a d\tau$ (The possibility of choosing the plus or minus sign derives from the invariance of the equations under time reversal).

• In the new parametrization, the line element reads

$$ds^2 = -dt^2 + e^{2b(t)} d\Omega^2$$

• Unfortunately, in general the solutions obtained cannot be written in terms of elementary functions of t.

• This is only possible when $Q_i = 0$, $\tau_1 = \tau_2$. Then

$$t - t_0 = \pm \log |\tau - \tau_1|$$

with t_0 an arbitrary integration constant.

• Choosing the minus sign in the previous expression, one obtains an expanding universe, with $e^{2b} = e^{2(t-t_0)/3\lambda}$ and $\phi = 0$, namely a de Sitter spacetime with vanishing scalar field.

• This is of course the unstable solution corresponding to the scalar sitting at the top of the potential.

• In the general case, the solutions have two qualitatively different branches if $\tau_1 = \tau_2$, or three if $\tau_1 \neq \tau_2$.

• We are only interested in those branches where t is a monotonic function of τ and the universe expands.

• Studying their behavior for $\tau \to \tau_{1,2}$ and $\tau \to \pm\infty$, we obtain other physically acceptable solutions:

- $Q_i > 0$, $\tau_1 = \tau_2$

$e^{2b} \sim e^{2\lambda t/3}$ for $t \to -\infty$; $\quad e^{2b} \sim t^{2/3}$ for $t \to \infty$

This solution describes a universe with de Sitter inflationary behavior for $t \to -\infty$ that gradually turns to a power-law expansion for $t \to \infty$.

- $Q_i < 0$, $\tau_1 = \tau_2$: two branches

$e^{2b} \sim e^{2\lambda t/3}$ for $t \to -\infty$; $\quad e^{2b} \sim (t - t_0)^{2\beta^2/3} \to 0$ for $t \to t_0$.

$e^{2b} \sim (t - t_0)^{2\beta^2/3}$, both at $t = t_0$ and at a later finite time.

Both branches describe a universe that initially expands and then recollapses.

If $\tau_1 \neq \tau_2$, the solutions are more complicated and present two acceptable branches:

- $Q_i = 0$

$e^{2b} \sim e^{2\lambda t/3}$ for $t \to -\infty$; $\quad e^{2b} \sim t^{2/3\beta^2}$ for $t \to \infty$.

$e^{2b} \sim (t - t_0)^{2\beta^2/3}$ for $t \to t_0$; $\quad e^{2b} \sim t^{2/3\beta^2}$ $t \to \infty$.

In the first case the behavior is qualitatively similar to the case in which $q \neq 0$ and $\tau_1 = \tau_2$.

- $Q_i > 0$

$e^{2b} \sim (t - t_0)^{2\beta^2/3}$ for $t \to t_0$; $\quad e^{2b} \sim t^{2/3}$ for $t \to \infty$.

$e^{2b} \sim (t - t_0)^{2\beta^2/3}$ for $t \to t_0$; $\quad e^{2b} \sim t^{2/3\beta^2}$ for $t \to \infty$.

- $Q_i < 0$

$e^{2b} \sim (t - t_0)^{2\beta^2/3}$ for $t \to t_0$; $\quad e^{2b} \sim t^{2/3\beta^2}$ for $t \to \infty$.

$e^{2b} = (t - t_0)^{2\beta^2/3}$ both at $t = t_0$ and at a later finite time.

• The most interesting solutions are those that behave exponentially for $t \to -\infty$ and as a power law for $t \to \infty$. These are obtained for $Q_i > 0$, $\tau_1 = \tau_2$, or $Q_i = 0$, $\tau_1 \neq \tau_2$.

• They correspond to an initial configuration where the scalar field is at the top of the potential and then rolls down either to $-\infty$ or to $+\infty$. Their behavior is similar, but they differ for the exponent of the late power-law expansion which is $2/3$ in the first case, and $2/3\beta^2$ in the second.

• The exponential expansion lasts until $\tau = \tau_f \sim 1/q\beta$, namely $t - t_0 \sim 1/\lambda$. After this time the acceleration of the expansion becomes negative. At such time the scale factor e^{2b} is of order $\lambda^{-2/3}$.

• Denoting t_i the time at which the inflation starts, the scale factor therefore inflates by a factor $e^{-2\lambda(t_i - t_0)/3}$. Choosing $t_i - t_0$ negative, one can then obtain the desired amount of inflation.

QUESTIONS:

- How generic is the inflationary behavior?
- Does ordinary matter modify the behavior of the solutions?

Introduction of ordinary matter spoils the integrability.
One can however study the dynamical system.

- The Einstein equations become

$$\frac{\dot{a}^2}{3} - \dot{\phi}^2 = \frac{1}{2} V e^{2a} + \frac{1}{2}\rho e^{2a},$$
$$\ddot{a} = \frac{3}{2} V e^{2a} + \frac{3}{4}(\rho - p) e^{2a}$$

while the equation for the scalar is unchanged.
The usual continuity equation holds,

$$\dot{\rho} + (p + \rho)\dot{a} = 0$$

Hence, for $p = \omega\rho$, $\rho = \rho_0 e^{-(1+\omega)a}$.

For dust ($\omega = 0$) one has

$$\ddot{\psi} = \lambda^2 e^{2\psi} + \frac{3}{4}\rho_0 e^{(\psi - \beta^2\chi)/\gamma}, \qquad \ddot{\chi} = \lambda^2 e^{2\chi} + \frac{3}{4}\rho_0 e^{(\psi - \beta^2\chi)/\gamma},$$

$$\dot{\psi}^2 - \beta^2\dot{\chi}^2 = \lambda^2 e^{2\psi} - \beta^2\lambda e^{2\chi} + \frac{3\gamma}{2}\rho_0 e^{(\psi - \beta^2\chi)/\gamma}$$

We put the equations in the form of a dynamical system, defining

$$X = \dot{\eta}, \quad Y = \dot{\chi}, \quad Z = \lambda\, e^{\chi}, \quad W = \sqrt{\lambda}\, e^{\eta}$$

with $\eta = (\psi - \beta^2\chi)/2\gamma$. Then

$$\dot{X} = \frac{\alpha}{2}W^2 - \frac{\beta^2}{2\gamma}Z^2 + \frac{1}{2\gamma}Z^{2\beta^2}W^{4\gamma},$$
$$\dot{Y} = Z^2 + \alpha W^2, \qquad \dot{Z} = YZ$$

with $\alpha = \frac{3\rho_0}{4\lambda}$ and W implicitly defined as

$$2\alpha W^2 + \frac{1}{\gamma}Z^{2\beta^2}W^{4\gamma} = \frac{\beta^2}{\gamma}Z^2 + 4\gamma X^2 + 4\beta^2 XY - \beta^2 Y^2$$

• The global properties of the solutions can be deduced from the study of their behavior near the critical points. In particular, the critical points at finite distance correspond to the limit $\tau \to \pm\infty$ and lie on the two straight lines

$$X = \frac{\pm\beta\, Y}{2(1 \pm \beta)}, \qquad Z = 0$$

• Near the critical points, for $\tau \to \pm\infty$,

$$a \sim \log t \sim \frac{\pm\beta\, Y_0}{1 \pm \beta}\, \tau, \qquad \frac{\sqrt{3}\,\phi}{\beta} \sim \frac{Y_0}{1 \pm \beta}\, \tau$$

where t is the cosmic time and goes to infinity in this limit. Hence, $e^{2b} \sim t^{2/3}$, $e^{2\varphi} \sim t^{\pm 2/\beta}$ as $t \to \infty$.

The behavior of the solutions for small t can instead be investigated considering the critical points at infinity. These can be studied by defining new variables

$$u = \frac{1}{X}, \qquad y = \frac{Y}{X}, \qquad z = \frac{Z}{X}, \qquad w = \frac{W}{X}$$

and considering the limit $u \to 0$. This is attained for $\tau \to \tau_0$, where τ_0 is a finite constant. The field equations become

$$u' = -\left(\frac{\alpha}{2}w^2 - \frac{\beta^2}{2\gamma}z^2 + \frac{1}{2\gamma}z^{2\beta^2}w^{4\gamma}u^{-2\gamma}\right)u$$

$$y' = z^2 + \alpha w^2 - \left(\frac{\alpha}{2}w^2 - \frac{\beta^2}{2\gamma}z^2 + \frac{1}{2\gamma}z^{2\beta^2}w^{4\gamma}u^{-2\gamma}\right)y$$

$$z' = -\left(\frac{\alpha}{2}w^2 - \frac{\beta^2}{2\gamma}z^2 + \frac{1}{2\gamma}z^{2\beta^2}w^{4\gamma}u^{-2\gamma} - y\right)z$$

where a prime denotes $u\, d/d\tau$, with the constraint

$$4\gamma + 4\beta^2 y - \beta^2 y^2 + \frac{\beta^2}{\gamma}z^2 - 2\alpha w^2 = \frac{1}{\gamma}z^{2\beta^2}w^{4\gamma}u^{-2\gamma}$$

• One obtains a set of critical points. From their values, one can deduce the behavior of the metric functions for $\tau \to \tau_0$.
In particular,

a) $t \sim |\tau - \tau_0|^3 \to 0; \quad e^{2b} \sim t^{4/3}, \quad e^{2\varphi} \sim \text{const}.$

b) $t \sim |\tau - \tau_0|^{1/\gamma} \to 0; \quad e^{2b} \sim t^{2\beta^2/3}, \quad e^{2\varphi} \sim t^{-2/\gamma}.$

c) $t \sim \log|\tau - \tau_0| \to -\infty; \quad e^{2b} \sim e^{\pm 2t/3}, \quad e^{2\varphi} \sim \text{const}.$

d) $t \sim |\tau - \tau_0|^{-\beta^2/\gamma} \to \infty; \quad e^{2b} \sim t^{2/3\beta^2}, \quad e^{2\varphi} \sim t^{2/\gamma}.$

• Some of these points correspond to $t \to 0$, other to $t \to \pm\infty$.

• However, the possible asymptotic behaviors of the solutions essentially coincide with those of the exact solutions.

• The study of the linearized equations shows that points a) and c) are saddle points for the trajectories at infinity, while points b) and d) are nodes.

• In particular, this implies that the solutions with exponential expansion, associated to point c), require specific initial conditions.

CONCLUSIONS

• We have described an exactly solvable model of gravity minimally coupled to a scalar field, with some solutions presenting an initial inflationary period followed by a late power-law expansion.

• Although these solutions are not generic, the model may be useful to discuss some topics as for example the generation of inhomogeneities.